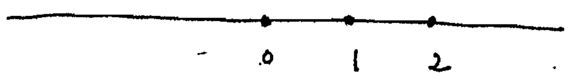
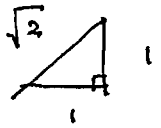
Calculus 9/16

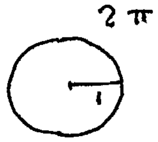
1.1 Real numbers

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$ ↖ ruler & compass



$\mathbb{Q}$ is not enough for geometry
(Euclid 300 BC, Pythagoras 570-495 BC)





"Number continuum" vs. Dedekind cut 1831-1916

$\mathbb{Q} = A \cup B$

$a \in A,\ b \in B \Rightarrow a < b$

Decimal
Binary > fractions

p-adic system: $x = c_0 + 0.c_1 c_2 c_3 \dots\ ;\ 0 \le c_i \le p-1$

$(p \in \mathbb{N},\ p \geq 2)$ $$c_0 + \frac{c_1}{p} + \dots + \frac{c_n}{p^n} \le x \le c_0 + \frac{c_1}{p} + \dots + \frac{c_n}{p^n} + \underline{\frac{1}{p^n}}$$

Ex. $x \in \mathbb{Q} \Leftrightarrow$ finite or periodic

$p = 60$
Babylonian
5000 - 500 BC

Postulate (axiom) of Nested Intervals:

$$I_n = [a_n, b_n] := \{ x \in \mathbb{R} \mid a_n \le x \le b_n \}$$

if $I_1 \supset I_2 \supset I_3 \supset \cdots$ and $b_n - a_n \longrightarrow 0$,

then $\bigcap_{n=1}^{\infty} I_n \neq \phi$ in $\mathbb{R}$, and it is unique.

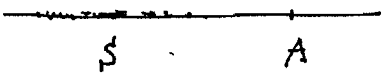
This is wrong for $(a_n, b_n) = (0, \frac{1}{n})$.

Least Upper Bound Axiom:

let $S \subset \mathbb{R}$ with $S \le A$ (ie $s \le A\ \forall s \in S$)

then the smallest upper bound

$A_0 = \operatorname{lub} S \equiv \sup S \in \mathbb{R}$ exists.



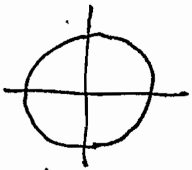
Ex. This LUB axiom can be proved using axiom of nested intervals or simply the decimal fractions (eg. using p-adic $p = 2$)

1.2 Functions & Continuity

$f: A \to B$, $f(A) = \{ b \in B \mid b = f(a) \text{ for some } a \in A \}$

domain — range = image

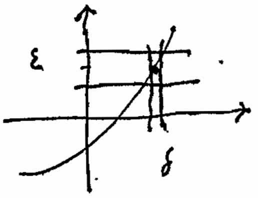
eg. $x^2 + y^2 = 1 \Rightarrow y = \pm\sqrt{1-x^2}$



this "multi-valued" functions defines 2 "single-valued" functions.

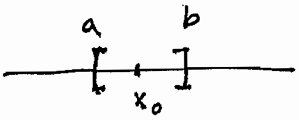
Consider $f: \underset{\mathbb{R}}{\overset{\cap}{A}} \to \mathbb{R}$ and A is in general a union of intervals (a,b), $(a,b]$, etc.

Def'n: f is conti at $x_0 \in A$ iff $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ (which may depends on x_0) such that $|x - x_0| < \delta$ and $x \in A$ $\Rightarrow |f(x) - f(x_0)| < \varepsilon$.



eg. $f(x) = cx^n$

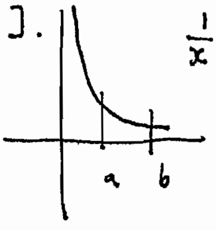
$$|f(x) - f(x_0)| = |c| \cdot |x - x_0| \cdot |x^{n-1} + \cdots + x_0^{n-1}|$$
$$\leq cn(\max\{|a|,|b|\})^{n-1} |x - x_0|$$



enough to take $\delta < \varepsilon / cn \cdot \max(|a|,|b|)^{n-1}$.

