Complex Analysis II

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Final Reports Week I

[1] June 9 黃哲宏 Big Picard Theorem

[2] June 11 李昱陞 Modular Forms and Moduli Problem

[3] June 11 林肱慶 (Confluent) Hypergeometric Functions

Picard's Big Theorem

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Hurwitz's Theorem (Ahlfors.p178)

For $n = 1, 2, \cdots$ $\Omega \xrightarrow[\text{holo.}]{f_n} \mathbb{C}\setminus\{0\}$,

$f_n \to f$ $\underbrace{\text{uniformly on every } K \subseteq_{\text{cpt.}} \Omega}$ $\Rightarrow \Omega \xrightarrow[\text{holo.}]{f} \mathbb{C}\setminus\{0\}$ or $f \equiv 0$

or, equivalently on a nbd. of each $p \in \Omega$

since each K is covered by finitely many such nbd.s & conversely each $p \in \Omega$ is covered by a closed disk; in this case we say "locally uniformly" or "normally".

Proof Suppose that $f \not\equiv 0$, so the zeros are iso.

Fix $z_0 \in \Omega$ Take $r > 0$ so that $f(z)$ is defined and $\neq 0$ on $0 < |z - z_0| \leq r$. Thus $\inf_{z \in C} |f(z)| > 0$, where $C = \{|z - z_0| = r\}$ So $\frac{1}{f_n} \to \frac{1}{f}$ and $f_n' \to f'$ uniformly on C Hence

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \lim_{n\to\infty} \frac{1}{2\pi i}\int_C \frac{f_n'(z)}{f_n(z)}dz = 0$$

and thus $f(z_0) \neq 0$

Theorem (Ahlfors.p226 with the spherical metric)

If $\Omega \xrightarrow[\text{holo.}]{f_n} \mathbb{C}^*$ & $f_n \to f$ normally, then $\Omega \xrightarrow[\text{holo.}]{f} \mathbb{C}^*$

The spherical metric: By $\mathbb{C}^* \hookrightarrow \mathbb{R}^3$ we have

$ds = \frac{2|dz|}{1+|z|^2} = \frac{2|dz^{-1}|}{1+|z|^{-2}}$ Then for $p, q \in \mathbb{C}^*$,

$\underline{d(p,q)} := \inf_{\gamma} \int_\gamma \frac{2|dz|}{1+|z|^2}$ γ: path from p to q

arc length of geodesics

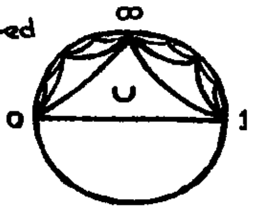
Proof We know that f is cts. w.r.t. d

If $f(z_0) \neq \infty$, then in a nbd. of z_0, f is bdd. and hence $f_n \neq \infty$ for all large enough n, thus f is analytic in the nbd.

If $f(z_0) = \infty$, by considering $\frac{1}{f}$ and $\frac{1}{f_n}$, we conclude that $\frac{1}{f}$ is analytic near z_0, hence f is mero. near z_0 Moreover, if f_n are holo., then $\frac{1}{f} \equiv 0$ by Hurwitz's thm hence $f \equiv \infty$

We know that there exists an universal cover of the twice-punctured plane by the unit disk:

$$\mathbb{D} \cong \mathbb{H} \xrightarrow{\lambda} \mathbb{C}\setminus\{0,1\}$$



Montel's Theorem: If Ω is any region and $\mathcal{F} = \{f \mid f$ is an analytic function on Ω with $f(\Omega) \subseteq \mathbb{C}\setminus\{0,1\}\}$, then $\mathcal{F}$ is normal w.r.t. $\mathbb{C}^*$

Since normality is a local property, we may assume our region to be the unit disk $\mathbb{D}$ Suppose that $\{f_n\}$ is a sequence in $\mathcal{F}$ Fix $z_0 \in \Omega$ Passing to subsequence if necessary, we may assume that $f_n(z_0) \to \alpha \in \mathbb{C}^*$ We could lift $\{f_n\}$ to a sequence $\{g_n : \mathbb{D} \to \mathbb{D}\}$, with all $g_n(z_0)$ lie in a fundamental domain U, which is (uniformly) bounded, hence normal But it is not immediate that $\{f_n\}$ would inherit normality Our goal is to find a subsequence of $\{f_n\}$ which is uniformly bounded on compact sets under some assumption on $\{f_n\}$

Proof Consider WLOG $f_n : \mathbb{D} \xrightarrow[\text{holo.}]{} \mathbb{C}\setminus\{0,1\}$ $(n = 1, 2, \cdots)$ If $\{f_n\}$ has no subseq. conv. normally to a const., then nor does the lifted seq. $\{g_n : \mathbb{D} \to \mathbb{D}\}^{(i)}$, which however has a subseq. $\{h_n\}$ conv. normally to some $h \neq$ const. So $h : \mathbb{D} \xrightarrow[\text{holo.}]{} \mathbb{D}$ by Hurwitz's thm. Hence $\{\lambda \circ h_n\}$ is locally bdd.$^{(ii)}$ □

(i) If $g_{n_k} \to a$ normally, then for all $K \subseteq_{\text{cpt}} \mathbb{D}$ & $\varepsilon > 0$, $g_{n_k}(K) \subset \lambda^{-1}(B_\varepsilon(\lambda(a)))$, hence $f_{n_k}(K) \subset B_\varepsilon(\lambda(a))$ for all large enough k. (Note $\lambda(a) = \alpha$)

(ii) For all $K \subseteq_{\text{cpt}} \mathbb{D}$, $h(K) \subset B_\varepsilon(0)$ for some $0 < \varepsilon < 1$, hence, for all large enough n, $h_n(K) \subset B_\varepsilon(0)$, hence $(\lambda \circ h_n)(K) \subset \lambda(\overline{B_\varepsilon(0)})$ (Note $\lambda^{-1}(\infty) \subset \partial\mathbb{D}$)

Picard's Big Theorem: If f has an isolated essential singularity at z_0 then in any small neighborhood of z_0 f attains every complex value infinitely many times with at most one exception.

The idea is to "zoom into" z_0 By the normality test we may pass to a limit and thus conclude that z_0 is either removable or a pole

Proof Let $z_0 = 0$ and define $f_n(z) = f(2^{-n}z)$ for every n large enough that f_n is defined and analytic on $0 < |z| < 2$ Then $f_{n_k}(z) \to F(z)$ uniformly on $\frac{1}{2} \le |z| \le 1$, where either $F \equiv \infty$ or F is analytic In the latter case, f is bounded near 0 by the maximum principle, hence 0 is a removable singularity In the former case, we conclude in the same way that 0 is a pole.

Preliminaries & Related Topics

A family of functions $\mathcal{F}$ is normal iff every seq. in $\mathcal{F}$ contains a subseq. which conv. normally (iff $\overline{\mathcal{F}}$ is cpt. Ahlfors. p221)

Theorem (Arzelà-Ascoli) (Ahlfors p222) A family $\mathcal{F}$ of cts. funcs from $\Omega \subset \mathbb{C}$ to a metric space S is normal iff (i) $\mathcal{F}$ is locally equicts. & (ii) $\forall z \in \Omega, \exists K \underset{\text{cpt.}}{\subseteq} S$ s.t. $\{f(z) | f \in \mathcal{F}\} \le K$

Corollary A family of mero. func.s is normal iff it is locally equicts. ($S = \mathbb{C}^*$)

For all mero. func. f $f^\# := \frac{2|f'|}{1+|f|^2}$ is called the spherical derivative.

Theorem (Marty) (Ahlfors p226) A family $\mathcal{F}$ of mero. func.s is normal iff $\{f^\# | f \in \mathcal{F}\}$ is locally bdd.

Remark Normality is a local property.

Theorem (Montel) (Ahlfors p224) A family of holo. func.s is normal w.r.t. $\mathbb{C}$ (i.e., with $S = \mathbb{C}$) iff it is locally bdd.

With the holo. uni. cov $\mathbb{D} \cong \mathbb{H} \xrightarrow{\lambda} \mathbb{C} \setminus \{0,1\}$, we start from the following two facts:

$$\begin{cases} \text{A } \overset{\text{(uni.)}}{\text{bdd.}} \text{ family of holo. func.s is normal} \\ \text{A bdd. entire func. reduces to a const.} \end{cases}$$

Suppose now that $f : \mathbb{C} \underset{\text{holo.}}{\longrightarrow} \mathbb{C} \setminus \{0,1\}$. $\exists F : \mathbb{C} \underset{\text{holo.}}{\longrightarrow} \mathbb{D}$ s.t. $\lambda \circ F = f$, i.e.

$$\begin{tikzcd} & \mathbb{D} \arrow[d, "\lambda"] \\ \mathbb{C} \arrow[ur, "F"] \arrow[r, "f"'] & \mathbb{C} \setminus \{0,1\} \end{tikzcd}$$

Then F = const. so f = const. and this proves

Picard's Little Theorem: An entire function omitting 2 values reduces to a constant.

Montel's Theorem: A family of holo. funcs omitting 2 values is normal w.r.t. $\mathbb{C}^*$.

Q: Is it true that "what forces entire func.s to be const. also makes holo. familes normal"?
— Bloch's heuristic principle

Zalcman's Principle: Suppose that P is a property of mero. func.s satisfying

(i) If $\langle f\ \Omega \rangle \in P$, then $\langle f|_{\Omega'}, \Omega' \rangle \in P$ for all $\Omega' \subset \Omega$

(ii) If $\langle f, \Omega \rangle \in P$ and $\varphi(z) = az + b$ with $a, b \in \mathbb{C}$, $a \ne 0$ then $\langle f \circ \varphi, \varphi^{-1}(\Omega) \rangle \in P$

(iii) If $\langle f_n, \Omega_n \rangle \in P$ $(n = 1, 2, \cdots)$ with $\Omega_1 \subset \Omega_2 \subset \cdots$ and $\bigcup_{n=1}^{\infty} \Omega_n = \mathbb{C}$, then ($f_n \to f$ normally on $\mathbb{C} \Rightarrow \langle f, \mathbb{C} \rangle \in P$)

Then ($\langle f, \mathbb{C} \rangle \in P \Rightarrow f$ = const.) if and only if ($\mathcal{F} := \{f | \langle f, \Omega \rangle \in P\}$ is normal for every $\Omega \subset \mathbb{C}$)

Conjecture: Let $\{U, \quad U_n\}$ be an open cover for $\mathbb{D} \setminus \{0\}$ by connected sets and let $f_j : U_j \underset{\text{holo.}}{\longrightarrow} \mathbb{C}$ with $df_i = df_j$ on $U_i \cap U_j$ Then the df_j's glue together to a mero. 1-form on $\mathbb{D}$.

Remark They do glue to a holo. 1-form $g\,dz$ on $\mathbb{D} \setminus \{0\}$ (trivially) If $\operatorname{Res}_{z=0} g = 0$, then $\exists f : \mathbb{D} \setminus \{0\} \underset{\text{holo.}}{\longrightarrow} \mathbb{C}$ with $df = g\,dz$, so $f = f_j + c_j$ for some c_j = const. ($j = 1, \cdots, n$) hence f is mero. on $\mathbb{D}$ by Picard's Big Theorem, so is $g\,dz$

Recall $\bullet\ \theta(z,\tau) = \sum e^{\pi i n^2\tau} e^{2\pi i n z}$, $\theta(z+b,\tau) = \theta(z,\tau)\quad b \in \mathbb{N}$

$$\theta(z+a\tau,\tau) = e^{-\pi i a^2\tau} e^{-2\pi i a z}\theta(z,\tau) \quad a \in \mathbb{N}$$

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For $a, b \in \mathbb{R}$ $\quad (S_a f)(z) := f(z+a)$

$$(T_a f)(z) := e^{\pi i a^2\tau} e^{2\pi i a z} f(z+a\tau)$$

Let $\theta_{00}(z,\tau) = \theta(z,\tau)$, $\theta_{01}(z,\tau) = S_{\frac{1}{2}}\cdot T_0\,\theta(z,\tau)$, $\theta_{10}(z,\tau) = S_0\cdot T_{\frac{1}{2}}\,\theta(z,\tau)$

$$\theta_{11}(z,\tau) = S_{\frac{1}{2}}\cdot T_{\frac{1}{2}}\,\theta(z,\tau)$$

$$\varphi_2 : E_\tau \longrightarrow \mathbb{P}^3$$

$$z \longmapsto (\theta_{00}(2z), \theta_{01}(2z), \theta_{10}(2z), \theta_{11}(2z)) = (X_0, X_1, X_2, X_3)$$

$\operatorname{Im}\varphi_2$ is the curve $C_\tau = \begin{cases} \theta_{00}^2 X_0^2 = \theta_{01}^2 X_1^2 + \theta_{10}^2 X_2^2 \\ \theta_{00}^2 X_3^2 = \theta_{10}^2 X_1^2 - \theta_{01}^2 X_2^2 \end{cases}$ $\quad * \ \theta_{00} = \theta_{00}(0)$, etc

- Jacobi 's identity : $\theta_{00}^4 = \theta_{01}^4 + \theta_{10}^4$

Action on θ

For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, ab, cd even, and assume $c \geq 0$

Consider $\theta((c\tau+d)y, \tau)$

$$+1 \longmapsto e^{-\pi i c^2\tau} e^{-2\pi i c(c\tau+d)y}\theta((c\tau+d)y,\tau)$$

let $\psi(y,\tau) = e^{\pi i c(c\tau+d)y^2}\theta((c\tau+d)y,\tau)$

$$+1 \longmapsto \psi(y,\tau)$$

$$+\frac{a\tau+b}{c\tau+d} \longmapsto e^{\pi i \frac{a\tau+b}{c\tau+d} - 2\pi i y}\psi(y,\tau)$$

$$\frac{\psi(y+\frac{a\tau+b}{c\tau+d},\tau)}{\theta((c\tau+d)y+a\tau+b,\tau)} = e^{\pi i c(\cancel{c}\tau+d)y^2 + 2\pi i c(a\tau+b)y + \pi i c\frac{(a\tau+d)^2}{c\tau+d}}$$

$$-2\pi i y$$

$$\frac{\theta((c\tau+d)y+a\tau+b,\tau)}{\psi(y,\tau)} = e^{-\pi i a^2\tau - 2\pi i a(c\tau+d)y - \pi i \cancel{c}(c\tau+d)y^2}$$

$$c(a\tau+b)^2 - a^2(c\tau+d)\tau = (2abc - a^2d)\tau + cb^2$$

$$= (a\tau+b) - ab(c\tau+d)$$

$\Rightarrow \psi(y,\tau) = \varphi(\tau)\,\theta(y, \frac{a\tau+b}{c\tau+d})$

$\theta(z,\tau)$ is normalized by $\int_0^1 \theta(z,\tau)dz = 1$

$$\therefore \varphi(\tau) = \int_0^1 \psi(y,\tau)dy = \int_0^1 e^{\pi i c(c\tau+d)y^2}\sum e^{\pi i n^2\tau} e^{2\pi i n(c\tau+d)y}dy$$

$$= \sum e^{-\pi i n^2\frac{d}{c}}\int_0^1 e^{\pi i(cy+n)^2(\tau+\frac{d}{c})}dy \qquad (\text{ if } c=0 \quad \varphi(\tau)=1)$$

$$= \sum_{1\leq n\leq c} e^{-\pi i n^2\frac{d}{c}}\int_{-\infty}^{\infty} e^{\pi i c^2y^2(\tau+\frac{d}{c})}dy$$

take Re > 0

when $\tau = it^{7^0} - \frac{d}{c}$, $\int_{-\infty}^{\infty} e^{\pi i c^2y^2(\tau+\frac{d}{c})}dy = \frac{1}{ct^2}$

$\therefore$ from analytic continuation $\int_{-\infty}^{\infty} e^{\pi i c^2y^2(\tau+\frac{d}{c})}dy = \dfrac{1}{c\cdot(\tau+\frac{d}{c}/i)^{\frac{1}{2}}}$

let $z = (c\tau+d)y \Rightarrow \theta(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}) = \dfrac{c^{\frac{1}{2}}\cdot(i)^{-\frac{1}{2}}}{S_{d.c}}(c\tau+d)^{\frac{1}{2}}e^{\pi i c\frac{z^2}{c\tau+d}}\theta(z,\tau)$

and $S_{d.c}$ is actually $c^{\frac{1}{2}}$ times some 8th root of unity

For odd prime p, the Legendre symbol is defined as $\left(\frac{a}{p}\right) := \begin{cases} a \equiv x \ \text{mod}\ p \\ -1 & \text{o.w} \end{cases}$

As a generalization, for odd $n = p_1^{d_1} \cdots p_K^{d_K}$, the Jacobi symbol is

$$\left(\frac{a}{n}\right) := \left(\frac{a}{p_1}\right)^{d_1} \cdots \left(\frac{a}{p_K}\right)^{d_K}$$

Some property : $\left(\frac{-1}{n}\right) = (-1)^{\frac{n-1}{2}}$, $\left(\frac{2}{n}\right) = (-1)^{\frac{n^2-1}{8}}$

$$\left(\frac{m}{n}\right)\left(\frac{n}{m}\right) = (-1)^{\frac{n-1}{2} \cdot \frac{m-1}{2}} \text{ for m.n odd positive and coprime}$$

Thm. $\theta\left(\frac{z}{c\tau+d}, \frac{a\tau+d}{c\tau+d}\right) = \zeta (c\tau+d)^{\frac{1}{2}} e^{\pi i c \frac{z^2}{c\tau+d}} \theta(z,\tau) \cdots (F)$

- if c is even, d is odd, $\zeta = (i)^{\frac{d-1}{2}} \left(\frac{c}{|d|}\right)$
- if c is odd, d is even $\zeta = (i)^{-\frac{c}{2}} \left(\frac{d}{c}\right)$

Pf. For $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ b even. $\theta(z, \tau+b) = \theta(z,\tau)$ — (1)

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $\theta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = (i)^{-\frac{1}{2}} \tau^{\frac{1}{2}} e^{\pi i \frac{z^2}{\tau}} \theta(z,\tau)$ (2) since $S_{0,1} = 1$

