Complex Analysis II, Final Reports

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Sum of 8 Squares   by 黃庭瀚

Introduction.

Fermat. every odd prime $p \equiv 1 \pmod 4$ can be written as
$p = x^2+y^2$, $x, y \in \mathbb{N}$ ($\mathbb{Z}$) , which was proved by Euler in 1747

Lagrange (in 1770) any $n \in \mathbb{N}$ can be written as $n = w^2+x^2+y^2+z^2$
where $w, x, y, z \in \mathbb{Z}$

These facts are not enough, we are now interested in the number of ways to factor $n \in \mathbb{N}$ into sum of k squares.

let $r_k(n) = \#$ of ways $n = x_1^2 + \cdots x_k^2$, $x_i \in \mathbb{Z}$

example. $5 = (\pm 2)^2 + (\pm 1)^2 = (\pm 1)^2 + (\pm 2)^2 \Rightarrow r_2(5) = 8$

Main Tools

Generating Function

for a sequence $\{F_n\}_0^\infty$, study the function F s.t. $F(z) = \sum_{n=0}^{\infty} F_n z^n$

example. $p(n)$, partition function, # of ways to partition n

$1 = 1$, $p(1) = 1$, $2 = 0+2 = 1+1$, $p(2) = 2$, $3 = 0+3 = 1+2 = 1+1+1$, $p(3) = 3$

under the convention that $p(0) = 1$, we have $\sum_0^\infty p(n) x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$

recall. $\vartheta_3(z, \tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z} = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2\pi i n z}$, $q = e^{\pi i \tau}$, $\tau \in \mathbb{H}$.

$\theta(\tau) = \theta(0, \tau) = \sum_{n=-\infty}^{\infty} q^{n^2}$, which satisfies

i. $\theta(\tau + 2) = \theta(\tau)$

ii. $\theta(\tau) = (i/\tau)^{1/2} \theta(-1/\tau)$

iii. $\theta(\tau) = \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n-1})(1+q^{2n-1})$ which implies
$\theta(\tau) \longrightarrow 1$ as $\operatorname{Im}\tau \longrightarrow \infty$

iv. $\theta(1 - 1/\tau) = (\tau/i)^{1/2} \sum_{n=-\infty}^{\infty} e^{\pi i (n+1/2)^2 \tau} = (\tau/i)^{1/2} (2e^{\pi i \tau/4} + \quad)$

i.e. $\theta(1-1/\tau) \sim (\tau/i)^{1/2} 2 e^{\pi i \tau/4}$ as $\operatorname{Im}\tau \longrightarrow \infty$

the behavior at $\theta(\tau)$ at "cusp point" $z=1$ is bounded

Sum of 2 Squares

Definition. $d_1(n) = \#$ of divisors (>0) of n of the form $4k+1$
$d_3(n) = \quad \sim \quad 4k+3$

Theorem. for $n \geq 1$, $r_2(n) = 4(d_1(n) - d_3(n))$

Observation. $\theta^2(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} \sum_{m=-\infty}^{\infty} q^{m^2} = \sum_{n,m=-\infty}^{\infty} q^{n^2+m^2} = \sum_{k=0}^{\infty} r_2(k) q^k$

Claim. to prove the Theorem is equivalent to prove that

$$\theta^2(\tau) = 2\sum_{n=-\infty}^{\infty} \frac{1}{q^n + q^{-n}} = 1 + 4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \qquad (A)$$

first, by writing $(1+q^{2n})^{-1} = (1-q^{2n})/(1-q^{4n})$
and $(1-q^{4n})^{-1} = \sum_{m=0}^{\infty} q^{4nm}$

(A) can be written as

$$1+4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} = 1+4\sum_{n=1}^{\infty} \frac{q^n - q^{3n}}{1-q^{4n}}$$

$$= 1+4\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} q^{n+4nm} - q^{3n+4nm} = 1+4\sum_{n=1}\sum_{m=0} q^{(4m+1)n} - q^{(4m+3)n}$$

$$= 1 + 4\sum_{k=1}^{\infty} (d_1(k) - d_3(k)) q^k$$ which is the desired result.

now, let $C(\tau) = 2\sum_{n=-\infty}^{\infty} \frac{1}{q^n + q^{-n}} = \sum_{n=-\infty}^{\infty} \frac{1}{\cos(n\pi\tau)}$

we find that

i. $C(\tau+2) = C(\tau)$ (note. $\theta^2(\tau)$ also satisfies these property)

ii. $C(\tau) = (i/\tau)\, C(-1/\tau)$

iii $C(\tau) \to 1$ as $\operatorname{Im}\tau \to \infty$

iv. $C(1-1/\tau) \sim 4(\tau/i)\, e^{\pi i\tau/2}$ as $\operatorname{Im}\tau \to \infty$

proof. i. is trivial.

ii. by Stein Chap 4. have $\sum_{n=-\infty} \frac{1}{\cosh(\pi n t)} = \frac{1}{t}\sum_{n=-\infty} \frac{1}{\cosh(\pi n/t)}$
let $\tau = it$, get ii.

iii. is also clearly by definition.

iv. by analytic continuation, $C(1-1/\tau) = (\tau/i)\sum_{n=-\infty} \frac{1}{\cos(\pi(n+\frac{1}{2})\tau)}$

the "main" part is $n = -1, 0$

$$\Rightarrow C(1-1/\tau) = 4(\tau/i)\, e^{\pi i\tau/2} + O(|\tau| e^{-3\pi t/2}), \quad t = \operatorname{Im}\tau$$

i.e. $C(1-1/\tau) \sim 4(\tau/i)\, e^{\pi i\tau/2}$ as $\operatorname{Im}\tau \to \infty$

then by our knowledge of modular forms,
we have $C(\tau)/\theta^2(\tau) = 1$ on $\mathbb{H}$ #

Sum of 4 Squares

Definition. $\sigma_1^*(n) = \sum$ divisors of n not divisible by 4

Theorem. $r_4(n) = 8\,\sigma_1^*(n)$

recall. Forbidden Eisenstein Series $E_2(\tau) = \sum_{m,n} \frac{1}{(m\tau+n)^2}$

Definition. $E_2^*(\tau) = \sum_m \sum_n \frac{1}{(\frac{m\tau}{2}+n)^2} - \sum_m \sum_n \frac{1}{(m\tau+\frac{n}{2})^2}$ $\qquad \zeta(2) = \pi^2/6$

recall in Stein. Chap 9. $F(\tau) = \sum_m \sum_n \frac{1}{(m\tau+n)^2} = 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n\tau}$ (B)

thus $E_2^*(\tau) = F(\tau/2) - 4F(2\tau)$

Claim. to prove the theorem is equivalent to prove that

$$\theta^4(\tau) = -\pi^{-2} E_2^*(\tau)$$

observe that $\sigma_1^*(n) = \begin{cases} \sigma_1(n) & \text{if } 4 \nmid n \\ \sigma_1(n) - 4\sigma(n/4) & \text{if } 4 \mid n \end{cases}$

then $E_2^*(\tau) = 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma_1(n) e^{\pi i n\tau} - 4\left(2\zeta(2) - 8\pi^2 \sum_{m=1}^{\infty} \sigma_1(m) e^{4\pi i m\tau}\right)$

$= -\pi^2 - 8\pi^2 \sum \sigma_1^*(n) e^{\pi i \tau}$ hence the claim is proved.

Prop.

i. $E_2^*(\tau+2) = E_2^*(\tau)$

ii. $E_2^*(\tau) = -\tau^2 E_2^*(-1/\tau)$

iii. $E_2^*(\tau) \longrightarrow -\pi^2$ as $\mathrm{Im}\,\tau \longrightarrow \infty$

iv. $E_2^*(1-1/\tau) = O(\tau^2 e^{\pi i \tau})$ as $\mathrm{Im}\,\tau \longrightarrow \infty$

proof let $\tilde{F}(\tau) = \sum_n \sum_m \frac{1}{(m\tau+n)^2}$, and Dedekind η function

$\eta(\tau) = q^{1/12} \prod_{n=1}^{\infty} (1-q^{2n})$ with the property $\eta(-1/\tau) = (\tau/i)^{1/2} \eta(\tau)$

we then find that $\frac{\eta'(\tau)}{\eta(\tau)} = \frac{\pi i}{12} - 2\pi i \sum_{n=1}^{\infty} \frac{n q^{2n}}{1-q^{2n}} = \frac{\pi i}{12} - 2\pi i \sum_{n=1}^{\infty} \sigma_1(n) q^{2n}$

$\Rightarrow (\eta'/\eta)(\tau) = (i/4\pi) F(\tau)$ $\qquad$ the lemma follows.

Lemma

$F(-1/\tau) = \tau^2 \tilde{F}(\tau)$

$F(\tau) - \tilde{F}(\tau) = 2\pi i/\tau$

$F(-1/\tau) = \tau^2 F(\tau) - 2\pi i \tau$

ii. $E_2^*(-1/\tau) = F(-1/(2\tau)) - 4F(-2/\tau)$

$= (4\tau^2 F(2\tau) - 4\pi i \tau) - 4((\tau^2/4) F(\tau/2) - \pi i \tau)$

$= -\tau^2 (F(\tau/2) - 4F(2\tau)) = -\tau^2 E_2^*(\tau)$.

iii. $E_2^*(\tau) = F(\tau/2) - 4F(2\tau)$, use (B)

iv. $E_2^*(1-1/\tau) = F(1/2 - 1/2\tau) - 4F(2 - 2/\tau)$

note that $F(\frac{\tau-1}{2\tau}) = (\frac{2\tau}{\tau-1})^2 F(\frac{2\tau}{1-\tau}) - 2\pi i \frac{2\tau}{1-\tau}$

and $F(\frac{2\tau}{1-\tau}) = F(-2 + 2/1-\tau)$,

$= F(2/1-\tau) = (\frac{1-\tau}{2})^2 F(\frac{\tau-1}{2}) - 2\pi i (\frac{\tau-1}{2})$

$\Rightarrow F(1/2 - 1/2\tau) = \tau^2 F(\frac{\tau-1}{2}) - \frac{2\pi i \, 2\tau}{1-\tau} - 2\pi i (\frac{2\tau}{\tau-1})^2 (\frac{\tau-1}{2})$

and $F(2 - 2/\tau) = F(-2/\tau) = (\frac{\tau^2}{4}) F(\tau/2) - \pi i \tau$

Hence $E_2^*(1-1/\tau) = \tau^2 [F(\tau-1/2) - F(\tau/2)]$

again by (B), we see that $E_2^*(1-1/\tau) \sim -16\pi^2 \tau^2 e^{\pi i \tau} + \cdots$

then $(E_2^*(\tau)/\theta^4(\tau)) = C = -\pi^2$ ☆

Sum of 8 squares

It is much simplier than 4 sum. Since $E_4(\tau)$ converges absolutely, while $E_2(\tau)$ does not.

Definition $\sigma_3^*(n) = \begin{cases} \sigma_3(n) = \sum_{d|n} d^3 & \text{if } n \text{ odd} \\ \sum_{d|n} (-1)^d d^3 = \sigma_3^e(n) - \sigma_3^o(n) & \text{if } n \text{ even.} \end{cases}$

Theorem $r_8(n) = 16\, \sigma_3^*(n)$

Claim it is equivalent to show that $\theta^8(\tau) = 48\pi^{-4} E_4^*(\tau)$

where $E_4^*(\tau) = \sum_{m,n}^* (m\tau+n)^{-4}$, which sums over m,n with opposite parity (i.e. $(m,n) = (\text{odd}, \text{even})$ or $(\text{even}, \text{odd})$)

we now try to construct $E_4^*(\tau)$ by $E_4(\tau)$

a clever choice gives

$E_4^*(\tau) + 2^{-4} E_4(\frac{\tau-1}{2}) = \sum_{m,n}^* (m\tau+n)^{-4} + \sum_{m,n} [(2n-m) + m\tau]^{-4}$

the second Σ is in fact the summation over m, n with Same parity $\Rightarrow E_4^*(\tau) + 2^{-4} E_4(\frac{\tau-1}{2}) = E_4(\tau)$

And recall that $E_4(\tau) = 2\zeta(4) + (2\pi)^4/3 \sum_{k=1}^{\infty} \sigma_3(k) e^{2\pi i k\tau}$ (C)

$\zeta(4) = \pi^4/90$ then $E_4^*(\tau) = 2\zeta(4) + (2\pi)^4/3 \sum \sigma_3(k) e^{2\pi i k\tau} - 2^{-4} \left(2\zeta(4) + (2\pi)^4/3 \sum \sigma_3(k) e^{\pi i k(\tau-1)}\right)$

$= 48^{-1}\pi^4 + \pi^4/3 \left(16 \sum_{k=1}^{\infty} \sigma_3(k) q^k - \sum_{k=1}^{\infty} (-1)^k \sigma_3(k) q^k\right)$

consider coefficient of q^n in $(\sim)$

for n odd, it is just $\sigma_3(n)$

n even, is $16\, \sigma_3(n/2) - \sigma_3(n)$

observe that $8\sigma_3(n/2) = \sigma_3^e(n)$, hence the coefficient is

$$2\sigma_3^e(n) - \left(\sigma_3^e(n) + \sigma_3^o(n)\right)_\infty = \sigma_3^e(n) - \sigma_3^o(n) = \sigma_3^*(n)$$

$$\Rightarrow E_4^*(\tau) = 48^{-1}\pi^4\left(1 + \sum_{n=1}^{\infty} 16\,\sigma_3^*(n) q^n\right) \quad *$$

Prop. i. $E_4^*(\tau+2) = E_4^*(\tau)$

ii. $E_4^*(\tau) = \tau^{-4} E_4^*(-1/\tau)$

iii. $(48/\pi^4) E_4^*(\tau) \longrightarrow 1$ as $\operatorname{Im}\tau \to \infty$

iv. $E_4^*(1 - 1/\tau) = O(\tau^4 e^{2\pi i \tau})$ as $\operatorname{Im}\tau \longrightarrow \infty$

proof. all are similar as before, with the aid of (C)

Hence we prove that $\theta^8(\tau) = 48\pi^{-4} E_4^*(\tau)$ $\quad *$

remark. for a function $F(\tau)$ on $\mathbb{H}$ with

$F(\tau+2) = F(\tau)$, $F(-1/\tau) = F(\tau)$, $F(\tau)$ bounded

then F is constant.

method in Stein. consider the behavior at cusp points 1 & ∞

for ∞, let $g(z) = F(\tau)$, $z = e^{\pi i \tau}$, since F has period 2

$g(z)$ is well-defined. Boundedness then implies $g(0) = F(\infty)$ is a removable singularity. Hence $g(0) = \lim_{\operatorname{Im}\tau\to\infty} F(\tau) < \sup_{\mathbb{H}} F(\tau)$. by max. principle.

for 1. first consider the map $\mu(\tau) = 1/1-\tau$, $\mu^{-1}(\tau) = 1 - 1/\tau$

let $Q(\tau) = F(1 - 1/\tau) = F(\mu^{-1}(\tau))$, study the behavior when $\tau \to \infty$

observe that $\mu^{-1} T_n \mu = (-ST_{-2})^n$ where $n \in \mathbb{Z}$

$T_n(\tau) = \tau + n$ i.e. $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $S(\tau) = -1/\tau$ i.e. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow F$ is invariant under $\mu^{-1} T_n \mu$.

$$Q(\tau) = F(\mu^{-1}(\tau)) = F(\mu^{-1} T_n \mu^{+1} \mu^{-1}(\tau)) = F(\mu^{-1} T_n(\tau)) = Q(T_n(\tau))$$

i.e. $Q(\tau)$ has period 1, Again let $h(z) = Q(\tau)$, $z = e^{2\pi i \tau}$

again $h(0) = Q(\infty)$ is a removable singularity

$$\Rightarrow Q(\infty) = \lim_{\operatorname{Im}\tau\to\infty} F(1 - 1/\tau) < \sup_{\mathbb{H}} F(\tau)$$

Hence we find that $F(\tau)$ reaches its max. inside the domain

$\Rightarrow F(\tau)$ is constant by max. principle.

The Concept of a Fundamental Group and a Covering Space 古習丞 1

§ Homotopic Equivalence, Path, Path-Class, and the Fundamental Group.

<u>Def1.</u> Let $\varphi_0, \varphi_1 \in C(X,Y)$ We say φ_0, φ_1 are homotopic if $\exists\, \varphi \in C(X \times I, Y)$ such that $\begin{cases} \varphi(\cdot, 0) = \varphi_0(\cdot) \\ \varphi(\cdot, 1) = \varphi_1(\cdot) \end{cases}$ We shall denote it by $\varphi_0 \simeq_\varphi \varphi_1$

Obviously, "$\simeq$" is an equivalence relation

We say that φ_0 and φ_1 are homotopic relative to the subset $A \subseteq X$ if $\varphi_0 \simeq_\varphi \varphi_1$ and $\varphi(a,t) = \varphi_0(a) = \varphi_1(a)$ $\forall a \in A$ $\forall t \in I$

<u>Def2.</u> A path γ in X is a continuous map $\gamma: I (= [0,1]) \to X$ We call $\gamma(0)$ be its initial point, and $\gamma(1)$ its terminal point

A loop or a closed path is a path γ with $\gamma(0) = \gamma(1)$ Two paths γ, δ are equivalent, denoted by $\gamma \sim \delta$, if $\gamma \simeq \delta$ relative to $\{0,1\}$

For a path γ in X, we let $\gamma^{-1}: \begin{matrix} I \to X \\ t \mapsto \gamma(1-t) \end{matrix}$ (also a path in X) be its inverse path. For $x_0 \in X$, let $\varepsilon_{x_0}: \begin{matrix} I \to X \\ t \mapsto x_0 \end{matrix}$ (also a path in X) be the constant path at x_0 For two paths $\gamma_0\ \gamma_1$ with $\gamma_0(1) = \gamma_1(0)$, in X, their product is defined to be $\gamma_0\, \gamma_1(t) = \begin{cases} \gamma_0(2t) & 0 \le t \le \frac{1}{2} \\ \gamma_1(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$ (again a path in X)

<u>Def3.</u> For any path γ in X, let $[\gamma] = \{\gamma' \mid \gamma' \sim \gamma\}$ be the path class of γ.

<u>Facts:</u> 1) $\gamma_0 \sim \gamma_1$, $\delta_0 \sim \delta_1 \Rightarrow \gamma_0\, \delta_0 \sim \gamma_1\, \delta_1$. (c.f Massey. lemma 3.1)

Hence $[\gamma]\,[\delta] := [\gamma\, \delta]$ is well-defined

2) In general, $(\gamma_0\, \gamma_1) \cdot \gamma_2 \neq \gamma_0\, (\gamma_1\, \gamma_2)$. (c.f Massey. Lemma 3.2

However, $(\gamma_0\, \gamma_1)\, \gamma_2 \sim \gamma_0\, (\gamma_1\, \gamma_2)$, i.e., $([\gamma_0] \cdot [\gamma_1]) \cdot [\gamma_2] = [\gamma_0]([\gamma_1] \cdot [\gamma$

3) Let x, y be the initial and terminal point, respectively, of γ Then $\begin{cases} \varepsilon_x\, \gamma \sim \gamma \\ \gamma\, \varepsilon_y \sim \gamma \\ \gamma\, \gamma^{-1} \sim \varepsilon_x \\ \gamma^{-1} \cdot \gamma \sim \varepsilon_y \end{cases}$, i.e $\begin{matrix} [\varepsilon_x]\,[\gamma] = [\gamma] \\ [\gamma]\,[\varepsilon_y] = [\gamma] \\ [\gamma]\,[\gamma^{-1}] = [\varepsilon_x] \\ [\gamma^{-1}] \cdot [\gamma] = [\varepsilon_y]. \end{matrix}$

(c.f Lemma 3.3, 3.4)

4) Let $\gamma_0 \sim \gamma_1$ be paths in X, and $\varphi \in C(X,Y)$. Then $\varphi \circ \gamma_0$, $\varphi \circ \gamma_1$ are paths in Y with $\varphi \circ \gamma_0 \sim \varphi \circ \gamma_1$ Hence it makes sense to define $\varphi_*([\gamma]) := [\varphi \circ \gamma]$, where the map φ_* has the following properties

(a) $\varphi_*([\gamma][\delta]) = \varphi_*[\gamma] \quad \varphi_*[\delta]$

(b) $\varphi_*[\varepsilon_x] = [\varepsilon_{\varphi(x)}]$ (c) $\varphi_*[\gamma^{-1}] = (\varphi_*[\gamma])^{-1}$

(d) For $\psi \in C(Y, Z)$, $(\varphi\psi)_* = \varphi_*\psi_*$

(e) $id_*[\gamma] = [\gamma]$

(c.f page 63)

We call φ_* the homomorphism induced by φ

Def 4 Let $\pi(X,x) := \{[\alpha] \mid \alpha : \text{a loop in } X \text{ with } \alpha(0) = x\}$

By facts, it is readily seen that $\pi(X,x)$ is a group. Indeed,

① $([\gamma_0][\gamma_1])[\gamma_2] = [\gamma_0]\cdot([\gamma_1]\cdot[\gamma_2])$ (fact 2)

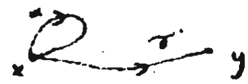
② $[\varepsilon_x]$ serves as the identity (fact 3)

③ $[\gamma][\gamma^{-1}] = [\gamma\gamma^{-1}] = [\varepsilon_x] \rightsquigarrow [\gamma]^{-1} = [\gamma^{-1}]$ (fact 3)

This group is called the fundamental group of X at the based point x

Prop 1 If X is arcwise connected, then $\pi(X,x) \cong \pi(X,y)$ $\forall\, x, y \in X$

Pf. $\Phi :$
$$\begin{aligned} \pi(X,x) &\longrightarrow \pi(X,y) \\ [\alpha] &\longmapsto [\gamma^{-1}\cdot\alpha\,\gamma] \end{aligned}$$



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(c.f Theorem 3.5.)