Def'n: f is uniformly conti if δ is indep of x_0.

eg. $f(x) = 1/x$ uniform on $[a,b]$ but not $(0,b]$.



$$\left| \frac{1}{x} - \frac{1}{x_0} \right| = \frac{|x - x_0|}{|x x_0|} \leq \frac{|x - x_0|}{a^2}$$

suffice to take $\delta < a^2 \varepsilon$ for any given $\varepsilon > 0$.

but on $A = (0, b]$, $\left| \frac{1}{x} - \frac{1}{x_0} \right| = \frac{1}{x_0} \nearrow \infty$

For any $\delta > 0$, let $x_0 < \delta$, $x = x_0/2$ (so $|x - x_0| < \delta$).

eg. Hölder / Lipschitz continuity

$|f(x_1) - f(x_2)| \leq L |x_1 - x_2|^\alpha$ for some $0 < \alpha \leq 1$.

$|\sqrt[n]{x_1} - \sqrt[n]{x_2}| \leq |x_1 - x_2|^{1/n}$ (why?). Try $n=2$ first.

Intermediate Value Theorem.

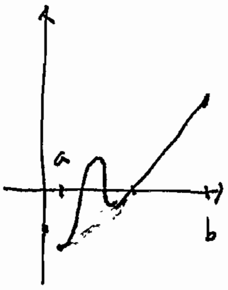
Let $f : [a, b] \to \mathbb{R}$ be conti. If $f(a) < 0$, $f(b) > 0$ then $\exists\ c \in (a, b)$ st. $f(c) = 0$.

Pf: Let $c = \text{lub}\ \underbrace{\{ x \in [a, b] \mid f(x) < 0 \}}_{} =: S$

claim: $f(c) = 0$:

If $f(c) > 0$, then $f(x) > 0$ on $[c-\delta, c+\delta]$ for some $\delta > 0$ (see below).

But then $c-\delta$ is an upper bound for S and $c-\delta < c$ $\times$



Similarly, if $f(c) < 0$, then $f(x) < 0$ on $[c-\delta, c+\delta]$ for some $\delta > 0$.

But then $f(c+\delta) < 0$ $\times$ Thus $f(c) = 0$. $\square$

Basic Fact: f conti, $f(x_0) > 0 \Rightarrow f(x) > 0$ on a neighborhood of x_0, ie. some $(x_0 - \delta, x_0 + \delta)$.

Pf: Let $\varepsilon = \frac{1}{2} f(x_0) > 0$, then $\exists\ \delta > 0$ st

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon = \frac{f(x_0)}{2}$$

ie. $0 < \frac{1}{2} f(x_0) < f(x) < \frac{3}{2} f(x_0)$. $\square$.

Remark: May use closed interval: eg. $[x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}]$.

Monotone functions.

A continuous monotonic fcn has a conti inverse fcn.

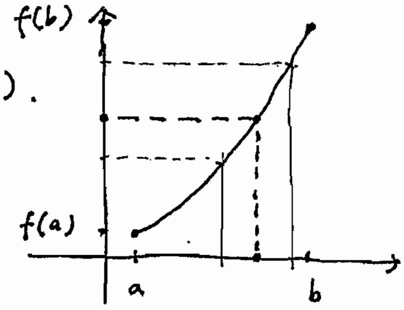
Pf: Say. let f be monotonic increasing ($f \nearrow$)

then $\forall \eta \in [f(a), f(b)]$, $\exists!$

$\xi \in [a, b]$ st. $\eta = f(\xi)$, Write $\xi = g(\eta)$.

The continuity of g

follows easily from the picture. $\square$



Calculus 9/21

1.3 The Elementary functions.

① $x \mapsto + - \times \div$ poly. rational, $y = \sqrt[n]{R(x)}$

more generally, algebraic functions $F(x, y(x)) = 0$.

② Trigonometric & inverse

③ Exponential & Logarithm

$>$ Q: Do we really know them?

let $a > 0$, $y = a^x$ how to define this? (for $x \in \mathbb{R}$)

for $x = p/q \in \mathbb{Q}$, $a^x = a^{p/q} = \sqrt[q]{a^p}$ ok.

in general, need "limits" to define a^x as conti fcn. (later)

$x = \log_a y$ the inverse fcn.

Compound (composite) functions:

$$A \xrightarrow[f]{} B \xrightarrow[g]{} C \quad \text{get } g\circ f : A \to C$$

$$g\circ f(x) \equiv gf(x) = g(f(x)).$$

Fact: If f is conti at x_0, g is conti at $f(x_0)$ then $g\circ f$ is conti at x_0.