∴ thm holds in these cases, we then use induction on $c + |d|$

(i) If $|d| > c$, choose ± s.t. $|d \pm 2c| < d$

replace τ by $\tau \pm 2$ in (F) and use (1)

$$\Rightarrow \theta\left(\frac{z}{c\tau+(d\pm 2c)}, \frac{a\tau+(b\pm 2a)}{c\tau+(d\pm 2c)}\right) = \zeta (c\tau+(d\pm 2c))^{\frac{1}{2}} e^{\pi i c \frac{z^2}{c\tau+(d\pm 2c)}} \theta(z,\tau)$$

compare ζ for (c, d) and $(c, d\pm 2c)$:

if c is odd, $(i)^{-\frac{c}{2}}\left(\frac{d\pm 2c}{c}\right) = (i)^{-\frac{c}{2}}\left(\frac{d}{c}\right)$

if c is even, $d > 0$, $d - 2c > 0$

$$\left(\frac{c}{d-2c}\right)\left(\frac{2}{d-2c}\right) = \left(\frac{d}{d-2c}\right) = \left(\frac{d-2c}{d}\right)(-1)^{\frac{d-1}{2}\frac{d-2c-1}{2}} = \left(\frac{c}{d}\right)\left(\frac{2}{d}\right)(-1)^{\frac{d-1}{2} \cancel{+} \frac{d-1}{2}\frac{d-2c-1}{2}}$$

$$\Rightarrow \left(\frac{c}{d-2c}\right) = (-1)^{\frac{c(c+1)}{2}}\left(\frac{c}{d}\right) \Rightarrow (i)^{\frac{d-2c-1}{2}}\left(\frac{c}{d-2c}\right) = (i)^{\frac{d-1}{2}}\left(\frac{c}{d}\right)$$

the other cases ($d-2c<0$, $d<0$) are similar

(ii) If $|d| < c$, replace τ by $-\frac{1}{\tau}$ and z by $\frac{z}{\tau}$ in (F), and use (2)

$$\Rightarrow \theta\left(\frac{z}{d\tau-c}, \frac{b\tau-a}{d\tau-c}\right) = \zeta\left(-\frac{c}{\tau}+d\right)^{\frac{1}{2}} e^{\pi i c \frac{z^2}{-c\tau+d\tau^2}} (i)^{-\frac{1}{2}} \tau^{\frac{1}{2}} e^{\pi i \frac{z^2}{\tau}} \theta(z,\tau)$$

$$= \zeta (i)^{-\frac{1}{2}} (d\tau - c)^{\frac{1}{2}} e^{\pi i d \frac{z^2}{d\tau-c}} \theta(z,\tau)$$

compare ζ for $(c\ d)$ and $(d, -c)$

* if $d < 0$, $(d\tau - c)^{\frac{1}{2}}$ is actually taken $(-d\tau + c)^{\frac{1}{2}}$ and compare $(-d, c)$ instead

if c is odd, $d > 0$, $(i)^{-\frac{c}{2}}\left(\frac{d}{c}\right)(i)^{-\frac{1}{2}} = (i)^{\frac{-c-1}{2}}\left(\frac{d}{c}\right)$

if c is even, $d > 0$, $(i)^{\frac{d-1}{2}}\left(\frac{c}{d}\right)(i)^{-\frac{1}{2}} = (i)^{-\frac{d}{2}}\left(\frac{c}{d}\right)(i)^{d-1} = (i)^{-\frac{d}{2}}\left(\frac{-c}{d}\right)$

the other cases ($d<0$) are similar

Modular form Let $SL_2(\mathbb{Z}) \curvearrowright \mathbb{C} \times \mathbb{H}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times (z, \tau) \longmapsto \left(\frac{z}{c\tau+d}\ \frac{a\tau+b}{c\tau+d}\right)$

the action normalizes the lattice action $(n_1, n_2) \times z \longmapsto z + n_1\tau + n_2$

i.e. $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z}) \curvearrowright \mathbb{C} \times \mathbb{H}$, $(\vec{n}, \gamma)(\vec{m}, \delta) = (\vec{n}\delta + \vec{m}, \gamma\delta)$

For the generator $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ of $SL_2(\mathbb{Z})$

$$\theta_{00}(z,\tau+1)=\theta_{01}(z,\tau) \qquad \theta_{00}(\tfrac{z}{\tau},-\tfrac{1}{\tau})=(-i\tau)^{\frac12}e^{\frac{\pi i z^2}{\tau}}\theta_{00}(z,\tau)$$

$$\theta_{01}(z,\tau+1)=\theta_{00}(z,\tau) \qquad \theta_{01}(\tfrac{z}{\tau}-\tfrac{1}{\tau})=(-i\tau)^{\frac12}e^{\frac{\pi i z^2}{\tau}}\theta_{10}(z,\tau)$$

$$\theta_{10}(z,\tau+1)=e^{\frac{\pi i}{4}}\theta_{10}(z,\tau) \qquad \theta_{10}(\tfrac{z}{\tau},-\tfrac{1}{\tau})=(-i\tau)^{\frac12}e^{\frac{\pi i z^2}{\tau}}\theta_{01}(z,\tau)$$

$$\theta_{11}(z,\tau+1)=e^{\frac{\pi i}{4}}\theta_{11}(z,\tau) \qquad \theta_{11}(\tfrac{z}{\tau},-\tfrac{1}{\tau})=-i(-i\tau)^{\frac12}e^{\frac{\pi i z^2}{\tau}}\theta_{11}(z,\tau)$$

the left side are by direct computation, the right side is by (2)

Def We say f defined on $\mathbb{H}$ is a modular form of weight k and level 4 if

(a) $f(\frac{a\tau+b}{c\tau+d})=(c\tau+d)^k f(\tau)$ for $\begin{pmatrix}a&b\\c&d\end{pmatrix}=\Gamma(4)$

(b) · $\exists$ c,d s.t $|f(\tau)|\le c$ for $\mathrm{Im}\,\tau>d$

· $\forall\, p/g\in\mathbb{Q}$, $\exists$ $c_{p,g}$, $d_{p,g}$ s.t. $|f(\tau)|\le c_{p,g}|\tau-p/g|^{-k}$ — horocircle

for $|\tau-p/g-i d_{p,g}|<d_{p,g}$

Prop. θ_{00}^2, θ_{01}^2, θ_{10}^2 are modular forms of weight 1 and level 4

Pf. (a) follows from $\zeta=(i)^{\frac{d-1}{2}}(\frac{c}{|d|})=\pm 1$

(b) As $\mathrm{Im}\,\tau\to\infty$, $\theta_{00}(z,\tau)=1+O(e^{-\pi\mathrm{Im}\tau})$

And if $\gamma\in\Gamma(4)$ s.t. $\gamma(\infty)=p/g$, then $\gamma^{-1}=\begin{pmatrix}x&x\\-g&p\end{pmatrix}$

$|f(\tau)|=|f(\gamma^{-1}\tau)(-g\tau+p)^{-k}|\le C'\cdot|\tau-p/g|^{-k}$ if f is bounded at i

for $\tau\in\gamma(\{\mathrm{Im}\,\tau>d\})$, a horocircle

since $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ send θ_{00}, θ_{01}, θ_{10} to each other

if we can check that they are bounded at ∞, we are done

and we know as $\mathrm{Im}\,\tau\to\infty$, $\theta_{01}(z,\tau)=1+O(e^{-\pi\mathrm{Im}\tau})$

$\theta_{10}(z,\tau)=O(e^{-\pi\mathrm{Im}\frac{\tau}{4}})$

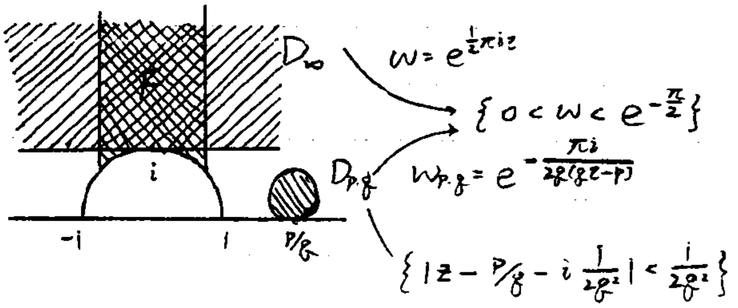
$\mathbb{H}/\Gamma(4)$ Consider the map $\Psi_2:\mathbb{H}/\Gamma(4)\longrightarrow\mathbb{P}^2$ (X_0,X_1,X_2)

$$\tau\longmapsto(\theta_{00}^2(0,\tau),\theta_{01}^2(0,\tau),\theta_{10}^2(0,\tau))$$

$$\tau+1\longmapsto(X_1,X_0,iX_2),\qquad -\tfrac{1}{\tau}\longmapsto(X_0,X_2,X_1)$$

$\mathrm{Im}\,\Psi_2$ lies in $A:\{X_0^2=X_1^2+X_2^2\}\setminus\{(0,1,\pm i),(1,\pm1,0),(1,0,\pm1)\}$

Goal. Expand Ψ_2 to the cusps.



W and $W_{p,g}$ provide coordinate for the cusps in $\mathbb{H}/\Gamma(4)$: $0,\frac12,1,2,3,\infty$

(7)

Let γ_i, $1 \le i \le 6$ s.t. $\gamma_i(\infty)$ are the six cusps

and $\delta_j = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}$ $1 \le j \le 4$ are the representative of the stabilizer at ∞

over $\Gamma(4)$ $\Rightarrow$ $\gamma_i \delta_j$ are representative of $SL_2(\mathbb{Z})/\Gamma(4)$

and $\mathbb{H}/\Gamma(4) = \bigcup_{i,j} \gamma_i \delta_j F$

the closure $\widetilde{\mathbb{H}/\Gamma(4)}$ is compact Hausdorff with the cusps

At ∞, $\Theta_{00} = 1 + 2\sum w^{2n^2}$, $\Theta_{01} = 1 + 2\sum (-1)^n w^{2n^2}$, $\Theta_{10} = 2w^{\frac12} \sum w^{2(n^2+n)}$

$\Rightarrow$ Θ_{00}^2, Θ_{01}^2, Θ_{10}^2 are holomorphic at ∞, and hence on all cusps

Thm. $\widetilde{\Psi}_2 : \widetilde{\mathbb{H}/\Gamma(4)} \longrightarrow A$ is an isomorphism

Pf. $\widetilde{\Psi}_2$ is a covering, and $(1,1,0)$ is mapped only by ∞

since $\frac{d}{dw} \frac{\Theta_{10}^2}{\Theta_{00}^2}\Big|_{w=0} \neq 0$, $\widetilde{\Psi}_2$ is an isomorphism

Tori as fiber

Consider $\Phi : \mathbb{C} \times \mathbb{H} \longrightarrow \mathbb{P}^3$ $\quad \overset{(X_0, X_1, X_2, X_3)}{\shortparallel}$

$(z, \tau) \longmapsto (\Theta_{00}(2z,\tau), \Theta_{01}(2z,\tau), \Theta_{10}(2z,\tau), \Theta_{11}(2z,\tau))$

we have $(\frac14 \mathbb{Z})^2 \rtimes SL_2(\mathbb{Z}) \curvearrowright \mathbb{C} \times \mathbb{H}$ and equivariant by Φ

i.e. $(\frac14, 0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \times (X_0, X_1, X_2, X_3) \longmapsto (X_2, -iX_3, X_0, -iX_1)$

$(0, \frac14, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \times (X_0, X_1, X_2, X_3) \longmapsto (X_1, X_0, X_3, -X_2)$

$(0\ 0\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) \times (X_0, X_1, X_2, X_3) \longmapsto (X_1, X_0, \lambda X_3, \lambda X_2)$ $\quad \lambda = e^{\frac{\pi i}{4}}$

$(0, 0, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) \times (X_0, X_1, X_2, X_3) \longmapsto (X_0, X_2, X_1, -iX_3)$

Prop. Let $\Gamma^* = \{ (m, n, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(4),\ m \equiv \frac{c}{8},\ n \equiv \frac{b}{8} \bmod 1 \} \subset (\frac14 \mathbb{Z})^2 \rtimes SL_2($

then $\Gamma^* \lhd (\frac14 \mathbb{Z})^2 \rtimes SL_2(\mathbb{Z})$ and $\Phi(z,\tau) = \Phi(z',\tau')$ iff $(z',\tau') = \gamma(z,\tau)$ for $\gamma \in \Gamma$

and $\mathrm{Im}\,\Phi \subset F : \{X_0^4 + X_3^4 = X_1^4 + X_2^4\}$

Pf (1) $\Gamma^* \lhd (\frac14 \mathbb{Z})^2 \rtimes SL_2(\mathbb{Z})$: direct verification

- $(\vec{n}, \gamma)(\vec{m}, \delta) = (\vec{n}\delta + \vec{m}, \gamma\delta)$, $\gamma = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

  if $(\vec{n}, \gamma)$, $(\vec{m}, \delta) \in \Gamma^*$, $m_1 a + m_2 c \sim \frac{za + yc}{8} \sim \frac{z}{8}$

  similarly $m_1 b + m_2 d \sim \frac{y}{8}$, and since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{c}{8}, \frac{b}{8}$ mod 1

  is a homomorphism from $\Gamma(4)$, Γ^* is a subgroup

- $(-\vec{m}\delta^{-1}, \delta^{-1})(\vec{n}, \gamma)(\vec{m}, \delta) = (-\vec{m}\delta^{-1}\gamma\delta + \vec{n}\delta + \vec{m}, \delta^{-1}\gamma\delta)$

  $\delta^{-1}\gamma\delta = \begin{pmatrix} & way + bw^2 - cy^2 - dyw \\ -zax - bz^2 + cx^2 + xzd & \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$

  if $(\vec{m}\ \delta) \in \Gamma^*$ $\Rightarrow$ $-m_1 \overset{1+4p}{a'} - m_2 \overset{4q}{c'} + n_1 x + n_2 z + n_1 \sim n_1 x + n_2 z \sim \frac{cx + bz}{8}$

  and $\frac{-zax - bz^2 + cx^2 + xzd}{8} \sim \frac{bz + cx}{8} + \frac{zx(d-a)}{8} \sim \frac{bz + cx}{8}$

(2) the edges of $\mathbb{H}/\Gamma(4)$ are identified by some γ_i's $i = 1, \dots 6$

which generate $\Gamma(4)$ and are conjugate to $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

$\therefore$ $\Gamma(4)$ is the least normal subgroup containing $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

(3) Γ^* is the least normal subgroup containing $\mathbb{Z}^2$ and $(0, \frac12\ \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix})$

Pf (4) Φ is invariant under $\mathbb{Z}^2$ and $(0, -\frac{1}{2}, \begin{pmatrix}1 & 4\\ 0 & 1\end{pmatrix})$ $\therefore$ invariant over Γ^*

(5) If $\Phi(z, \tau) = \Phi(z', \tau') = P \Rightarrow P \in C_\tau \cap C_{\tau'}$, which implies $\Psi_2(\tau) = \Psi_2(\tau')$

we have $C_\tau : \begin{cases} a_0 X_0^2 = a_1 X_1^2 + a_2 X_2^2 \cdots (1) \\ a_0 X_3^2 = a_1 X_2^2 - a_2 X_1^2 \cdots (2) \end{cases}$ $a_0 = \theta_{00}^2$ $a_1 = \theta_{01}^2$, $a_2 = \theta_{10}^2$

$(1) \times a_1 + (2) \times a_2$
$(1) \times a_2 - (2) \times a_1$ $\Rightarrow \begin{cases} a_0 X_1^2 = a_1 X_0^2 + a_2 X_3^2 \cdots (3) \\ a_0 X_2^2 = a_2 X_0^2 - a_1 X_3^2 \cdots (4) \end{cases}$

by $(1) \times X_1^2 + (2) \times X_2^2$
$(1) \times X_2^2 - (2) \times X_1^2$ $\Rightarrow \begin{cases} a_0 = \lambda(X_1^4 + X_2^4) \\ a_1 = \lambda(X_0^2 X_1^2 - X_2^2 X_3^2) \\ a_2 = \lambda(X_0^2 X_2^2 + X_1^2 X_3^2) \end{cases}$

or when $X_1 = X_2 = 0$, by (1), (3) $\Rightarrow \begin{cases} a_0 = \mu(X_0^2 X_2^2 - X_1^2 X_3^2) \\ a_1 = \mu(X_1^2 X_2^2 - X_0^2 X_3^2) \\ a_2 = \mu(X_0^4 - X_1^4) \end{cases}$

and hence $\exists\, \gamma \in \Gamma^*$ s.t. $(z', \tau') = \gamma(z, \tau)$

(6) From $(1)^2 + (2)^2$ and $a_0^2 = a_1^2 + a_2^2$, we have $X_0^4 + X_3^4 = X_1^4 + X_2^4$

Let $\pi : F \longrightarrow A$ as in the proof, we have:

Cor. $F \subset \mathbb{P}^3$ has a fiber structure over $A \subset \mathbb{P}^2$

Let $F_0 = \mathrm{Im}\,\Phi \Rightarrow \pi(F_0) = A_0 = A \setminus \{\text{cusps}\}$ and $F_0 \cong \mathbb{C} \times \mathbb{H}/\Gamma^*$

i.e.
$$\begin{tikzcd} \mathbb{C} \times \mathbb{H} \arrow[r] \arrow[d] & \mathbb{C} \times \mathbb{H}/\Gamma^* \arrow[r, "\Phi", "\sim"'] \arrow[d] & F_0 \arrow[r, hook] \arrow[d] & F \subset \mathbb{P}^3 \arrow[d, "\pi"] \\ \mathbb{H} \arrow[r] & \mathbb{H}/\Gamma(4) \arrow[r, "\Psi_2", "\sim"'] & A_0 \arrow[r, hook] & A \subset \mathbb{P}^2 \end{tikzcd}$$

Rmk. Let $\alpha = (\frac{1}{4}, 0, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix})$, $\beta = (0, \frac{1}{4}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix})$

their action on F induces action α^τ, β^τ on $E_\tau = \mathbb{C}/\Lambda_\tau$

Prop. $\tau' = \gamma(\tau)$ for $\gamma \in \Gamma(4)$ iff $\exists\, E_\tau \xrightarrow[\sim]{f} E_{\tau'}$ s.t.

$$\begin{tikzcd} E_\tau \arrow[r, "f"] \arrow[d, shift right, "\alpha^\tau"'] \arrow[d, shift left, "\beta^\tau"] \arrow[dr, phantom, "\circlearrowright"] & E_{\tau'} \arrow[d, shift right, "\alpha^{\tau'}"'] \arrow[d, shift left, "\beta^{\tau'}"] \\ E_\tau \arrow[r, "f"] & E_{\tau'} \end{tikzcd}$$

Pf. f can be lifted to the universal cover $\mathbb{C} \xrightarrow{\hat{f}} \mathbb{C}$

where $\tilde{f}(z) = Lz + M \Rightarrow L = c\tau' + d$, $L\tau = a\tau' + b \in \Lambda_{\tau'}$

$f \circ \alpha^{\tau'} = \alpha^\tau \circ f \iff L(z + \frac{1}{4}) + M = Lz + M + \frac{1}{4} \mod \Lambda_{\tau'}$

$\iff \frac{c\tau' + d - 1}{4} \in \Lambda_{\tau'} \iff c \equiv 0,\ d \equiv 1 \mod 4$

similarly $a \equiv 1$, $b \equiv 0 \mod 4$ and $\tau = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\tau'$, $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma(4)$

For $a \in A_0$, we associate $\pi^{-1}(a)$ with α^τ, β^τ, where $\Psi_2(\tau) = a$

then $(a, \alpha^\tau, \beta^\tau)$ are non-isomorphic for different a in the sense of the prop

which interpret A_0 as $\left\{ \begin{array}{l} \text{moduli space of complex tori} \\ \text{with } \alpha, \beta, \text{ up to isomorphism} \end{array} \right\}$

Reference D. Mumford Tata Lectures on Theta I

(c)

復變期末報告：(Confluent) Hypergeometric functions

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Foreward

This report follows the structure of Ch14 and Ch16 in "A Course of Modern Analysis" by Whittaker & Watson. The first part will be Ch14: Hypergeometric function, and the second one will be Ch16: Confluent Hypergeometric functions. To keep it clean, I'll relegate tedius proofs to the appendices, so interested readers may check it by themselves. Where I'll talk about in class, I'll mark it with red pen in my report.