RMK: 1) Whether or not $\pi(X,x)$ is abelian, finite, nilpotent, free is independent to the point x, and thus depends only on the space X itself, provided X is arcwise-connected.

2) The homomorphism is not canonical.

§ The effect of a continuous map on the fundamental group

A continuous map induces a group homomorphism $\varphi_* : \pi(X,x) \to \pi(Y,\varphi(x))$, and if $\varphi : X \to Y$ is a homeomorphism, then φ_* is an isomorphism

Prop 2 Let $\varphi_0 \simeq \varphi_1 : X \to Y$ relative to $\{x\} \subseteq X$. Then

$\varphi_{0*} = \varphi_{1*} : \pi(X,x) \to \pi(Y,\varphi_0(x))$, i.e., the induced homomorphisms are the same

Pf Assume $\varphi_0 \simeq_F \varphi_1$. Let G
$$\begin{aligned} I\times I &\longrightarrow Y \\ (x,y) &\longmapsto F(\alpha(x),y) \end{aligned}$$

$\Rightarrow \varphi_0\circ\alpha \sim_G \varphi_1\circ\alpha \Rightarrow \varphi_{0*}[\alpha] = [\varphi_0\circ\alpha] = [\varphi_1\circ\alpha] = \varphi_{1*}[\alpha]$

(c.f Massey, Theorem 4.1) *

Def 5. A subset $A \subseteq X$ is called a retract of X if there exists a continuous map $r: X \longrightarrow A$ (a retraction) such that $r(a) = a \ \forall a \in A$

Let $i: A \hookrightarrow X$ be the inclusion map. Then $r \circ i = id_A$, so $r_* i_* = id_{\pi(A,a)}$ (by fact 4). Hence, i_* is 1-1, and r_* is onto

Def 6. A subset $A \subseteq X$ is called a deformation retract if there exists a retraction $r: X \to X$ of A such that $r \simeq id_X$

Prop 3. If A is a deformation retract of X, then the inclusion map $i: A \hookrightarrow X$ induces a group isomorphism $i_*: \pi(A,a) \xrightarrow{\simeq} \pi(X,a)$.

pf $i_* r_* = (i \circ r)_* = (id_X)_* = id_{\pi(X,a)}$ ☆

RMK: 1) $X \simeq_\varphi Y$ (homeo) $\Rightarrow \pi(X,x) \xrightarrow{\simeq} \pi(Y, \varphi(x))$

(Actually, we have a stronger version: $X \overset{f}{\simeq} Y \Rightarrow \pi(X) \simeq \pi(Y)$, c.f. Theorem 8.3)

2) $A \subseteq X$ is a deformation retract $\Rightarrow \pi(A,a) \simeq \pi(X,a)$

e.g. 1) Let $C \subseteq \mathbb{R}^3$ be a cylinder, $M \subseteq \mathbb{R}^3$ be a Möbius band. Then $\pi(S^1) \cong \pi(C) \cong \pi(M)$

2) Any star-shaped subset $\Delta \subseteq \mathbb{R}^n$ is "contractible to a point" (simply-connected), and its fundamental group is trivial.

§ Covering Space and its lifting property.

Axiom: We assume our topological spaces be arcwise-connected and locally arcwise-connected, except for those specially mentioned.

Def 7. Let $X \in Ob(top)$. A covering space $(\tilde{X}, p)$ of X consists of $\tilde{X} \in Ob(top)$ and a continuous map $p: \tilde{X} \to X$ satisfying:

Each point $x \in X$ has an arcwise-connected open neighborhood U such that each arc-component of $p^{-1}(U)$ is homeomorphic (by the restriction of p on the component) to U. In this case, we call such U an elementary neighborhood of the point x, and p a projection.

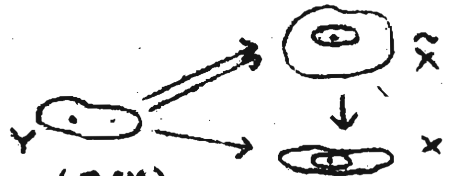
Obviously, if g is a path in $\tilde{X}$, then pg is a path in X; and if $g_0 \sim g_1$ are paths in $\tilde{X}$, it follows that $pg_0 \sim pg_1$. Conversely,

<u>Question</u> 1) If f is a path in X, does there exists a path g in $\tilde{X}$ such that $pg = f$? 2) If g_0, g_1 are paths in $\tilde{X}$ s.t $pg_0 \sim pg_1$, does it follow that $g_0 \sim g_1$?

We shal see that both questions turn out to be affirmative.

<u>Lemma 1</u>: Let $(\tilde{X}, p)$ be a covering space of X, and Y a connected space. Given $f_0, f_1 \in C(Y, \tilde{X})$ s.t $pf_0 = pf_1$, then the set $\{y \in Y \mid f_0(y) = f_1(y)\}$ is either empty or Y

pf Y is connected, so it suffices to prove it clopen. (c.f lemma 3.2).



(open)

(closed)

<u>Theorem 1</u>: Let $(\tilde{X}, p)$ be a covering space of X, $\tilde{x}_0 \in \tilde{X}$, and $x_0 = p(\tilde{x}_0)$. Then for any path $f: I \to X$ with initial point x_0, there exists a unique path $g: I \to \tilde{X}$ with initial at $\tilde{x}_0$ s.t $pg = f$. (cf lemma).

Pf. <u>Existence</u>: If the image $f(I)$ of the path is contained in some elementary neighborhood, then the existence of the lifting is trivial. Otherwise, we may apply Lebegue's number lemma to remedy.

<u>Uniqueness.</u> It follows immediately by Lemma 1 ∗

Corollary 1: Let $(\tilde{X}, p)$ be a covering space of X, and $x, y \in X$. Then $\mathrm{card}(p^{-1}(x)) = \mathrm{card}(p^{-1}(y))$. (c.f Massey lemma 3.4).

Theorem 2: Let $g_0, g_1: I \to \tilde{X}$ be paths in $\tilde{X}$ with the same initial point at $\tilde{x}_0$, If $pg_0 \sim pg_1$, then $g_0 \sim g_1$ In particular, g_0 and g_1 have the same terminal point (c.f lemma 3.3)

Pf. The assumption "$pg_0 \sim pg_1$" implies that there exists a conti

$$F: I \times I \to X \text{ s.t } \begin{cases} F(\cdot, 0) = pg_0(\cdot), & F(0,t) = pg_0(0) = pg_1(0) \\ F(\cdot, 1) = pg_1(\cdot) & F(1, t) = pg_0(1) = pg_1(1) \end{cases}$$

We shall prove that there exists a unique continuous mapping $G: I \times I \to \tilde{X}$ s.t $pG = F$ and $G(0,0) = \tilde{x}_0$. Again Lebesgue's number lemma applies dividing $I \times I$ into small rectangles (finite), each of which is mapped into ~~some elementary~~ neighborhood X by F. Hence we may lift F to G ~~starting from~~ $[0,s_1] \times [0,t_1]$ so that $G(0,0) = \tilde{x}_0$, and so on. Here, the uniqueness of G is guaranteed by Lemma 1. By theorem 1, $G(\cdot, 0)$ is the unique lifting of the path pg_0, so $G(\cdot,0) = g_0(\ )$. Similarly, $G(\cdot,1) = g_1(\cdot)$, $G(0,I) = \{\tilde{x}_0\}$ and $G(1,I) = \{\tilde{x}_1\}$ such that $p(\tilde{x}_1) = pg_0(1) = pg_1(1)$ *

Corollary 2. Let $\tilde{x}_0 \in \tilde{X}$ and $x_0 = p(\tilde{x}_0)$. Then p induces a monomorphism
$p_*: \pi(\tilde{X}, \tilde{x}_0) \hookrightarrow \pi(X, x_0)$ (c.f. Theorem 4.1)

NOTATION: $f: (X, x) \to (Y, y)$ means $f \in C(X,Y)$ and $f(x) = y$

Theorem 3. Let $(\tilde{X}, p)$ be a covering space of X, Y be arcwise connected and locally arcwise-connected and $\varphi: (Y, y_0) \longrightarrow (X, x_0)$. Then there exists a lifting $\tilde{\varphi}: (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ such that the following diagram commutes iff and only if $\varphi_*(\pi(Y, y_0)) \subseteq p_*(\pi(\tilde{X}, \tilde{x}_0))$

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\tilde{\varphi}"] \arrow[r, "\varphi"'] & (X, x_0) \end{tikzcd} \quad > \quad \begin{tikzcd} & \pi(\tilde{X}, \tilde{x}_0) \arrow[d, "p_*"] \\ \pi(Y, y_0) \arrow[ur, "\tilde{\varphi}_*"] \arrow[r, "\varphi_*"'] & \pi(X, x_0) \end{tikzcd}$$

Pf ($\Rightarrow$) Since the induced diagram also commutes, $\mathrm{im}\,\varphi_* \subseteq \mathrm{im}\,p_*$

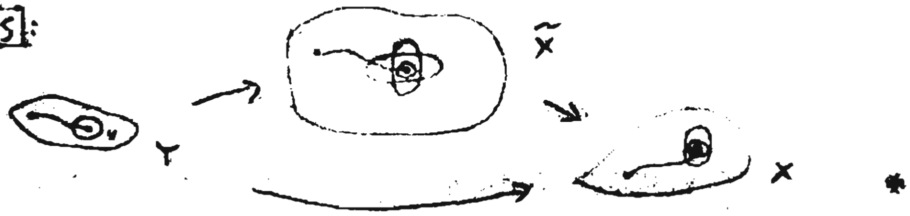
RMK: By coro 2, p_* is a monomorphism, so the existence of $\tilde{\varphi}_*$ s.t the induced diagram commutes is actually equivalent to the necessary condition. Hence the converse makes sense.

($\Leftarrow$) We shall construct such $\tilde{\varphi}$. For each $y \in Y$, we may choose a path f with $\begin{cases} f(0) = y_0 \\ f(1) = y \end{cases}$, and consider the lifting g (of φf) in $\tilde{X}$. Define $\tilde{\varphi}(y) = g(1)$, i.e. the terminal point of the lifting g

$\boxed{\text{Well-Defined}}$: The choice $\tilde{\varphi}(y)$ does not depend on the path, for if $f \sim f'$, then they induce the same point by theorem 2. Hence, it only depends on the path-class of f. Suppose α β are two ~~different~~ path-classes from y_0 to y. Then $\alpha\beta^{-1} \in \pi(Y, y_0)$ and $\varphi_*(\alpha\beta^{-1})$ may be lifted to a loop-class in $\tilde{X}$. Hence the liftings of $\varphi_*(\alpha)$

and the lifting of $\varphi_*(\beta)$ end at the same point

$\boxed{\text{Continuous}}$:



§ Homomorphisms of Covering Spaces

Def 8 Let $(\tilde{X}_1, p_1)$, $(\tilde{X}_2, p_2)$ be covering spaces of X. A homomorphism of $(\tilde{X}_1, p_1)$ into $(\tilde{X}_2, p_2)$ is a continuous map $\varphi: \tilde{X}_1 \to \tilde{X}_2$ such that $\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{\varphi} & \tilde{X}_2 \\ & \searrow_{p_1} & \downarrow p_2 \\ & & X \end{array}$. A homomorphism φ is an isomorphism if there exists a homomorphism ψ such that $\varphi\circ\psi$ and $\psi\circ\varphi$ are identities.

An automorphism is an isomorphism of a covering space into itself.

The set of all automorphisms of a covering space $(\tilde{X}, p)$, denoted by $A(\tilde{X}, p)$, is a group.

Corollary 3. Let $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ be covering spaces of X with $x_0 = p_1(\tilde{x}_1) = p_2(\tilde{x}_2)$. Then there exists a homomorphism $\varphi: \tilde{X}_1 \to \tilde{X}_2$ such that $\varphi(\tilde{x}_1) = \tilde{x}_2$, if and only if $p_{1*}\pi(\tilde{X}_1, x_1) \subseteq p_{2*}\pi(\tilde{X}_2, x_2)$

pf. A special case of theorem 3 (c.f lemma 6.3)

Theorem 4: Let $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ be covering spaces of X, and φ a homomorphism of $\tilde{X}_1$ into $\tilde{X}_2$. Then $(\tilde{X}_1, \varphi)$ is a covering space of $\tilde{X}_2$

Pf ① Let $x \in X$, and U_1 [U_2, resp.] be an elementary neighborhood of X for the covering $(\tilde{X}_1, p_1)$ [$(\tilde{X}_2, p_2)$, resp] Let U be the arc-component of $U_1 \cap U_2$ that contains x

② φ maps X_1 onto X_2: Let $x_1 \in \tilde{X}_1$ be a based point, $\left\{ \begin{array}{l} x_2 = \varphi(x_1) \\ x_0 = p_1(x_1) \end{array} \right.$
For each $y \in \tilde{X}_2$, consider a path β in $\tilde{X}_2$ from x_2 to y, and the (unique) lifting α of $p(\beta)$ into $\tilde{X}_1$ starting from x_1. Since $\varphi(\alpha)$ is obviously a lifting path of $p_2(\beta)$ into $\tilde{X}_2$ which starts from x_2, we have $\varphi(\alpha) = \beta$ ($\Rightarrow \varphi(\alpha(1)) = y$) by the uniqueness assertion *

Prop 4. Let φ_0 and φ_1 be homomorphisms of $(\tilde{X}_1, p_1)$ into $(\tilde{X}_2, p_2)$. If $\varphi_0(x) = \varphi_1(x)$ for some $x \in \tilde{X}_1$ then $\varphi_0 \equiv \varphi_1$

pf A direct consequence of Lemma 1 (c.f Massey lemma 6.1)

Coro 4.1: The group $A(\tilde{X}, p)$ operates ~~without fixed~~ points on $\tilde{X}$ (c.f lem 1.2)

Coro 4.2: $p_{1*}\pi(\tilde{X}_1, \tilde{x}_1) = p_{2*}\pi(\tilde{X}_2, \tilde{x}_2)$ with $p_1(\tilde{x}_1) = p_2(\tilde{x}_2)$ if and only if there exists an isomorphism φ between $\tilde{X}_1, \tilde{X}_2$ s.t $\varphi(\tilde{x}_1) = \tilde{x}_2$

pf A direct consequence of that automorphisms act without fixed points and corollary 3

Coro 4.3: Let $(\tilde{X}, p)$ be a covering space of X and $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x)$

There exists an automorphism $\varphi \in A(\tilde{X}, p)$ s.t $\varphi(\tilde{x}_1) = \tilde{x}_2$ if and only if $p_*\pi(\tilde{X}, \tilde{x}_1) = p_*\pi(\tilde{X}, \tilde{x}_2)$ (c.f. Massey coro. 6.5).

Prop 5. (c.f Theorem 4.2) Let $(\tilde{X}, p)$ be a covering space of X and $x_0 \in X$. Then the subgroups $p_*\pi(\tilde{X}, \tilde{x})$ for $\tilde{x} \in p^{-1}(x_0)$ are exactly a conjugacy class of subgroups of $\pi(X, x_0)$

pf Let $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$ and a path-class γ from $\tilde{x}_0$ to $\tilde{x}_1$.

Then
$$\begin{array}{ccc} \pi(\tilde{X}, \tilde{x}_0) & \xhookrightarrow{p_*} & \pi(X, x_0) \\ \downarrow \wr\, u & \circlearrowright & \downarrow v \\ \pi(\tilde{X}, \tilde{x}_1) & \xhookrightarrow{p_*} & \pi(X, x_0) \end{array},$$

where $u(\alpha) = \gamma^{-1}\alpha\gamma$. Here $v \circ p_*(\beta) = p_*(\gamma^{-1}\beta\gamma) = (p_*\gamma)^{-1} p_*\beta\, (p_*\gamma)$

The converse is obvious (c.f Theorem 4.1 Massey

Coro 5.1: $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ are isomorphic, if and only if for some (thus any, by prop 5) $\tilde{x}_1 \in \tilde{X}_1$, $\tilde{x}_2 \in \tilde{X}_2$ such that $p_1(\tilde{x}_1) = p_2(\tilde{x}_2) = x_0$, $p_{1*}\pi(\tilde{X}_1, \tilde{x}_1)$ and $p_{2*}\pi(\tilde{X}_2, \tilde{x}_2)$ belong to the same conjugacy class in ~~$\pi(X, x_0)$~~

pf ($\Rightarrow$) Directly by Coro 4.2

($\Leftarrow$) Choose $\tilde{x}_1' \in p^{-1}(p(\tilde{x}_1))$ s.t $p_{1*}\pi(\tilde{X}_1, \tilde{x}_1') = p_{2*}\pi(\tilde{X}_2, \tilde{x}_2)$ by prop 4, and apply coro 4.2.

Coro 5.2: Any universal covering spaces are isomorphic.

§ The group action "$p^{-1}(x) \curvearrowleft \pi(X,x)$".

Def. 8 Let $\alpha \in \pi(X,x)$ act on $\tilde{x} \in p^{-1}(x)$ by $\tilde{x}\,\alpha$ being the terminal point of the lifting path starting from $\tilde{x}$. The result is well-defined by Theorem 1 and 2. Indeed, $(\tilde{x}.\alpha).\beta = \tilde{x}\,(\alpha\beta)$ and $\tilde{x}\,1 = \tilde{x}$

Facts: 5. The action is transitive, i.e., for any $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x)$, there exists an $\alpha \in \pi(X,x)$ s.t. $\tilde{x}_0.\alpha = \tilde{x}_1$

6) For any $\tilde{x} \in p^{-1}(x)$, $\mathrm{Stab}(\tilde{x}) = p_*\pi(\tilde{X},\tilde{x}) \leq \pi(X,x)$

Hence $\pi(X,x)/p_*\pi(\tilde{X},\tilde{x}) \leftrightarrow p^{-1}(x)$ and

$$[\pi(X,x) : p_*\pi(\tilde{X},\tilde{x})] = \mathrm{card}(p^{-1}(x))$$

*7) Let $\varphi \in A(\tilde{X},p)$, $\tilde{x} \in p^{-1}(x)$, and $\alpha \in \pi(X,x)$, $\varphi(\tilde{x}.\alpha) = (\varphi\tilde{x}).\alpha$

Def 9: A right G-space S is a set which the group G act on the right. Let E_1, E_2 be right G-spaces. The map $f: E_1 \to E_2$ is a G-space homomorphism if $f(x\,g) = f(x).g$, an isomorphism if there exists a homomorphism h s.t. fh and hf are identities, and an automorphism if f is an isomorphism if $E_1 = E_2$

Facts: Assume $E \curvearrowleft G$ is transitive

8) Let $x_0 \in E$. $\mathrm{Stab}(x_0) \leq G$ and $G/\mathrm{Stab}(x_0) \simeq E$ (as a G-space).
(Notice that a different choice of x_0 gives rise to a conjugacy subgroup)

*9) If for any $x, y \in E$ s.t. $\mathrm{Stab}(x) = \mathrm{Stab}(y)$, there exists $\varphi \in A$ s.t. $\varphi(x) = y$, then $A = \mathrm{Aut}(E)$. The converse is obvious

*10) Let $x_0 \in E$ and $H := \mathrm{Stab}(x_0)$. Then $\mathrm{Aut}(E) \simeq N(H)/H$ as a group. (cf. Massey Appendix B).

Theorem 5 $A(\tilde{X},p) \simeq \mathrm{Aut}(p^{-1}(x_0)) \simeq N(p_*\pi(\tilde{X},\tilde{x}))/p_*\pi(\tilde{X},\tilde{x})$ $\forall \tilde{x} \in \tilde{X}$ s.t. $p(\tilde{x}) = x_0$

pf The second isomorphism follows from fact 10. For the first: If $\varphi \in A(\tilde{X},p)$, then $\varphi|_{p^{-1}(x_0)}$ is an automorphism of $p^{-1}(x_0)$ by fact 5. Moreover, by Coro 5.2, φ is completely determined by its restriction $\varphi|_{p^{-1}(x_0)}$, so the map $\varphi \mapsto \varphi|_{p^{-1}(x)}$ is 1-1. That it is onto follows from Coro 5.3 and fact 9. That it preserves the group operation is immediate

RMK: The first isomorphism is natural by the nature of the covering space. The second is itself right generally for transitive G-spaces

Def. 10 A covering space $(\tilde{X}, p)$ is regular if $p_*\pi(\tilde{X}, \tilde{x}) \trianglelefteq \pi(X, x)$

Note that it is independent to the choice $\tilde{x} \in p^{-1}(x)$ by prop. 4

RMK If $(\tilde{X}, p)$ is a regular covering space, then $A(\tilde{X}, p) \simeq \pi(X,x) / p_*\pi(\tilde{X}, \tilde{x})$ for any $x \in X$, $\tilde{x} \in p^{-1}(x)$. In particular, the universal covering space $(\hat{X}, \hat{p})$ is regular ($\because 0 \trianglelefteq G$), and thus $A(\hat{X}, \hat{p}) \simeq \pi(X, x)$

§ Regular Covering Spaces and Quotient Spaces

Prop 6. The automorphism group $A(\tilde{X}, p)$ operates transitively on $p^{-1}(x)$ $\forall x \in X$, iff and only iff $(\tilde{X}, p)$ is a regular covering space

pf By coro 4.3, $p_*\pi(\tilde{X}, \tilde{x}_1) = p_*\pi(\tilde{X}, \tilde{x}_2)$ $\forall \tilde{x}_1, \tilde{x}_2 \in p^{-1}(x)$.