1.4 + 1.5 Sequences and Mathematical induction.

Q: How to get a formula for $1^k + 2^k + \cdots + n^k$?

1.6 + 1.7 Limits (of a sequence)

Intuitive method: (Sandwich)

Example 1: $a_n = \frac{n^2-1}{n^2+n+1}$ expect "$a_n \to 1$ as $n \to \infty$"

$r_n := 1 - a_n = \frac{n+2}{n^2+n+1}$ then $0 < r_n < \frac{2n}{n^2} = \frac{2}{n}$ for $n > 2$

Hence $r_n \to 0$. We write $\lim_{n\to\infty} a_n = 1$.

Example 2: $a_n = \sqrt[n]{p}$; say $p > 1$

let $\sqrt[n]{p} = 1 + r_n \Rightarrow p = (1+r_n)^n \geqslant 1 + nr_n \Rightarrow 0 < r_n \leqslant \frac{p-1}{n}$
$\to 0$ as $n \to \infty$.

Q: How about $a_n = \sqrt[n]{n}$? (Hint : bi-nomial exp. to the 2nd term)

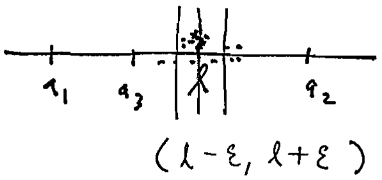
Example 3. $a_n = \dfrac{n}{\alpha^n}$ for $\alpha > 1$.

$\exists\ n_0$ st. $\alpha^{n/2} > n$, hence $a_n = \dfrac{n}{\alpha^{n/2}} \cdot \dfrac{1}{\alpha^{n/2}} < \dfrac{1}{\alpha^{n/2}} \to 0$.

Formal definition of limit :

$$\lim_{n\to\infty} a_n = l \text{ iff } \forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N} \text{ st. } n > N \Rightarrow |a_n - l| < \varepsilon .$$

In this case, we say $\{a_n\}$ conv. to l otherwise we say a_n is divergent.



4 Rules of limits :

If $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} b_n = b$, then

a) $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n = a \pm b$

b) $\lim\limits_{n\to\infty} (a_n b_n) = ab$

c) If $b \neq 0$, then $\lim\limits_{n\to\infty} (a_n / b_n) = a/b$

Pf of b) : $|a_n b_n - ab| = |a_n b_n - a b_n + a b_n - ab|$

$\leq |a_n - a| \cdot |b_n| + |a| \cdot |b_n - b|$

may assume that $|b_n| \leq M, \forall n$.

Given $\varepsilon_1 > 0$, $\exists N$ st $|a_n - a| < \varepsilon_1$, $|b_n - b| < \varepsilon_1$ $\forall n > N$

and then $|a_n b_n - ab| \leq (M + |a|)\, \varepsilon_1$

For any $\varepsilon > 0$, simply choose ε_1 small st $(M + |a|)\, \varepsilon_1 < \varepsilon$.

pf of c) : Hint : Only need to do $\lim\limits_{n\to\infty} 1/b_n = 1/b$. (Ex.)

Reformulation of conti. functions.

(1) f is conti at x_0 iff $\lim\limits_{x\to x_0} f(x) = f(x_0)$.

(2) Here $\lim\limits_{x\to x_0} f(x)$ is the same as $\lim\limits_{n\to\infty} f(x_n)$, for all $x_n \to x_0$. To be conti.

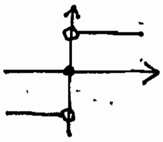
Calculus 9/23

1.8 Limit for one variable functions

Def'n : $f: A \to \mathbb{R}$, $A \subset \mathbb{R}$, $\lim\limits_{x\to\xi} f(x) = \eta$ iff $\forall \varepsilon > 0$, $\exists \delta > 0$

st. $|x-\xi| < \delta$ & $x \in A$ $\Rightarrow$ $|f(x) - \eta| < \varepsilon$.

eg. $f(x) = \sin\frac{1}{x}$, $A = \mathbb{R}\setminus\{0\}$, $f(x) = x\sin\frac{1}{x}$ removable discontinuity or $f(x) = \operatorname{sgn} x$



It is clear that f is conti at $\xi \Leftrightarrow \lim\limits_{x\to\xi} f(x) = f(\xi)$.