1st Part: Hypergeometric

Hypergeometric functions are defined as

$$F(a,b;c;z) := 1+\sum_{n=1}^{\infty}\frac{a(a+1)\cdots(a+n-1)\cdot b(b+1)\cdots(b+n-1)}{c(c+1)\cdots(c+n-1)\quad n!}z^n =: 1+\sum_{n=1}^{\infty}\frac{(a,n)(b,n)}{(c,n)(1,n)}z^n. \quad \text{whenever } |z|<1.$$

Many functions of importance can be expressed in terms of hypergeometric functions, such as

$$\begin{cases}(1+z)^n = F(-n,\beta;\beta;z)\\ \log(1+z) = zF(1,1;2;-z)\\ e^z = \lim\limits_{\beta\to\infty}F(1,\beta;1;z/\beta).\end{cases}$$

Value of $F(a,b;c;1)$ when $Re(c-a-b)>0$

We can see that for $0\le x<1$, ← ☆ Appendix 1 for proof.

$$c\big(c-1-(2c-a-b-1)x\big)F(a,b;c;x)+(c-a)(c-b)xF(a,b;c+1;x) = c(c-1)(1-x)F(a,b;c-1;x)$$
$$= c(c-1)\Big\{1+\sum_{n=1}^{\infty}(u_n-u_{n-1})x^n\Big\}.$$

where u_n denotes the n-th coefficient in $F(a,b;c-1;x)$.

Let $x\to1^-$.

For the right hand side, by Abel, tends to 0 if $1+\sum_{n=1}^{\infty}(u_n-u_{n-1})=0$ if $\lim\limits_{n\to\infty}u_n=0$, which is true when $Re(c-a-b)>0$.

For the left hand side, with more careful analysis on the coefficients of $F(a,b;c;x)$, $F(a,b;c+1;x)$, which can be found at §2.38, the limit exists for both term and is

$$c(a+b-c)F(a,b,c;1)+(c-a)(c-b)F(a,b;c+1;1).$$

∴ We have recurrence formula

$$F(a,b,c;1)=\frac{(c-a)(c-b)}{c(c-a-b)}F(a,b;c+1;1)$$

$$\Rightarrow\ F(a,b,c;1)=\prod_{n=0}^{m-1}\frac{(c-a+n)(c-b+n)}{(c+n)(c-a-b+n)}F(a,b;c+m;1).\ \text{-----}\ (1).$$

coeff. for $F(a,b;c+m;x)$

$$\text{Now, }\ |F(a,b;c+m;1)-1|\le\sum_{n=1}^{\infty}|u_n(a,b,c+m)|\le\sum_{n=1}^{\infty}\frac{(|a|,n)(|b|,n)}{(m-|c|,n)(1,n)}=\frac{|a||b|}{m-|c|}\sum_{n=0}^{\infty}\frac{(|a|-1,n)(|b|-1,n)}{(m-|c|-1,n)(1,n+1)}$$
$$<\frac{|ab|}{m-|c|}\sum_{n=0}^{\infty}u_n(|a|-1,|b|-1,m-1-|c|)$$

The series converges if $Re\big(m-1-|c|-(|a|-1)-(|b|-1)\big)>0$ iff $Re\big(m-(|a|+|b|+|c|-1)\big)$. which is true for m large. Meanwhile, $\frac{|ab|}{m-|c|}\to0$ as $m\to\infty$.

P2.

$\Rightarrow$. $F(a,b;m+c;1) \to 1$ as $m\to\infty$.

Now, $$\prod_{n=0}^{m-1}\frac{(c-a+n)(c-b+n)}{(c+n)(c-a-b+n)} = \frac{\Gamma(c-a+m)}{\Gamma(c-a)}\frac{\Gamma(c-b+m)}{\Gamma(c-b)}\frac{\Gamma(c)}{\Gamma(c+m)}\frac{\Gamma(c-a-b)}{\Gamma(c-a-b+m)}$$
$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{\Gamma(c-a+m)}{\Gamma(c+m)}\frac{\Gamma(c-b+m)}{\Gamma(c-a-b+m}\right].$$

$\boxed{\text{Fact}}$: $\log\Gamma(z+a) = (z+a-\frac{1}{2})\log z - z + \frac{1}{2}\log(2\pi) + o(1)$ as $|z|\to\infty$ by §13.6, which I omit here.

$$\therefore \frac{\Gamma(c-a+m)\Gamma(c-b+m)}{\Gamma(c+m)\Gamma(c-a-b+m)} = \exp\Big(\log m\big(m+c-a-\tfrac{1}{2}+m+c-b-\tfrac{1}{2}-\cancel{m}-c+\tfrac{1}{2}-m-c+a+b+\tfrac{1}{2}\big)$$
$$-(m+m-\cancel{m}-m) + 0\cdot\tfrac{1}{2}\cancel{\log(2\pi)} + o(1)\Big) = \exp(o(1)).$$

$\Rightarrow$ Substituting the results above into (1), and we get
$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

5 min.

Kummer's 24 solutions

Solution of Riemann's P-equation by Hypergeometric Functions.

Recall that $F(a,b;c;z)$ is a solution in $P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & ;z \\ 1-c & b & c-a-b & \end{Bmatrix}$, and P has the following transformation

(1) $$\left(\frac{z-a}{z-b}\right)^k\left(\frac{z-c}{z-b}\right)^l P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & ;z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} a & b & c & \\ \alpha+k & \beta-k-l & \gamma+l & ;z \\ \alpha'+k & \beta'-k-l & \gamma'+l & \end{Bmatrix},\quad k,l\in\mathbb{C}.$$

(2) $$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & ;z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} \tilde{a} & \tilde{b} & \tilde{c} & \\ \alpha & \beta & \gamma & ;\tilde{z} \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$ where $\tilde{z}=f(z)$ with f being a Möbius transform and $(a,b,c)\xrightarrow{f}(\tilde{a},\tilde{b},\tilde{c})$.

Thus, for a general 2nd order linear differential equation with 3 regular singularities
$$0 = \frac{d^2u}{dz^2} + \left\{\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c}\right\}\frac{du}{dz} + \left\{\frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-c)(b-a)}{z-b} + \frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right\}\frac{u}{(z-a)(z-b)(z-c)}$$

The solutions are characterized by Rieman's P-equation
$$\boxed{P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & ;z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}.}$$

Thus, by the transformation rules,
$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & ;z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = \left(\frac{z-a}{z-b}\right)^\alpha\left(\frac{z-c}{z-b}\right)^\gamma P\begin{Bmatrix} a & b & c & \\ 0 & \beta+\alpha+\gamma & 0 & ;z \\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & \end{Bmatrix}.$$
$$= \left(\frac{z-a}{z-b}\right)^\alpha\left(\frac{z-c}{z-b}\right)^\gamma P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & \beta+\alpha+\gamma \overset{\shortparallel}{\,} a & 0 & ;\left(\frac{(z-a)(c-b)}{(z-b)(c-a)}\right) \\ \underset{1-c}{\underset{\shortparallel}{\alpha'-\alpha}} & \underset{b}{\underset{\shortparallel}{\beta'+\alpha+\gamma}} & \gamma'-\gamma & \end{Bmatrix}.$$

Thus, a solution to $P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & ;z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$ will be
$$u_1 = \left(\frac{z-a}{z-b}\right)^\alpha\left(\frac{z-c}{z-b}\right)^\gamma F\left(\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma;\ 1+\alpha-\alpha';\ \frac{(z-a)}{(z-b)}\frac{(c-b)}{(c-a)}\right).$$

However, in transforming process, we can interchange α with α' / γ with γ', so this gives us 4 solutions by hypergeometric functions
$$u_2 = \left(\frac{z-a}{z-b}\right)^{\alpha'}\left(\frac{z-c}{z-b}\right)^\gamma F\left\{\alpha'+\beta+\gamma,\ \alpha'+\beta'+\gamma;\ 1+\alpha'-\alpha;\ \frac{(z-a)}{(z-b)}\frac{(c-b)}{(c-a)}\right).$$
$$u_3 = \left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma'} F\left\{\alpha+\beta+\gamma',\ \alpha+\beta'+\gamma';\ 1+\alpha-\alpha';\ \frac{(z-a)}{(z-b)}\frac{(c-b)}{(c-a)}\right)$$
$$u_4 = \left(\frac{z-a}{z-b}\right)^{\alpha'}\left(\frac{z-c}{z-b}\right)^{\gamma'} F\left\{\alpha'+\beta+\gamma',\ \alpha'+\beta'+\gamma';\ 1+\alpha'-\alpha;\ \frac{(z-a)}{(z-b)}\frac{(c-b)}{(c-a)}\right).$$

Moreover, in sending (a,b,c) onto $(0,\infty,1)$, we have five other permutations

(b,c,a), (c,a,b), (a,c,b), (c,b,a), (b,a,c).

That gives us $5\times4=20$ more solutions. Combined with the first 4, and we have 24 solutions to that general equation. These 24 sol'ns are due to Kummer.

Specifically, if we set $\left(\alpha,\alpha',\beta,\beta',\gamma,\gamma',\frac{(z-a)}{(z-b)}\frac{(c-b)}{(c-a)}\right)$ as $(0,1-C,A,B,0,C-A-B)$, then we have 24 solutions of the hypergeometric function satisfied by $F(A,B,C;x)$ $(=u_1)$.

3 min.

Relations between particular solutions of the hypergeometric equation.

As is mentioned above, we have constructed 24 particular solutions, all of which belongs to

$$P\begin{Bmatrix} a & b & c & \\ 0 & A & 0 & ;z \\ 1-C & B & C-A-B & \end{Bmatrix}$$ of dimension 2.

However, as the dimension of the solution space is only 2, any 3 solutions are related linearly.

For simplicity and demonstration, I only present the first 4 solutions in the first group

$$y_1=F(A,B;C;x).$$
$$y_2=(-x)^{1-C}F(A-C+1,B-C+1,2-C;x)$$
$$y_3=(1-x)^{C-A-B}F(C-B,C-A;C;x)$$
$$y_4=(-x)^{1-C}(1-x)^{C-A-B}F(1-B,1-A;2-C;x)$$

$$\left(\begin{aligned} &y_{17}=F(A,B;A+B-C+1;1-x)\\ &y_{18}=(1-x)^{C-A-B}F(C-B,C-A;C-A-B+1;1-x)\\ &y_{21}=(-x)^{-B}F(A,A-C+1;A-B+1;x^{-1})\\ &y_{22}=(-x)^{-A}F(B,B-C+1;B-A+1;x^{-1})\end{aligned}\right)$$

and I $\boxed{\text{claim}}$ $y_1=y_3$, $y_2=y_4$ if $|\arg(1-x)|<\pi$.

(i) $\exists\ C_1,C_2,C_3\in\mathbb{C}$ s.t. $C_1y_1+C_2y_2+C_3y_3=0$.

First, we look at the constant coefficient : $C_1+C_3=0$

$$x\text{ - coefficient : }\frac{AB}{C}\cdot C_1+(-1)^{1-C}\cdot C_2+\Big(\underbrace{\overset{\text{from }F(C-B,C-A;C;x)}{\frac{(C-B)(C-A)}{C}}-\overset{\text{from }(1-x)^{C-A-B}}{(C-A-B)}}_{\overset{\shortparallel}{\frac{AB}{C}}}\Big)C_3=0$$

$$\Rightarrow\ -C_1=C_3\ ,\ C_2=0\ \Rightarrow\ y_1=y_3.$$

(ii) $\exists\ d_1,d_2,d_4\in\mathbb{C}$ s.t. $d_1y_1+d_2y_2+d_4y_4=0$

constant coeff. : $d_1=0$.

x-coeff. : $(-1)^{1-C}d_2+(-1)^{1-C}\cdot1\cdot d_4=0\Rightarrow d_2=-d_4\Rightarrow y_2=y_4$.

Remark

"Intra-group" relations are quite easy to obtain as the expansion will have the same form (e.g. all in x) while "Inter-group" relations are much more complicated. Hence it's beneficial for us to examine hypergeometric functions from a different point of view, namely integral representation. The next section is about Barnes' contour integral.

10 min.

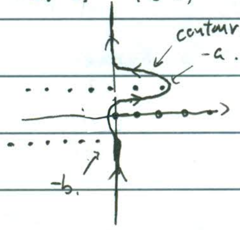
Barnes' contour integrals for the hypergeometric functions

Consider $\frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s ds$, where the contour is curved s.t.

the poles of $\Gamma(a+s)\Gamma(b+s)$ $(-a-n, -b-m, n,m\in\mathbb{N})$ lies on the left of it

while the ones of $\Gamma(-s)$ $(n\in\mathbb{N})$ lies on the right of it.

i.e.



The idea is to construct hypergeometric functions in terms of residues, but first we check the integrand $\left|\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s\right| = \exp\left(\mathrm{Re}\left[\log\Gamma(a+s)+\log(\Gamma(b+s))+\log\Gamma(-s)-\log\Gamma(c+s)+s\log(-z)\right]\right)$

By the "fact" mentioned in P.2, $\log\Gamma(a+s)+\log\Gamma(b+s)+\log\Gamma(-s)-\log\Gamma(c+s)) \doteq \log s\,(a+b-c-1)+\log(2\pi)+o(1)$.

$\therefore$ The integrand is of order $O(|s|^{(a+b-c-1)}\exp(-\arg(-z)\,\mathrm{Im}(s)-\pi|I(s)|))$,

so as long as $|\arg(-z)|<\pi-\delta$, the integrand decays exponentially, and so the contour integral is an analytic function in z. ← The integral exists.

Now, with $\Gamma(-s)\Gamma(1+s)=\frac{\pi}{-\sin\pi s}$, consider

$\frac{1}{2\pi i}\int_C \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(1+s)}\frac{(-z)^s\pi}{\sin\pi s}ds$ where C is the right semi-circle with radius $N+\frac{1}{2}$.

The integrand is of the order $O(N^{a+b-c-1})\frac{(-z)^s}{\sin\pi s}$, and for $s=(N+\frac{1}{2})e^{i\theta}$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, $|z|<1$,

$$\frac{(-z)^s}{\sin\pi z} = 2i\frac{\exp((N+\frac{1}{2})e^{i\theta}\log(-z))}{e^{i\pi(N+\frac{1}{2})e^{i\theta}}-e^{-i\pi(N+\frac{1}{2})e^{i\theta}}} = O\Big(\exp\big((N+\tfrac{1}{2})\log|-z|\underbrace{\cos\theta}_{\text{dominate for } |\theta|\leq\frac{\pi}{4}} - (N+\tfrac{1}{2})\arg(-z)\underbrace{\sin\theta}_{\text{dominate for } |\theta|>\frac{\pi}{4}} - (N+\tfrac{1}{2})\pi\underbrace{|\sin\theta|}\big)\Big)$$

If $\log|-z|<0$ (i.e. $|z|<1$),

then $\begin{cases} O(\exp((N+\frac{1}{2})\log|-z|\cos\theta) \\ O(\exp[(N+\frac{1}{2})(\pi-|\arg(-z)|)|\sin\theta|] \end{cases}$ both has exponential decay.

$\therefore \frac{1}{2\pi i}\int_C \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(1+s)}\frac{(-z)^s\pi}{\sin\pi s}ds \to 0$ as $N\to\infty$. ← Integral on semi-circle tends to 0.

Now, $\frac{1}{2\pi i}\Big[\int_{-\infty}^{\infty i} - \underbrace{\lim\int_C}_{0}\Big] = \sum$ residue of integrand $= \sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)}\frac{\cancel{(-1)^n}}{\cancel{\cos\pi n}}z^n = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}F(a,b;c;z)$.

if $|z|<1$.

$\therefore \frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s ds = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}F(a,b;c;z)$ if $|z|<1$.

3 min.

Continuation of hypergeometric series using "the other side" of Barnes' integral.

(Connection between z and z^{-1}).

In the previous section, we used the right semi-circle when $|z|<1$.

Now, when $|z|>1$, we will use the left semi-circle.

Let D be the left semi-circle. Then for $s=(N+\frac{1}{2})e^{i\theta}$, $\frac{\pi}{2}<\theta<\frac{3\pi}{2}$

$\underset{\text{perimeter of } D}{O(N}\cdot\exp[(N+\frac{1}{2})\log|-z|\cos\theta]) \to 0$ if $\log|-z|>0$ ($\because \cos\theta<0$).

$\therefore$ For $|z|>1$, $\underbrace{\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s ds}_{\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}F(a,b;c;z)} = \sum$ residues at $-a-n$ $+\sum$ residues at $-b-n$

$$= \sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(1-c+a+n)}{\Gamma(1+n)\Gamma(1-b+a+n)}\frac{\sin(c-a-n)\pi}{\cos n\pi\,\sin(b-a-n)\pi}(-z)^{-a-n}$$

$$+ \sum_{n=0}^{\infty}\frac{\Gamma(b+n)\Gamma(1-c+b+n)}{\Gamma(1+n)\Gamma(1-a+b+n)}\frac{\sin(c-b-n)\pi}{\cos n\pi\,\sin(a-b-n)\pi}(-z)^{-b-n}$$

$= \frac{\Gamma(a)\Gamma(a-b)}{\Gamma(a-c)}(-z)^{-a}F(a,1-c+a;1-b+a;z^{-1}) + \Gamma(b)\Gamma(b-a)$ … $-b$

$$\therefore \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}F(a,b;c;z) = \frac{\Gamma(a)\Gamma(a-b)}{\Gamma(a-c)}(-z)^{-a}F(a,1-c+a;1-b+a;z^{-1}) + \frac{\Gamma(b)\Gamma(b-a)}{\Gamma(b-c)}(-z)^{-b}F(b,1-c+b;1-a+b;z^{-1})$$

when $|z|>1$, $|\arg(-z)|<\pi$ (principal value).

As demonstration, we have the following Corollary (from Barnes' integral)

$\boxed{\text{Corollary}}$ Putting $b=c$, we'll have

$$\Gamma(a)(1-z)^{-a} = \Gamma(a)F(a,b;b;z) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\Gamma(a+s)\Gamma(-s)(-z)^{s}\,ds.$$

Barnes' Lemma.

$\boxed{\text{☆Lemma}}$

$$I := \frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s)\,ds = \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)}$$

where the contour separates the singularities of $\Gamma(a+s)$ and $\Gamma(b+s)$ from $\Gamma(\gamma-s)\Gamma(\delta-s)$.

i.e.

(pf). If we set C : right semi-circle with radius ρ and when $\rho\to\infty$, set ρ such that C avoids the poles of $\Gamma(\gamma-s)$ and $\Gamma(\delta-s)$ (Rigorously, choose ρ s.t. the distance of C to those poles have lower bound).

$$\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s) = \frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(1-\gamma+s)\Gamma(1-\delta+s)}\frac{\pi^2}{\sin(\gamma-s)\pi\,\sin(\delta-s)\pi} = O(|s|^{\alpha+\beta+\gamma+\delta-2}\exp(-2\pi|\operatorname{Im}(s)|)).$$

as $|s|\to\infty$. $\Rightarrow$ I is an analytic function.