$\Leftrightarrow$ By prop 5, $p_*\pi(\tilde{X}, \tilde{x}_1) \trianglelefteq \pi(X, x) \Leftrightarrow (\tilde{X}, p)$ is regular ✱

As a result, $\tilde{X} / A(\tilde{X}, p) \simeq X$ (natural homeo) if $(\tilde{X}, p)$ regular.

Question: Let Y be a topological space, and G a group of homeomorphisms of Y. Let $p : Y \to Y/G$ be the natural projection. When does (Y, p) be a regular covering space of Y/G with $A(Y, p) = G$?

If this is the case, it is clear by coro 4.1 that G operates on Y without fixed points. Moreover, for each point $y \in Y$, there is a neighborhood U of y such that the sets $\{\varphi(U) \mid \varphi \in G\}$ are pairwise disjoint. Such a group of homeomorphisms is said to be properly discontinuous. Surprisingly,

Prop 7. Let G be a properly discontinuous group of homeomorphisms of Y, and $p : Y \to Y/G$ denote the natural projection. Then (Y, p) is a regular covering space of Y/G, and $G = A(Y, p)$

pf Let $x \in Y/G$, and choose a point y such that $p(y) = x$. By assumption, there is a neighborhood N of y such that $\{\varphi(N) \mid \varphi \in G\}$ are pairwise disjoint. Since Y is locally arcwise connected, there exists a locally arcwisely connected neighborhood $V \subseteq N$. Let $U = p(V)$

Claim: U is an elementary neighborhood of X. p is open $\Rightarrow U$ is open; p continuous $\Rightarrow U$ arcwise continuous. p maps V injectively onto U, and since p is open, we have that $p : V \xrightarrow{\sim} U$ (homeo). For any other arc-component W of $p^{-1}(U)$, $\exists \varphi \in G$ s.t. $\varphi(V) = W$. $p\varphi = p$, and since $\varphi : V \xrightarrow{\sim} W$, we have $p|_W : W \xrightarrow{\sim} U$ ✱

Thus (Y, p) is a covering space Y/G. Obviously, $\forall \varphi \in G$, φ is an automorphism of Y, so $G \leq A(Y, p)$. If $G \subsetneq A(Y,p)$, then $\exists \psi \in A(Y,p)$ s.t. ψ sends some point to that any other $\tau \in G$ does not (since automorphism acts without fixed point), a contradiction to that the action is transitive

§. Galois Correspondence between Covering Spaces and Subgroups of the Fund'l Grp

Lemma: Let X be a topo space having a universal covering space (Y, q). Then for any $G \leq \pi(X, x)$, there exists a covering space $(\tilde{X}, p)$ of X s.t. $p_* \pi(\tilde{X}, \tilde{x}) = G$

pf. Choose $y \in p^{-1}(x)$ and a subgroup H of $A(Y, q)$ by
$H = \{\varphi \mid \varphi(y) = y.\alpha \text{ for some } \alpha \in G\}$, which is naturally isomorphic to G by $\begin{matrix} H \xrightarrow{\sim} G \\ \varphi \mapsto \alpha^{-1} \end{matrix}$

Well-defined: $\varphi(y) = y\alpha^{-1} = y.\beta^{-1} \Rightarrow y\alpha^{-1}\beta = y \Rightarrow \alpha^{-1}\beta = id$, $\because Y$ simply connected

Bijection: $\pi(X, x) \cong A(Y, q)$

Homomorphism: $\varphi_1\varphi_2(y) = \varphi_1(y.\beta^{-1}) = (y.\beta^{-1}).\alpha^{-1} = y.(\alpha\beta)^{-1}$

Since $H \leq A(Y, q)$, it is a properly discontinuous group of homeomorphisms of Y. Let $\tilde{X} = Y/H$

$$\begin{tikzcd} Y \arrow[r, "\gamma"] \arrow[d, "q"'] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{X} = Y/H \arrow[dl, "p : \text{ induced by } q"] \\ X & \end{tikzcd}$$

By the previous proposition, (Y, γ) is a covering space of $\tilde{X}$, and hence we assert that $(\tilde{X}, p)$ is a covering space of X:

Choose an elementary neighborhood U for $x_0 \in X$ (wrt (Y, q)). Then $p^{-1}(U) = \gamma(q^{-1}(U))$. In each arc-component of $p^{-1}(U)$, we choose a point $\tilde{x} \in \tilde{X}$, and an elementary neighborhood V ..

$\mathrm{Stab}(x_0) = G \Rightarrow p_* \pi(\tilde{X}, \tilde{x}) = G$ #

Our last work is to prove the existence of the universal covering space of X. First we develop a necessary condition. Assume $(\tilde{X}, p)$ is a universal covering of X. Let $x \in X$, $\tilde{x} \in p^{-1}(x)$, U be an elementary neighborhood of x and V be the component of $p^{-1}(U)$ which contains $\tilde{x}$. Then we have the following diagram:

$$\begin{tikzcd} \pi(V, \tilde{x}) \arrow[r] \arrow[d, "(p|_V)_*"', "\cong"] \arrow[dr, phantom, "\circlearrowright"] & \pi(\hat{X}, \tilde{x}) = 1 \arrow[d, "p_*"] \\ \pi(U, x) \arrow[r, hook, "i_*"'] & \pi(X, x) = 1 \end{tikzcd}$$

Since $V \cong U$ (homeo), $(p|_V)_*$ is an group isomorphism.
But then image$(i_*) = \{1\}$ Hence we have:
$\forall x \in X$, $\exists$ an elementary neighborhood U of x such that any closed path in U can be shrunk in X. Such property is called semilocally simply connected. It is also sufficient

Definition: Such U is called basic in X

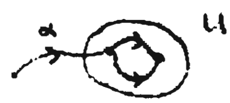
Theorem Let X be semilocally simply connected Then X has a universal covering space.

Pf choose a base point $x_0 \in X$, and define
$\tilde{X} := \{\alpha \mid$ a path class α in X initial at $x_0\}$, and
$p: \begin{array}{l} \tilde{X} \longrightarrow X \\ \alpha \longmapsto \alpha(1) \end{array}$ (i.e the terminal point of α)

By hypothesis, $\{U \subseteq X \mid U$: basic set$\}$ forms a basis for the topology of X Let $(\alpha, U) = \{\beta \in \tilde{X} \mid \beta = \alpha\,\alpha'$ for some path class α' in $U\} \subseteq \tilde{X}$ and $\{(\alpha, U) \mid \alpha \in \tilde{X}, U$: basic$\}$ a basis for the topology of $\tilde{X}$ Notice that $\gamma \in (\alpha, U) \cap (\beta, V) \Rightarrow \exists W$ basic set $(\gamma, W) \subseteq (\alpha, U) \cap (\beta, V)$

Claim 1 Let $\alpha \in \tilde{X}$ and U be a basic open neighborhood of $p(\alpha)$ Then $p|_{(\alpha, U)}$ maps (α, U) bijectively to U

Pf 1-1: $\beta \in (\alpha, U) \leadsto p(\beta) = p(\alpha \cdot \alpha') \in U$
For $\beta \neq \gamma \in (\alpha, U)$, $\begin{array}{l} p(\beta) = p(\alpha \cdot \alpha') \\ p(\gamma) = p(\alpha \cdot \alpha'') \end{array}$ If $p(\beta) = p(\gamma)$,
then $\alpha\,\alpha'(1) = \alpha\,\alpha''(1)$ But since U is simply connected $\leadsto \alpha\,\alpha' = \alpha\,\alpha'' \leadsto \beta = \gamma$ $\ast$



Onto: It follows directly from that U is arc-connected #

Claim 2 Let U be any basic set and $x \in U$ Then $p^{-1}(U) = \bigcup_{\lambda \in \Lambda} (\alpha_\lambda, U)$ where $\{\alpha_\lambda\ \lambda \in \Lambda\}$ consists of all path classes with initial x_0 and Terminal x Moreover, the sets (α_λ, U) are pairwise disjoint

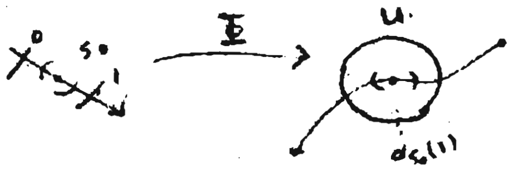
Pf "$\subseteq$" $\beta \in p^{-1}(U) \leadsto p(\beta) \in U \leadsto \beta$ terminals in $U \leadsto \beta \in (\beta, U)$
"$\supseteq$" $\beta \in (\alpha_\lambda, U) \leadsto \beta = \alpha_\lambda\,\alpha' \leadsto \beta$ terminals in $U \leadsto \beta \in p^{-1}(U)$.
Now if $\beta \in (\alpha_1, U) \cap (\alpha_2, U)$, then $\beta = \alpha_1\,\alpha_1' = \alpha_2\,\alpha_2'$
$\Rightarrow \alpha_1\,\alpha_1'\,\alpha_2'^{-1} = \alpha_2 \Rightarrow \alpha_1 = \alpha_2$ #

- It follows from Claim 2 that p is continuous
- By claim 1, $p|(\alpha,U)$ is open. since any open subset of (α,U) is a union of the sets of the form (β,V) ($V\subseteq U$)

  Hence $p|(\alpha,U) : (\alpha,U) \xrightarrow{\sim} U$ $\rightsquigarrow$ (α,U) is arcconnected
- By claim 2, each component of $p^{-1}(U)$ is disjoint to any others, so U is an elementary neighborhood

$\tilde{X}$ is arcwise connected:

Denote $\tilde{x_0} \in \tilde{X}$ to be the constant path class of x_0. Given any $\alpha \in \tilde{X}$, we claim that $\Phi : \begin{matrix} I \longrightarrow \tilde{X} \\ s \longmapsto \alpha_s \end{matrix}$ is continuous, and hence a path in $\tilde{X}$, where $\alpha_s(t) = \alpha(st)$. But this is immediate



$\tilde{X}$ is simply connected:

Finally, it suffices to prove that $p_*\pi(\tilde{X}, \tilde{x_0}) = 1$.

Let $\alpha \in \pi(X, x_0)$. We shall consider the action of α on $\tilde{x_0}$. Since the path $\left(\begin{matrix} I \to \tilde{X} \\ s \mapsto \alpha_s \end{matrix}\right)$ has initial at $\tilde{x_0}$ and terminal at α and thus is "the" lifting path of α. Hence by the definition of the action, we have $\tilde{x_0} \cdot \alpha = \alpha$

Therefore, $stab(\tilde{x_0}) = 1 \Leftrightarrow p_*\pi(\tilde{X}, \tilde{x_0}) = 1$ #

Classification of compact Surface

高尉庭

Recall

1. Definition of n-manifold

An n-dimensional manifold X is Hausdorff space and $\forall p \in X$, $\exists$ neighborhood U_p of p s.t. $U_p \underset{\text{homeomorphic}}{\simeq} \mathbb{D}^n$

where $\mathbb{D}^n := \{ x \in \mathbb{R}^n \ \ |x| < 1 \}$

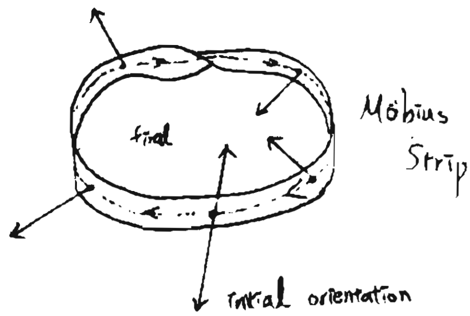
2. Orientable surface and Nonorientable Surface

A connected 2-manifold is orientable if every closed path is orientation preserving

Let γ be a closed path in X The orientation changes continueously along γ and goes back to initial point.

If the orientation reverse, we call γ orientation-reversing

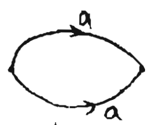
o.w. γ is orientation-preserving



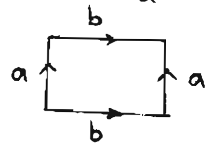
§ 1 Connected Sum and some example of compact surface.

△ Orientable surface

1 Sphere (S^2) : aa^{-1}



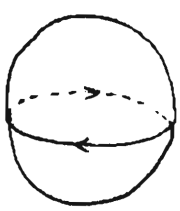
2 torus $aba^{-1}b^{-1}$

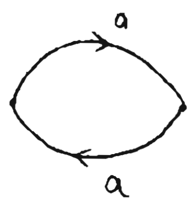


Nonorientable surface

3 projective plane $\mathbb{RP}^2 = S^2 /_{(x,y,z) \sim (-x,-y,-z)}$

S^2 aa





Connected Sum

S_1, S_2 are surface $S_1 \underset{\text{open}}{\supset} D_1 \sim$ disk , $h\ \partial D_1 \to \partial D_2$ homeo $\sim$
$S_2 \underset{\text{open}}{\supset} D_2 \sim$ disk

then $S_1 \# S_2 := (S_1 \setminus D_1) \cup_h (S_2 \setminus D_2)$

Some fact

① $S_1 \# S_2$ is independent to the choose of hole

② $S_1 \# S_2 \sim S_2 \# S_1$

③ $(S_1 \# S_2) \# S_3 \sim S_1 \# (S_2 \# S_3)$

④ S_1, S_2 are orientable, then so is $S_1 \# S_2$

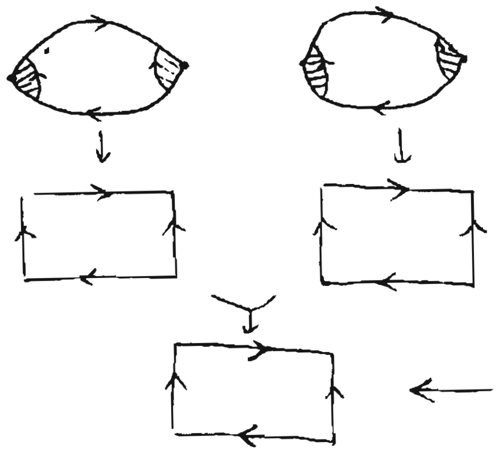
⑤ If either S_1 or S_2 is nonorientable, then $S_1 \# S_2$ is nonorientable.

Example

① $S_1 \# S^2 \sim S_1$

② If S_1, S_2 are proj , then $S_1 \# S_2$ is Klein Bottle.

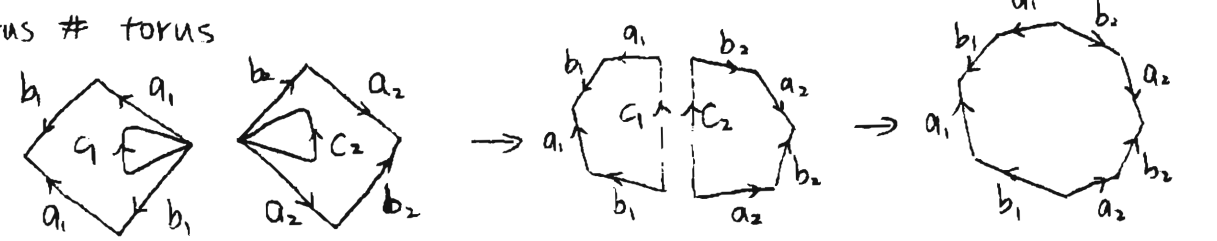
pf :



$\leftarrow$ Klein Bottle

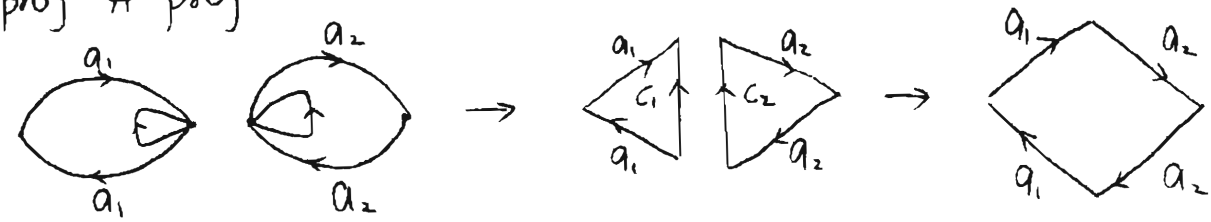
$(a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1})$

③ torus # torus



In general, $\#\, n$ tori $= a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots\cdots a_n b_n a_n^{-1} b_n^{-1}$

④ proj # proj



$(a_1 a_1 a_2 a_2)$

In general, $\# n$ proj $= a_1 a_1 a_2 a_2 \cdots a_n a_n$

§2 Triangulation of compact Surface

△ Def (A triangulation of a compact surface S)

$\exists$ a family $\{T_1, T_2, \cdots T_n\}$, T_i is closed subset of S and this family cover S, $T_i' \xrightarrow[\sim]{\varphi_i} T_i$, $T_i' \subset \mathbb{R}^3$ is triangle, and saytisfy the property $\forall i = 1 \sim n$

$\forall i \neq j$, either ① $T_i \cap T_j = \phi$ or ② $T_i \cap T_j$ = a single common vertex or ③ $T_i \cap T_j$ = an entire edge in common

△ Thm (T Rado, 1925)

Given any cpt surface S, then $\exists$ triangulation of S

△ Prop for cpt surface

① Each edge is an edge of exactly two triangles

② Let v be vertex of a triangle $\exists \{T_0, T_1, \cdots T_r\}$ s t $T_i \cap T_{i+1}$ = one edge, $T_0 = T_r$ $\forall i = \sim r$

§3 The Classification theorem for compact surface.

△ Thm Any compact surface S is
$$\begin{cases} ① \; S \sim S^2 \\ ② \; S \sim \# n \text{ tori} \\ ③ \; S \sim \# n \text{ proj} \end{cases}$$
for some $n \in \mathbb{N}$

pf **First Step** Show $S \sim$ polygon $P/_{\text{certain paired edge on } \partial P}$

let $\{T_i\}_{i=1}^{n}$ be triangulation of S s.t. T_i has an edge e_i in common with at least one of the triangles $T_1, \cdots T_{i-1}$ $(2 \leq i \leq n)$ Assume we get two set of triangle $\{T_1, \cdots T_k\}$ & $\{T_{k+1}, \cdots T_n\}$

$\Rightarrow S$ can divide into two disjoint closed set $\rightarrow\!\!\times$ ($\because S$ is connected)

Let $\mathbb{R}^2 \supset T_i' \xrightarrow[\sim]{\varphi_i} T_i$, and T_i' are disjoint $\forall i = 1 \sim n$,

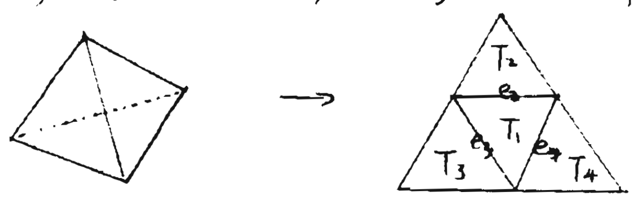
$T = \bigcup_{i=1}^{n} T_i' \subset \mathbb{R}^2$ Define φ $T' \to S$ by $\varphi|_{T_i'} = \varphi_i$

$\Rightarrow \varphi$ is conti and onto

Since T' is cpt and S is Hausdorff space

By Appendix $\Rightarrow S$ has quotient topology determined by φ

Identity $p_i \in T_i'$, $p_j \in T_j'$ if $\varphi(p_i) = \varphi(p_j)$ at some e_k



Let D denote the resulting quotient space of T', so the map $\varphi : T' \to S$ induces a map ψ $D \to S$

$\because D$ is cpt and S is Hausdorff space

$\therefore S$ has quotient topology determined by ψ

Claim: $D \sim$ a closed disk.

Using the fact that if $E_1, E_2 \sim$ closed disk, $A_1 \subset \partial E_1$

$A_2 \subset \partial E_2$ $\wedge$ $A_1, A_2 \sim [0,1]$, $A_1 \xrightarrow[\sim]{h} A_2$

then $E_1 \cup_h E_2 \sim$ closed disk.

By induction, we know $D \sim$ closed disk.

Second Step. Elimination of adjacent edges of the first kind

Define the certain pair of edge occur with both +1 & -1
we call it a pair of the first kind
o.w we call it a pair of the second kind

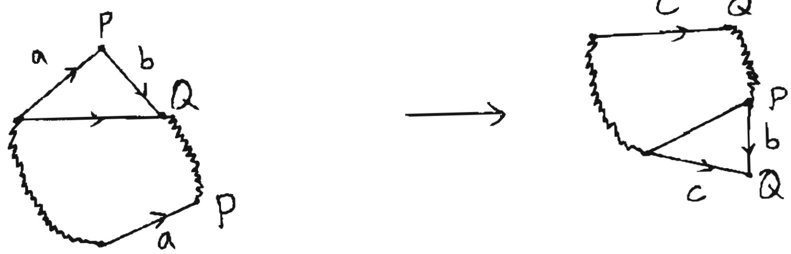
Process 1

Continue this process until all such pair are eliminated
or obtain a polygon with only two side (aa , aa^{-1})

Third Step Transformation to polygon s.t all vertices identity to a single vertex

Suppose there are at least two different equivalence classes of vertices The polygon must have an adjacent pair of vertices which is non-equivalence.