Theorem ! $\lim\limits_{x\to\xi} f(x) = \eta \Leftrightarrow \lim\limits_{x_n\to\xi} f(x_n) = \eta$

$\forall \{x_n\} \subset A$ with $\lim\limits_{n\to\infty} x_n = \xi$.

pf : " $\Rightarrow$ " Given $x_n \to \xi$

$\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon)$ st. $|x-\xi| < \delta \Rightarrow |f(x) - \eta| < \varepsilon$

for one such δ, $\exists N = N(\delta)$ st. $n > N \Rightarrow |x_n - \xi| < \delta$

Hence $|f(x_n) - \eta| < \varepsilon$ as desired.

" $\Leftarrow$ " If this is NOT TRUE, then $\exists \varepsilon > 0$

st. for any $\delta > 0$, $\exists x_n \in (\xi - \frac{1}{n}, \xi + \frac{1}{n})$

say $\delta = \frac{1}{n}$

but $|f(x_n) - \eta| \geq \varepsilon$. i.e. $\lim\limits_{n\to\infty} f(x_n) \neq \eta$ —×—

So conti. really means commutativity of f and lim :

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n).$$

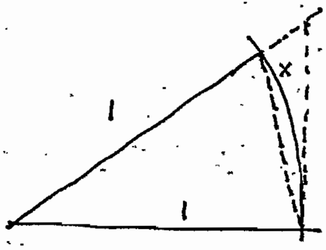
eg. 4 Rules + inverse + compound $\rightsquigarrow$ $f(x) = \sqrt[3]{x^2 + \tan^{-1}\sqrt{x}}$

Also $\dfrac{n^2-1}{n^2+n+1} = \dfrac{1 - \frac{1}{n^2}}{1 + \frac{1}{n} + \frac{1}{n^2}} \longrightarrow \frac{1}{1} = 1$. etc.

Similarly, $x \to \infty$ means $z = \frac{1}{x} \to 0$: $\lim\limits_{x\to\infty} \dfrac{x^2-1}{x^2+x+1} = 1$.

Elementary functions revisited :

poly . Trigonometric . exp



let the arc has length x .

This relie on Euclidean geom.

Example : $\lim_{x \to 0} \frac{\sin x}{x} = 1$

area $\Rightarrow \frac{1}{2} \sin x < \frac{1}{2} x < \frac{1}{2} \tan x$

$\Rightarrow 1 < \frac{x}{\sin x} < \frac{1}{\cos x}$, let $x \to 0$.

Cor. : $\lim_{x \to 0} \frac{1-\cos x}{x} = \lim_{x \to 0} \frac{2 \sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2} \cdot \frac{x}{4} = 0$

$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$ etc.

Example : The Natural Base $e = 2.718281828 \ldots$

$e := \lim_{n \to \infty} S_n$ where $S_n = 1 + \frac{1}{1!} + \text{``}\frac{1}{2!} + \cdots + \frac{1}{n!}\text{''} < 3$

$S_n \nearrow$ hence the limit exists.

$e \notin \mathbb{Q}$: $n > m \Rightarrow S_n \leq S_m + \frac{1}{(m+1)!} \frac{1}{1-\frac{1}{m+1}} = S_m + \frac{1}{m} \frac{1}{m!}$

$n \to \infty \Rightarrow S_m < e \leq S_m + \frac{1}{m} \frac{1}{m!}$

i.e. $m! S_m < m! e \leq m! S_m + \frac{1}{m}$ $\quad -\!\!\times\!\!-$ if $e = \frac{p}{m}$.

Q : How about $\pi \notin \mathbb{Q}$?

Fact : $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \equiv \lim_{n \to \infty} T_n$.

$T_n = \left(1 + \frac{1}{n}\right)^n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2} \cdot \frac{1}{n^2} + \cdots + \frac{n(n-1) \cdots 1}{n!} \cdot \frac{1}{n^n}$

$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right) < S_n < 3$

Also $T_n \nearrow$, hence $T = \lim_{n \to \infty} T_n$ exists .

In fact, for $m > n$, $T_m > 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{m}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{m}\right) \cdots \left(1 - \frac{n-1}{m}\right)$

Fix n, and let $m \to \infty$, get

$T \geq S_n > T_n$. Now let $n \to \infty$ get $T \geq e \geq T$.