If $\boxed{\underset{\sim\sim\sim\sim\sim\sim\sim\sim}{\operatorname{Re}(\alpha+\beta+\gamma+\delta-1)<0}}$. (Taking the perimeter of C into account),

then $\int_C \to 0$

$$\therefore I = \sum \text{residues of } \Gamma(\gamma-s) \text{ and } \Gamma(\delta-s) = \sum_{n=0}^{\infty}\frac{\Gamma(\alpha+\gamma+n)\Gamma(\beta+\gamma+n)}{\Gamma(1+n)\,\Gamma(1-\delta+\gamma+n)}\underbrace{\frac{\pi}{\cos n\pi\,\sin(\delta-\gamma-n)\pi}}_{}$$

$$+\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+\delta+n)\Gamma(\beta+\delta+n)}{\Gamma(1+n)\,\Gamma(1-\gamma+\delta+n)}\frac{\pi}{\cos n\pi\,\sin(\gamma-\delta-n)\pi}$$

$$\frac{\pi}{\sin(\delta-\gamma)\pi} \qquad\qquad \frac{\pi}{\sin(\gamma-\delta)\pi}$$

$\therefore$ By the formula $F(a,b;c;1) = \dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$, (when $\operatorname{Re}(c-a-b)>0$).

$$I = \frac{\pi}{\sin(\gamma-\delta)\pi}\left[\frac{\Gamma(\alpha+\delta)\Gamma(\beta+\delta)}{\Gamma(1-\gamma+\delta)}F(\alpha+\delta,\beta+\delta;1-\gamma+\delta;1) - \frac{\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(1-\delta+\gamma)}F(\alpha+\gamma,\beta+\gamma;1-\delta+\gamma;1)\right].$$

$$= \frac{\pi}{\sin(\gamma-\delta)\pi}\left[\frac{\Gamma(\alpha+\delta)\Gamma(\beta+\delta)}{\cancel{\Gamma(1-\gamma+\delta)}}\frac{\cancel{\Gamma(1-\gamma+\delta)}\Gamma(1-\alpha-\beta-\gamma-\delta)}{\Gamma(1-\gamma-\alpha)\Gamma(1-\gamma-\beta)} - \frac{\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\cancel{\Gamma(1-\delta+\gamma)}}\frac{\cancel{\Gamma(1-\delta+\gamma)}\Gamma(1-\alpha-\beta-\gamma-\delta)}{\Gamma(1-\delta-\alpha)\Gamma(1-\delta-\beta)}\right].$$

$\Gamma(s)\Gamma(1-s) = \frac{-\pi}{\sin\pi s}$

$$= \frac{\pi\,\Gamma(1-\alpha-\beta-\gamma-\delta)}{\sin(\gamma-\delta)\pi}\left\{\frac{\Gamma(\alpha+\delta)\Gamma(\beta+\delta)}{\Gamma(1-\gamma-\alpha)\Gamma(1-\gamma-\beta)} - \frac{\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(1-\delta-\alpha)\Gamma(1-\delta-\beta)}\right\}$$

$$\rightsquigarrow\ = \frac{\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)\sin(\alpha+\beta+\gamma+\delta)\pi\,\sin(\gamma-\delta)\pi}\left(\underset{\sim\sim\sim}{\sin(\alpha+\gamma)\pi\sin(\beta+\gamma)} - \sin(\alpha+\delta)\pi\sin(\beta+\delta)\pi\right).$$ 積化和差

$$\underbrace{\frac{1}{2}\left[\cos(\cancel{\alpha-\beta})\pi - \cos(\alpha+\beta+2\gamma)\pi\right] \quad \frac{1}{2}\left[\cos(\cancel{\alpha-\beta})\pi - \cos(\alpha+\beta+2\delta)\pi\right]}_{}.$$

和差化積

$$\sin(\alpha+\beta+\gamma+\delta)\pi\,\sin(\gamma-\delta)\pi$$

$$\therefore I = \frac{\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)} \text{ for } \operatorname{Re}(\alpha+\beta+\gamma+\delta-1)<0.$$

However, fixing other variables, say β,γ,δ, I is an analytic function on α $\Rightarrow$ $I = \dfrac{\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)}$ by analytic continuation.

For later use, we give an observation:

If we change $(\alpha,\beta,\gamma,\delta)$ into $(\alpha+k,\beta+k,\gamma-k,\delta-k)$ and integrate from $k-\infty i$ to $k+\infty i$, the result is still the same. where $k\in\mathbb{R}$. (by change of variables $t=s-k$.)

Connection between hypergeometric functions of $\boxed{z}$ and $\boxed{1-z}$.

↑ Mention as side remark.

By Barnes' contour integral and Barnes' Lemma, we have

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}F(a,b;c;z)=\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s ds$$

$$=\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\left\{\frac{1}{2\pi i}\int_{-k-\infty i}^{-k+\infty i}\Gamma(a+t)\Gamma(b+t)\Gamma(s-t)\Gamma(c-a-b-t)\,dt\right\}\frac{\Gamma(-s)(-z)^s}{\Gamma(c-a)\Gamma(c-b)}ds$$

Now, if $\overset{>0}{k}$ is chosen that s contour and t contour never intersect (so that $\Gamma(s-t)$ is regular), then the exponential decay on s and t ensures the interchangeability of two integrals.

$$\therefore\ \frac{\Gamma(c-a)\Gamma(c-b)\Gamma(a)\Gamma(b)}{\Gamma(c)}F(a,b;c;z)=\frac{1}{2\pi i}\int_{-k-\infty i}^{-k+\infty i}\Gamma(a+t)\Gamma(b+t)\Gamma(c-a-b-t)\underbrace{\left\{\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\Gamma(s-t)\Gamma(-s)(-z)^s ds\right\}}_{\overset{\shortparallel}{\Gamma(-t)(1-z)^t}\text{ by }\boxed{\text{Cor.}}\text{ on P.5.}}dt.$$

$$=\frac{1}{2\pi i}\int_{-k-\infty i}^{-k+\infty i}\Gamma(a+t)\Gamma(b+t)\Gamma(c-a-b-t)\Gamma(-t)(1-z)^t dt$$

Now, if $\boxed{|1-z|<1}$, $\boxed{|\arg(1-z)|<2\pi}$, then we may again use right semi-circles to aid ~~our~~ case

$$\frac{1}{2\pi i}\int_{-k-\infty i}^{-k+\infty i}\Gamma(a+t)\Gamma(b+t)\Gamma(c-a-b-t)\Gamma(-t)(1-z)^t dt=\sum\text{residues of }\Gamma(-t)+\sum\text{residues of }\Gamma(c-a-b-t)$$

$$=\Gamma(a)\Gamma(b)\Gamma(c-a-b)F(a,b;1-c+a+b;1-z)+\Gamma(c-b)\Gamma(c-a)\Gamma(a+b-c)(1-z)^{c-a-b}F(c-a,c-b;c-a-b+1;1-z)$$

$$\therefore\ \Gamma(c-a)\Gamma(c-b)\Gamma(a)\Gamma(b)F(a,b;c;z)=\Gamma(c)\Gamma(a)\Gamma(b)\Gamma(c-a-b)F(a,b;1-c+a+b;1-z)$$
$$+\Gamma(c)\Gamma(c-b)\Gamma(c-a)\Gamma(a+b-c)(1-z)^{c-a-b}F(c-a,c-b;c-a-b+1;1-z).$$

Should be at ~21

≈ 10 min.

In doing so, we know the nature of singularity of F at $z=1$. ($(1-z)^{c-a-b}$).

Solutions of Riemann's equation by a contour integral

Barnes' integral gives a representation of hypergeometric functions, but in my opinion, there might be some shortcomings: (1) It doesn't directly represent solutions of general Riemann's equation.

✓(2). $\Gamma(s)$, while well known to mathematicians, are still not elementary enough to be evaluated quickly.

✓(3). The contour is infinite (not compact), so manipulation might not be very easy.

In this section, we'll examine the solutions from the viewpoint of differential equations.

Given a Riemann's equation,

$$\frac{d^2u}{dz^2}+\left\{\frac{1-\alpha-\alpha'}{z-a}+\frac{1-\beta-\beta'}{z-b}+\frac{1-\gamma-\gamma'}{z-c}\right\}\frac{du}{dz}+\left\{\frac{\alpha\alpha'(a-b)(a-c)}{z-a}+\frac{\beta\beta'(b-a)(b-c)}{z-b}+\frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right\}\frac{u}{(z-a)(z-b)(z-c)}=0.$$

Replace u by I where $u=(z-a)^{\alpha}(z-b)^{\beta}(z-c)^{\gamma}I$ (kill the exponents at a,b,c)

Then, with some calculation ← ☆ Appendix 2 for proof.

I is a solution of

$$\frac{d^2I}{dz^2}+\left\{\frac{1+\alpha-\alpha'}{z-a}+\frac{1+\beta-\beta'}{z-b}+\frac{1+\gamma-\gamma'}{z-c}\right\}\frac{dI}{dz}+\frac{(\alpha+\beta+\gamma)\{(\alpha+\beta+\gamma+1)z+\sum a(\alpha+\beta'+\gamma'-1)\}}{(z-a)(z-b)(z-c)}I=0$$

where $\sum$ is a symmetric polynomial-like summation. i.e. $\sum a(\alpha+\beta'+\gamma'-1)=a(\alpha+\beta'+\gamma'-1)+b(\beta+\alpha'+\gamma'-1)$
$+c(\gamma+\alpha'+\beta'-1)$

Multiplying by $(z-a)(z-b)(z-c)$, we see I satisfies

$$\boxed{\begin{aligned}&Q(z)\frac{d^2I}{dz^2} - \{(\lambda-2)Q'(z)+R(z)\}\frac{dI}{dz} + \{\tfrac{1}{2}(\lambda-2)(\lambda-1)Q''(z)+(\lambda-1)R'(z)\}I=0\\ &\text{where } \begin{cases}\lambda = 1-\alpha-\beta-\gamma = \alpha'+\beta'+\gamma'\\ Q(z)=(z-a)(z-b)(z-c)\\ R(z)=\sum(\alpha'+\beta+\gamma)(z-b)(z-c) = (\alpha'+\beta+\gamma)(z-b)(z-c)+(\alpha+\beta'+\gamma)(z-a)(z-c)+(\alpha+\beta+\gamma')(z-a)(z-b)\end{cases}\end{aligned}}$$

Now, we $\boxed{\text{claim}}$ $I = \int_C (t-a)^{\alpha'+\beta+\gamma-1}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}(z-t)^{\alpha-\beta-\gamma}dt$.

where C is a compact contour satisfying some criterion I'll later explain.

Since C is compact, we may differentiate the integral inside the integration directly.

Then the equation becomes

$$\int_C (t-a)^{\alpha'+\beta+\gamma-1}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}(z-t)^{-\alpha-\beta-\gamma-2}K\,dt=0$$

where $K = (\lambda-2)\{Q(z)+(t-z)Q'(z)+\frac{1}{2}(t-z)^2Q''(z)\} + (t-z)\{\underbrace{R(z)+(t-z)R'(z)}\}$

$= (\lambda-2)\{\overset{\text{Taylor}}{Q(t)-(t-z)^3}\} + (t-z)\{R(t)-(t-z)^2\sum(\alpha'+\beta+\gamma)\}$ Taylor's expansion with z fixed

$= -(1+\alpha+\beta+\gamma)(t-a)(t-b)(t-c) + \underset{\uparrow}{\sum}(\alpha'+\beta+\gamma)(t-b)(t-c)(t-z)$

symmetric sum

It turns out that $\boxed{(t-a)^{\alpha'+\beta+\gamma-1}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}(z-t)^{-\alpha-\beta-\gamma-2}K = \frac{dV}{dt}}$

where $V = (t-a)^{\alpha'+\beta+\gamma}(t-b)^{\alpha+\beta'+\gamma}(t-c)^{\alpha+\beta+\gamma'}(z-t)^{-(1+\alpha+\beta+\gamma)}$ ← multi-valued

∴ The integral solves the equation satisfied by I iff $\int_C \frac{dV}{dt}dt=0$.

That is, V assumes the same value at endpoints of C

$\boxed{\text{Remark}}$ Even if C is closed, in general it is still not true as V is multi-valued.

Note that $V = (t-a)^{\alpha'+\beta+\gamma-1}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}(z-t)^{-\alpha-\beta-\gamma}U$

where $U = (t-a)(t-b)(t-c)(z-t)^{-1}$: one-valued.

∴ V assumes the same value iff $\underbrace{(t-a)^{\alpha'+\beta+\gamma-1}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}(z-t)^{\alpha-\beta-\gamma}}_{\text{integrand of } I.}$ assumes the same value

∴ $(z-a)^{\alpha}(z-b)^{\beta}(z-c)^{\gamma}\int_C (t-a)^{\alpha'+\beta+\gamma-1}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}(z-t)^{-\alpha-\beta-\gamma}dt$ satisfies Riemann's equation

~31'

iff the integrand assumes the same value at endpoints of C.

2min

☆ Ex. Hypergeometric function $F(a,b;c;z)$ is in $P\begin{Bmatrix}0 & \infty & 1 & \\ 0 & a & 0 & ;z\\ 1-c & b & c-a-b & \end{Bmatrix}$

If we set $(z-a)^{\alpha}(1-\frac{z}{b})^{\beta}(z-c)^{\gamma}\int_C (t-a)^{\beta+\gamma+\alpha'+1}(1-\frac{t}{b})^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}(t-z)^{-\alpha-\beta-\gamma}dt$.

and let $b\to\infty$. Then we'd naively believe that

$\int_C t^{a-c}(t-1)^{c-b-1}(t-z)^{-a}dt$ is hypergeometric function.

If $Re(c) > Re(b) > 0$, then $V = t^{1+a-c}(t-1)^{c-b}(t-z)^{1-a}$ is 0 at 1 and ∞. ☆

$\therefore \int_1^\infty t^{a-c}(t-1)^{c-b-1}(t-z)^{-a}dt$ should be hypergeometric function.

Setting $u=t^{-1}$, then $\underbrace{\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-uz)^{-a}du}$ might be it.

In fact, it is true by direct check.

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F(a,b;c;z)=\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-uz)^{-a}du.$$

$\boxed{\text{Remark}}$ Setting $z=1$, RHS is Bessel function and thus we'll get $F(a,b;c;1)$ again. ┘

Determining integrals with exponent α at a.

↑ Side remark.

Given an equation, the integral

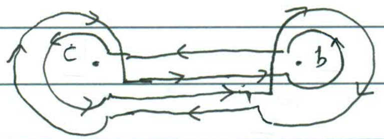
$$I=(z-a)^{\alpha}(z-b)^{\beta}(z-c)^{\gamma}\int_C (t-a)^{\alpha'+\beta+\gamma-1}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}(t-z)^{-\alpha-\beta-\gamma}dt.$$

satisfies the equation, provided C is suitable.

Singularities of the integrand are a,b,c,z.

Choose an ingenious contour (b^+,c^+,b^-,c^-); and such contour satisfies our requirement that the integrand assumes the same value.

i.e.



The contour surrounds b counter-clockwise, c counter-clockwise, then b clockwise, c clockwise.

In this case, monodromy effect will be cancelled out.

Now, we want to construct I so that I is $(z-a)^{\alpha}$ locally around a.

Take $B_\varepsilon(a)$ s.t. $B_\varepsilon(a)$ doesn't contain c or b. Then the contour can be chosen so that t never enters $B_\varepsilon(a)$.

Now, choose $\arg(z-a)$ to be less than π and $\arg(z-b)$, $\arg(z-c)$ so that it reduces to $\arg(a-b)$, $\arg(a-c)$ as $z\to a$. At the starting point of contour C, fix $\arg(t-a)$, $\arg(t-b)$ and choose $\arg(t-z)$ s.t. $\arg(t-z)\to\arg(t-a)$ as $z\to a$.

Then $(z-b)^{\beta}=(a-b)^{\beta}\left\{1+\beta\left(\frac{z-a}{a-b}\right)+\cdots\right\}$. ← z is small perturbation of a.

$(z-c)^{\gamma}=(a-c)^{\gamma}\left\{1+\gamma\left(\frac{z-a}{a-c}\right)+\cdots\right\}$.

and $(t-z)^{-\alpha-\beta-\gamma}=(t-a)^{-\alpha-\beta-\gamma}\left\{1-(\alpha+\beta+\gamma)\frac{a-z}{t-a}+\cdots\right\}$. ε is small s.t. all these series converge uniformly.

$$\therefore I=(z-a)^{\alpha}(z-b)^{\beta}(z-c)^{\gamma}\int_C (t-a)^{\beta+\gamma+\alpha'-1}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}(t-z)^{-\alpha-\beta-\gamma}dt.$$

can be expanded into a series of exponent α around a.

$$\therefore I=\underbrace{(a-b)^{\beta}(a-c)^{\gamma}}_{\uparrow\ \text{constant.}}\underset{\uparrow\ \text{exponent } \alpha}{P^{(\alpha)}}\underbrace{\int_N^{(b^+,c^+,b^-,c^-)}(t-a)^{\alpha'+\beta+\gamma-1}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}dt}_{\uparrow\ \text{constant}}$$

$(z-a)^{\alpha}\{1+\Sigma' C_n(z-a)^n\}$

In a similar way, we can construct solutions with exponent $\alpha,\alpha',\beta,\beta',\gamma,\gamma'$ respectively.

Relations between contiguous hypergeometric functions

Let $p^{(\alpha)}$ denote the (multi-valued) solution to the Riemann's equation with exponent α around a.
Similarly, $p^{(\alpha')}$, $p^{(\beta)}$, $p^{(\beta')}$, $p^{(\gamma)}$, $p^{(\gamma')}$ can be defined.

Let P be a constant multiple of any of the six functions above.
Let $P_{l+1,m-1}(z)$ denote the function replacing the exponents l, m by $l+1, m-1$.
Such function is called A "contiguous function" of P.
There are $6\times5=30$ contiguous functions of P
It will be shown shortly that any two contiguous functions of P and P have a linear relation, the coefficients being polynomials in z.

Clearly, there are $\frac{1}{2}\times30\times29=435$ relations. To demonstrate, take $P(z)$ in the form

$$P(z)=(z-a)^{\alpha}(z-b)^{\beta}(z-c)^{\gamma}\int_C (t-a)^{\alpha'+\beta+\gamma-1}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}(t-z)^{-\alpha-\beta-\gamma}\,dt$$

where C is the "double circuit" type contour described in P.8.

First, we have from this integral that $\int_C \frac{d}{dt}\left((t+b)^{\alpha'+\beta+\gamma}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}(t-z)^{-\alpha-\beta-\gamma}\right)dt=0.$
as it assumes the same value.

$$\Rightarrow (\alpha'+\beta+\gamma)P+(\alpha+\beta'+\gamma-1)P_{\alpha'+1,\beta'-1}+(\alpha+\beta+\gamma'-1)P_{\alpha'+1,\gamma'-1}=\frac{(\alpha+\beta+\gamma)}{z-b}P_{\beta+1,\gamma'-1}$$

by differentiating term-by-term.
By symmetry, we can see that RHS equals to $\frac{(\alpha+\beta+\gamma)}{z-c}P_{\beta'-1,\gamma+1}$.