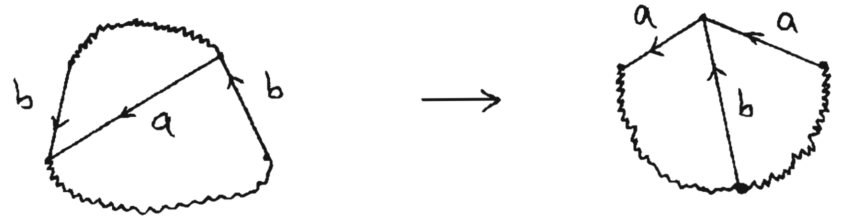
Process 2



By this process, we can add # Q and reduce # P
Using the process 1 & 2, we can get a polygon s.t all vertices are to be identified to a single vertex

Fourth Step Make any pair of edge of the second kind adjacent.

Process 3

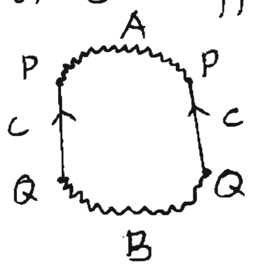


Continue this process until all pair of edge of the second kind adjacent.

① If there are no pair of first kind, then the symbol of the polygon must be the form $a_1 a_1 \cdots a_n a_n \Rightarrow S \sim \# n$ proj

② else, we must have the form $c \cdots d \cdot c^{-1} \cdots d^{-1} \cdots$

pf of ② Suppose no such d

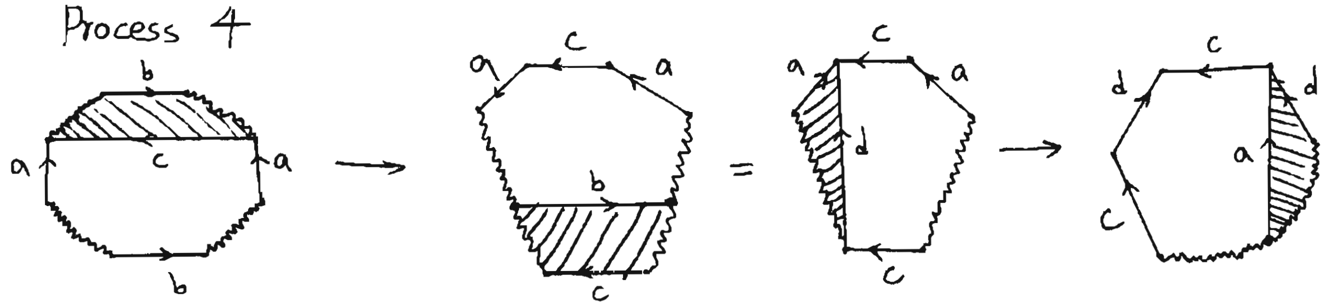


$\Rightarrow$ all edge in $\begin{matrix}A\\B\end{matrix}$ is identified with $\begin{matrix}A\\B\end{matrix}$

$\Rightarrow P \neq Q \quad \rightarrow\!\!\!\!\times$ (Third Step)

Fifth Step Pairs of the first kind

Process 4



Continue this process until all pair of first kind are the form $aba^{-1}b^{-1}$

If there are no pair of second kind, the symbol of polygon must be $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} \Rightarrow S \sim \# n$ tori

It remains to treat the case in which there are pairs of first kind and second kind at this stage.

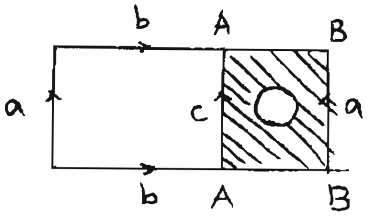
Key point:

* Lemma: (torus) # (proj) $\sim$ # 3 proj

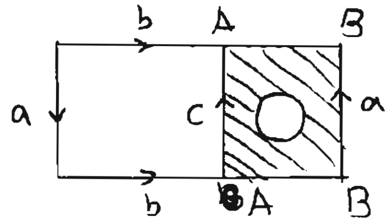
pf of lemma

By example 2 we only need to prove (proj) # (torus) $\sim$ (proj) # (Klein)

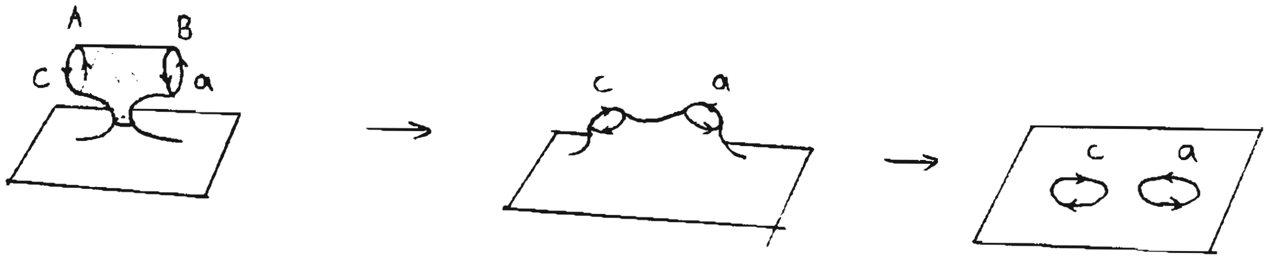
At first, cosider the connected sum of cylinder and Mobius strip



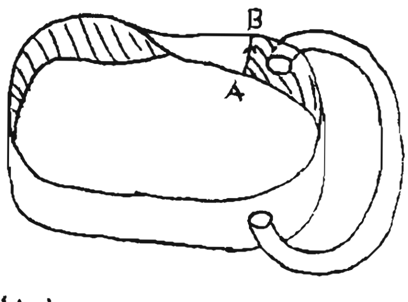
torus



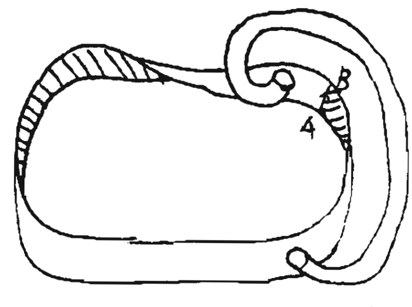
Klein Bottle



So we can discover (Möbius) # (torus) $\sim$ (Möbius) # (Klein)



(Möbius) # (torus)



(Möbius) # (Klein)

By the fact (proj) $\sim$ (Möbius) + (disk) / boundary

$\Rightarrow$ we have proven lemma

Hence the case $\begin{cases} m & \text{pair of second kind} \\ n & \text{quadruple of first kind} \end{cases} \rightarrow \#(m+2n)$ proj

$\Rightarrow S \sim \#(m+2n)$ proj

Q.E.D

Another statement of theorem:

Any cpt orientable surface $\sim \begin{cases} \text{sphere} \\ \# n \text{ tori} \end{cases}$

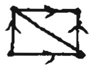
Any cpt nonorientable surface $\sim \begin{cases} \text{proj} \ \# \ (\text{orientable}) \\ \text{Klein Bottle} \ \# \ (\text{orientable}) \end{cases}$

§4 Uniquessness of Classification

In the end, we need to check $\begin{cases} \text{sphere} \\ \# n \text{ tori} \\ \# n \text{ proj} \end{cases}$ are topology different.

Using Eular charactistic χ

$\chi(S^2) = 2$

$\chi(\text{torus}) = 0$  $1+2-3=0$

$\chi(\text{proj}) = 1$  $2+2-3=1$

Fact: $\chi(S)$ is independent to the choose of triangulation

Prop $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) \quad 2$

Using prop above, we can get

Surface	Eular charactistic
S^2	2
$\# n$ tori	$2-2n$
$\# n$ proj	$2-n$
(proj) $\#$($\# n$ tori)	$1-2n$
(Klein) $\#$($\# n$ tori)	$-2n$

△ Thm S_1, S_2 are cpt surface, then

$S_1 \sim S_2 \Leftrightarrow$ both are orientable or nonorientable and have same Eular charactistic

Reference: ① William S Massey, Algebraic Topology: An Introduction (1967)

§ Appendix

△ Def (Quotient Topology)

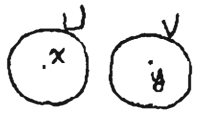
Let X be a topology space Y be a set. $f\ X \to Y$

The quotient topology determined by f is defined as follow:

A set $U \subset Y$ is open $\Leftrightarrow$ $f^{-1}(U)$ is open on X

△ Def (Hausdorff Space)

Let X be a topology space. X is Hausdorff space if for any $x, y \in X \wedge x \neq y$, $\exists\, U, V \underset{\text{open}}{\subseteq} X$ s.t. $x \in U$, $y \in V$ and $U \cap V = \phi$



△ Prop. (i) The closed subset of cpt set is cpt

(ii) The cpt subset of Hausdorff space is closed

(iii) Y be quotient topology determined by $f\ X \to Y$

If X is $\begin{cases} \text{cpt} \\ \text{connected} \\ \text{arcwise connected} \end{cases}$, then so is Y

(iv) Z be top sp $g\ Y \to Z$ conti $\Leftrightarrow$ $g \circ f$ is conti

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, dashed, "g\circ f"] & \\ Y \arrow[r, "g"'] \arrow[ur, phantom, "\circlearrowleft"] & Z \end{tikzcd}$$

pf I only want to prove (ii), if $W \underset{\text{closed}}{\subseteq} X$, $p \in X \setminus W$, $\alpha \in W$

Take $U_\alpha, V_\alpha \underset{\text{open}}{\subset} X$ s.t $p \in U_\alpha$, $\alpha \in V_\alpha \wedge U_\alpha \cap V_\alpha = \phi$

$\because$ W is cpt $\Rightarrow \exists\, i = 1 \sim n$ s.t $W \subset \bigcup_{i=1}^{n} V_{\alpha_i} \Rightarrow U := \bigcap_{i=1}^{n} U_i$ is open and $U \cap W \subset U \cap \bigcup_{i=1}^{n} V_i = \phi \Rightarrow X \setminus W$ is open $\Rightarrow W$ is closed

△ Thm. The bijective conti map from cpt top sp. X to Haus sp. Y is homeomorphism

pf. Only need to show f is closed map If $W \underset{\text{closed}}{\subseteq} X$, then W is cpt.

$\Rightarrow f(W)$ is cpt $\Rightarrow f(W)$ is closed, since Y is Haus sp. ✱

Frobenius Method for ODEs with Regular Singular Points

李自然 2015.6.18

INTRODUCTION

An ODE of the form

$$y^{(n)} + a_1(x)\,y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1}(x)\,y' + a_n(x)\,y = 0, \tag{1}$$

where each $a_k(x)$ is analytic in a deleted neighborhood of $x=0$ and has a pole at $x=0$ whose order is at most k ($a_k(x)$ may be analytic at $x=0$), is said to have a regular singular point at $x=0$. We shall use the so-called "Frobenius method" to solve all such ODEs.

By the transformation $Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$ where $Y_j = x^{j-1} y^{(j-1)}$ $(1 \le j \le n)$, we find that

$$x Y_j' = (j-1)\,Y_j + Y_{j+1} \quad (1 \le j < n) \quad \text{and}$$
$$x Y_n' = (n-1)\,Y_n - x a_1 Y_n - \cdots - x^n a_n Y_1,$$

so that this transformation transforms (1) into $xY' = A(x)\,Y$, where

$$A(x) = \begin{pmatrix} 0 & 1 & & & \\ 0 & 1 & 1 & & \\ 0 & 0 & 2 & 1 & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \cdots & (n-2) & 1 \\ -x^n a_n & -x^{n-1} a_{n-1} & \cdots & & -x a_1 + (n-1) \end{pmatrix}. \tag{2}$$

Hence, we convert the original scalar equation into a matrix ODE. We shall investigate the general matrix ODEs of the form $xY' = A(x)\,Y$ (which is called having a regular singular point at $x=0$ if A is analytic near $x=0$), and then use the results to solve the scalar equations having regular singular points.

MATRIX ODEs WITH REGULAR SINGULAR POINTS

Lemma: Let $A \in M_{m\times m}(\mathbb{C})$, $B \in M_{n \times n}(\mathbb{C})$ and let $C, X \in M_{m\times n}(\mathbb{C})$. Then,

$AX - XB = C$ has a unique solution $X \Leftrightarrow A, B$ have no common eigenvalues.

(Pf.) Observe first that

$AX - XB = C$ has a unique solution $\Leftrightarrow AX - XB = 0$ has a unique solution ($X=0$), since if we regard $X \in \mathbb{C}^{mn}$, then $AX - XB$ (in $M_{m\times n}(\mathbb{C})$) $= MX$ (in $\mathbb{C}^{mn}$) for some $M \in M_{mn}(\mathbb{C})$, and thus $\exists!\, X$ such that $MX = C$ if and only if M is invertible if and only if $\exists!\, X$ such that $MX = 0$.

(1) If A and B have a common eigenvalue λ, so do A and B^t; let $v, w \neq 0$ so that $Av = \lambda v$ and $B^t w = \lambda w$; then $X = v w^t \neq 0$ solves $AX - XB = 0$. (See p.2.)

(Pf. of Lemma, cont'd.)

(2) Conversely, suppose that $AX-XB=0$ has a nonzero solution $X=X_0\neq 0$. Let J_A, J_{B^t} be the Jordan canonical forms (upper-triangular!) of A and B^t respectively, and let Q, R be invertible matrices so that $A=Q^{-1}J_AQ$ and $B^t=R^{-1}J_{B^t}R$. Then $AX-XB=0 \Leftrightarrow J_A(QXR^t)-(QXR^t)J_{B^t}^t=0$, and $X\neq 0 \Leftrightarrow QXR^t\neq 0$. Hence, may assume first that in the equation $AX-XB=0$ having a nonzero solution X_0, both A and B^t are in Jordan canonical forms. Now, write $A=\bigoplus_{i=1}^r V_i$ and $B^t=\bigoplus_{j=1}^s W_j$, where each V_i is a Jordan block of size m_i with eigenvalue λ_i, and each W_j is a Jordan block of size n_j with eigenvalue μ_j. Let $\{e_1,\cdots,e_m\}$ be the standard basis for $\mathbb{R}^m$ and let $\{u_1,\cdots,u_n\}$ be the standard basis for $\mathbb{R}^n$. Write $X_0=(x_{ij})=\sum_{i,j}x_{ij}e_iu_j^t$. Since $X_0\neq 0$, without loss of generality, may assume $x_{ij}\neq 0$ for some $1\le i\le m_1$ and $1\le j\le n_1$. Let

$\alpha=\max\{i \mid x_{ij}\neq 0,\ 1\le i\le m_1\}$ and $\beta=\max\{j \mid x_{\alpha j}\neq 0,\ 1\le j\le n_1\}$.

Write $X_0=\sum_{j=1}^{n_1}\sum_{i=1}^{m_1}x_{ij}e_iu_j^t+\sum_{i>m_1 \text{ or } j>n_1}x_{ij}e_iu_j^t=X_0'+X_0''$ and observe that $\mathbb{C}[M]$ is a commutative ring for any square matrix M, we have

$$\begin{aligned}
0&=(A-\lambda_1)^{\alpha-1}(AX_0-X_0B)(B-\mu_1)^{\beta-1}\\
&=A(A-\lambda_1)^{\alpha-1}X_0'(B-\mu_1)^{\beta-1}-(A-\lambda_1)^{\alpha-1}X_0'(B-\mu_1)^{\beta-1}B+(A-\lambda_1)^{\alpha-1}(AX_0''-X_0''B)(B-\mu_1)^{\beta-1}\\
&=A(x_{\alpha\beta}e_1u_1^t)-(x_{\alpha\beta}e_1u_1^t)B+\sum_{i>m_1 \text{ or } j>n_1}\tilde{x}_{ij}e_iu_j^t\\
&=(\lambda_1-\mu_1)x_{\alpha\beta}e_1u_1^t+\sum_{i>m_1 \text{ or } j>n_1}\tilde{x}_{ij}e_iu_j^t.
\end{aligned}$$

Since $x_{\alpha\beta}\neq 0$ and since $\{e_iu_j^t\}$ forms a linearly independent set, we get $\lambda_1=\mu_1$. Hence, A and B^t have a common eigenvalue, and thus A and B have a common eigenvalue. #

<u>Theorem 1</u>: Suppose that $A(x)=\sum_{r=0}^{\infty}A_rx^r\in M_{n\times n}(\mathbb{C})[[x]]$ where no two eigenvalues of A_0 differ by a positive integer. Then there exists a $P(x)=\sum_{r=0}^{\infty}P_rx^r\in M_{n\times n}(\mathbb{C})[[x]]$ with $P_0=I$, so that the transformation $Y=PZ$ reduces the ODE $xY'=A(x)Y$ to the form $xZ'=A_0Z$ (formally in $M_{n\times n}(\mathbb{C})[[x]]$).

(Pf.) First, $Y=PZ$ transforms $xY'=A(x)Y$ into $xZ'=(P^{-1}AP-xP^{-1}P')Z$:

$$Y'=P'Z+PZ';\ xY'=A(x)Y \Rightarrow x(P'Z+PZ')=A(x)PZ \Rightarrow xZ'=P^{-1}(AP-xP')Z.$$

The above transformation needs justification, namely the existence of P^{-1} in $M_{n\times n}(\mathbb{C})[[x]]$. We shall, however, accept this first.

Now, we try to solve $\begin{cases} xP'=AP-PA_0 \\ P_0=I\end{cases}$; if a solution P exists, then $P^{-1}\in M_{n\times n}(\mathbb{C})[[x]]$ (since $P_0=I\in M_{n\times n}(\mathbb{C})^{\times}$) and hence $P^{-1}AP-xP^{-1}P'=A_0$, so that the theorem will be proved. (See p.3)

(Pf. of Thm. 1, cont'd.)

Substitute $P = \sum P_r x^r$ and $A = \sum A_r x^r$ in $xP' = AP - PA_0$ ($P_0 = I$), we get $\sum_{r=1}^{\infty} r P_r x^r = \sum_{r=0}^{\infty}\left(\sum_{s=0}^{r} A_{r-s}P_s\right)x^r - \sum_{r=0}^{\infty} P_r A_0 x^r$; comparing the coefficients shows that

$$rP_r = \left(\sum_{s=0}^{r} A_{r-s}P_s\right) - P_r A_0 \Leftrightarrow (A_0 - rI)P_r - P_r A_0 = -\sum_{s=0}^{r-1} A_{r-s}P_s \quad (r>0). \tag{3}$$

Since A_0 has no two eigenvalues differing by a positive integer, for every positive integer r, $A_0 - rI$ and A_0 have no common eigenvalues ; thus, the lemma and (3) guarantee that $P_1, P_2, \cdots$ can be solved inductively (and uniquely), and hence the required P exists in $M_{n\times n}(\mathbb{C})[[x]]$. #

<u>Theorem 2</u>: The ODE $xZ' = A_0 Z$ ($A_0 \in M_{n\times n}(\mathbb{C})$) has a fundamental solution $Z = x^{A_0}$ ($x^{A_0} := e^{A_0 \log x} := \sum_{r=0}^{\infty} \frac{(A_0 \log x)^r}{r!}$) ; that is, all solutions (in $n\times 1$ matrices) Z of this ODE are of the form $Z = x^{A_0}C$ for some $C \in \mathbb{C}^n$.

(Pf.) Since $\frac{d}{dx}(x^{A_0}) = \sum_{r=1}^{\infty} \frac{1}{x}\frac{A_0^r (\log x)^{r-1}}{(r-1)!} = \frac{1}{x}A_0 x^{A_0}$, $x\frac{d}{dx}(x^{A_0}) = A_0 x^{A_0}$, so that $Z = x^{A_0}$ solves $xZ' = A_0 Z$. If Z_1 is an $n\times 1$ matrix function solving $xZ' = A_0 Z$,

$(x^{-A_0}Z_1)' = -\frac{1}{x}A_0 x^{-A_0}Z_1 + x^{-A_0}Z_1' = -\frac{1}{x}A_0 x^{-A_0}Z_1 + x^{-A_0}\left(\frac{1}{x}A_0 Z_1\right) = 0$

$\Rightarrow x^{-A_0}Z_1 = C \in \mathbb{C}^n \Rightarrow Z_1 = x^{A_0}C$. #

<u>Remark</u>: In the above proof, "$Z_1 = x^{A_0}C$" means that they define the same global analytic function as the solution to the ODE $xZ' = A_0 Z$. The above proof works since we can first choose a small open disk not containing 0 and then define a branch of "log" on it ; then we can deduce that $Z_1 = x^{A_0}C$ on that disk, and hence continue this solution to all of $\mathbb{C}\setminus\{0\}$.

<u>Theorem 3</u>: Let $F(x) = \sum_{r=0}^{\infty} F_r x^r$ be an $n\times n$ matrix function holomorphic for $|x| < x_0$, and let $Z = \sum_{r=0}^{\infty} a_r x^r \in \mathbb{C}^n[[x]]$ solving $xZ' = FZ$ formally. Then $\sum_{r=0}^{\infty} a_r x^r$ converges for $|x| < x_0$ and thus represents a solution to $xZ' = FZ$.

(Pf.) Substitute the power series expansion for F and Z into $xZ' = FZ$ and comparing the coefficients, we get

$$F_0 a_0 = 0 \; ; \quad (F_0 - rI)a_r = -\sum_{s=1}^{r} F_s a_{r-s} \quad (r>0). \tag{4}$$

Since F_0 has finitely many eigenvalues, there exists a smallest non-negative integer k so that $\det(F_0 - rI) = 0$ ($\Leftrightarrow (F_0 - rI)^{-1}$ exists) $\forall\, r > k$.

Introduce the matrix norm $\|A\|$ as follows: If $A = (a_{ij})$, define $\|A\| = \max_i \sum_{j=1}^{n} |a_{ij}|$. Then it is easy to verify that $\|A+B\| \le \|A\| + \|B\|$ and $\|AB\| \le \|A\|\|B\|$ whenever $A+B$ or AB is defined.