- This shows that it is very important to compare a_n, a_m esp. when the limit is unknown , though exists.

Calculus 9/28

ch.1 Supplement

ℝ continuity : nested intervals $[a_n, b_n]$

$a_1 \le a_2 \le \cdots \le b_2 \le b_1$, $b_n - a_n \to 0$

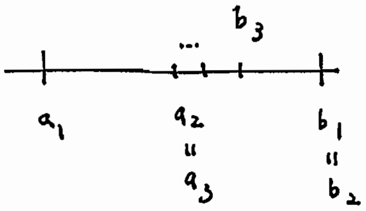
contains a unique real number

Compactness (Weierstrass)

Theorem: Every bounded (infinite) sequence in ℝ has a convergent subsequence.

pf: let $\{x_n\} \subset [a_1, b_1]$

construct $[a_2, b_2]$ ... to contain so many elements of $\{x_n\}$



Now pick $x_{n_k} \in [a_k, b_k]$

then $x_{n_1}, x_{n_2}, x_{n_3}, \cdots$ converges. □

Def'n : A set is compact if every seq has a conv. sub.seq (in the set)

Cor : An interval is cpt $\Longleftrightarrow$ it is $[\cdot, \cdot]$.

Cor : Every monotone sequence converges if bounded.

More generally, we have the lub / glb axiom. (Done).

Completeness (Cauchy).

Theorem: $x_1, x_2, x_3, \cdots$ conv. iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ st. $|x_n - x_m| < \varepsilon$ for all $n, m \geq N$.

This is "intrinsic conv. test" when the "lim" is unknown.

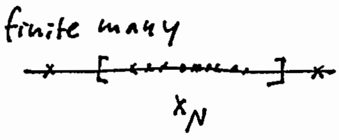
pf: $\Rightarrow$ if $x = \lim_{n \to \infty} x_n$ exists, then given any $\varepsilon > 0$

$\exists N$ st $|x_n - x| < \frac{\varepsilon}{2}$, $|x_m - x| < \frac{\varepsilon}{2}$ for $n, m \geq N$

so, $|x_n - x_m| < |x_n - x| + |x_m - x| < \varepsilon$.

$\Leftarrow$ Pick $\varepsilon = 1$, then $\exists N$, st. $|x_n - x_N| < 1$ $\forall n \geq N$

Then $\exists A$ st. $\{x_i\} \subset [-A, A]$



This $\Rightarrow \exists$ conv. sub.seq.

$\Rightarrow$ the whole seq. conv. (Exercise). □

Example : Def of $a^x = \lim_{n\to\infty} a^{r_n}$ where $\underset{\mathbb{Q}}{\underset{\uparrow}{r_n}} \to \underset{\mathbb{R}}{\underset{\uparrow}{x}}$

Cauchy's test :

$$|a^{r_n} - a^{r_m}| = |a^{r_m}(a^{\delta} - 1)| \leq M \cdot |a^{\delta} - 1| .$$

Fundamental Theorems on conti. functions.

I. Theorem : Every $f : [a,b] \to \mathbb{R}$ conti $\Rightarrow$ unif. conti.

pf : If not, then $\exists\, \varepsilon > 0$ st. for $\delta = 1/n$, $\exists\, x_n, \xi_n \in [a,b]$

$$|x_n - \xi_n| < \frac{1}{n} \quad \text{but} \quad |f(x_n) - f(\xi_n)| \geq \varepsilon \quad (*).$$

$\{x_n\}$ has a subsequence conv. to η, $\lim_{i\to\infty} x_{n_i} = \eta$

then $\lim_{i\to\infty} \xi_{n_i} = \eta$ too. This contradicts to $(*)$. $\square$

II. Intermediate Value Theorem. (Done).

III. Theorem : Every conti $f : [a,b] \to \mathbb{R}$ has a maximum.

pf : Step 1. f is bounded.

2 possible pfs : $\left\langle \begin{array}{l} \text{By I.} \\ \text{By } f(x_n) > n;\ \lim_{n\to\infty} \text{ get } \times \end{array} \right.$

Step 2 : Let $M = \text{lub of } \{f(x) \mid x \in [a,b]\} =: S$

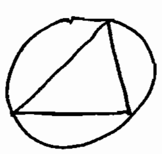
either $M \in S$ (done, $M = f(x_0)$ for some x_0)

or $\exists\, x_n \in [a,b]$, $\lim_{n\to\infty} f(x_n) = M$.

but then $\exists$ sub. seq. $x_{n_i} \to \xi$

By continuity of f : $f(\xi) = \lim_{i\to\infty} f(x_{n_i}) = M$. $\square$

Example : largest triangle in a circle.