With cyclical interchange between (a,α,α'), (b,β,β'), and (c,γ,γ'), we can get six linear relations in total.

Next, by writing $t-a=(t-b)+(b-a)$,

$$\begin{aligned}P(z)&=(z-a)^{\alpha}(z-b)^{\beta}(z-c)^{\gamma}\int_C (t-a)^{\beta+\gamma+\alpha'-1}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}(t-z)^{-\alpha-\beta-\gamma}dt\\&=(z-a)^{\alpha}(z-b)^{\beta}(z-c)^{\gamma}\int_C (t-a)^{\underline{\beta+\gamma+\alpha'-2}}(t-b)^{\underline{\alpha+\beta'+\gamma}}(t-c)^{\alpha+\beta+\gamma'-1}(t-z)^{-\alpha-\beta-\gamma}dt\\&\quad+(b-a)P_{\alpha'-1}. = P_{\alpha'-1,\beta'+1}+(b-a)P_{\alpha'-1}.\end{aligned}$$

($P_{\alpha'-1}$ isn't a solution of Riemann's equation)

Writing $t-a=(t-c)+(c-a)$,

$$P(z)=P_{\alpha'-1,\gamma'+1}+(c-a)P_{\alpha'-1}$$

$$\Rightarrow (c-b)P(z)+(a-c)P_{\alpha'-1,\beta'+1}+(b-a)P_{\alpha'-1,\gamma'+1}=0.$$

Changing $t-a$ into $t-b$, $t-c$, we have 3 more in total.

Writing $(t-z)=(t-a)-(z-a)$, we have

$$P=\frac{1}{z-b}P_{\beta+1,\gamma'-1}-(z-a)^{\alpha+1}(z-b)^{\beta}(z-c)^{\gamma}\int_C (t-a)^{\alpha'+\beta+\gamma-1}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}(t-z)^{-\alpha-\beta-\gamma-1}dt$$

Cyclically, we have

$$\frac{1}{(z-a)}\left[P-\frac{1}{z-b}P_{\beta+1,\gamma'-1}\right]=\frac{1}{(z-b)}\left[P-\frac{1}{(z-c)}P_{\gamma+1,\alpha'-1}\right]=\frac{1}{(z-c)}\left[P-\frac{1}{(z-a)}P_{\alpha+1,\beta'-1}\right].$$

Now, following these relation, we may reach any two of the 30 contiguous functions in finite steps. Thus there's a corollary:

Corollary Let "neighboring" functions of P be with exponents $\alpha+p, \alpha'+q, \beta+r, \beta'+s, \gamma+t, \gamma'+u$ where p,q,r,s,t,u be integers s.t. $p+q+r+s+t+u=0$.

Then any two such functions and P are related linearly.

2nd part

Confluent Hypergeometric Functions.

In the first part, we have examined some properties of hypergeometric functions. Through such functions, we're able to know more about some elementary functions ($(1-z)^{-\alpha}$ comes to mind). However, there are still a lot of functions that can't be expressed merely by hypergeometric functions due to its irregular singularity at some point (at ∞). Luckily, we can examine it through "confluence" of hypergeometric functions.

3 min.

Confluence of two singularities of Riemann's equation

Consider the equation corresponding to $P\begin{Bmatrix} 0 & \infty & c & \\ \frac{1}{2}+m & -c & c-k & ;z \\ \frac{1}{2}-m & 0 & k & \end{Bmatrix}$ and let $c\to\infty$.

It can be shown that the limiting equation would be ← ☆ See Appendix 3 for proofs.

$$\frac{d^2u}{dz^2}+\frac{du}{dz}+\left(\frac{k}{z}+\frac{\frac{1}{4}-m^2}{z^2}\right)u=0 \quad \text{—— (A).}$$

Setting $u=e^{-\frac{1}{2}z}W_{k,m}(z)$, we have

$$\frac{d^2W}{dz^2}+\left\{-\frac{1}{4}+\frac{k}{z}+\frac{\frac{1}{4}-m^2}{z^2}\right\}W=0 \quad \text{—— (B).}$$

Remark Around 0, the rational function has only double pole, so the solution is regular singular at 0. Around ∞, note that if $k\neq 0$, then ∞ is NOT regular singular.

When $2m$ is NOT an integer, two solutions of (β), regular near 0, are given by

$$\begin{cases} M_{k,m}(z) = z^{\frac{1}{2}+m} e^{-\frac{1}{2}z}\left\{1 + \frac{\frac{1}{2}+m-k}{1!\,(2m+1)} z + \frac{(\frac{1}{2}+m-k)(\frac{3}{2}+m+k)}{2!\,(2m+1)(2m+2)} z^2 + \cdots\right\} = z^{\frac{1}{2}+m} e^{-\frac{1}{2}z} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}+m-k,n)}{(1,n)\,(2m+1,n)} z^n. \\ M_{k,-m}(z) = z^{\frac{1}{2}-m} e^{-\frac{1}{2}z} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}-m-k,n)}{(1,n)\,(2m+1,n)} z^n. \end{cases}$$

The two solutions certainly form a fundamental set of solutions of (β) as their behaviors are different near 0.

Kummer's formulae

Concerning $M_{k,m}$, there are two formulae given by Kummer.

$$(\text{I}).\ z^{-\frac{1}{2}-m} M_{k,m}(z) = (-z)^{-\frac{1}{2}-m} M_{-k,m}(-z)$$

$$(\text{II}).\ M_{0,m}(z) = z^{\frac{1}{2}+m} \sum_{p=0}^{\infty} \frac{1}{2^{4p}\, p!\, (m+1,p)} z^{2p}$$

Remark Formula (I) connects solutions from different equations.

(pf). of (I): The $z^{-\frac{1}{2}-m}$, $(-z)^{-\frac{1}{2}-m}$ are just to eliminate the multi-valued part of $M_{k,m}$ and $M_{-k,m}$.

$\therefore$ It is to say $e^{-z} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}+m-k,n)}{(1,n)\,(2m+1,n)} z^n = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}+m+k,n)}{(1,n)\,(2m+1,n)} (-1)^n z^n$

For the LHS, the coefficient for z^n is

$$\sum_{j=0}^{n} \frac{(-1)^{n-j}}{(n-j)!} \frac{(\frac{1}{2}+m-k,j)}{(1,j)\,(2m+1,j)} = \frac{1}{n!} \sum_{j=0}^{n} \frac{(\frac{1}{2}+m-k,j)\,(-n,j)}{(1,j)\,(2m+1,j)} \cdot (-1)^{n-j} \cdot (-1)^{j}$$

$$= \frac{(-1)^n}{n!} \sum_{j=0}^{\infty} \frac{(\frac{1}{2}+m-k,j)(-n,j)}{(1,j)\,(2m+1,j)} = \frac{(-1)^n}{n!} F(\tfrac{1}{2}+m-k, -n; 2m+1; 1)$$

$(-n,l)=0\ \forall\, l \geq n+1$

$$= \frac{(-1)^n}{n!} \frac{\Gamma(2m+1)\,\Gamma(m+k+\frac{1}{2}+n)}{\Gamma(m+\frac{1}{2}+k)\,\Gamma(2m+1+n)} = \frac{(-1)^n}{n!} \frac{(\frac{1}{2}+m+k,n)}{(2m+1,n)}$$, which is exactly the coeff. of RHS.

of (II): $M_{0,m}(z) = z^{m+\frac{1}{2}} e^{-\frac{1}{2}z} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}+m,n)}{(1,n)(2m+1,n)} z^n$

$\therefore$ Coefficient of $z^{n+m+\frac{1}{2}}$: $\sum_{j=0}^{n} \frac{(-1)^j}{j!\,2^j} \frac{(m+\frac{1}{2})\cdots(m+\frac{1}{2}+n-j-1)}{(n-j)!\,(2m+1)\cdots(2m+n-j)}$

$$= \frac{(m+\frac{1}{2})\cdots(m+n-\frac{1}{2})}{n!\,(2m+1)\cdots(2m+n)} \sum_{j=0}^{n} \frac{(-n)\cdots(-n+j-1)\,(-2m-n)\cdots(-2m-n+j-1)}{(-m-n+\frac{1}{2})\cdots(-m-n+j-\frac{1}{2})} \cdot \frac{1}{2^j}$$

$$= \frac{(m+\frac{1}{2})\cdots(m+n-\frac{1}{2})}{n!\,(2m+1)\cdots(2m+n)} F(-n, -2m-n, -m-n+\tfrac{1}{2}; \tfrac{1}{2})$$

$$= \frac{(m+\frac{1}{2})\cdots(m+n-\frac{1}{2})}{n!\,(2m+1)\cdots(2m+n)} F(-\tfrac{1}{2}n, -m-\tfrac{1}{2}n, -m-n+\tfrac{1}{2}, 1)$$ ← See Appendix 4 for proofs

By formula for $F(a,b;c;1)$ derived in P.2,

the value is $\frac{(m+\frac{1}{2})\cdots(m+n-\frac{1}{2})}{n!\,(2m+1)\cdots(2m+n)} \frac{\Gamma(-n-m+\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-m-\frac{1}{2}n)\,\Gamma(\frac{1}{2}-\frac{1}{2}n)}$

$$= \frac{(-1)^n\,\Gamma(\frac{1}{2}-m)\,\Gamma(\frac{1}{2})}{n!\,(2m+1)\cdots(2m+n)\,\Gamma(\frac{1}{2}-m-\frac{1}{2}n)\,\Gamma(\frac{1}{2}-\frac{1}{2}n)}$$

When n is odd, $\frac{1}{\Gamma(\frac{1}{2}-\frac{1}{2}n)} = 0$. For n is even, $(n=2p)$, it is

P.12.

$$\frac{\Gamma(\frac{1}{2}-m)\Gamma(\frac{1}{2})}{(2p)!\,(2m+1)\cdots(2m+2p)\,\Gamma(\frac{1}{2}-m-p)\Gamma(\frac{1}{2}-p)}$$

$$= \frac{\Gamma(\frac{1}{2}-m)(-\frac{1}{2})(-\frac{3}{2})\cdots(-p+\frac{1}{2})}{(2p)!\,2^{2p}\,(m+\frac{1}{2})(m+1)(m+\frac{3}{2})\cdots(m+p-\frac{1}{2})(m+p)\,\Gamma(\frac{1}{2}-m-p)}$$

$$= \frac{(-m-\frac{1}{2})(-m-\frac{3}{2})\cdots(-m-p+\frac{1}{2})(-1)(-3)\cdots(-2p+1)}{(2p)!\,2^{3p}\,(m+\frac{1}{2})(m+\frac{3}{2})\cdots(m+p-\frac{1}{2})(m+1)\cdots(m+p)} = \frac{1\cdot 3\cdot 5\cdots(2p-1)}{(2p)!\,2^{3p}\,(m+1)\cdots(m+p)} = \frac{1}{2^{4p}\,p!\,(m+1)\cdots(m+p)}.$$

That proves (II).

5 min.

Definition of the function $W_{k,m}(z)$

The function $M_{k,m}$ and $M_{k,-m}$ constitute a fundamental set of solutions of (B), but when $2m$ is an integer, they're not. Thus, it is more convenient to consider the solutions in integral representation.

Using the formula $I = (z-a)^{\alpha}(z-b)^{\beta}(z-c)^{\gamma}\int_C (t-a)^{\alpha'+\beta+\gamma-1}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}(z-t)^{-\alpha-\beta-\gamma}\,dt$ and confluence $b=\infty$, $c\to\infty$ and multiply by $e^{\frac{1}{2}z}$ ← ☆Appendix 5 for proofs.

With a constant multiple, we define

$$W_{k,m}(z) - \frac{1}{2\pi i}\Gamma(k+\frac{1}{2}-m)\,e^{-\frac{1}{2}z} z^{k}\int_{\infty}^{(0^+)}(-t)^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m} e^{-t}\,dt$$

where the contour looks like

The contour is curved so that $t=-z$ is outside it (to avoid monodromy)

The integral is single-valued once we chose the branch so that

$-\pi < \arg(-t) < \pi$ ($\arg(-t) \to -\pi$ at the begining, $\to \pi$ at the end)

and $\arg(1+\frac{t}{z}) \to 0$ as $t\to\infty$. (Not $2k\pi$ or so)

Then it is an analytic function in z as the contour is more or less compact.

Write $v = \int_{\infty}^{(0^+)}(-t)^{-k-\frac{1}{2}+m}(1+\frac{t}{z})^{k-\frac{1}{2}+m}e^{-t}\,dt$

Then $\frac{d^2v}{dz^2} + \left(\frac{2k}{z}-1\right)\frac{dv}{dz} + \frac{\frac{1}{4}-m^2+k(k-1)}{z^2}v = 0$

Thus, $e^{-\frac{1}{2}z}z^{k}v$ will satisfy (B).

Thus, $W_{k,m}(z) = \frac{-1}{2\pi i}\Gamma(k+\frac{1}{2}-m)e^{-\frac{1}{2}z}z^{k}\int_{\infty}^{(0^+)}(-t)^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m}e^{-t}\,dt$

is a solution for (B).

Transform contour integral to infinite integral (Only as side remark).

If $k-\frac{1}{2}-m \leq -1 \in \mathbb{Z}$, i.e. $-k-\frac{1}{2}+m \in \mathbb{N}$, then the integral will be 0 by Cauchy's theorem.

To overcome such difficulty, when $\Re(k-\frac{1}{2}-m)\leq 0$ and is not an integer, we transform the contour integral to an infinite integral

$$-\frac{1}{2\pi i}\Gamma(k+\tfrac{1}{2}-m)e^{-\frac{1}{2}z}z^{k}\int_{\infty}^{(0^+)}(-t)^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m}e^{-t}\,dt.$$

$\left(\Gamma(k+\frac{1}{2}-m)=\frac{1}{\Gamma(\frac{1}{2}-k+m)}\frac{\pi}{\sin(\pi(k+\frac{1}{2}-m))}\right)$

$$= -\frac{1}{2\pi i}\frac{\pi}{\Gamma(\frac{1}{2}-k+m)\sin(\pi(k+\frac{1}{2}+m))}e^{-\frac{1}{2}z}z^{k}\left[-\int_{0}^{\infty}e^{-\pi i(-k-\frac{1}{2}+m)}\underset{\underset{|-t|}{\shortparallel}}{|t|}^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m}e^{-t}dt + \int_{0}^{\infty}e^{\pi i(-k-\frac{1}{2}+m)}t^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m}e^{-t}dt\right].$$

initial $\arg(-t)$

$$= \frac{z^{k}e^{-\frac{1}{2}z}}{-2i}\frac{\left(e^{\pi i(-k-\frac{1}{2}+m)}-e^{-\pi i(-k-\frac{1}{2}+m)}\right)}{\Gamma(\frac{1}{2}-k+m)\sin(\pi(k+\frac{1}{2}-m))}\int_{0}^{\infty}t^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m}e^{-t}dt$$

$$= \frac{\cancel{-}\sin(-\pi+\cancel{\pi(k+\frac{1}{2}-m)})e^{-\frac{1}{2}z}z^{k}}{\Gamma(\frac{1}{2}-k+m)\sin(\cancel{\pi(k+\frac{1}{2}-m)})}\int_{0}^{\infty}t^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m}e^{-t}dt$$

$$= \frac{e^{-\frac{1}{2}z}z^{k}}{\Gamma(\frac{1}{2}-k+m)}\int_{0}^{\infty}t^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m}e^{-t}dt.\ (=W_{k,m}(z))$$

This formula enables us to compute $W_{k,m}(z)$ when $m+\frac{1}{2}-k \in \mathbb{N}$.

$\therefore W_{k,m}$ is defined for all k, m, z except for $z<0$.

Expressions of functions by $W_{k,m}(z)$.

In this section, we'll see that numerous functions can be expressed by $W_{k,m}$.

(I) Error function (Theory of probability).

$$\mathrm{Erfc}(x)=\int_{x}^{\infty}e^{-t^2}dt. \quad ,x\in\mathbb{R}.$$

Let $t=x^2(w^2-1)$, and then $w=s/x$ in the integral for $W_{-\frac{1}{4},\frac{1}{4}}(x^2)$,

we have
$$W_{-\frac{1}{4},\frac{1}{4}}(x^2)=x^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}\int_{0}^{\infty}\left(1+\frac{t}{x^2}\right)^{-\frac{1}{2}}e^{-t}dt.$$
$$=x^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}\int_{1}^{\infty}(\underbrace{1+(w^2-1)}_{\overset{\shortparallel}{w^2}})^{-\frac{1}{2}}e^{-x^2(w^2-1)}2x^2w\,dw.$$
$$=2x^{\frac{3}{2}}e^{\frac{1}{2}x^2}\int_{x}^{\infty}e^{-s^2}\frac{1}{x}ds. = 2x^{\frac{1}{2}}e^{\frac{1}{2}x^2}\int_{x}^{\infty}e^{-s^2}ds.$$

$$\therefore \mathrm{Erfc}(x)=\frac{1}{2}x^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}W_{-\frac{1}{4},\frac{1}{4}}(x^2).$$

P.14

$\boxed{\text{Remark}}$ Conduction of heat uses $\int_a^b e^{-t^2 - x^2/t^2}\,dt$ which relates to Error function, and so with $W_{k,m}$.

(II) Incomplete Gamma function.: (Discussed by Legendre)

$$\gamma(n,x) = \int_0^x t^{n-1} e^{-t}\,dt.$$

Set $t = s - x$ in the integral of $W_{\frac{1}{2}(n-1),\frac{1}{2}n}(x)$, and

$$W_{\frac{1}{2}(n-1),\frac{1}{2}n}(x) = e^{-\frac{1}{2}x} x^{\frac{1}{2}(n-1)} \int_0^\infty \left(1+\frac{t}{x}\right)^{n-1} e^{-t}\,dt.$$

$$= e^{-\frac{1}{2}x} x^{\frac{1}{2}(n-1)} \int_x^\infty \left(\frac{s}{x}\right)^{n-1} e^{-s} e^{x}\,ds.$$

$$= e^{\frac{1}{2}x} x^{-\frac{1}{2}n+\frac{1}{2}} \int_x^\infty s^{n-1} e^{-s}\,ds.$$

~41 min.

$$\therefore \quad \gamma(n,x) = \Gamma(n) - \int_x^\infty s^{n-1} e^{-s}\,ds = \Gamma(n) - e^{-\frac{1}{2}x} x^{\frac{1}{2}(n-1)} W_{\frac{1}{2}(n-1),\frac{1}{2}n}(x).$$

3 min.

(III). Logarithmic-integral function (Discussed by Gauss)

When $|\arg\{-\log z\}| < \pi$, define

$$li(z) = \int_0^z \frac{dt}{\log t}$$

Let $s - \log z = u$, and $u = -\log t$ in the integral for $W_{-\frac{1}{2},0}(-\log z)$,

$$W_{-\frac{1}{2},0}(-\log z) = z^{\frac{1}{2}} (-\log z)^{-\frac{1}{2}} \int_0^\infty \left(1 + \frac{s}{-\log z}\right)^{-1} e^{-s}\,ds$$

$$= z^{\frac{1}{2}} (-\log z)^{-\frac{1}{2}} \int_{\log z}^\infty \left(\frac{u}{-\log z}\right)^{-1} e^{-u} \cdot \frac{1}{z}\,du$$

$$= z^{-\frac{1}{2}} (-\log z)^{\frac{1}{2}} \int_z^0 \frac{t}{-\log t} \cdot \frac{-1}{t}\,dt$$

$$= -z^{-\frac{1}{2}} (-\log z)^{\frac{1}{2}}\, li(z)$$

$$\Rightarrow \quad li(z) = -(-\log z)^{-\frac{1}{2}} z^{\frac{1}{2}} W_{-\frac{1}{2},0}(-\log z).$$

☆☆☆ 12 min.