Now, we claim that there exists a $c>0$, independent of r, so that $\|(F_0 - rI)^{-1}\| \le c$ $\forall\, r > k$. To prove this, observe that for $r>k$, we have $(F_0 - rI)^{-1} = \frac{\operatorname{adj}(F_0 - rI)}{\det(F_0 - rI)}$; since $\det(F_0 - rI)$ is a polynomial in r of degree n, (See p.4)

(Pf. of Thm. 3, cont'd)

and since all entries of $\mathrm{adj}(F_0 - rI)$ are polynomials in r of degree at most $n-1$, we conclude that $\lim_{r\to\infty}(F_0 - rI)^{-1} = 0$; hence, $\|(F_0 - rI)^{-1}\|$ is bounded $\forall r > k$, and thus the required c exists.

By the claim, (4) implies

$$a_r = -(F_0 - rI)^{-1}\sum_{s=1}^{r} F_s a_{r-s} \Rightarrow \|a_r\| \le c\sum_{s=1}^{r}\|F_s\|\|a_{r-s}\| \quad (r>k). \tag{5}$$

Define $\phi(x) = \sum_{r=1}^{\infty}\|F_r\| x^r$ $(|x| < x_0)$. ϕ is well-defined (i.e. holomorphic in $|x| < x_0$) since $\|F_r\| \le \sum_{i,j}|(F_r)_{ij}|$ and $\sum_{r=0}^{\infty}\left(\sum_{i,j}|(F_r)_{ij}|\right)x^r$ converges for $|x| < x_0$ by the convergence of $F(x)$ in $|x| < x_0$. Note that $\phi(0) = 0$.

Introduce the "scalar majorizing function"

$$\hat{y}(x) = \frac{1}{1 - c\phi(x)}\left(\|a_0\| + \sum_{s=1}^{k}\left(\|a_s\| - c\sum_{t=1}^{s}\|F_t\|\|a_{s-t}\|\right)x^s\right)$$

$$\Rightarrow \hat{y}(x) = c\phi(x)\hat{y}(x) + \|a_0\| + \sum_{s=1}^{k}\left(\|a_s\| - c\sum_{t=1}^{s}\|F_t\|\|a_{s-t}\|\right)x^s. \tag{6}$$

Since $1 - c\phi(0) = 1 \neq 0$, $\hat{y}(x)$ is holomorphic in $|x| < x_1$ for some $x_1 \le x_0$, and hence we can rewrite $\hat{y}(x) = \sum_{r=0}^{\infty}\hat{a}_r x^r$ for $|x| < x_1$. Substitute the expansions of $\hat{y}$ and ϕ in (6), we get

$$\sum_{r=0}^{\infty}\hat{a}_r x^r = c\sum_{r=0}^{\infty}\left(\sum_{t=1}^{r}\|F_t\|\hat{a}_{r-t}\right)x^r + \|a_0\| + \sum_{s=1}^{k}\left(\|a_s\| - c\sum_{t=1}^{s}\|F_t\|\|a_{s-t}\|\right)x^s \quad (|x| < x_1);$$

comparing the coefficients of the above identity, we get (by some calculations)

$$\hat{a}_r = \|a_r\| \quad (0 \le r \le k); \qquad \hat{a}_r = c\sum_{t=1}^{r}\|F_t\|\hat{a}_{r-t} \quad (r > k). \tag{7}$$

By (5)(7) and induction, we can easily show that $\hat{a}_r \ge \|a_r\|$ $\forall r \ge 0$.
Since $\sum_{r=0}^{\infty}\hat{a}_r x^r$ converges for $|x| < x_1$, so does $\sum_{r=0}^{\infty}\|a_r\| x^r$ and $\sum_{r=0}^{\infty} a_r x^r$ (by the inequality $\hat{a}_r \ge \|a_r\|$). Since the original ODE $xZ' = FZ$ has no singular point in $0 < |x| < x_0$ (i.e. $\frac{1}{x}F(x)$ has no singular point in $0 < |x| < x_0$), by the existence and uniqueness theorem of matrix ODEs, $\sum_{r=0}^{\infty} a_r x^r$ actually converges in $|x| < x_0$ (see Appendix for the details). #

<u>Theorem 4</u>: If $A(x) = \sum_{r=0}^{\infty} A_r x^r$ is analytic in $|x| < x_0$, where no two eigenvalues of A_0 differ by a positive integer, then the ODE $xY' = A(x)Y$ has a (analytic) fundamental solution of the form $Y = P(x)x^{A_0}$, where $P(0) = I$ and $P(x)$ is analytic in $|x| < x_0$ (so that all solutions of this ODE are of the form $P(x)x^{A_0}C$ for some $C \in \mathbb{C}^n$).

(Pf.) By Theorem 1, $\exists P(x)$: square matrix function satisfying $\begin{cases} xP' = AP - PA_0 \\ P_0 = I \end{cases}$, so that $Y = PZ$ reduces $xY' = A(x)Y$ to $xZ' = A_0 Z$. Regarding P as a vector function and applying Theorem 3, we find that $P(x)$ is analytic in $|x| < x_0$. Since $\det P(0) = 1$ and since now $\det P(x)$ is continuous in $|x| < x_0$, there is a small disk D not containing 0, so that $\det P(x) \neq 0$ in D and thus $P(x)^{-1}$ exists in D. (See p.5.)

(Pf. of Thm. 4, cont'd.)

Now, Y solves $xY' = A(x)Y$ in $D \Leftrightarrow Z = P^{-1}Y$ solves $xZ' = A_0 Z$ in D $\Leftrightarrow Z = x^{A_0} C$ in D for some $C \in \mathbb{C}^n$ (Theorem 2) $\Leftrightarrow Y = P(x)x^{A_0}C$ in D for some $C \in \mathbb{C}^n$. (We assume that A is an $n\times n$ matrix function.) By continuation, we conclude that $xY' = A(x)Y$ has a fundamental solution $Y = P(x)x^{A_0}$ (in $0 < |x| < x_0$). #

<u>Theorem 5</u>: Theorem 4 still holds even if we drop the assumption "no two eigenvalues of A_0 differ by a positive integer" in it.

(Pf.) First, we may assume that in the ODE $xY' = A(x)Y$ given in Theorem 4, $A_0 = A(0)$ is already in Jordan canonical form, for: if J is the Jordan canonical form of A_0 and if $T \in GL_n(\mathbb{C})$ so that $J = T^{-1}A_0T$ (assume $A_0 \in M_{n\times n}(\mathbb{C})$), then $Y = TZ$ transforms $xY' = A(x)Y$ into $xZ' = B(x)Z$, where $B(x) = T^{-1}A(x)T \Rightarrow B(0) = J$. Now, under our first assumption, consider the "shearing transformation" $Y = S(x)W$ as follows: if $A(x) = \begin{pmatrix} J_{11} + x\Psi_{11}(x) & x\Psi_{12}(x) \\ x\Psi_{21}(x) & J_{22} + x\Psi_{22}(x)\end{pmatrix}$ where J_{22} is a Jordan block with eigenvalue λ, then we define $S(x) = \begin{pmatrix} I_{n-p} & 0 \\ 0 & xI_p\end{pmatrix}$ where p is the size of J_{22}, and hence $Y = SW$ transforms $xY' = A(x)Y$ into $xW' = C(x)W$, where

$$C(x) = S^{-1}AS - xS^{-1}S' = \begin{pmatrix} J_{11} + x\Psi_{11}(x) & x^2\Psi_{12}(x) \\ \Psi_{21}(x) & J_{22} - I_p + x\Psi_{22}(x)\end{pmatrix} \quad \left(S^{-1} = \begin{pmatrix} I_{n-p} & 0 \\ 0 & \frac{1}{x}I_p\end{pmatrix}\right).$$

Thus, $C(0) = \begin{pmatrix} J_{11} & 0 \\ \Psi_{21}(0) & J_{22} - I_p\end{pmatrix}$ and $J_{22} - I_p$ is a Jordan block of size p with eigenvalue $\lambda - 1$. Hence, after a finite steps of such shearing transformations, the equation $xY' = A(x)Y$ can be reduced to $x\tilde{Y}' = \tilde{A}\tilde{Y}$ where $\tilde{A}(0)$ has no two eigenvalues differing by a positive integer. Then, Theorem 4 implies that $\tilde{Y}$ has a fundamental solution of the form $\tilde{P}(x)x^{\tilde{A}_0}$, and so does Y. #

FROBENIUS METHOD FOR SCALAR ODEs WITH REGULAR SINGULAR POINTS

We turn back to solve the ODE (1) with regular singular points at $x=0$. First, use the method given in the introduction section to convert (1) into $xY' = A(x)Y$ where $A(x)$ is of the form (2). By our previous discussions, it is essential to find the eigenvalues of $A_0 = A(0)$; that is, we have to find the characteristic polynomial $\varphi(\lambda) := \det(\lambda I - A_0)$ and then solve $\varphi(\lambda) = 0$ (the so-called "indicial equation" of (1)).

<u>Fact</u>: Let α_k be the x^{-k}-coefficient of $a_k(x)$ of (1) (which is assumed to have a regular singular point at $x=0$). Then

$$\varphi(\lambda) = \lambda(\lambda-1)\cdots(\lambda-n+1) + \lambda(\lambda-1)\cdots(\lambda-n+2)\alpha_1 + \lambda(\lambda-1)\cdots(\lambda-n+3)\alpha_2 + \cdots + \lambda(\lambda-1)\alpha_{n-2} + \lambda\alpha_{n-1} + \alpha_n.$$

(Pf. of Fact.)

Let Δ_{n-k} be the determinant of the $(n-k)\times(n-k)$ matrix derived from $\lambda I - A_0$ by deleting the first k columns and the first k rows of $\lambda I - A_0$. Then, the cofactor expansion of Δ_{n-k} along the first row of it gives the recursive relation $\Delta_{n-k} = (\lambda - k)\Delta_{n-k-1} + \alpha_{n-k}$. Solving this recursive relation (with the aid of $\Delta_1 = \lambda + \alpha_1 - (n-1)$) gives the desired formula for $\varphi(\lambda) = \Delta_n$. #

Now, we discuss the solution to (1) which is assumed to have a regular singular point at $x=0$. Let $A(x)$ be defined as in (2) and let $A_0 = A(0)$, and assume that $A(x)$ is analytic in $|x| < x_0$. Let J_0 be the Jordan canonical form of A_0, say $J_0 = T^{-1}A_0T$, $T \in GL_n(\mathbb{C})$.

<u>I. No two eigenvalues of A_0 differ by an integer</u>

This is the simplist case. By Theorem 4, (and the pf. of Thm 5) $xY' = A(x)Y$ has a fundamental solution of the form $Y = TP(x)x^{J_0}$ where $P(0) = I$ and $P(x)$ is analytic in $|x| < x_0$. As in the introduction section, if $Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$ solves $xY' = A(x)Y$, then we can derive (by induction) that $\begin{cases} Y_1^{(n)} + a_1(x)Y_1^{(n-1)} + \cdots + a_{n-1}(x)Y_1' + a_n(x)Y_1 = 0 \\ Y_j = x^{j-1}Y_1^{(j-1)} \quad (1 \le j \le n) \end{cases}$.

Hence, the column independence of $TP(x)x^{J_0}$ implies the independence of the n entries of the first row of $TP(x)x^{J_0}$, and hence a solution basis for (1) is given by the n entries of the first row of $TP(x)x^{J_0}$, namely $\{y_1, \cdots, y_n\}$ with $y_j = \pi_j(x)x^{\lambda_j}$ $(1 \le j \le n)$, if $J_0 = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ (so that $x^{J_0} = \begin{pmatrix} x^{\lambda_1} & & \\ & \ddots & \\ & & x^{\lambda_n} \end{pmatrix}$).
($\pi_j(x)$ is analytic in $|x| < x_0$ $\forall j$)

<u>II. No two eigenvalues of A_0 differ by a positive integer</u>

In this case, we can still apply Theorem 4 directly, but we have to discuss the structure of x^{J_0} in some details. Write $A = \bigoplus_{i=1}^{r} V_i$ where each V_i is a Jordan block. Let V be a Jordan block in $\{V_1, \cdots, V_r\}$ with eigenvalue λ of size k, say $V = \underbrace{\begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}}_{k}$. Then

$$x^V = e^{V\log x} = x^\lambda \begin{pmatrix} 1 & \log x & \frac{1}{2!}(\log x)^2 & \cdots & \frac{1}{(k-1)!}(\log x)^{k-1} \\ & 1 & \log x & & \frac{1}{(k-2)!}(\log x)^{k-2} \\ & & 1 & & \vdots \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

and then the entries of $(\pi_1(x) \cdots \pi_k(x))x^V$ (where each $\pi_j(x)$ is holomorphic in $|x| < x_0$) give k independent solutions to (1). Collecting all such solutions gives n linearly independent solutions (a solution basis) to (1).

<u>Remark</u>: If we define $(y_1 \cdots y_k) = (\pi_1(x) \cdots \pi_k(x))x^V$ where $\pi_j(x)$ and V are as above, we can derive that $y_j = \pi_j x^\lambda + y_{j-1}(\log x) - y_{j-2}\frac{(\log x)^2}{2!} + \cdots + (-1)^j y_1 \frac{(\log x)^{j-1}}{(j-1)!}$ for each $1 \le j \le k$.

<u>Remark</u>: If A_0 has an eigenvalue λ with multiplicity m, then actually J_0 has exactly one Jordan block corresponding to λ (and the block size is m). (This can be proved directly by observing the powers of $A_0 - \lambda I$.)

III. A_0 has two eigenvalues differ by a positive integer

This case is much more complicated. By means of successive coordinate and shearing transformations, we can reduce $xY' = A(x)Y$ to $xW' = C(x)W$ via $Y = SW$ (where $S(x)$ is entire and invertible when $x \neq 0$) such that C_0 is in Jordan canonical form and no two eigenvalues of C_0 differ by a positive integer, as in Theorem 5. Then we can apply the method in II. and deduce that (1) has a solution basis given by the entries of $(\pi_1(x)\cdots\pi_n(x))\, x^{C_0}$. However, the structure of C_0 is more complicated, unlike the simple structure given in the second remark of II.

Example: Solve $x^3y''' + xy' - y = 0$.

(Sol.) The indicial equation of this ODE is $\lambda(\lambda-1)(\lambda-2) + \lambda - 1 = 0$, i.e. $(\lambda-1)^3 = 0$. So the roots of this indicial equation are $\lambda = 1, 1, 1$. By the discussions in II (together with the remarks), this ODE has a solution basis $\{y_1, y_2, y_3\}$ given by $y_1 = (\sum a_r x^r)x$, $y_2 = (\sum b_r x^r)x + y_1(\log x)$ and $y_3 = (\sum c_r x^r)x + y_2 \log x - y_1 \frac{(\log x)^2}{2!}$. After some calculations, we find that y_1 can be chosen to be $y_1 = x$, and y_2 can be chosen to be $y_2 = x + x\log x$. So $y_3 = (\sum c_r x^r)x + (x + x\log x)\log x - x\frac{(\log x)^2}{2!} = (\sum c_r x^r)x + x\log x + \frac{1}{2}x(\log x)^2$. But then we find that $x(\log x)^2$ solves this ODE. Hence a solution basis for this ODE is given by $\{x = y_1,\ x\log x = y_2 - y_1,\ x(\log x)^2\}$. All solutions to this ODE are of the form $c_1 x + c_2 x\log x + c_3 x(\log x)^2$, $c_1, c_2, c_3 \in \mathbb{C}$. #

APPENDIX

The existence and uniqueness theorem of matrix ODEs:

Consider the ODE $y' = A(x)y$ where $A(x)$ is an $n\times n$ matrix function holomorphic in a simply connected region $S \subset \mathbb{C}$. Then the IVP $\begin{cases} y' = A(x)y \\ y(x_0) = y_0 \end{cases}$ $(x_0 \in S)$ possesses a unique solution y in S.

(Pf.) Since S is simply connected (and is open), we only need to prove this theorem for compact subsets K of S (so that $\operatorname{Int} K \neq \phi$), and then consider continuations. Let $M > 0$ so that $\|A(x)\| \le M$ $\forall x \in K$ (see p.3 for the def. of $\|\cdot\|$). Suppose that D is an open disk in K with radius $\rho < \frac{1}{2M}$, centered at x_0. Consider $\{y_n\}$ defined by $y_0(x) \equiv y_0$ and $y_{n+1}(x) = y_0 + \int_{x_0}^{x} A(t)y_n(t)\,dt$ (the integration path lies in D). Let $m_n := \sup_{x\in D}\|y_{n+1}(x) - y_n(x)\|$. Then $m_{n+1} \le \rho M m_n < \frac{1}{2}m_n$. Thus $y_n \to y$ uniformly in D for some y. Then also $Ay_n \to Ay$ uniformly in D. Thus, letting $n\to\infty$ in the iteration relation for $\{y_n\}$, we get $y(x) = y_0 + \int_{x_0}^{x} A(t)y(t)\,dt$, so that y solves $\begin{cases} y' = A(x)y \\ y(x_0) = y_0 \end{cases}$. If $\tilde{y}$ also solves $\begin{cases} \tilde{y}' = A(x)\tilde{y} \\ \tilde{y}(x_0) = y_0 \end{cases}$, then similarly, for $d = \sup_D \|y - \tilde{y}\|$, we have $d \le \frac{1}{2}d \Rightarrow d = 0$. Hence the solution y is unique in D. (See p.8.)

(Pf. of Thm., cont'd.)

Now, $\forall x_1 \in S$, we may choose a path γ whose image $\Gamma \subset S$ connects x_0 and x_1. Then, since Γ is compact, the distance between Γ and $\mathbb{C} \setminus S$ is positive, and hence we can choose a compact set $K \subset S$ so that $\Gamma \subset K$ and such that the distance between Γ and $\mathbb{C} \setminus K_1$ is greater than $\frac{1}{2M}$ (M as before). Then, we can use finitely many balls of radius $\rho < \frac{1}{2M}$ covering Γ so that no balls are disjoint from all the others. Then, applying the previous result, we can continue the solution y from x_0 to x_1. The continuation is unique since S is simply connected. #

Remark: By this theorem, we can explain the last sentence of the proof of Thm.3 in more details.

In the proof of Theorem 3, we have proved that $z = \sum a_r x^r$ converges in $|x| < x_1$. Let the open sectors I and II be as in Fig.1. By the existence and uniqueness theorem of matrix ODEs, we can find a holomorphic function z_1 on I so that $z = z_1$ in a small disk lying in $\{|x| < x_1\} \cap I$ and hence $z = z_1$ in $\{|x| < x_1\} \cap I$ by the identity principle. Then we have extended z to $z^* := \begin{cases} z & \text{in } |x| < x_1 \\ z_1 & \text{in } I \end{cases}$.

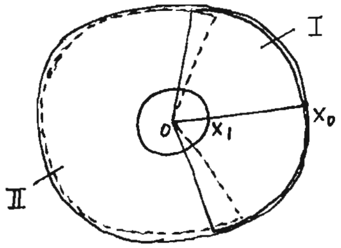


Fig.1

Similarly, we can extend z^* to $z^{**} := \begin{cases} z^* & \text{in } \{|x| < x_1\} \cup I \\ z_2 & \text{in } II \end{cases}$ for some z_2.

Now, z^{**} is analytic in $|x| < x_0$, and $z^{**} = z$ in $|x| < x_1 \Rightarrow \sum a_r x^r$ converges in $|x| < x_0$. #

REFERENCE

Wasow, Asymptotic Expansions for Ordinary Differential Equations (Section 4, 5 and 17 are the main references.) (1965, Interscience edition)

APPENDIX II: ALTERNATIVE APPROACH TO FROBENIUS METHOD 2015.9.12

Here is an alternative approach to solve an ODE having a regular singular point at 0. Such an ODE has the general form

$$Ly=0,$$

where L stands for the linear operator $x^n\frac{d^n}{dx^n}+x^{n-1}P_1(x)\frac{d^{n-1}}{dx^{n-1}}+\cdots+xP_{n-1}(x)\frac{d}{dx}+P_n(x)$ (each $P_j(x)$ is holomorphic in a neighborhood of $x=0$).

To solve the equation $Ly=0$ for $y=y(x)$, we may assume that

$$y=y(x,\lambda)=\sum_{r=0}^{\infty}c_r x^{\lambda+r}.$$

Then, we find that $Ly(x,\lambda)=\sum_{r=0}^{\infty}c_r x^{\lambda+r}f(x,\lambda+r)$, where

$$f(x,\lambda+r)=[\lambda+r]_n+[\lambda+r]_{n-1}P_1(x)+\cdots+[\lambda+r]_1P_{n-1}(x)+P_n(x) \text{ and } [x]_n=x(x-1)\cdots(x-n+1).$$

Rewrite $f(x,\lambda+r)=\sum_{s=0}^{\infty}f_s(\lambda+r)x^s$. Then

$$Ly(x,\lambda)=\sum_{r=0}^{\infty}[c_rf_0(\lambda+r)+c_{r-1}f_1(\lambda+r-1)+\cdots+c_0f_r(\lambda)]x^{\lambda+r}.$$

Hence, $Ly(x,\lambda)=0 \Leftrightarrow c_rf_0(\lambda+r)+c_{r-1}f_1(\lambda+r-1)+\cdots+c_0f_r(\lambda)=0\ \forall r\geq 0$.

Consider the equations

$$c_rf_0(\lambda+r)+c_{r-1}f_1(\lambda+r-1)+\cdots+c_0f_r(\lambda)=0. \qquad (\mathcal{E}_r)$$

It can be seen that, given c_0, then we can find $c_1,c_2,\cdots\in\mathbb{C}(\lambda)$ so that each $(\mathcal{E}_r)$, $r>0$, is satisfied; more precisely,

$$c_N=\frac{c_0F_N(\lambda)}{f_0(\lambda+1)f_0(\lambda+2)\cdots f_0(\lambda+N)} \text{ for some } F_N(\lambda)\in\mathbb{C}[\lambda],\ N>0.$$

For the equation $(\mathcal{E}_0)$, namely $c_0f_0(\lambda)=0$, the equation $f_0(\lambda)=0$ is exactly the so-called "indicial equation;" if $\lambda\in\mathbb{C}$ such that $f_0(\lambda)=0$ but $f_0(\lambda+N)\neq 0$ for all $N\in\mathbb{N}$, then $y=y(x,\lambda)=\sum_{r=0}^{\infty}c_rx^{\lambda+r}$ ($\neq 0$ if we choose $c_0\neq 0$) solves the ODE $Ly=0$ formally, thus analytically by the previous discussions. However, we would like to give another proof for the fact that $y(x,\lambda)=\sum_{r=0}^{\infty}c_rx^{\lambda+r}$ is analytic in the variable x, as a corollary of the following lemma (which tells us more):

Lemma*: Suppose that $P_1(x),\cdots,P_n(x)$ all converge in $|x|<\Gamma$ ($\Gamma>0$), and $c_1,c_2,\cdots\in\mathbb{C}(\lambda)$ are chosen (c_0 fixed) so that each $(\mathcal{E}_r)$, $r>0$, is satisfied, as above. Let Λ be the set of roots of the indicial equation $f_0(\lambda)=0$ (counting multiplicity), and suppose that $\Lambda_0=\{\lambda_0,\lambda_1,\cdots,\lambda_{\alpha-1}\}\subset\Lambda$ satisfies (i) $\lambda_0\geq\lambda_1\geq\cdots\geq\lambda_{\alpha-1}$, (ii) $\lambda_i-\lambda_j\in\mathbb{Z}$ $\forall 0\leq i<j\leq\alpha-1$ and (iii) $\lambda-\lambda'\notin\mathbb{Z}$ $\forall\lambda\in\Lambda_0,\lambda'\in\Lambda\setminus\Lambda_0$. Let $N=\lambda_0-\lambda_{\alpha-1}\in\mathbb{Z}_{\geq 0}$ and let $f^*(\lambda)=f_0(\lambda+1)\cdots f_0(\lambda+N)$. Also let $c_r^*=c_r^*(\lambda)=f^*(\lambda)c_r$. Then, in $|x|<\Gamma$ and in $\lambda\in B(\Lambda_0,\delta)=\{x\in\mathbb{C}:|x-x_0|<\delta \text{ for some } x_0\in\Lambda_0\}$ for some $\delta>0$, the function $\sum_{r=0}^{\infty}c_r^*(\lambda)x^r$ is analytic in both variables, and its convergence is uniform in compact subsets.

(Pf. of Lemma*.)

First observe that $f(x,\lambda+r)=\sum_{s=0}^{\infty} f_s(\lambda+r)x^s=[\lambda+r]_n+[\lambda+r]_{n-1}P_1(x)+\cdots+P_n(x)$ converges in $|x|<\Gamma$, and hence $\frac{\partial}{\partial x}f(x,\lambda+r)=\sum_{s=0}^{\infty}(s+1)f_{s+1}(\lambda+r)x^s$ also converges in $|x|<\Gamma$. Let $\varepsilon>0$ be arbitrary and let $R=\Gamma-\varepsilon$. Let $M(\lambda+r)$ be the least upper bound for $\frac{\partial}{\partial x}f(x,\lambda+r)$ on $|x|=R$. By Cauchy's integral theorem,

$$(s+1)f_{s+1}(\lambda+r)=\frac{1}{2\pi i}\int_R \frac{\frac{\partial}{\partial x}(x,\lambda+r)}{z^{s+1}}dx \Rightarrow |f_{s+1}(\lambda+r)|\le|(s+1)f_{s+1}(\lambda+r)|\le\frac{M(\lambda+r)}{R^s}.$$

Next, observe that $f_0(\lambda+r+1)\ne 0\ \forall r\ge N,\ \lambda\in B(\Lambda_0,\delta)$, if we choose $\delta>0$ small. In this case, all c_r^* are finite in $\lambda\in B(\Lambda_0,\delta)$, and for all $r\ge N$,

$$c_{r+1}^*=-\frac{1}{f_0(\lambda+r+1)}\left(c_r^*f_1(\lambda+r)+c_{r-1}^*f_2(\lambda+r-1)+\cdots+c_0^*f_{r+1}(\lambda)\right)$$

$$\Rightarrow |c_{r+1}^*|\le\frac{1}{|f_0(\lambda+r+1)|}\left(|c_r^*|M(\lambda+r)+|c_{r-1}^*|M(\lambda+r-1)R^{-1}+\cdots+|c_0^*|M(\lambda)R^{-r}\right)=:\mathcal{C}_{r+1}.$$

Thus, for every $r>N$,

$$\mathcal{C}_{r+1}=\frac{|c_r|M(\lambda+r)}{|f_0(\lambda+r+1)|}+\frac{|f_0(\lambda+r)|}{|f_0(\lambda+r+1)|}R^{-1}\mathcal{C}_r \Rightarrow \frac{\mathcal{C}_{r+1}}{\mathcal{C}_r}\le\frac{M(\lambda+r)}{|f_0(\lambda+r+1)|}+\frac{|f_0(\lambda+r)|}{|f_0(\lambda+r+1)|}R^{-1}.$$

Define $A_{N+1}, A_{N+2},\cdots$ by $A_{N+1}=\mathcal{C}_{N+1}$ and $\frac{A_{r+1}}{A_r}=\frac{M(\lambda+r)}{|f_0(\lambda+r+1)|}+\frac{|f_0(\lambda+r)|}{|f_0(\lambda+r+1)|}R^{-1}$ $(r>N)$. Then it is easy to see that $|c_{r+1}^*|\le\mathcal{C}_{r+1}\le A_{r+1}\ \forall r\ge N$.

Next, observe that $\frac{\partial}{\partial x}f(x,\lambda+r)=[\lambda+r]_{n-1}P_1'(x)+\cdots+P_n'(x)$ is a polynomial of degree $n-1$ in $\lambda+r$. Hence, as $r\to\infty$,

$$M(\lambda+r)=\sup_{|x|=R}\left|\frac{\partial}{\partial x}f(x,\lambda+r)\right|=O(r^{n-1}) \text{ uniformly in } \lambda\in B(\Lambda_0,\delta).$$

Also observe that $\left|\frac{f_0(\lambda+r)}{f_0(\lambda+r+1)}\right|\to 1$ as $r\to\infty$, uniformly in $\lambda\in B(\Lambda_0,\delta)$.

Thus, $\frac{A_{r+1}}{A_r}=\frac{O(r^{n-1})}{|f_0(\lambda+r+1)|}+\left|\frac{f_0(\lambda+r)}{f_0(\lambda+r+1)}\right|R^{-1}\to R^{-1}$ as $r\to\infty$, uniformly in $\lambda\in B(\Lambda_0,\delta)$.

From this and from the fact $|c_{r+1}^*|\le\mathcal{C}_{r+1}\le A_{r+1}$, the ratio test implies that $\sum_{r=0}^{\infty}c_r^*x^r$ is analytic in $|x|<R=\Gamma-\varepsilon$ and converges uniformly in $\lambda\in B(\Lambda_0,\delta)$. This implies the conclusion of Lemma*. #

We return to solve our equation $Ly=0$. Use the notations and assumptions in Lemma*, we find that for $y(x,\lambda)=\sum_{r=0}^{\infty}c_rx^{\lambda+r}$ and $\bar{y}(x,\lambda)=\sum_{r=0}^{\infty}c_r^*x^{\lambda+r}=f^*(\lambda)y(x,\lambda)$,

$$Ly(x,\lambda)=c_0f_0(\lambda)x^\lambda \Rightarrow L\bar{y}(x,\lambda)=c_0F(\lambda)x^\lambda,\ F(\lambda)=f_0(\lambda)f^*(\lambda).$$

By Lemma*, $\bar{y}(x,\lambda)$ is analytic in $|x|<\Gamma$, $\lambda\in B(\Lambda_0,\delta)$; hence,

$$\left[L\left\{\frac{\partial^k}{\partial\lambda^k}\bar{y}(x,\lambda)\right\}\right]_{\lambda=\lambda_\mu}=\left[\frac{\partial^k}{\partial\lambda^k}\{L\bar{y}(x,\lambda)\}\right]_{\lambda=\lambda_\mu}=\left[\frac{\partial^k}{\partial\lambda^k}\{c_0F(\lambda)x^\lambda\}\right]_{\lambda=\lambda_\mu}=0$$

for $k=0,1,\cdots,m-1$, where m is the zero order of $F(\lambda)$ in $\lambda=\lambda_\mu\in\Lambda_0$. Hence, $\left\{\left[\frac{\partial^k}{\partial\lambda^k}\bar{y}(x,\lambda)\right]_{\lambda=\lambda_\mu}\ \middle|\ k=0,1,\cdots,m-1\right\}$ is a set of solutions to $Ly=0$.

Having the rough picture above, we can start to solve $Ly=0$. Assume that in Λ_0 (here and below, we continue using the notations and assumptions in Lemma*), we have $\lambda_0=\lambda_1=\cdots=\lambda_{i-1}>\lambda_i=\cdots$.† Then it is easy to see that $F(\lambda)=f_0(\lambda)f^*(\lambda)=f_0(\lambda)f_0(\lambda+1)\cdots f_0(\lambda+N)$ has zero order i at $\lambda=\lambda_0$. It follows that

$$\left\{\left[\frac{\partial^k}{\partial\lambda^k}\bar{y}(x,\lambda)\right]_{\lambda=\lambda_0}=:y_k \;\middle|\; k=0,1,\cdots,i-1\right\} \quad \left(\bar{y}(x,\lambda)=\sum_{r=0}^{\infty}c_r^* x^{\lambda+r}\right)$$

is a set of solutions to $Ly=0$. By Lemma*, we can differentiate $\bar{y}(x,\lambda)=x^\lambda\sum_{r=0}^{\infty}c_r^*x^r$ at $\lambda=\lambda_0$ arbitrarily many times and get

$$\left[\frac{\partial^k}{\partial\lambda^k}\bar{y}(x,\lambda)\right]_{\lambda=\lambda_0}=\left[(\log x)^k\omega_0(x,\lambda)+\binom{k}{1}(\log x)^{k-1}\omega_1(x,\lambda)+\cdots+\omega_k(x,\lambda)\right]_{\lambda=\lambda_0},$$

where $\omega_k(x,\lambda)=\sum_{r=0}^{\infty}(c_r^*)^{(k)}(\lambda)\,x^{\lambda+r}$. (This series still converges in $\underset{x\neq0}{|x|<\Gamma}$ $\forall\lambda\in B(\Lambda_0,\delta)$.) If we choose $c_0\neq0$, then $\omega_0(x,\lambda_0)=\sum_{r=0}^{\infty}c_r^*(\lambda_0)x^{\lambda_0+r}\neq0$ (for $c_0^*(\lambda_0)\neq0$), and hence it can be seen that $\{y_0,y_1,\cdots,y_{i-1}\}$ are linearly independent solutions to $Ly=0$.

Next, consider the second subset of Λ_0 whose elements are identical, say $\lambda_i=\lambda_{i+1}=\cdots=\lambda_{j-1}>\lambda_j=\cdots$. In this case, $F(\lambda)$ has zero order j at $\lambda=\lambda_i$, and hence $\left\{\left[\frac{\partial^k}{\partial\lambda^k}\bar{y}(x,\lambda)\right]_{\lambda=\lambda_i}=:\tilde{y}_k \;\middle|\; k=0,1,\cdots,j-1\right\}$ is a set of solutions to $Ly=0$. If we pick $c_0\neq0$, $\omega_i(x,\lambda_i)$ is not identically zero, for $(c_0^*)^{(i)}(\lambda_i)\neq0$. Hence, if we set $y_k=\tilde{y}_k$ for $k=i,i+1,\cdots,j-1$, we can see that $\{y_i,y_{i+1},\cdots,y_{j-1}\}$ are linearly independent solutions to $Ly=0$.

Proceeding as above, we can find α solutions to $Ly=0$ corresponding to Λ_0; the solutions corresponding to the same "indices" (roots of the indicial equation $f_0(\lambda)=0$) in Λ_0 are linearly independent. Similarly we can find n solutions to $Ly=0$ corresponding to the indices in Λ; again, solutions corresponding to the same indices are linearly independent. It remains to show that these n solutions are linearly independent. We divide the argument into three steps.

<u>Step 1</u>: If $\psi_1(x),\cdots,\psi_n(x)$ are analytic in a neighborhood of $x=0$, and if $\psi_1(x)(\log x)^{n-1}+\cdots+\psi_n(x)\equiv0$ in this neighborhood, then $\psi_1(x)\equiv\cdots\equiv\psi_n(x)\equiv0$.

(Pf.) Suppose that not all $\psi_j(x)$, $1\le j\le n$, are identically zero. Then, after some cancellation of x-powers (if needed), may assume that $\psi_m(0)\neq0$ but $\psi_j(0)=0$ $\forall j\neq m$. Now, $\sum_{j=1}^{n}\psi_j(x)(\log x)^{n-j}\equiv0 \Rightarrow \psi_m(x)+\sum_{j\neq m}\psi_j(x)(\log x)^{m-j}\equiv0$ $\Rightarrow (x\in\mathbb{R}^+,\ x\to0)\ \psi_m(0)=0$ ⨳. #

<u>Step 2</u>: The α solutions $\{y_0,\cdots,y_{\alpha-1}\}$ associated to Λ_0 (as constructed above) are linearly independent.

(Pf.) Set $\Lambda_0=\{\lambda_0=\cdots=\lambda_{n_1-1}>\lambda_{n_1}=\cdots=\lambda_{n_2-1}>\lambda_{n_2}=\cdots>\lambda_{n_k}=\cdots=\lambda_{\alpha-1}\}$ and let $n_0=0$. (See p. 12.)

† When we write "$\lambda_a>\lambda_b$" for $\lambda_a,\lambda_b\in\Lambda_0$, we mean that $\lambda_a-\lambda_b>0$.

(Pf. of Step 2, cont'd.)

Observe that $c_0^*(\lambda) = f^*(\lambda)c_0$ $(c_0 \neq 0)$ has zero order n_j at $\lambda = \lambda_{n_j}$. This shows that $w_{n_j}(0, \lambda_{n_j}) = \left[\sum_{r=0}^{\infty}(c_r^*)^{(n_j)}(\lambda_{n_j})\, x^{\lambda_{n_j}+r}\right]_{x=0} = (c_0^*)^{(n_j)}(\lambda_{n_j}) \neq 0$ and $w_s(0, \lambda_{n_j}) = 0$ for $0 \leq s < n_j$. Now, assume that $\beta_0, \cdots, \beta_{\alpha-1} \in \mathbb{C}$ and that $\beta_0 y_0 + \cdots + \beta_{\alpha-1} y_{\alpha-1} \equiv 0$. Observe that $\left[\frac{\partial^{\alpha-1}}{\partial\lambda^{\alpha-1}} \bar{y}(x,\lambda)\right]_{\lambda=\lambda_{\alpha-1}=\lambda_{n_k}} =: y_{\alpha-1} = \sum_{j=0}^{\alpha-1}\binom{\alpha-1}{j}(\log x)^{\alpha-1-j}\, w_j(x, \lambda_{n_k})$.

If we view all y_k as "polynomials" of $(\log x)$ with analytic coefficients, in the sense of Step 1, then Step 1 shows that the sum of the analytic coefficients of $(\log x)^{\alpha-1-n_k}$ of the $\beta_j y_j$'s $(0 \leq j \leq \alpha-1)$ is identically zero; meanwhile, in these analytic coefficients, $\beta_{\alpha-1} w_{n_k}(x, \lambda_{n_k})$ (corresponding to the $\beta_{\alpha-1} y_{\alpha-1}$ term) possesses the "lowest" possible degree λ_{n_k} in x, hence must vanish $\Rightarrow \beta_{\alpha-1} = 0$. Similarly, $\beta_{\alpha-2} = \cdots = \beta_0 = 0$, and thus $\{y_0, \cdots, y_{\alpha-1}\}$ are linearly independent. #

Step 3: The n solutions to $Ly = 0$ constructed above are linearly independent and thus form a solution basis for $Ly = 0$.

(Pf.) Let $\Lambda = \Lambda_0 \dot{\cup} \Lambda_1 \dot{\cup} \cdots \dot{\cup} \Lambda_k$ where every two elements in the same Λ_j differ by an integer, but every two elements from distinct Λ_j's do not differ by an integer. Denote the corresponding solutions to Λ_j by $\{y_{j,1}, \cdots, y_{j,n_j}\}$. Now, suppose that $\beta_{s,t} \in \mathbb{C}$ so that $\sum_{j=0}^{k}\sum_{l=1}^{n_j} \beta_{j,l}\, y_{j,l} \equiv 0$. If we continue the variable x for r rounds around $x = 0$, we get $\sum_{j=0}^{k} \theta_j^r \sum_{l=1}^{n_j} \beta_{j,l}\, y_{j,l} \equiv 0$ where $\theta_0, \cdots, \theta_k \in \mathbb{C}$ are distinct, $r \in \mathbb{Z}_{\geq 0}$. This shows that $\sum_l \beta_{j,l}\, y_{j,l} \equiv 0$ $\forall j$, and thus $\beta_{j,l} = 0$ $\forall j, l$ by Step 2. #

Example: We shall use the materials developed in this section to solve the example $x^3y''' + xy' - y = 0$ in Page 7 again.

(Sol.) The indicial equation of this ODE is $(\lambda-1)^3 = 0$ (as in P.7). If we put $y = x^\lambda \sum_{r=0}^{\infty} c_r x^r$ into this ODE and solve $c_r = c_r(\lambda)$ formally in $\mathbb{C}(\lambda)$, and pick $c_0 = 1$ meanwhile, we find that all $c_r(\lambda)$, $r \in \mathbb{N}$, vanish. Thus, a solution basis for this ODE is given by $\left\{\left[\frac{\partial^k}{\partial\lambda^k}\left(x^\lambda \sum_{r=0}^{\infty} c_r x^r\right)\right]_{\lambda=1} \,\middle|\, k = 0,1,2\right\} = \left\{\left[\frac{\partial^k}{\partial\lambda^k} x^\lambda\right]_{\lambda=1} \,\middle|\, k=0,1,2\right\} = \{x, x\log x, x(\log x)^2\}$. #

REFERENCE FOR APPENDIX II

Ince, Ordinary Differential Equations, Chapter XVI. (1956, Dover edition)

Hecke Operators 陳學儀 B01201042.

Hecke Operators

Definition: (1) Let R be the set of lattices of $\mathbb{C}$, then we define the hecke operator $T(n)$.

$$T(n)\Gamma := \sum_{[\Gamma':\Gamma]=n} \Gamma' \quad \text{where } \Gamma \in R,\ \Gamma' \text{ be the sublattice of } \Gamma \text{ of index } n.$$
$n \in \mathbb{N}$

(2) homothety operators R_λ for $\lambda \in \mathbb{C}^*$

$$R_\lambda \Gamma := \lambda \Gamma$$

Recall: $[\Gamma':\Gamma]=n$ is equivalent to Γ/Γ' has order n. and thus $n\Gamma \subseteq \Gamma'$ $\forall\, \Gamma'$, which implies that the number of Γ' is equal to the number of subgroups of order n in $\Gamma/n\Gamma \simeq (\mathbb{Z}/n\mathbb{Z})^2$. So if n is prime, the number of such lattices is $n+1$.

Proposition (1). $R_\lambda R_\mu = R_{\lambda\mu}$ $\forall\, \lambda,\mu \in \mathbb{C}^*$

(2) $R_\lambda T(n) = T(n) R_\lambda$ $\forall\, \lambda \in \mathbb{C}^*$ $n \geq 1$.

(3) $T(m)T(n) = T(mn)$ for $\gcd(m,n)=1$.

(4). $T(p^n)T(p) = T(p^{n+1}) + p\,T(p^{n-1})R_p$ p prime, $n \geq 1$.

<pf>:

(1), (2) follows from definition.

(3). it suffices to show that $\forall\, \Gamma''$ of index mn, $\exists!\, \Gamma'$ s.t $\Gamma'' \subseteq \Gamma' \subseteq \Gamma$ with $[\Gamma:\Gamma']=n$ and $[\Gamma':\Gamma'']=m$, which follows from Γ/Γ'' of order mn decomposes uniquely into a direct sum of a group of order m and n.

(4). Let Γ be a lattice, then $T(p^n)T(p)\Gamma$, $T(p^{n+1})\Gamma$, $T(p^{n-1})R_p\Gamma$ are linear combinations of sublattice of index p^{n+1} (note that $R_p\Gamma$ is of index p^2)

Let Γ'' be a lattice of index p^{n+1}, and a, b, c be the correspondence coefficient, the we just need to prove that $a = b + pc$, obviously, $b=1$

(i) $\Gamma'' \not\subseteq p\Gamma$, then $c=0$, and a is the number of Γ' between Γ, Γ'' of index p. Then we consider $\Gamma/p\Gamma$, the image of Γ' in $\Gamma/p\Gamma$ is of index p, which contains the image of Γ''. And since $\Gamma'' \not\subseteq p\Gamma$, so $\Gamma' \neq 0$, so the image of Γ'' is also of index p, and then we pick the image of Γ'' in $\Gamma/p\Gamma$ as Γ', which is unique so $a=$

(ii) $\Gamma'' \subseteq p\Gamma$, then $c = 1$, and since the number of sub-lattices of index p is $p+1$, so $a = p+1$.