The asymptotic expansion of $W_{k,m}(z)$ when $|z|$ is large

To examine the asymptotic behavior of $W_{k,m}$, we first examine

$$\left(1+\frac{t}{z}\right)^{\lambda} = 1 + \frac{\lambda t}{z} + \cdots + \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!}\frac{t^n}{z^n} + R_n(t,z)$$

where

$$R_n(t,z) = \frac{\lambda(\lambda-1)\cdots(\lambda-n)}{n!}\left(1+\frac{t}{z}\right)^{\lambda} \int_0^{t/z} u^n (1+u)^{-\lambda-1}\,du.$$

$\boxed{\text{Remark}}$: $\int_{\infty}^{(0^+)} (-t)^{z-1} e^{-t}\,dt = \left[e^{\pi i(z-1)} - e^{-\pi i(z-1)}\right]\int_0^{\infty} (t)^{z-1} e^{-t}\,dt$

$= 2i\sin(z-1)\pi\,\Gamma(z). = -2i\sin z\pi\,\Gamma(z).$, so it is also fine to derive from original form.

$$W_{k,m}(z) = \frac{e^{-\frac{1}{2}z} z^k}{\Gamma(\frac{1}{2}-k+m)}\int_0^{\infty} t^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m} e^{-t}\,dt.$$

$$= \frac{e^{-\frac{1}{2}z} z^k}{\Gamma(\frac{1}{2}-k+m)}\left\{\sum_{l=0}^{n}\frac{(k-\frac{1}{2}+m)\cdots(k-\frac{1}{2}+m-l+1)}{l!\,z^l}\Gamma(-k-\frac{1}{2}+m+l) + \int_0^{\infty} t^{-k-\frac{1}{2}+m} R_n(t,z) e^{-t}\,dt\right\}$$

$$= e^{-\frac{1}{2}z} z^k \left\{\sum_{l=0}^{n}\frac{(k-\frac{1}{2}+m)\cdots(k-\frac{1}{2}+m-l+1)(\frac{1}{2}-k+m)\cdots(\frac{1}{2}-k+m+(l-1))}{l!\,z^l} + \int_0^{\infty}\frac{t^{-k-\frac{1}{2}+m} R_n(t,z) e^{-t}\,dt}{\Gamma(-k+\frac{1}{2}+m)}\right\}$$

$$= e^{-\frac{1}{2}z} z^k \left\{\sum_{l=0}^{n}\frac{(m^2-(k-\frac{1}{2})^2)(m^2-(k-\frac{3}{2})^2)\cdots(m^2-(k-l+\frac{1}{2})^2)}{l!\,z^l} + \frac{1}{\Gamma(-k+\frac{1}{2}+m)}\int_0^{\infty} t^{-k-\frac{1}{2}+m} R_n(t,z) e^{-t}\,dt\right\}$$

provided $\operatorname{Re}(n-k-\frac{1}{2}+m)>0$.

Now, if $|\arg z| \le \pi-\alpha$, $|z|>1$, then

$$\begin{cases} 1 \le |1+\frac{t}{z}| \le 1+t & \text{if } \operatorname{Re}(z)\ge 0. \\ |1+t/z| \ge |\sin\alpha| & \text{if } \operatorname{Re}(z)\le 0 \end{cases}$$

The part where $\operatorname{Re}(z)\ge 0$ is straightforward.

If $\operatorname{Re}(z)\le 0$, set $z = re^{i\theta}$ where $r>1$, $\frac{\pi}{2}\le|\theta|\le(\pi-\alpha)$.

Norm^2 of $1+\frac{t}{z} = \left|1+\frac{t}{r}\cos\theta - i\frac{t}{r}\sin\theta\right|^2 = 1+\frac{t^2}{r^2}\cos^2\theta+\frac{2t}{r}\cos\theta+\frac{t^2}{r^2}\sin^2\theta$

$= \frac{t^2}{r^2}+\frac{2t}{r}\cos\theta+1. =: g(t).$

$0=g'(t)=\frac{2t}{r^2}+\frac{2}{r}\cos\theta$ iff $t=-r\cos\theta\ (>0)$.

$\therefore\ g(-r\cos\theta) = \cos^2\theta-2\cos^2\theta+1 = \sin^2\theta\ge\sin^2\alpha$ ← minimum

$\therefore\ |1+t/z|\ge|\sin\alpha|$ for $\operatorname{Re}(z)\le 0$.

$\therefore\ |R_n(t,z)| \le \left|\frac{\lambda(\lambda-1)\cdots(\lambda-n)}{n!}\right|(1+t)^{|\lambda|}\frac{1}{(\sin\alpha)^{|\lambda|}}\underbrace{\int_0^{|t/z|} u^n(1+u)^{|\lambda|}\,du}. \le (1+t)^{|\lambda|}\int_0^{|t/z|}u^n\,du$

$\Rightarrow |R_n(t,z)| \le \frac{|\lambda(\lambda-1)\cdots(\lambda-n)|}{n!}(1+t)^{2|\lambda|}\frac{1}{(\sin\alpha)^{|\lambda|}}\left|\frac{t}{z}\right|^{n+1}\Big/ n+1 = O(z^{-n-1}).$

← Depend on α.

$\therefore$ For $|z|>1$, $\left|\frac{1}{\Gamma(-k+\frac{1}{2}+m)}\int_0^{\infty} t^{-k-\frac{1}{2}+m} R_n(t,z)e^{-t}\,dt\right| = O\left\{\int_0^{\infty} t^{-k-\frac{1}{2}+m+n}(1+t)^{2|\lambda|}|z|^{-n-1}e^{-t}\,dt\right\}$

$= O(|z|^{-n-1})$ as $z\to\infty$.

$\boxed{\text{Remark}}$: Note that the estimate varies with upper bound of argument $\pi-\alpha$.

Integral $\to\infty$ as $\alpha\to 0$.

$\therefore$ For $|z|$ large,

$$W_{k,m} \sim e^{\frac{1}{2}z} z^{k} \left\{ 1 + \sum_{n=1}^{\infty} \frac{\{m^2-(k-\frac{1}{2})^2\} \cdots \{m^2-(k-n+\frac{1}{2})^2\}}{n!\, z^n} \right\}.$$

Second solution of the equation (B) for $W_{k,m}(z)$.

$W_{k,m}(z)$ satisfies $\frac{d^2W}{dz^2} + \left\{-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4}-m^2}{z}\right\}W = 0$.

Note that $W_{-k,m}(-z)$ satisfies

$$\frac{d^2w}{d(-z)^2} + \left\{-\frac{1}{4} + \frac{(-k)}{(-z)} + \frac{\frac{1}{4}-m^2}{(-z)^2}\right\}W = 0.$$

$$\frac{d^2w}{d(-z)^2} = \frac{d^2w}{dz^2}.$$

$\therefore$ $W_{-k,m}(-z)$ is another solution.

Now, $W_{k,m}(z) = e^{-\frac{1}{2}z} z^{k} \{1+O(z^{-1})\}$, and when $|\arg(-z)| < \pi$,

$W_{-k,m}(-z) = e^{-\frac{1}{2}z} (-z)^{-k} (1+O(z^{-1}))$.

They have different local behavior around 0. $\Rightarrow$ They form a fundamental set of solutions.

Contour integrals of the Mellin-Barnes type for $W_{k,m}(z)$

$\uparrow$

Mention as a side remark, and its use of asymptotic expansion

Consider $I = \frac{e^{-\frac{1}{2}z} z^{k}}{2\pi i} \int_{-\infty i}^{\infty i} \frac{\Gamma(s)\Gamma(-s-k-m+\frac{1}{2})\Gamma(-s-k+m+\frac{1}{2})}{\Gamma(-k-m+\frac{1}{2})\Gamma(-k+m+\frac{1}{2})} z^{s}\, ds$ — (C),

where $|\arg z| < \frac{3}{2}\pi$ and neither of the number $k \pm m + \frac{1}{2}$ is a positive integer or 0.

The contour is curved so that poles of $\Gamma(s)$ and $\Gamma(-s-k-m+\frac{1}{2})\Gamma(-s-k+m+\frac{1}{2})$ are on opposite sides.

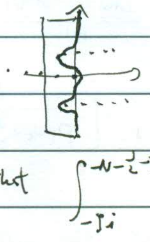
Note that $\Gamma(s)\Gamma(-s-k-m+\frac{1}{2})\Gamma(-s-k+m+\frac{1}{2}) = o\left(\exp\left((s-\frac{1}{2})(\frac{\pi}{2}i + \log|s|) - s + (-2s-2k)(\frac{3}{2}\pi i + \log|s|)\right)\right)$.

$+2s$

$= o\left(\exp\left((-\frac{3}{2}\pi)|s| - \log|s|(-2k-\frac{1}{2})\right)\right) \to 0$ as $|s| \to \infty$.

Now, choose N s.t. poles of $\Gamma(-s-k-m+\frac{1}{2})\Gamma(-s-k+m+\frac{1}{2})$ lies on the right of $R(s) = -N-\frac{1}{2}$.

Then consider the contour $\pm \zeta i$, $-N-\frac{1}{2} \pm \zeta i$



Given any N, as $\zeta \to \infty$, we have that $\int_{-\zeta i}^{-N-\frac{1}{2}-\zeta i}$, $\int_{\zeta i}^{-N-\frac{1}{2}+\zeta i} \to 0$.

$\therefore$ By Cauchy's Theorem,

$$\frac{e^{-\frac{1}{2}z}z^{k}}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma(s)\Gamma(-s-k-m+\frac{1}{2})\Gamma(-s-k+m+\frac{1}{2})}{\Gamma(-k-m+\frac{1}{2})\Gamma(-k+m+\frac{1}{2})}z^{s}ds.$$

$$=e^{-\frac{1}{2}z}z^{k}\left\{\sum_{n=0}^{N}\text{residue of }\Gamma(s)+\frac{1}{2\pi i}\int_{-N-\infty i}^{-N+\infty i}\frac{\Gamma(s)\Gamma(-s-k-m+\frac{1}{2})\Gamma(-s-k+m+\frac{1}{2})}{\Gamma(-k-m+\frac{1}{2})\Gamma(-k+m+\frac{1}{2})}z^{s}ds\right\}.$$

z^s ensures that as $N\to\infty$, $|z|^{-N-\frac{1}{2}}\to 0$.

$$\therefore I=e^{-\frac{1}{2}z}z^{k}\sum\frac{\Gamma(n-k-m+\frac{1}{2})\Gamma(n-k+m+\frac{1}{2})}{n!\,\Gamma(-k-m+\frac{1}{2})\Gamma(-k+m+\frac{1}{2})}(-1)^{n}z^{-n}.$$

$$=e^{-\frac{1}{2}z}z^{k}\sum_{n=0}^{\infty}\frac{\{m^{2}-(k-\frac{1}{2})^{2}\}\cdots\{m^{2}-(k-n+\frac{1}{2})^{2}\}}{n!\,z^{n}}.$$

That is, asymptotically I is just like $W_{k,m}(z)$.

Furthermore, substitute $\int_{-\infty i}^{\infty i}\Gamma(s)\Gamma(-s-k-m+\frac{1}{2})\Gamma(-s-k+m+\frac{1}{2})z^{s}ds$ for v

in $z^{2}\frac{d^{2}v}{dz^{2}}+2kz\frac{dv}{dz}+(k-m-\frac{1}{2})(k+m-\frac{1}{2})v-z^{2}\frac{dv}{dz}$, we have

$$(-s-k-m+\tfrac{1}{2})(-s-k+m+\tfrac{1}{2})$$

$$\int_{-\infty}^{\infty}\left[s(s-1)+2ks+(k-m-\tfrac{1}{2})(k+m-\tfrac{1}{2})\right]\Gamma(s)\Gamma(-s-k-m+\tfrac{1}{2})\Gamma(-s-k+m+\tfrac{1}{2})z^{s}ds$$

$$-\int_{-\infty}^{\infty}\Gamma(s+1)\Gamma(-s-k-m+\tfrac{1}{2})\Gamma(-s-k+m+\tfrac{1}{2})z^{s+1}ds. \leftarrow t=s+1$$

$$=\left(\int_{-\infty i}^{\infty i}-\int_{1-\infty i}^{1+\infty i}\right)\left(\Gamma(s)\Gamma(-s-k-m+\tfrac{3}{2})\Gamma(-s-k+m+\tfrac{3}{2})z^{s}ds.\right.$$

There is no poles between the area enclosed

($\because$ The contour is curved to have these poles lie to the right of it.
As the contour moves to $(1-\infty i)\to(1+\infty i)$, the poles are moved as well).

$\therefore$ It equals 0 by Cauchy's Theorem

$\therefore$ I satisfies (B).

$\Rightarrow I=AW_{k,m}(z)+BW_{-k,m}(-z)$, and letting $|z|\to\infty$, $R(z)>0$,

we must have $A=1$, $B=0$.

$\therefore$ $I=W_{k,m}(z)$. for $|\arg z|<\pi$. I is defined for $|\arg z|<\frac{3}{2}\pi$,

so it is in fact an analytic continuation of $W_{k,m}(z)$.

ions between
m and $M_{k,\pm m}$

Recall that $M_{k,m} := z^{\frac{1}{2}+m} e^{-\frac{1}{2}z} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}+m-k, n)}{n!(2m+1, n)} z^n$ is a solution of (B), so we can express $W_{k,m}$ in terms of $M_{k,m}$ and $M_{k,-m}$: fund. set of solutions of (B).

From the previous section, we know that

$$W_{k,m}(z) = \frac{e^{-\frac{1}{2}z} z^k}{2\pi i} \int_{-\infty i}^{\infty i} \frac{\Gamma(s)\Gamma(-s-k-m+\frac{1}{2})\Gamma(-s-k+m+\frac{1}{2})}{\Gamma(-k-m+\frac{1}{2})\Gamma(-k+m+\frac{1}{2})} z^s ds.$$

Note that $\Gamma(s)\Gamma(-s-k-m+\frac{1}{2})\Gamma(-s-k+m+\frac{1}{2}) = \frac{\Gamma(s)\pi^2}{\Gamma(\frac{1}{2}+s+k+m)\Gamma(\frac{1}{2}+s+k-m)} \frac{1}{\cos(\pi(s+k+m))\cos(\pi(s+k-m))}$.

exponentially large near real axis with large

exponentially large when $I(s)$ is large.

$re^{i\theta}$

∴ Right semi-circle contour tends to 0 as $r \to \infty$.

$$\therefore W_{k,m}(z) = e^{-\frac{1}{2}z} z^k \left\{ \sum_{n=0}^{\infty} \frac{\pi\Gamma(-k-m+\frac{1}{2}+n)}{n!\,\Gamma(-2m+1+n)} \frac{z^{-m-k+\frac{1}{2}+n}}{\cos(\pi(-2m+\frac{1}{2}+n))} \Big/ \Gamma(-k-m+\frac{1}{2})\Gamma(-k+m+\frac{1}{2}) + \right.$$

$$\left. \sum_{n=0}^{\infty} \frac{\pi\Gamma(-k+m+\frac{1}{2}+n)}{n!\,\Gamma(2m+1+n)} \frac{z^{m-k+\frac{1}{2}+n}}{\cos(\pi(2m+\frac{1}{2}+n))} \Big/ \Gamma(-k-m+\frac{1}{2})\Gamma(-k+m+\frac{1}{2}) \right.$$

$\sin(\pi(-2m+n))$

$$= z^{\frac{1}{2}-m} e^{-\frac{1}{2}z} \left\{ \sum_{n=0}^{\infty} \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)} \frac{(\frac{1}{2}+m+k, n)}{n!(2m-1, n)} z^n + \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)} \frac{(\frac{1}{2}-m+k, n)}{(n!)(1-2m, n)} z^n. \right.$$

$$= \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)} M_{k,m}(z) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)} M_{k,-m}(z).$$

olic Cylinder
tions. Weber's
quation.

Consider $w = z^{-\frac{1}{2}} W_{k,-\frac{1}{4}}(\frac{1}{2}z^2)$, then w satisfies

$\frac{d}{zdz}\left(\frac{d}{zdz}(wz^{\frac{1}{2}})\right) + \left\{-\frac{1}{4} + \frac{2k}{z^2} + \frac{3/4}{z^4}\right\} wz^{\frac{1}{2}} = 0$ which follows from (B) and cleaning up the mess from $\frac{1}{2}z^2$.

$$\frac{d}{zdz}(wz^{\frac{1}{2}}) = \frac{1}{2}\frac{1}{z^{3/2}} w + \frac{1}{z^{1/2}} w', \quad \frac{d}{zdz}\left(\frac{d}{zdz}(wz^{\frac{1}{2}})\right) = -\frac{3}{4}\frac{1}{z^{5/2}} w + \cancel{\frac{1}{2}\frac{1}{z^{3/2}} w'} - \cancel{\frac{1}{2}\frac{1}{z^{3/2}} w'} + \frac{1}{z^{3/2}} w''$$

$$\therefore 0 = w'' - \frac{3}{4} w + \left\{-\frac{1}{4}z^2 + 2k + \frac{3}{4}\right\} w = w'' + \left\{2k - \frac{1}{4}z^2\right\} w.$$

∴ The function $D_n(z) := 2^{\frac{1}{2}n+\frac{1}{4}} z^{-\frac{1}{2}} W_{\frac{1}{2}n+\frac{1}{4}, -\frac{1}{4}}(\frac{1}{2}z^2)$.

satisfies $\frac{d^2 D_n}{dz^2} + \{n + \frac{1}{2} - \frac{1}{4}z^2\} D_n(z) = 0$.

Remark: D_n is associated with the parabolic cylinder in harmonic analysis. This equation is called Weber's equation.

From the previous section, we have that

$$D_n(z) = 2^{\frac{1}{2}n+\frac{1}{4}} z^{-\frac{1}{2}} W_{\frac{1}{2}n+\frac{1}{4},-\frac{1}{4}}\left(\frac{1}{2}z^2\right) = \frac{\Gamma(\frac{1}{2})\, 2^{\frac{1}{2}n+\frac{1}{4}} z^{-\frac{1}{2}}}{\Gamma(\frac{1}{2}-\frac{1}{2}n)} M_{\frac{1}{2}n+\frac{1}{4},-\frac{1}{4}}\left(\frac{1}{2}z^2\right) + \frac{\Gamma(-\frac{1}{2})\, 2^{\frac{1}{2}n+\frac{1}{4}} z^{-\frac{1}{2}}}{\Gamma(-\frac{1}{2}n)} M_{\frac{1}{2}n+\frac{1}{4},\frac{1}{4}}\left(\frac{1}{2}z^2\right)$$

However, $z^{\frac{1}{2}} M_{\frac{1}{2}n+\frac{1}{4},-\frac{1}{4}}\left(\frac{1}{2}z^2\right) = 2^{-\frac{1}{4}} e^{-\frac{1}{4}z^2} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2}n,k)}{(\frac{1}{2},k)k!}\left(\frac{1}{2}z^2\right)^k.$

$$z^{-\frac{1}{2}} M_{\frac{1}{2}n+\frac{1}{4},\frac{1}{4}}\left(\frac{1}{2}z^2\right) = 2^{-\frac{3}{4}} z e^{-\frac{1}{4}z^2} \sum_{k=0}^{\infty} \frac{(\frac{1}{2}-\frac{1}{2}n,k)}{(\frac{3}{2},k)\,k!}\left(\frac{1}{2}z^2\right)^k.$$

and these are one-valued functions of z and entire.

$\Rightarrow$ D_n is an entire function in z.

and in previous section, we have the asymptotic expansion of $W_{k,m}(z)$

$$W_{k,m}(z) \sim e^{-\frac{1}{2}z} z^k \left\{ 1 + \sum_{n=1}^{\infty} \frac{[m^2-(k-\frac{1}{2})^2] \cdots [m^2-(k-n+\frac{1}{2})^2]}{n!\, z^n} \right\} \quad \text{for } |\arg z| < \pi, \ |z| \to \infty$$

Also, $W_{k,m}$ can be continued to I, so this holds for $|\arg z| < \frac{3}{2}\pi$

$$\Rightarrow D_n(z) = 2^{\frac{1}{2}n+\frac{1}{4}} z^{-\frac{1}{2}} W_{\frac{1}{2}n+\frac{1}{4},-\frac{1}{4}}\left(\frac{1}{2}z^2\right)$$

$\sim e^{\frac{1}{4}z^2} z^n \left(1 - \frac{n(n-1)}{2z^2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4z^2} + \cdots\right)$. Because of $\frac{1}{2}z^2$ in the argument

$W(\frac{1}{2}z^2)$, we have to restrict $|\arg z| < \frac{3}{4}\pi$.

e second
lution of
eber's equation

Weber's equation is $\frac{d^2 D_n}{dz^2} + (n + \frac{1}{2} - \frac{1}{4}z^2) D_n(z) = 0$.

Thus, $D_{-n-1}(\pm iz)$ satisfies

$$0 = -\frac{d^2 D_{-n-1}}{dz^2} + \left(-n-1+\frac{1}{2}+\frac{1}{4}z^2\right) D_{-n-1}(\pm iz) = -\left(\frac{d^2 D_{-n-1}}{dz^2} + \left(n+\frac{1}{2}-\frac{1}{4}z^2\right) D_{-n-1}\right).$$

Also, reiterating this transformation again, $D_n(-z)$ is also a solution

From the previous section, we know that

$D_n(z)$ and $D_{-n-1}(iz)$, for $-\frac{3}{4}\pi < \arg z < \frac{1}{4}\pi$ (where asymptotic expansion works),

They have different behavior (z^n and z^{-n-1})

$\Rightarrow$ They're lin. indep.

Relation between $D_n(z)$, $D_{-n-1}(\pm iz)$

We must have that

$$D_n(z) = a D_{-n-1}(iz) + b D_{-n-1}(-iz) \quad , a, b \in \mathbb{C} \quad \text{as they all solve Weber's equation.}$$

Examining the first two coefficients, we have that

$$\frac{\Gamma(\frac{1}{2}) z^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}-\frac{1}{2}n)} + \frac{\Gamma(-\frac{1}{2}) z^{\frac{1}{2}n-\frac{1}{2}}}{\Gamma(-\frac{1}{2}n)} z + \cdots \qquad \leftarrow \text{From expansion by } M_{k,m}, M_{k,-m}$$

$$= a\left\{ \frac{\Gamma(\frac{1}{2}) z^{\frac{1}{2}n-\frac{1}{2}}}{\Gamma(1+\frac{1}{2}n)} + \frac{\Gamma(-\frac{1}{2}) z^{-\frac{1}{2}n-1}}{\Gamma(\frac{1}{2}+\frac{1}{2}n)} iz + \cdots \right\}$$

$$+ b\left\{ \frac{\Gamma(\frac{1}{2}) z^{-\frac{1}{2}n-\frac{1}{2}}}{\Gamma(1+\frac{1}{2}n)} - \frac{\Gamma(-\frac{1}{2}) z^{-\frac{1}{2}n-1}}{\Gamma(\frac{1}{2}+\frac{1}{2}n)} iz + \cdots \right\}.$$

$$\Rightarrow \begin{pmatrix} \frac{\Gamma(\frac{1}{2}) z^{-\frac{1}{2}n-\frac{1}{2}}}{\Gamma(1+\frac{1}{2}n)} & \frac{\Gamma(\frac{1}{2}) z^{-\frac{1}{2}n-\frac{1}{2}}}{\Gamma(1+\frac{1}{2}n)} \\ i\Gamma(-\frac{1}{2}) z^{-\frac{1}{2}n} \sim & -i \sim \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\Leftrightarrow \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{\Gamma(1+\frac{1}{2}n) 2^{n+\frac{1}{2}}}{\Gamma(\frac{1}{2}-\frac{1}{2}n)} \\ \frac{-i\Gamma(\frac{1}{2}+\frac{1}{2}n) 2^{n+\frac{1}{2}}}{\Gamma(-\frac{1}{2}n)} \end{pmatrix}. \quad \Leftrightarrow \quad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} (2\pi)^{-\frac{1}{2}} \Gamma(n+1) e^{\frac{1}{2}n\pi i} \\ (2\pi)^{-\frac{1}{2}} \Gamma(n+1) e^{-\frac{1}{2}n\pi i} \end{pmatrix}.$$

Note that $\Gamma(1+\frac{1}{2}n) / \Gamma(\frac{1}{2}-\frac{1}{2}n) = \Gamma(1+\frac{1}{2}n)\Gamma(\frac{1}{2}+\frac{1}{2}n) \, \pi / \sin\pi(\frac{1}{2}+\frac{1}{2}n)$

$$\left(\Gamma(\tfrac{3}{2})\Gamma(1) = \tfrac{1}{2}\sqrt{\pi} \quad , \Gamma(2)\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\sqrt{\pi} \right.$$

$$\Gamma(1+\tfrac{1}{2}n)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}n) = \tfrac{1}{2}n \, \Gamma(\tfrac{1}{2}+\tfrac{1}{2}(n-1))\Gamma(1+\tfrac{1}{2}(n-1)) \Big)$$

$$\Rightarrow \Gamma(1+\tfrac{1}{2}n)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}n) = \frac{\sqrt{\pi}}{2^n} n! = \frac{\sqrt{\pi}}{2^n}\Gamma(n+1)$$

Similarly, $\dfrac{\Gamma(\frac{1}{2}+\frac{1}{2}n)}{\Gamma(-\frac{1}{2}n)} = \dfrac{\sqrt{\pi}}{2^n} \dfrac{\Gamma(n+1)\pi}{\sin(\pi(1+\frac{1}{2}n))}$.

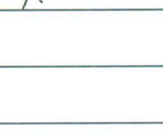
$$\Rightarrow D_n(z) = \frac{\Gamma(n+1)}{\sqrt{2\pi}} \left[e^{\frac{1}{2}n\pi i} D_{-n-1}(iz) + e^{-\frac{1}{2}n\pi i} D_{-n-1}(-iz) \right].$$

General asymptotic expansion of $D_n(z)$

The asymptotic expansion of $D_n(z)$ can only be obtained for $|\arg z| < \frac{3}{4}\pi$,

but now with help from $D_{-n-1}(iz)$, $D_{-n-1}(-iz)$, we may see more.

$D_n(-z)$



If $\frac{5}{4}\pi > \arg z > \frac{1}{4}\pi$, we may set $-z$ and $-iz$ between $\pm\frac{3}{4}\pi$

Write iz for z, and we get

$$D_n(z) = e^{n\pi i} D_n(-z) + \frac{\sqrt{2\pi}}{\Gamma(-n)} e^{\frac{1}{2}(n+1)\pi i} D_{-n-1}(-iz)$$

Then, applying asymptotic expansion to $D_n(-z)$ and $D_{-n-1}(-iz)$, we have

$$D_n(z) \sim e^{-\frac{1}{4}z^2} z^n \left\{ 1 - \frac{n(n-1)}{2z^2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4 z^4} + \cdots \right\} - \frac{\sqrt{2\pi}}{\Gamma(-n)} e^{n\pi i} e^{\frac{1}{4}z^2} z^{-n-1} \left\{ 1 + \frac{(n+1)(n+2)}{2z^2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4 z^4} + \cdots \right\}.$$

Now, if $-\frac{1}{4}\pi > \arg z > -\frac{5}{4}\pi$, we have

$$D_n(z) = e^{-n\pi i} D_n(-z) + \frac{\sqrt{2\pi}}{\Gamma(-n)} e^{-\frac{1}{2}(n+1)\pi i} D_{-n-1}(iz).$$

As D_n is single-valued, we can now obtain the asymptotic behavior for all direction.

Contour integral for $D_n(z)$

Consider $\int_{\infty}^{(0^+)} e^{-zt-\frac{1}{2}t^2}(-t)^{-n-1}dt$ where $|\arg(-t)| \le \pi$. It is single-valued, and

$$\left\{\frac{d^2}{dz^2} - z\frac{d}{dz} + n\right\}\int_{\infty}^{(0^+)} e^{-zt-\frac{1}{2}t^2}(-t)^{-n-1}dt = \int_{\infty}^{(0^+)}(t^2+zt+n)e^{-zt-\frac{1}{2}t^2}(-t)^{-n-1}dt$$

$$= \int_{\infty}^{(0^+)} \frac{d}{dt}\left(e^{\cancel{-z}t-\frac{1}{2}t^2}(-t)^{-n}\right)dt = 0.$$

Which is satisfied by $e^{\frac{1}{4}z^2}D_n(z)$.

$$\Rightarrow e^{-\frac{1}{4}z^2}\int_{\infty}^{(0^+)} e^{-zt-\frac{1}{2}t^2}(-t)^{-n-1}dt = a D_n(z) + b D_{-n-1}(iz)$$

$e^{-\frac{1}{4}z^2}E_n(z). =$

Note that $E_n(0) = \int_{\infty}^{(0^+)} e^{-\frac{1}{2}t^2}(-t)^{-n-1}dt$, $E_n'(0) = \int_{\infty}^{(0^+)} e^{-\frac{1}{2}t^2}(-t)^{-n}dt.$

$$E_n(0) = -2i\sin(n+1)\pi\int_0^{\infty} e^{-\frac{1}{2}t^2}t^{-n-1}dt$$

$$= 2^{-\frac{1}{2}n} i\sin n\pi \int_0^{\infty} e^{-u}u^{\frac{1}{2}n-1}du \quad \swarrow \left(\frac{1}{2}t^2 = u \Rightarrow du = tdt \Rightarrow dt = \frac{du}{\sqrt{2u}}. \quad e^{-\frac{1}{2}t^2}t^{-n-1}dt = e^{-u}2^{\frac{n}{2}-\frac{1}{2}}u^{-\frac{1}{2}n-\frac{1}{2}-\frac{1}{2}}\cdot 2^{-\frac{1}{2}}du.\right).$$

$$= 2^{-\frac{1}{2}n} i\sin n\pi\, \Gamma(-\tfrac{1}{2}n).$$

$$E_n'(0) = -2i\sin(n+1)\pi\int_0^{\infty} e^{-\frac{1}{2}t^2}t^{-n}dt$$

$$= -2^{\frac{1}{2}-\frac{1}{2}n} i\sin(n\pi)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}n)$$

Comparing coefficients, we have

$$b=0,\quad a = \frac{\Gamma(\frac{1}{2}-\frac{1}{2}n)}{\Gamma(\frac{1}{2})2^{\frac{1}{2}n}}2^{-\frac{1}{2}n} i\sin(n\pi)\Gamma(-\tfrac{1}{2}n) = 2i\,\Gamma(-n)\sin n\pi$$

$$\Rightarrow D_n(z) = \frac{\Gamma(n+1)}{-2\pi i}e^{-\frac{1}{4}z^2}\int_{\infty}^{(0^+)} e^{-zt-\frac{1}{2}t^2}(-t)^{-n-1}dt.$$

Recurrence formulae for $D_n(z)$.

$$0 = \int_{\infty}^{(0^+)}\frac{d}{dt}\left(e^{-zt-\frac{1}{2}t^2}(-t)^{-n-1}\right)dt = \int_{\infty}^{(0^+)}\left\{-z(-t)^{-n-1} + (-t)^{-n} + (n+1)(-t)^{-n-2}\right\}e^{-zt-\frac{1}{2}t^2}dt$$

$$\therefore 0 = \frac{(n+1)\Gamma(n+1)}{-2\pi i}e^{-\frac{1}{4}z^2}\int_{\infty}^{(0^+)} e^{-zt-\frac{1}{2}t^2}(-t)^{-n-2}dt + \frac{e^{-\frac{1}{4}z^2}\Gamma(n+1)}{-2\pi i}\int_{\infty}^{(0^+)} e^{-zt-\frac{1}{2}t^2}(-t)^{-n-1}dt - z\frac{\Gamma(n+1)}{-2\pi i}e^{-\frac{1}{4}z^2}\int e^{-zt-\frac{1}{2}t^2}(-t)^{-n-1}dt$$

$$= D_{n+1}(z) + nD_{n-1}(z) - zD_n(z) = D_{n+1}(z) - zD_n(z) + nD_{n-1}(z)$$

Also, $$D_n'(z) = -\frac{\Gamma(n+1)}{2\pi i}\frac{d}{dz}\left(\left(e^{-\frac{1}{4}z^2}\right)\int_{\infty}^{(0^+)} e^{-zt-\frac{1}{2}t^2}(-t)^{-n-1}dt\right)$$

$$= -\frac{\Gamma(n+1)}{2\pi i}\left((-\tfrac{1}{2}z)e^{-\frac{1}{4}z^2}\int_{\infty}^{(0^+)} e^{-zt-\frac{1}{2}t^2}(-t)^{-n-1}dt + e^{-\frac{1}{4}z^2}\int_{\infty}^{(0^+)} e^{-zt-\frac{1}{2}t^2}(-t)^{-n}dt\right)$$

$$= -\frac{1}{2}zD_n(z) + nD_{n-1}(z)$$

$\therefore$ We have
$$\begin{cases} D_{n+1}(z) - zD_n(z) + nD_{n-1}(z) = 0 \\ D_n'(z) + \frac{1}{2}zD_n(z) - nD_{n-1}(z) = 0 \end{cases}$$

P. 22.

roperties of $D_n(x)$
hen n is an integer

When $n \in \mathbb{Z}$, write

$$D_n(z) = -\frac{n!\, e^{-\frac{1}{4}z^2}}{2\pi i}\int^{(0^+)} \frac{e^{-zt-\frac{1}{2}t^2}}{(-t)^{n+1}}\,dt.$$

Write $t = v-z$, then we get

$$D_n(z) = (-1)^n \boxed{\frac{n!\, e^{\frac{1}{4}z^2}}{2\pi i}\int^{(z^+)} \frac{e^{-\frac{1}{2}v^2}}{(v-z)^{n+1}}\,dv.}$$

$$= (-1)^n e^{\frac{1}{4}z^2} \frac{d^n}{dz^n}\left(e^{-\frac{1}{2}v^2}\right)$$ by Cauchy's integral formula.

Also, if $m \neq n$ are integers, then

$$D_n(z)D_m''(z) - D_m(z)D_n''(z) + (m-n)D_m(z)D_n(z)$$

$$= -D_n(z)\{m+\tfrac{1}{2}\cancel{-z^2}\}D_m + D_m\{n+\tfrac{1}{2}\cancel{-z^2}\}D_n + (m-n)D_m(z)D_n(z) = 0.$$

$$\therefore (m-n)\int_{-\infty}^{\infty} D_m(z)D_n(z)\,dz = \left[D_n(z)D_m'(z) - D_m(z)D_n'(z)\right]\Big|_{-\infty}^{\infty}$$

$$= \left[D_n(z)\left(-\tfrac{1}{2}zD_m + mD_{m-1}\right) - D_m\left(-\tfrac{1}{2}zD_n + nD_{n-1}\right)\right]\Big|_{-\infty}^{\infty}$$

$$= \left(mD_n(z)D_{m-1} - nD_mD_{n-1}\right)\Big|_{-\infty}^{\infty} = 0$$

since when n, m are integers the asymptotic expansion

$e^{-\frac{1}{4}z^2}z^n\left\{1 - \frac{n(n-1)}{2z^2} + \cdots\right\}$ ← terminates within finite term.

$\therefore$ At infinity, D_m is dominated by $e^{-\frac{1}{4}z^2} \to 0$.

$$\Rightarrow \int_{-\infty}^{\infty} D_mD_n\,dz = 0.$$

When $m=n$, we have

$$(n+1)\int_{-\infty}^{\infty}[D_n(z)]^2\,dz = \int_{-\infty}^{\infty} D_n\{D_{n+1}' + \tfrac{1}{2}zD_{n+1}\}\,dz.$$

$$= \underbrace{D_nD_{n+1}\Big|_{-\infty}^{\infty}}_{0} + \int_{-\infty}^{\infty}\left(\tfrac{1}{2}zD_{n+1}D_n - D_{n+1}D_n'\right)dz = \int_{-\infty}^{\infty} \underbrace{zD_n}_{D_{n+1}+nD_{n-1}}D_{n+1} + \overset{0}{n D_{n+1}D_{n-1}}$$

$$= \int_{-\infty}^{\infty}(D_{n+1})^2.$$

$$\therefore \int_{-\infty}^{\infty} D_n^2 = n!\int_{-\infty}^{\infty} D_0^2 = n!\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz = (2\pi)^{\frac{1}{2}}n!$$

$\therefore$ If f can be expanded into $f(z) = a_0D_0 + a_1D_1 + \cdots$,

then $a_n = \frac{1}{(2\pi)^{\frac{1}{2}}n!}\int_{-\infty}^{\infty} D_n(t)f(t)\,dt.$

Appendix 1

Show that $c(c-1-(2c-a-b-1)x)F(a,b;c;x)+(c-a)(c-b)xF(a,b;c+1;x)$.

$$=c(c-1)(1-x)F(a,b;c-1;x).$$

(pf). It suffices to compare the coefficients of x^n.

x^n: of $c((c-1)-(2c-a-b-1)x)F(a,b;c;x)$

is $$c(c-1)\frac{(a,n)(b,n)}{(c,n)(1,n)}-c(2c-a-b-1)\frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)}+(c-a)(c-b)\frac{(a,n-1)(b,n-1)}{(c+1,n-1)(1,n-1)}$$

$$=c(c-1)\frac{(a,n-1)(b,n-1)}{(c-1,n-1)(1,n-1)}\left[\frac{(a+n-1)(b+n-1)(c-1)}{(c+n-1)(c+n-2)n}-\underbrace{\frac{(2c-a-b-1)}{c+n-2}}_{\parallel}+\underbrace{\frac{(c-a)(c-b)}{(c+n-2)(c+n-1)}}_{\parallel}\right].$$

$$-2+\frac{2(n-2)+a+b+1}{c+n-2}\qquad 1+\frac{-(2n-3)c-(n-2)(n-1)-c(a+b)+ab}{(c+n-2)(c+n-1)}.$$

$$=c(c-1)\frac{(a,n-1)(b,n-1)}{(c-1,n-1)(1,n-1)}\left[-1+\frac{c((a+n-1)(b+n-1)+n(2(\cancel{n-2})+a+b\cancel{+1}-(2\cancel{n-3})-\cancel{(a+b)}))}{(c+n-2)(c+n-1)n}+\frac{(-(a+n-1)(b+n-1)+n(n-1)(a+b+1+2(n-2))+abn-n(n-1)(n-2))}{(c+n-2)(c+n-1)n}\right].$$

$$(-(a+n-1)(b+n-1)+\underbrace{n(n-1)}(a+b+1+\cancel{2(n-2)})+\underbrace{abn}-\cancel{n(n-1)(n-2)}+n(n-1)(n-2)).$$

$$n(ab+(n-1)(a+b)+\underbrace{(n-1)+(n-1)(n-2)}_{\substack{\parallel\\(n-1)^2}})=n(a+n-1)(b+n-1))$$

$$=c(c-1)\frac{(a,n-1)(b,n-1)}{(c-1,n-1)(1,n-1)}\left[-1+\frac{(c+\cancel{n-1})(a+n-1)(b+n-1)}{n(c+n-2)(c+\cancel{n-1})}\right].=c(c-1)\left[\underset{\substack{\uparrow\\u_n}}{\frac{(a,n)(b,n)}{(c-1,n)(1,n)}}-\underset{\substack{\uparrow\\u_{n-1}}}{\frac{(a,n-1)(b,n-1)}{(c-1,n-1)(1,n-1)}}\right].$$

Appendix 2.