And thus $a = b + pc$ ☆

Corollary (1) $T(p^n)$ $n > 1$ are polynomials in $T(p)$ and R_p.

(2) The algebra generated by the R_λ and the $T(p)$, p prime is commutatives and contains all the $T(n)$.

Lemma Let $S_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b\ d \in \mathbb{Z},\ ad = n \quad a \geq 1,\ 0 \leq b < d \right\}$

if $\sigma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in S_n$, define $\Gamma_\sigma = \mathbb{Z}w_1' + \mathbb{Z}w_2'$ $w_1' = aw_1 + bw_2$ $w_2' = dw_2$.

be sublattice of $\Gamma = \mathbb{Z}w_1 + \mathbb{Z}w_2$.

then f $S_n \to \Gamma(n) := \{\Gamma' : [\Gamma\ \Gamma'] = n\}$ is a bijection

$\sigma \mapsto \Gamma_\sigma$

<pf>. Obviously $\Gamma_\sigma \in \Gamma(n)$ since $\det(\sigma) = n$.

Conversely, let $\Gamma' \in \Gamma(n)$, we can assume that $\Gamma' = \mathbb{Z}w_1' + \mathbb{Z}w_2'$ with $w_1' = aw_1 + bw_2$, $w_2' = dw_2$, $a \geq 1$, $d \geq 1$, $0 \leq b < d$.

then we just need to show that $ad = n$

Consider $\Gamma / \Gamma' + \mathbb{Z}w_2$ and $\mathbb{Z}w_2 / \mathbb{Z}w_2 \cap \Gamma'$

then $\quad \overset{\text{is}}{\mathbb{Z}/a\mathbb{Z}} \qquad \overset{\text{is}}{\mathbb{Z}/d\mathbb{Z}}$

and $0 \to \mathbb{Z}w_2 / \mathbb{Z}w_2 \cap \Gamma' \to \Gamma/\Gamma' \to \Gamma / \Gamma' + \mathbb{Z}w_2 \to 0$ is an exact sequence.

thus $ad = n$. ☆.

Modular functions and Lattice functions.

Definition (1). f is a modular function of weight $2k$ if f is mero. on $\mathbb{H} \cup \{\infty\}$ and $f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right)$ $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

(2) If f is holo. on $\mathbb{H} \cup \{\infty\}$, then f is a modular form.

(3). If $f(\infty) = 0$, then f is a cusp form.

Definition (4). Let F be a function on $\mathcal{R}$ (the set of lattices of $\mathbb{C}$).
We say that F is of weight $2k$ if
$$F(\lambda\Gamma) = \lambda^{-2k} F(\Gamma), \quad \Gamma \in \mathcal{R}, \ \lambda \in \mathbb{C}^*$$

Then we can see that $F(\lambda\omega_1, \lambda\omega_2) = \lambda^{-2k} F(\omega_1, \omega_2)$.
Moreover, $F(\omega_1, \omega_2)$ is invariant by the action of $SL_2(\mathbb{Z})$.
Also, $\omega_2^{2k} F(\omega_1, \omega_2)$ depends only on $z = \omega_1/\omega_2$,
so there exists a function f on $\mathbb{H}$ s.t. $F(\omega_1, \omega_2) = \omega_2^{-2k} f(\omega_1/\omega_2)$.
then we see that $f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

Conversely, for a modular function f, we can define $F(\omega_1, \omega_2) = \omega_2^{-2k} f(\omega_1/\omega_2)$ and get a lattice function.
Thus, modular function $f \xleftrightarrow{1-1}$ lattice function F.

Action of $T(n)$ on lattice function F

Let F be a lattice function of wt. $2k$. Then by definition.
$$R_\lambda F = \lambda^{-2k} F \quad \forall \lambda \in \mathbb{C}^*$$
$$T(n)(F(\Gamma)) = F(T(n)\Gamma).$$
$\Rightarrow R_\lambda(T(n)F) = T(n)(R_\lambda F) = \lambda^{-2k}(T(n)F)$. thus $T(n)F$ is also of wt. $2k$
then we can get
(1) $T(m)T(n)F = T(mn)F$ if $(m,n) = 1$.
(2) $T(p)T(p^n)F = T(p^{n+1})F + p^{1-2k}T(p^{n-1})F$ p prime, $n \geq 1$

Action of $T(n)$ on modular function f

Let f be a modular function of wt. $2k$, corresponds to a lattice function F then we define $T(n)f$ as $n^{2k-1}T(n)F$

Definition
$$T(n)f(z) := n^{2k-1} T(n) F(z,1)$$
$$= n^{2k-1} \sum_{\sigma \in S_n} F(az+b, d) = n^{2k-1} \sum_{\substack{a \geq 1 \\ ad = n \\ 0 \leq b < d}} d^{-2k} f\left(\frac{az+b}{d}\right)$$

And thus $T(n)f$ is also a modular function of wt. $2k$

Proposition: (1) $T(m)T(n)f = T(mn)f$ if $(m,n)=1$.

(2). $T(p)T(p^n)f = T(p^{n+1})f + p^{2k-1}T(p^{n-1})f$ if p is prime, $n \geq 1$.

Thus if f is mero on $\mathbb{H}$, so is $T(n)f$
" holo "

Behavior at Infinity

We suppose that f is a modular function i.e., mero at ∞

let $f(z) = \sum_{m=-N}^{\infty} c(m) q^m$ be its Laurent expansion w.r.t. $q = e^{2\pi i z}$

Proposition $T(n)f = \sum_{m=-N'}^{\infty} \gamma(m) q^m$ with $\gamma(m) = \sum_{\substack{a|(n,m) \\ a \geq 1}} a^{2k-1} c\left(\frac{mn}{a^2}\right)$.

<pf>. By def. $T(n)f(z) = n^{2k-1} \sum_{\substack{ad=n,\, a\geq 1 \\ 0 \leq b < d}} d^{-2k} \sum_{m=-N}^{\infty} c(m) e^{2\pi i m \left(\frac{az+b}{d}\right)}$

if $d \mid m$, $\sum_{0 \leq b < d} e^{2\pi i b \cdot \frac{m}{d}} = d$, o.w. $= 0$

take $m = m/d$

then. $T(n)f(z) = n^{2k-1} \sum_{\substack{ad=n \\ a\geq 1}} d^{-2k+1} \sum_{m'=-N'}^{\infty} c(m'd)\, q^{am'} = \sum_{m=-N'}^{\infty} q^m \sum_{\substack{a|(n,m) \\ a\geq 1}} \left(\frac{n}{d}\right)^{2k-1} c\left(\frac{md}{a}\right)$.

Corollary: (1) $\gamma(0) = \sigma_{2k-1}(n)\, c(0)$, $\gamma(1) = c(n)$.

(2). if $n = p$, p prime

$$\gamma(m) = \begin{cases} c(pm) & \text{if } m \not\equiv 0 \pmod p. \\ c(pm) + p^{2k-1} c\left(\frac{m}{p}\right) & \text{if } m \equiv 0 \pmod p. \end{cases}$$

(3) if f is a modular form (resp. a cusp form), so is $T(n)f$.

Thus, $T(n)$ act on M_k and M_k^0, which is commute and satisfy

$T(m)T(n) = T(mn)$ if $(m,n)=1$.

$T(p)T(p^n) = T(p^{n+1}) + p^{2k-1}T(p^{n-1})$. if p is prime, $n \geq 1$.

Eigenfunctions of T(n).

Let $f(z)$ be a modular form of wt. $2k$, $\not\equiv 0$, we assume that $f(z)$ is an eigenfunction of all the $T(n)$,

i.e., $\exists\ \lambda(n) \in \mathbb{C}$ s.t $T(n)f = \lambda(n) f \quad \forall\ n \geq 1$.

Theorem: $f(z) = \sum_{n=0}^{\infty} c(n) q^n$, f is an eigenfunction of $T(n)$, then

(1). $c(1) \neq 0$ o.w. $f \equiv 0$.

(2). if f is normalized by $c(1)=1$, then $c(n) = \lambda(n) \quad \forall\ n > 1$.

<pf>

$T(n)f(z) = \sum_{m=0}^{\infty} \gamma(m) q^m = \sum_{m=0}^{\infty} \lambda(n)\, c(m) q^m$ so $\gamma(m) = \lambda(n)\, c(m)$.

by Cor. $\gamma(1) = c(n) = \lambda(n) c(1)$, so if $c(1) = 0$, $c(n) = 0 \quad \forall n$.

and if $c(1) = 1$, $c(n) = \lambda(n)$.

Corollary: (1) 2 modular forms of wt. $2k$, $k>0$, which are eigenfunctions of $T(n)$ with the same eigenvalues $\lambda(n)$, and which are normalized by $c(1)=1$, coincide

(2). If $c(1) = 1$ for some eigenfunction $f(z) = \sum_{n=1}^{\infty} c(n) q^n$

then $c(m)\,c(n) = c(mn)$ if $(m,n) = 1$

$c(p)\,c(p^n) = c(p^{n+1}) + p^{2k-1} c(p^{n-1})$ p prime.

<pf> (2) since $\lambda(m)\lambda(n) f = T(m)\,T(n) f = T(mn) f = \lambda(mn) f$ and $\lambda(n) = c(n)$.

$$T(p)\,T(p^n) f = \lambda(p)\lambda(p^n) f = T(p^{n+1}) f + p^{2k-1}\, T(p^{n-1}) f$$
$$= \lambda(p^{n+1}) f + p^{2k-1} \lambda(p^{n-1}) f$$

Dirichlet Series.

Definition. $\Phi_f(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$, $f = \sum_{n=0}^{\infty} c(n) q^n$ be an eigenfunction.

Proposition: $\Phi_f(s)$ converges absolutely for $Re(s) > 2k$

<pf>. By Corollary on page. 9 $c(n) = O(n^{2k-1})$. and thus $\Phi_f(s)$ converges absolutely for $Re(s) > 2k$.

Theorem $\Phi_f(s) = \prod_{p \in P} \dfrac{1}{1 - c(p)p^{-s} + p^{2k-1-2s}}$

<pf>. since $c(m)c(n) = c(mn)$ if $(m,n) = 1$.

thus $\sum_{n=1}^{\infty} \dfrac{c(n)}{n^s} = \prod_p \left(\sum_{n=0}^{\infty} \dfrac{c(p^n)}{p^{ns}} \right)$.

consider $\underbrace{\left(\sum_{n=0}^{\infty} c(p^n) \cdot T^s \right)\left(1 - c(p)T + p^{2k-1}T^2 \right)}_{(*)}$ where $T = p^{-s}$

for the coefficient of T^{n+1}

$n \geq 1$: $c(p^{n+1}) - c(p^n)c(p) + c(p^{n-1})p^{2k-1} = 0$

$n = 0$: $c(p) - c(p) = 0$

$n = -1$: $c(1) = 1$.

thus $(*) = 1$. ※

Remark: Conversely, if $\Phi_f(s) = \prod_{p \in P} 1/(1 - c(p)p^{-s} + p^{2k-1-2s})$, then

(1) $c(m)c(n) = c(mn)$ if $(m,n) = 1$.

(2) $c(p)c(p^n) = c(p^{n+1}) + p^{2k-1}c(p^{n-1})$. p prime $n > 1$.

<pf> $1/(1 - c(p)p^{-s} + p^{2k-1-2s}) = 1 + c(p)p^{-s} + \cdots = \sum_{n=0}^{\infty} \dfrac{c(p^n)}{p^{ns}}$

and thus (1) is obvious. for (2),

$1 = \left(\sum_{n=0}^{\infty} \dfrac{c(p^n)}{p^{ns}} \right)\left(1 - c(p)p^{-s} + p^{2k-1-2s} \right)$ after compute will get (2).

Analytic Continuation of Dirichlet Series

Theorem: $\Phi_f(s)$ can be continued analytically beyond the line $Re(s) = 2k$ with the following properties:

(1) If f is a cusp form, then $\Phi_f(s)$ is an entire function of s

(2) If not, $\Phi_f(s)$ has only one simple pole at $s = 2k$, with residue $(-1)^k f(\infty)(2\pi)^{2k} / \Gamma(2k)$.

(3) Let $X_f(s) = (2\pi)^{-s}\Gamma(s)\Phi_f(s)$, then it satisfies the func eq.

$X_f(s) = (-1)^k X_f(2f - s)$.

<pf> Claim $X_f(s) = \int_0^\infty [f(iy) - f(\infty)] y^{s-1} dy$ for $Re(s) > 2k$

$$(2\pi)^{-s}\Gamma(s)\Phi_f(s) = \frac{1}{(2\pi)^s}\left(\int_0^\infty e^{-t} t^{s-1} dt\right)\left(\sum_{n=1}^\infty \frac{c(n)}{n^s}\right) = \frac{1}{(2\pi)^s}\int_0^\infty \sum_{n=1}^\infty \frac{c(n)}{n^s} t^s e^{-t}\frac{dt}{t}$$

$$= \int_0^\infty \sum_{n=1}^\infty c(n)\left(\frac{t}{2\pi n}\right)^s e^{-t}\frac{dt}{t} = \int_0^\infty \sum_{n=1}^\infty c(n) y^s e^{-2\pi n y}\frac{dy}{y} = \int_0^\infty [f(iy) - f(\infty)] y^{s-1} dy$$

Since $f(-1/z) = z^{2k} f(z)$ so $f(i/y) = (iy)^{2k} f(iy)$

$$X_f(s) = \int_1^\infty [f(iy) - f(\infty)] y^{s-1} dy + \underline{\int_0^1 [f(i/y)(iy)^{-2k} - f(\infty)] y^s \frac{dy}{y}} \quad (**)$$

$$(**) = \int_0^1 f(i/y)(-1)^k y^{-2k+s-1} dy - f(\infty)\left.\frac{y^s}{s}\right|_0^1$$

$$\overset{\parallel}{(-1)^k\int_\infty^1 f(iw)\, w^{2k-s+1}\left(\frac{-1}{w^2}\right)dw} \qquad \overset{\parallel}{f(\infty)/s} \quad \text{since } Re(s) > 2k > 0$$

$$= (-1)^k \int_1^\infty f(iw)\, w^{2k-s-1} dw = (-1)^k \int_1^\infty [f(iw) - f(\infty)] w^{2k-s-1} dw + (-1)^k\int_1^\infty f(\infty) w^{2k-s-1} dw$$

$$\int_1^\infty f(\infty) w^{2k-s-1} dw = f(\infty)\left.\frac{w^{2k-s}}{2k-s}\right|_1^\infty = f(\infty)/s-2k \quad \text{since } Re(2k-s) < 0$$

$$\Rightarrow X_f(s) = \int_1^\infty [f(iy) - f(\infty)]\left(y^s + (-1)^k y^{2k-s}\right)\frac{dy}{y} + f(\infty)\left[\frac{-1}{s} + \frac{(-1)^k}{s-2k}\right]$$

Since $f(iy) - f(\infty) = \sum_{n=1}^\infty c(n) e^{-2\pi n y} = e^{-2\pi n y}\left(\sum_{n=0}^\infty c(n+1) e^{-2\pi n y}\right)$ decrease exponentially

thus RHS is meaningful $\forall\, s \in \mathbb{C}$.

Moreover, if f is a cusp form i.e., $f(\infty) = 0$, $X_f(s)$ is entire.

then $\Phi_f(s) = (2\pi)^s X_f(s)/\Gamma(s)$ is also entire.

Otherwise, $X_f(s)$ has simple poles at $s = 0,\ 2k$

with residue $-f(\infty),\ f(\infty)(-1)^k$

And since $\Gamma(s)$ has simple poles at $s = 0, -1, -2, \ldots$,

so $\Phi_f(s)$ has an unique simple pole at $s = 2k$ with residue $(-1)^k f(\infty)(2\pi)^{2k}/\Gamma(2k)$

$$(-1)^k X_f(2k-s) = \int_1^\infty [f(iy) - f(\infty)]\left((-1)^k y^{2k-s} + y^s\right)\frac{dy}{y}$$

$$+ f(\infty)\left[\frac{(-1)^{k+1}}{2k-s} + \frac{}{-s}\right] = X_f(s)$$

✡.

The Petersson scalar product.

Definition Let f, g be two cusp forms of wt $2k$

(1) $\mu(f, g) = f(z)\overline{g(z)}\, y^{2k-2}\, dx\, dy$ $\quad x = Re(z) \quad y = Im(z)$.

(2) $\langle f, g \rangle = \int_D \mu(f, g) = \int_D f(z)\overline{g(z)}\, y^{2k-2}\, dx\, dy$ $\quad D$ fund. domain.

Fact (1) $\mu(f, g)$ is invariant by $SL_2(\mathbb{Z})$.

(2) $\langle T(n)f, g \rangle = \langle f, T(n)g \rangle$

Thus $T(n)$ are hermitian operators w.r.t $\langle f, g \rangle$.
Since $T(n)$ commute with each other, there exists an orthogonal basis of M_k^0 made of eigenvectors of $T(n)$ also, the eigenvalues of $T(n)$ are real numbers.

Reference Serre. A Course in Arithmetic.

Appendix

Theorem If f is a cusp form of wt. $2k$, $f(z) = \sum_{n=0}^{\infty} a_n q^n$, then $a_n = O(n^k)$

<pf> $a_0 = 0$, so $|f(z)| = O(q) = O(e^{-2\pi y})$ $\quad y = \mathrm{Im}(z)$ as $y \to \infty$

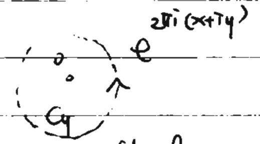
let $\phi(z) = |f(z)|\, y^k$, then it is invariant under $SL_2(\mathbb{Z})$.

- $\phi(z+1) = |f(z+1)|\, y^k = |f(z)|\, y^k$
- $\phi(-1/z) = |f(-1/z)|\,(\mathrm{Im}(-1/z))^k = |z|^{2k}|f(z)|\, y^k \frac{1}{|z|^{2k}} = \phi(z)$.

Also $\phi(z)$ is conti. on fund. domain of $SL_2(\mathbb{Z})$ and $\lim_{y\to\infty} \phi(z) = 0$,

this implies ϕ is bounded on whole $\mathbb{H}$

$\Rightarrow |f(z)| \le M y^{-k} \quad \forall z \in \mathbb{H}$.



Then we fix $y > 0$, and vary x between $0, 1$

then $$a_n = \frac{1}{2\pi i}\int_{C_y} \frac{f(z)}{q^{n+1}}\, dq = \frac{2\pi i}{2\pi i}\int_0^1 \frac{f(x+iy)}{q^{n+1}}\, q\, dx = \int_0^1 \frac{f(x+iy)}{q^n}\, dx$$

$$\Rightarrow |a_n| \le \frac{|f(x+iy)|}{|e^{2\pi i(x+iy)n}|} \le M y^{-k} e^{2\pi n y} \quad \forall y > 0$$

pick $y = 1/n$, then $|a_n| \le e^{2\pi} M n^k$ $\quad$ ※

Recall We know that $E_{2k}(z) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$

then $$\frac{a_n}{n^{2k-1}} = 2\frac{(2\pi i)^{2k}}{(2k-1)}\frac{\sigma_{2k-1}(n)}{n^{2k-1}} \quad n \ge 1$$

$$= A\left(\sum_{d|n}\left(\frac{d}{n}\right)^{2k-1}\right) = A\sum_{d|n}\frac{1}{d^{2k-1}} \le A\sum_{d=1}^{\infty}\frac{1}{d^{2k-1}} = A\,\zeta(2k-1) < \infty$$

$$\Rightarrow a_n = O(n^{2k-1})$$

Corollary: If f is not a cusp, $a_n = O(n^{2k-1})$, weight $2k$

<pf> then we consider $$f(z) - \frac{E_{2k}(z)}{2\zeta(2k)}\cdot a_0 = \sum_{n=0}^{\infty} a_n q^n - a_0 - \frac{1}{\zeta(2k)}\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{k=1}^{\infty}\sigma_{2k-1}(n) q^n$$

$$= \sum_{n=1}^{\infty} b_n q^n$$ and is cusp form.

then $b_n = O(n^k)$ $\quad a_n = O(n^{2k-1})$.

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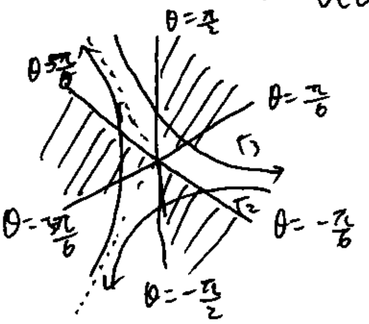
The Asymptotic of Airy Function

Consider the differential equation $y'' = sy$, it has an irregular singular point at $s = \infty$ So the solution near ∞ has a monomial term, exponential term and asymptotic term

To solve the equation, assume $y = \int_\Gamma v(t) e^{st} dt$ and we can solve that $v(t) = e^{-\frac{t^3}{3}}$ and Γ has three kinds.