Let u satisfy $$\frac{d^2u}{dz^2}+\left\{\frac{1-\alpha-\alpha'}{z-a}+\frac{1-\beta-\beta'}{z-b}+\frac{1-\gamma-\gamma'}{z-c}\right\}\frac{du}{dz}+\left\{\frac{\alpha\alpha'(a-b)(a-c)}{z-a}+\frac{\beta\beta'(b-c)(b-a)}{z-b}+\frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right\}\frac{u}{(z-a)(z-b)(z-c)}=0.$$

Set $u=(z-a)^\alpha(z-b)^\beta(z-c)^\gamma I$, then I satisfies

$$\frac{d^2I}{dz^2}+\left\{\frac{1+\alpha-\alpha'}{z-a}+\frac{1+\beta-\beta'}{z-b}+\frac{1+\gamma-\gamma'}{z-c}\right\}\frac{dI}{dz}+\frac{(\alpha+\beta+\gamma)\{(\alpha+\beta+\gamma+1)z+\sum a(\alpha+\beta'+\gamma'-1)\}}{(z-a)(z-b)(z-c)}I=0.$$

(pf). $$\frac{d^2I}{dz^2}=(z-a)^{-\alpha}(z-b)^{-\beta}(z-c)^{-\gamma}\frac{d^2u}{dz^2}-2\left(\frac{\alpha}{z-a}+\frac{\beta}{z-b}+\frac{\gamma}{z-c}\right)(z-a)^{-\alpha}(z-b)^{-\beta}(z-c)^{-\gamma}\frac{du}{dz}+\left(\frac{\alpha(\alpha+1)}{(z-a)^2}+\frac{\beta(\beta+1)}{(z-b)^2}+\frac{\gamma(\gamma+1)}{(z-c)^2}+\frac{2\alpha\beta}{(z-a)(z-b)}+\frac{2\alpha\gamma}{(z-a)(z-c)}+\frac{2\beta\gamma}{(z-b)(z-c)}\right)(z-a)^{-\alpha}(z-b)^{-\beta}(z-c)^{-\gamma}u.$$

$$\frac{dI}{dz}=(z-a)^{-\alpha}(z-b)^{-\beta}(z-c)^{-\gamma}\frac{du}{dz}-\left(\frac{\alpha}{z-a}+\frac{\beta}{z-b}+\frac{\gamma}{z-c}\right)(z-a)^{-\alpha}(z-b)^{-\beta}(z-c)^{-\gamma}u.$$

Now, if $\frac{d^2I}{dz^2}+f(z)\frac{dI}{dz}+g(z)I=0$, then $\frac{d^2I}{dz^2}+f(z)\frac{dI}{dz}$ must "kill" all $\frac{du}{dz}$ terms as $\frac{du}{dz}$ and u are functionally independent.

$$\therefore\ \frac{dI}{dz}:\left(\frac{1-\alpha-\alpha'}{z-a}+\frac{1-\beta-\beta'}{z-b}+\frac{1-\gamma-\gamma'}{z-c}\right)+2\left(\frac{\alpha}{z-a}+\frac{\beta}{z-b}+\frac{\gamma}{z-c}\right)=\frac{1+\alpha-\alpha'}{z-a}+\frac{1+\beta-\beta'}{z-b}+\frac{1+\gamma-\gamma'}{z-c}.$$

$$I:\left(\frac{\alpha\alpha'(a-b)(a-c)}{z-a}+\frac{\beta\beta'(b-c)(b-a)}{z-b}+\frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right)\frac{1}{(z-a)(z-b)(z-c)}-\left(\boxed{\frac{\alpha(\alpha+1)}{(z-a)^2}}+\boxed{\frac{\beta(\beta+1)}{(z-b)^2}}+\frac{\gamma(\gamma+1)}{(z-c)^2}+\frac{2\alpha\beta}{(z-a)(z-b)}+\frac{2\alpha\gamma}{(z-a)(z-c)}+\frac{2\beta\gamma}{(z-b)(z-c)}\right)$$

$$+\left(\frac{\alpha}{z-a}+\frac{\beta}{z-b}+\frac{\gamma}{z-c}\right)\left(\frac{1+\alpha-\alpha'}{z-a}+\frac{1+\beta-\beta'}{z-b}+\frac{1+\gamma-\gamma'}{z-c}\right)$$

$$=\left(\frac{\alpha\alpha'(a-b)(a-c)}{z-a}+\frac{\beta\beta'(b-a)(b-c)}{z-b}+\frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right)\frac{1}{(z-a)(z-b)(z-c)}-\sum\frac{\alpha\alpha'}{(z-a)^2}-\sum\frac{\alpha(\beta'-1)+\beta(\alpha'-1)}{(z-a)(z-b)}$$

$$=\frac{1}{(z-a)(z-b)(z-c)}\left[\sum\frac{1}{z-a}\underset{\substack{\uparrow\\f_a(a)}}{\underline{(\alpha\alpha'(a-b)(a-c))}}-\sum\underset{\substack{\uparrow\\f_a(z)}}{\frac{\alpha\alpha'(z-b)(z-c)}{(z-a)}}-\sum[\alpha(\beta'-1)+\beta(\alpha'-1)](z-c)\right].$$

Appendix P.2.

$$= \frac{1}{(z-a)(z-b)(z-c)}\left[-\sum \alpha\alpha'\Big(\underbrace{(a-b)+(a-c)}_{f_a'(a)} + \underbrace{(z-a)}_{\frac{1}{2}f_a''(a)(z-a)^2 / z-a}\Big) - \sum\big(\beta(\gamma'-1)+\gamma(\beta'-1)\big)(z-a)\right]$$

$$= \frac{1}{(z-a)(z-b)(z-c)}\left[\sum\Big(-\alpha\alpha'-\big(\beta(\gamma'-1)+\gamma(\beta'-1)\big)\Big)z - \sum\Big(\alpha\alpha'(a-b-c) + a\big(\beta(\gamma'-1)+\gamma(\beta'-1)\big)\Big)\right].$$

$$\begin{cases} \sum\Big(-\alpha\alpha'-\big(\beta(\gamma'-1)+\gamma(\beta'-1)\big)\Big) = \sum\alpha(-1+\alpha+\beta+\gamma) + 2\sum\alpha = \sum\alpha(1+\alpha+\beta+\gamma) = (\alpha+\beta+\gamma)(1+\alpha+\beta+\gamma). \\ -\sum\Big[\big(\alpha\alpha'(a-b-c)+a(\beta(\gamma'-1)+\gamma(\beta'-1)\big)\Big] = \sum a\big(\underset{\overset{\parallel}{\alpha(\alpha+\beta+\gamma+\beta'+\gamma'-1)}}{-\alpha\alpha'+\beta\beta'+\gamma\gamma'} + \beta(\gamma'-1)+\gamma(\beta'-1)\big). \end{cases}$$

$$= \sum a(\alpha+\beta+\gamma)(\alpha+\beta'+\gamma'-1).$$

$$= \frac{(\alpha+\beta+\gamma)}{(z-a)(z-b)(z-c)}\left[(\alpha+\beta+\gamma+1)z + \sum a(\alpha+\beta'+\gamma'-1)\right]$$ ※

Appendix 3.

Consider $P\begin{Bmatrix} 0 & \infty & c & \\ \frac{1}{2}+m & -c & c-k & ;z \\ \frac{1}{2}-m & 0 & k & \end{Bmatrix}$ and its corresponding equation. As $c\to\infty$, show that the limiting equation is $\frac{d^2u}{dz^2}+\frac{du}{dz}+\left(\frac{k}{z}+\frac{\frac{1}{4}-m^2}{z^2}\right)u=0$. Let $u=e^{-\frac{1}{2}z}W_{k,m}(z)$, then $W_{k,m}$ satisfies

$$\frac{d^2W}{dz^2}+\left\{-\frac{1}{4}+\frac{k}{z}+\frac{\frac{1}{4}-m^2}{z^2}\right\}W=0.$$

(pf) The equation corresponds to $\frac{d^2u}{dz^2}+\left\{\frac{1-c}{z-c}\right\}\frac{du}{dz}+\left\{\frac{\frac{1}{4}-m^2}{z^2}+\frac{k(c-k)}{(z-c)^2}+\frac{\delta}{z}-\frac{\delta}{z-c}\right\}u=0$ ----(1).

At $z=\infty$, set $w=\frac{1}{z} \Rightarrow \begin{cases}\frac{du}{dz}=\frac{dw}{dz}\frac{du}{dw}=-w^2\frac{du}{dw}. \\ \frac{d^2u}{dz^2}=w^4\frac{d^2u}{dw^2}+2w^3\frac{du}{dw}\end{cases}$

For the constant coefficient $q(w)=q^{-2}w^{-2}+q^{-1}w^{-1}+\cdots$, we expect $q^{-2} \overset{\text{exponents.}}{=} c\cdot 0 = 0$.

$$\frac{1}{w^4}\left\{\left(-\frac{1}{4}-m^2\right)w^2+\frac{k(c-k)}{(1-cw)^2}w^2+\delta w-\frac{\delta}{1-wc}w\right\} = \underbrace{\left(\frac{1}{4}-m^2\right)w^{-2}+k(c-k)w^{-2}-\delta c w^{-2}}_{\overset{\shortparallel}{0}}+o(w^{-1})$$

$$\Rightarrow \delta = k+\frac{1}{c}\left(\frac{1}{4}-m^2-k^2\right).$$

$\therefore$ As $c\to\infty$, (1) becomes $\frac{d^2u}{dz^2}+\frac{du}{dz}+\left\{\frac{\frac{1}{4}-m^2}{z^2}+\frac{k}{z}\right\}u=0$.

Now, let $w=h(z)u(z)$ s.t. w satisfies $\frac{d^2w}{dz^2}+\square\, w=0$.

i.e. $\frac{d^2w}{dz^2}=hu''+2h'u'+h''u=h(-u'-\square u)+2h'u'+\square u=\square u$.

$\therefore h'=\frac{1}{2}h \Rightarrow h=e^{\frac{1}{2}z}$.

$\therefore u=e^{-\frac{1}{2}z}w$ transforms the equation.

Appendix P.3

Appendix 4. Show that $F(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};x)=F(\alpha,\beta;\alpha+\beta+\frac{1}{2};4x(1-x)\}$ for $|x|<\frac{1}{2}$.

(pf). $F(\underset{\overset{\uparrow}{a}}{2\alpha},\underset{\overset{\uparrow}{b}}{2\beta};\underset{\overset{\uparrow}{c}}{\alpha+\beta+\frac{1}{2}};x)$ satisfies $z(1-z)\frac{d^2u}{dz^2}+\{c-\underset{\overset{\|}{2c}}{(a+b+1)}z)\frac{du}{az}-4\alpha\beta u=0$

It suffices to show that $\overset{\overset{f(z).}{\|}}{F}(\alpha,\beta;\alpha+\beta+\frac{1}{2};4z(1-z)\}$ also satisfies the equation, for then analytic functions with constant coefficient 1 is unique.

$$\frac{df}{dz}=(4-8z)\frac{\alpha\beta}{\alpha+\beta+\frac{1}{2}}F(\alpha+1,\beta+1;\alpha+\beta+\frac{3}{2};4z(1-z))$$

$$\frac{d^2f}{dz^2}=-8\frac{\alpha\beta}{\alpha+\beta+\frac{1}{2}}F(\alpha+1,\beta+1;\alpha+\beta+\frac{3}{2}:4z(1-z))+\underbrace{(4-8z)^2}_{\|}\frac{\alpha(\alpha+1)\beta(\beta+1)}{(\alpha+\beta+\frac{1}{2})(\alpha+\beta+\frac{3}{2})}F(\alpha+2,\beta+2;\alpha+\beta+\frac{5}{2};4z(1-z))$$

$$16(4z^2-4z+1)=16-64z(1-z).$$

Idea: Expand the series w.r.t. $z(1-z)$ and argue each coeff. $=0$.

$$\therefore (z(1-z))^k:\ -8\frac{\alpha\beta}{\alpha+\beta+\frac{1}{2}}\frac{(\alpha+1,k-1)(\beta+1,k-1)}{(k-1)!\,(c+1,k-1)}4^{k-1}+16\frac{\alpha(\alpha+1)\beta(\beta+1)}{c(c+1)}\frac{(\alpha+2,k-1)(\beta+2,k-1)}{(k-1)!\,(c+2,k-1)}4^{k-1}$$

$$-64\frac{(\alpha,k)(\beta,k)}{(k-2)!\,(c,k)}4^{k-2}+4(1-\cancel{2z})\frac{(\alpha,k+1)(\beta,k+1)}{k!\,(c+1,k)}4^k+8z\frac{(\alpha,k+1)(\cancel{\beta,k+1})}{\cancel{k!}(c+1,k)}4^k$$

$$-16\frac{(\alpha,k)(\beta,k)}{(k+1)!\,(c+1,k+1)}4^{k-1}-4\alpha\beta\frac{(\alpha,k)(\beta,k)}{(k-1)!\,(c,k)}4^k \qquad (-8z(1-2z)=-8(2z(1-z)-z)).$$

$$=4^k\cdot 2\frac{(\alpha,k)(\beta,k)}{k!\,(c,k+1)}\Big[-k(c+k)+\underbrace{2(\alpha+k)(\beta+k)k-2k(k-1)(c+k)+2c(\alpha+k)(\beta+k)}_{2(\alpha+k)(\beta+k)(c+k)}-2kc(c+k)-2\alpha\beta\,c(c+k)\Big].$$

$$=4^k\cdot 2\frac{(\alpha,k)(\beta,k)}{k!\,(c,k+1)}(c+k)\Big[\cancel{-k}+2(\alpha+\cancel{k})(\beta+k)\underbrace{-2k\cancel{(k-1)}}_{\cancel{-2k^2}+\cancel{k}}\underset{\overset{\|}{-(2\alpha+2\beta+1)\cancel{k}}}{-2kc}-2\cancel{\alpha}\beta\Big]=0.$$

$\therefore$ $f(z)=F(\alpha,\beta;\alpha+\beta+\frac{1}{2};4z(1-z))$ really satisfies the equation and is analytic, has the same constant coeff. as $F(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};z)\Rightarrow F(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};z)=F(\alpha,\beta;\alpha+\beta+\frac{1}{2};4z(1-z))$.

Appendix 5. Show that $W_{k,m}$ can be obtained from the integral representation of hypergeometric functions.

(pf). $P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & ;z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$ has solution.

$$(z-a)^{\alpha}(z-b)^{\beta}(z-c)^{\gamma}\int_D (t-a)^{\alpha'+\beta+\gamma-1}(t-b)^{\alpha+\beta'+\gamma-1}(t-c)^{\alpha+\beta+\gamma'-1}(z-t)^{-\alpha-\beta-\gamma}dt$$

$$=(-1)^{-\alpha-\beta-\gamma}(-b)^{\alpha+\beta+\gamma+\beta'-\gamma}(-c)^{\alpha+\beta+\gamma+\gamma'-1}(z-a)^{\alpha}(1-\frac{z}{b})^{\beta}(1-\frac{z}{c})^{\gamma}\int_{-D}(-t-a)^{\alpha'+\beta+\gamma-1}(1+\frac{t}{b})^{\alpha+\beta'+\gamma-1}(1+\frac{t}{c})^{\alpha+\beta+\gamma'-1}(1+\frac{t}{z})^{-\alpha-\beta-\gamma}z^{-\alpha-\beta-\gamma}dt$$

where we factored out some constants and changed the orientation of D.

Since P is a linear space, we may drop out all factors, set $a=0$, $b\to\infty$, $(\alpha,\alpha',\beta,\beta',\gamma,\gamma')$
$=(\frac{1}{2}-m,\frac{1}{2}+m,-c,0,c-k,k)$.

Thus the representation becomes

$$z^{\frac{1}{2}+m}\cdot(1+0)^{-c}\left(1-\frac{z}{c}\right)^{c-k}\int_{-D}(-t)^{+m-k-\frac{1}{2}}(1+0)^{\alpha+\beta+\gamma-1}\left(1+\frac{t}{c}\right)^{-c+(\frac{1}{2}+m+k)}\left(1+\frac{t}{z}\right)^{m+k-\frac{1}{2}}z^{m+k-\frac{1}{2}}dt.$$

$$=1^{\alpha+\beta+\gamma-1}z^{k}\left(1-\frac{z}{c}\right)^{c}\cdot\left(1-\frac{z}{c}\right)^{-k}\int_{-D}(-t)^{m-k-\frac{1}{2}}\left(1+\frac{t}{c}\right)^{-c}\cdot\left(1+\frac{t}{c}\right)^{(\frac{1}{2}+m+k)}\left(1+\frac{t}{z}\right)^{m+k-\frac{1}{2}}dt.$$

Taking out $1^{\alpha+\beta+\gamma-1}$ and letting $c\to\infty$, we have

$$e^{-z}\cdot z^{k}\int_{-D}(-t)^{m-k-\frac{1}{2}}\left(1+\frac{t}{z}\right)^{m+k-\frac{1}{2}}e^{-t}dt.$$

Multiplying $-\frac{1}{2\pi i}\Gamma(k+\frac{1}{2}-m)e^{\frac{1}{2}z}$; and replace $-D$ with ,

$$W_{k,m}(z)=\frac{-1}{2\pi i}\Gamma(k+\tfrac{1}{2}-m)e^{-\frac{1}{2}z}z^{k}\int_{\infty}^{(0^{+})}(-t)^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m}e^{-t}dt.$$