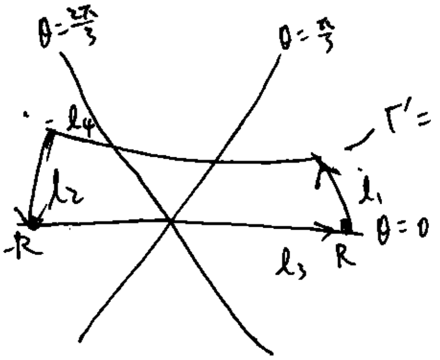
(cf Wasow p124 - p125)



Now defined $Ai(s) = \frac{1}{2\pi i} \int_{\Gamma_1} e^{-\frac{t^3}{3} + st} dt$

Claim: $Ai(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(x^3 + sx)} dx$

Thus $\forall$ such Γ_1, we get a unique answer and thus $Ai(s)$ is well-defined. Since the exponential decay of $\left| e^{-\frac{t^3}{3} + st} \right|$ along Γ_1, $Ai(s)$ is entire $\forall s \in \mathbb{C}$



$\Gamma' = \frac{\Gamma_1}{i}$

$$\frac{1}{2\pi i} \int_{\Gamma_1} e^{-\frac{t^3}{3} + st} dt = \frac{1}{2\pi} \int_{\frac{\Gamma_1}{i}} e^{i(\frac{z^3}{3} + sz)} dz$$

It suffices to prove that $\int_{l_1} e^{i(\frac{z^3}{3} + sz)} dz$, $\int_{l_2} e^{i(\frac{z^3}{3} + sz)} dz \to 0$ as $R \to \infty$

Since on l_1, l_2, $y < \sqrt{3}\, c\, x$ for some $x \in (0,1)$, $\left| e^{i(\frac{z^3}{3} + sz)} \right| = O(e^{-d y x^2})$. $\left| \int_{l_1} + \int_{l_2} \left| e^{i(\frac{z^3}{3} + sz)} \right| dz \right| = \int_0^{\frac{\pi}{3}} R\, O(e^{-d R^3 \cos^2\theta \sin\theta}) d\theta$

$$= \int_0^{\delta = \sin^{-1}(\frac{1}{R^{2.5}})} R\, O(e^{-dR^3 \cos^2\theta \sin\theta}) d\theta + \int_\delta^{\frac{\pi}{3}} R\, O(e^{-dR^3 \cos^2\theta \sin\theta}) d\theta$$

$$= O(\tfrac{1}{R^{\frac{1}{2}}}) + O(e^{-d'R}) \longrightarrow 0 \text{ as } R \to \infty$$

So $Ai(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\frac{z^3}{3} + sz)} dz$

Now we can use Laplace's method to find the monomial term and exponential term, and the concept of saddle point helps us to find the asymptotic expansion

Laplace's method:

Consider $\int_a^b e^{-s\Phi(x)}\psi(x)\,dx$, where Φ,ψ are smooth, Φ attends it's minimum on (a,b), and $\Phi''(x)>0$ on $[a,b]$, then ~~When~~ when $Re(s)>0$, $|s|\to\infty$, $\int_a^b e^{-s\Phi(x)}\psi(x)\,dx = e^{-s\Phi(x_0)}\left[\frac{A}{s^{\frac12}}+O\left(\frac{1}{|s|}\right)\right]$

where $A=\sqrt{2\pi}\,\frac{\psi(x_0)}{(\Phi''(x_0))^{\frac12}}$, Φ attends it's minimum at x_0

Lemma: For fixed $a>0$, $0>m>-1$, $Re(s)\geq 0$, $|s|\to\infty$,

$$\int_0^a e^{-sx}x^m\,dx = s^{-m-1}\Gamma(m+1)+O(1/|s|)$$

Proof. For $Re(s)\geq 0$, $s\neq 0$, $\int_0^N e^{-sx}x^m\,dx = \int_0^a e^{-sx}x^m\,dx + \int_a^N e^{-sx}x^m\,dx$.

Since $m>-1$, the first integral is analytic.

$\int_a^N e^{-sx}x^m\,dx = \frac{m}{s}\int_a^N e^{-sx}x^{m-1}\,dx - \frac{e^{-sx}\,x^m}{s}\Big|_a^N$. Since $Re(s)\geq 0$, $m-1<-1$, take $N\to\infty$, $\int_a^N e^{-sx}x^m\,dx$ is analytic when $Re(s)>0$ and continuous when $Re(s)=0$. So $\int_0^\infty e^{-sx}x^m\,dx$ is analytic when $Re(s)>0$, that should be $s^{-m-1}\Gamma(m+1)$ and it's valid for $Re(s)=0$. $\left|\int_a^N e^{-sx}x^m\right| \leq \left|\frac{m}{s}\right|\int_a^N x^{m-1}\,dx + \frac{N^m-a^m}{|s|} = O\left(\frac{1}{|s|}\right)$

Now back to Laplace's method. By let $\Phi^*(x)=\Phi(x)-\Phi(x_0)$ we may assume $\Phi(x_0)=0$. Write $\Phi(x)=\frac{(x-x_0)^2}{2}\Phi''(x_0)\varphi(x)$ where $\varphi(x)=1+O(x-x_0)$

Locally, we can change the variable $y=(x-x_0)(\varphi(x))^{\frac12}$, where $(\varphi(x))^{\frac12}$ is locally defined by $e^{\frac12\log(\varphi(x))}$ $\frac{dy}{dx}\Big|_{x_0}=$, $\frac{dx}{dy}=1+O(y)$.

So for $x_0\in[a',b']$, $b'-a'$ small, $\int_{a'}^{b'} e^{-s\Phi(x)}\psi(x)\,dx$

$= \int_\alpha^\beta e^{-s\frac{\Phi''(x_0)}{2}y^2}\psi(x_0)\,dy + O\left(\int_\alpha^\beta e^{-s\frac{\Phi''(x_0)}{2}y^2}|y|\,dy\right)$ Since $\psi(x)=\psi(x_0)+O(y)$. Let $y^2=z$, $dy=\frac12 z^{-\frac12}dz$, the integral equals to

$\psi(x_0)\int_0^{a_0} e^{-s\frac{\Phi''(x_0)}{2}z}z^{-\frac12}\,dz$ ~~$+O(e^{-s})$~~ $= s^{-\frac12}\left(\frac{2\pi}{\Phi''(x_0)}\right)^{\frac12}+O\left(\frac{1}{|s|}\right)$

There are still some part of the term $\int_{\bar{a}}^{\bar{b}} e^{-s\Phi(x)} \psi(x)\,dx$, where $\Phi(x)>0$ $|\Phi'(x)| > c > 0$ on $[\bar{a},\bar{b}]$. The integral equals to $-\frac{1}{s}\int_{\bar{a}}^{\bar{b}} d\left(e^{-s\Phi(x)}\right)\frac{\psi(x)}{\Phi'(x)}$

$$= \frac{1}{s}\int_{\bar{a}}^{\bar{b}} e^{-s\Phi(x)} \frac{d}{dx}\left(\frac{\psi(x)}{\Phi'(x)}\right)dx - \frac{1}{s} e^{-s\Phi(x)} \frac{\psi(x)}{\Phi'(x)}\Bigg|_{\bar{a}}^{\bar{b}} = O\left(\frac{1}{|s|}\right)$$

The rest is that $\int_0^r e^{-s\frac{\Phi''(x_0)}{2}y^2} y\,dy = O\left(\frac{1}{|s|}\right)$ The integral is

$$-\frac{1}{s\Phi''(x_0)}\int_0^r \frac{d}{dy}\left(e^{-s\frac{\Phi''(x_0)}{2}y^2}\right)dy = O\left(\frac{1}{|s|}\right)$$ Now the theorem is proved.

Back to Airy function $Ai(s)$, We have when $s>0$, $s\to\infty$,

(i) $Ai(-s) = \frac{1}{2\pi^{\frac{1}{2}}} s^{-\frac{1}{4}}\left[e^{i\left(\frac{2}{3}s^{\frac{3}{2}}-\frac{\pi}{4}\right)}\left(1+O\left(\frac{1}{s^{\frac{3}{2}}}\right)\right)+e^{-i\left(\frac{2}{3}s^{\frac{3}{2}}-\frac{\pi}{4}\right)}\left(1+O\left(\frac{1}{s^{\frac{3}{2}}}\right)\right)\right]$

(ii) $Ai(s) = \frac{1}{2\pi^{\frac{1}{2}}} s^{-\frac{1}{4}} e^{-\frac{2}{3}s^{\frac{3}{2}}}\left(1+O\left(1/s^{\frac{3}{2}}\right)\right)$

Note that $\int_a^R e^{i\left(\frac{x^3}{3}+sx\right)}dx = \int_a^R \frac{1}{i(x^2+s)}\,d\left(e^{i\left(\frac{x^3}{3}+sx\right)}\right)$

$$= \frac{1}{x^2+s}e^{i\left(\frac{x^3}{3}+sx\right)}\Bigg|_a^R + \int_a^R \frac{2x}{-i(x^2+s)^2}e^{i\left(\frac{x^3}{3}+sx\right)}dx,$$ the integral

$\int_a^\infty e^{i\left(\frac{x^3}{3}+sx\right)}dx$, $\int_{-\infty}^b e^{i\left(\frac{x^3}{3}+sx\right)}dx$ exists for real s.

(i): $Ai(-s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{x^3}{3}-sx\right)}dx$. let $x=s^{\frac{1}{2}}y$, the integral is

$$\frac{s^{\frac{1}{2}}}{2\pi}\int_{-\infty}^{\infty} e^{is^{\frac{3}{2}}\left(\frac{y^3}{3}-y\right)}dy$$ let $I_-(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\left(\frac{u^3}{3}-u\right)}du$.

The function $\frac{u^3}{3}-u$ has two critical points $\pm$ -1 is local maximal while 1 is local minimal. But it is pure imaginary. So we take both point Now $(-\infty,\infty) = (-\infty,-2]\cup[-2,0]\cup[0,2]\cup[2,\infty]$

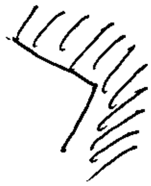
On $[0,2]$, set $v=-it$, $\Phi(x) = \frac{x^3}{3}-x$, $\Phi(1) = -\frac{2}{3}$, $\Phi''(1) = 2$,

We get $\int_0^2 \frac{1}{2\pi}e^{-v\Phi(x)}dx = \frac{1}{2\sqrt{\pi}}e^{-\frac{2}{3}it}\left(\frac{1}{(-it)^{\frac{1}{2}}}+O\left(\frac{1}{|t|}\right)\right)$

On $[-2,0]$, set $v=it$, $\Phi^*(x) = -\frac{x^3}{3}+x$, similarly we get

$$\int_{-2}^{0} \frac{e^{it(\frac{x^3}{3}-x)}}{2\pi}\,dx = \frac{1}{2\sqrt{\pi}}\, e^{\frac{2}{3}it}\left(\frac{1}{(it)^{\frac{1}{2}}} + O\left(\frac{1}{|t|}\right)\right).$$

$$\int_{-\infty}^{-2} e^{it\Phi(x)}\,dx = \lim_{N\to\infty}\frac{1}{it}\int_{-N}^{-2}\frac{d}{dx}\left(e^{it\Phi(x)}\right)\frac{dx}{x^2-1}$$
$$= \lim_{N\to\infty}\frac{1}{it}\left(\int_{-N}^{-2}\frac{2x}{(x^2-1)^2}\,e^{it\Phi(x)}\,dx + \frac{e^{it\Phi(x)}}{x^2-1}\Bigg|_{-N}^{-2}\right) = O\left(\frac{1}{|t|}\right)$$

Similarly, $\int_{2}^{\infty} e^{it\Phi(x)}\,dx = O\left(\frac{1}{|t|}\right)$. $O\left(\frac{1}{|t|}\right)$ means $e^{ist}\,O\left(\frac{1}{|t|}\right)$ for all real s' Note that in the expansion, square root is defined as $\sqrt{x} = e^{\frac{1}{2}\log x}$, where log is defined on  as single-valued. So we get $Ai(-s) =$

$$\frac{1}{2\pi^{\frac{1}{2}}s^{\frac{1}{4}}}e^{i\left(\frac{2}{3}s^{\frac{3}{2}}-\frac{\pi}{4}\right)}\left(1+O\left(\frac{1}{|s^{\frac{3}{2}}|}\right)\right) +$$

$$\frac{}{2\pi^{\frac{1}{2}}s^{\frac{1}{4}}}e^{-i\left(\frac{2}{3}s^{\frac{3}{2}}-\frac{\pi}{4}\right)}\left(1+O\left(\frac{1}{|s^{\frac{3}{2}}|}\right)\right)$$

Note that s is positive so I'm confused that why I write $|s^{\frac{3}{2}}|$ instead of $s^{\frac{3}{2}}$

$Ai(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\frac{x^3}{3}+sx)}dx$. Let $v^2=s$, $Ai(v^2) =$ $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\frac{x^3}{3}+v^2x)}dx$. Let $y=\frac{x}{v}$, $Ai(v^2) =$ $\frac{v}{2\pi}\int_{-\infty}^{\infty} e^{iv^3(\frac{y^3}{3}+y)}dy$. Now we have to seek a better contour, since $\frac{y^3}{3}+y$ has no critical point on $\mathbb{R}$. In fact, it has two critical point $\pm i$, so we seek for a contour containing i and $\frac{y^3}{3}+y$ on it is pure imaginary.

Let $y=u+iw$, $\frac{y^3}{3}+y = \frac{(u+iw)^3}{3}+(u+iw) = \frac{u^3-3uw^2+3u}{3} + i\frac{u^2w-w^3+w}{3}$. $u(u^2-3w^2+3)=0$ We take the contour $w^2=\frac{u^2}{3}+1$;

Note that the contour is desirable. So

$Ai(v^2) = \frac{v}{2\pi}\int_\Gamma e^{(\frac{8}{9}x^2+\frac{2}{3})(\frac{x^2}{3}+1)^{\frac{1}{2}}\cdot(-v^3)}dz$ Let $F(x) = (\frac{8}{9}x^2+\frac{2}{3})(\frac{x^2}{3}+1)^{\frac{1}{2}}$

$dz = d(x+iy) = dx(1+i\frac{dy}{dx})$. F is even, $\frac{dy}{dx} = \frac{x}{3y}$ is an odd function for x Since $e^{-v^3F(x)}$ decays exponentially,

$Ai(v^2) = \frac{v}{2\pi}\int_{-\infty}^{\infty} e^{-v^3F(x)}dx$ $F(x) = \frac{2}{3}+x^2+O(x^4)$ as $x\to 0$

So $F''(0)=2$. Take $c>0$, $\frac{1}{2\pi}\int_{-c}^{c} e^{-v^3F(x)}dx =$

$e^{-\frac{2}{3}v^3}\left[\frac{1}{2\pi^{\frac{1}{2}}v^{\frac{3}{2}}} + o\left(\frac{1}{|v|}\right)\right]$. $\int_c^{\infty} e^{-v^3F(x)}dx < \int_c^{\infty} e^{-v^3(\frac{2}{3}+\frac{8}{9}x^2)}dx$

$$= e^{-\frac{2}{3}v^3}\int_c^\infty e^{-\frac{8}{9}v^3x^2}\,dx = e^{-\frac{2}{3}v^3}\left(\int_c^\infty \frac{1}{-2sx}\, d\left(e^{-sx^2}\right)\right)$$

$$= e^{-\frac{2}{3}v^3}\left(\frac{-1}{2sx}e^{-sx^2}\Big|_c^\infty + \int_c^\infty -\frac{e^{-sx^2}}{2sx^2}\,dx\right)$$

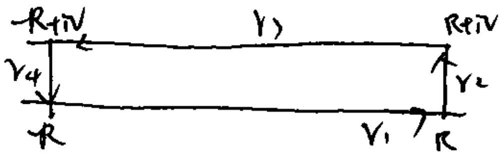
$$= e^{-\frac{2}{3}v^3}\left(O(e^{-\delta_1 s}) + \int_{c^2}^\infty -\frac{e^{-su}}{4su^{\frac{3}{2}}}\,du\right)$$

$= e^{-\frac{2}{3}v^3}\left(O(e^{-\delta s})\right)$. Similarly, $\int_{-\infty}^{-c} e^{-v^3F(x)}\,dx$ $= e^{-\frac{2}{3}v^3}O(e^{-\delta s})$. So $Ai(s) = \frac{1}{2\pi^{\frac{1}{2}}s^{\frac{1}{4}}}e^{\frac{2}{3}s^{\frac{3}{2}}}\left(1+O\left(\frac{1}{s^{\frac{3}{2}}}\right)\right)$.

All asymptotic term comes from $\int_{-c}^{c} e^{-v^3F(x)}\,dx$.

Back to the three solution of the differential equation $y''=sy$ Take proper $\Gamma_1, \Gamma_2, \Gamma_3$, the corresponding solutions u_1, u_2, u_3 satisfying that $u_1+u_2+u_3=0$ and $w^2u_2(w^2z) = u_1(z) = wu_3(wz)$, where $w = e^{\frac{2\pi}{3}}$ Since $u_1 = Ai$, we get $Ai(z) + wAi(wz) + w^2Ai(w^2z) = 0$ Now we consider another contour integral. Let $L_v = \{x+iv \mid x\in\mathbb{R}\}$ for real $v>0$

$$\int_{L_v} e^{i\left(\frac{z^3}{3}+v^2z\right)}\,dz = \int_{-\infty}^{\infty} e^{i\left(\frac{(x+iv)^3}{3}+v^2(x+iv)\right)}\,dx$$



$$= \int_{-\infty}^{\infty} e^{i\left(\frac{x^3}{3}+ivx^2+\frac{2}{3}iv^3\right)}\,dx$$

$= e^{-\frac{2}{3}v^3}\int_{-\infty}^{\infty} e^{-vx^2+\frac{1}{3}ix^3}\,dx$. $\left|\int_0^v e^{i\left(\frac{(R+it)^3}{3}+v^2(R+it)\right)}\,dt\right|$

$\le \int_0^v e^{\frac{1}{3}v^3-R^2t}\,dt \longrightarrow 0$ as $R\longrightarrow\infty$.

So $Ai(v^2) = \frac{e^{-\frac{2}{3}v^3}}{2\pi}\int_{-\infty}^{\infty} e^{-vx^2+\frac{1}{3}ix^3}\,dx = \frac{e^{-\frac{2}{3}v^3}}{\pi}\int_0^\infty e^{-vx^2}\cos\left(\frac{1}{3}x^3\right)dx$

Note that $Ai(v^2)$ is holomorphic on $Re(v)>0$, $\int_0^\infty e^{-vx^2}\cos(\frac{1}{3}x^3)\,dx$ converges uniformly on any compact set $\subset \{Re(v)>0\}$, so that is a holomorphic function on $Re(v)>0$. So the definition of $Ai(v^2) = \frac{1}{\pi} e^{-\frac{2}{3}v^3}\int_0^\infty e^{-vx^2}\cos(\frac{1}{3}x^3)\,dx$ is valid on $Re(v)>0$.

Let $t = x^2$, $Ai(v^2) = \frac{1}{2\pi} e^{-\frac{2}{3}v^3}\int_0^\infty e^{-vt}\cos(\frac{1}{3}t^{\frac{3}{2}})\frac{dt}{t^{\frac{1}{2}}}$

We can expand $\cos(\frac{1}{3}t^{\frac{3}{2}})$ and integrate term by term to get the asymptotic expansion (cf CTM 550 CH6, Watson's Lemma)

$$\frac{\cos(\frac{1}{3}t^{\frac{3}{2}})}{t^{\frac{1}{2}}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{2n}(2n)!}t^{3n-\frac{1}{2}} \qquad \int_0^\infty e^{-vt}\cdot\frac{(-1)^n}{3^{2n}(2n)!}t^{3n-\frac{1}{2}}\,dt$$

$$= \frac{\Gamma(3n+\frac{1}{2})(-1)^n}{3^{2n}(2n)!\,v^{3n+\frac{1}{2}}}$$ So we get $Ai(z) \sim \frac{1}{2\pi z^{\frac{1}{4}}} e^{-\frac{2}{3}z^{\frac{3}{2}}}\sum_{n=0}^{\infty}\frac{\Gamma(3n+\frac{1}{2})(-1)^n}{3^{2n}(2n)!\,z^{\frac{3}{2}n}}$.

The asymptotic is valid on any sector $\theta\in(-\pi+\delta, \pi-\delta)$ $\forall\ \delta>0$

Now for $\arg(z)\in(\frac{1}{3}\pi, \frac{5}{3}\pi)$, write $z = \varsigma e^{\pi i}$ and consider $Ai(z) = -wAi(wz) - w^2Ai(w^2z)$, we get

$$Ai(\varsigma e^{\pi i}) \sim \frac{1}{2\pi\varsigma^{\frac{1}{4}}} e^{\frac{2}{3}i\varsigma^{\frac{3}{2}}}\sum_{n=0}^{\infty}\frac{\Gamma(3n+\frac{1}{2})\,e^{-\frac{1}{4}(2n+1)\pi i}}{3^{2n}(2n)\ \varsigma^{\frac{3}{2}n}}$$

$$+ \frac{1}{2\pi\varsigma^{\frac{1}{4}}} e^{-\frac{2}{3}i\varsigma^{\frac{3}{2}}}\sum_{n=0}^{\infty}\frac{\Gamma(3n+\frac{1}{2})\,e^{\frac{1}{4}(2n+1)\pi i}}{3^{2n}(2n)!\ \varsigma^{\frac{3}{2}n}}$$

Remark 1. The Shearing in solving Airy's function showing that the asymptotic should be a power series of $x^{-\frac{1}{2}}$, not x^{-1}

Remark 2. We can't write $\frac{e^{i\theta}+e^{-i\theta}}{2} = \cos(\theta)$ in the first estimation of $Ai(-s)$ as $s\to\infty$

Remark 3. If we really solve the origin differential equation, after shearing, the Stoke's rays are $\theta = \frac{(2n+1)\pi}{6}$ For the solution Ai, $\theta = \pm\frac{\pi}{6}$ are "fake", so the asymptotic expansion is valid for $\theta\in(-\pi+\delta, \pi-\delta)$ (cf Wasow P.132)