Final Remark on $\mathbb{Q}$ & $\mathbb{R}$:

$\mathbb{Q}$ is countable (i.e. as a set $\mathbb{Q} \cong \mathbb{N}$)

But $\mathbb{R}$ is uncountable. (Hint: diagonal process.)

End of Ch. 1.

[2.1-2.2 Integrals: Calculus 9/30.

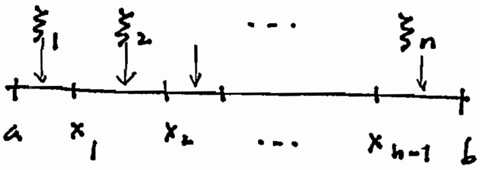
Analytic def'n: Let f be (conti) on [a, b]

Consider a subdivision: $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

denote $\Delta x_i = x_i - x_{i-1}$. Choose any $\xi_i \in [x_{i-1}, x_i]$

Form the Riemann Sum:

$$F_n = \sum_{i=1}^{n} f(\xi_i)\cdot \Delta x_i$$



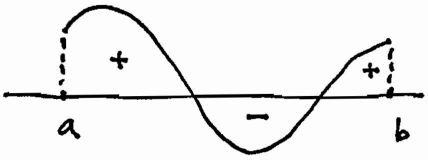
If the $\lim_{n\to\infty} F_n$ exists, indep of sub div & choices of ξ_i, as long $\Delta x_i \to 0$

we call it the (Riemann) integral, denoted by

$$\int_a^b f(x)\,dx.$$

Theorem of Existence:

The integral exists for f conti. or piece-wise conti.



Sign convention

Basic Examples

Ex 1. $f(x) = x^2$. Let $\Delta x_i = h = \frac{b-a}{n}$, $\xi_i = x_i$

$$\begin{aligned} F_n &= \sum_{i=1}^{n} (a+ih)^2 h \\ &= n a^2 h + 2ah^2 \sum_{i=1}^{n} i + h^3 \sum_{i=1}^{n} i^2 \\ &= a^2(b-a) + a(b-a)^2\left(1+\frac{1}{n}\right) + (b-a)^3\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \end{aligned}$$

$$\Rightarrow \int_a^b x^2\,dx = \frac{1}{3}(b^3 - a^3) = \frac{1}{3}x^3\Big]_a^b .$$

Ex 2. $f(x) = x^\alpha$, $\alpha \in \mathbb{Q} \setminus \{-1\}$. Let $q = \sqrt[n]{b/a}\ {}^{\wedge 1}$, $\overset{\xi_i}{\overset{\shortparallel}{x_i}} = a q^i$

$\Delta x_i = a q^{i-1}(q-1) \to 0$ as $n \to \infty$

$$\begin{aligned} F_n &= \sum_{i=1}^{n} (\xi_i)^\alpha \Delta x_i \\ &= a^{\alpha+1}\frac{q-1}{q}\sum_{i=1}^{n}\left(q^{1+\alpha}\right)^i \qquad \shortparallel\ q^{\alpha+1}\frac{q^{n(\alpha+1)}-1}{q^{\alpha+1}-1} \\ &= \left(b^{\alpha+1} - a^{\alpha+1}\right) q^\alpha \frac{q-1}{q^{\alpha+1}-1} \end{aligned}$$

What is $\lim_{n\to\infty} \frac{q-1}{q^{\alpha+1}-1}$? i.e. $q \to 1$.

Separate cases $\alpha \in \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$?

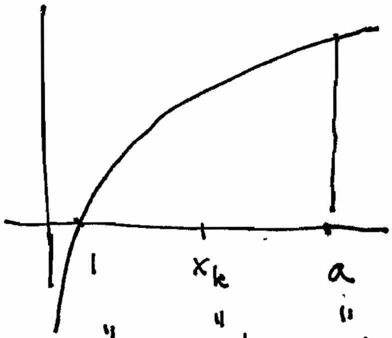
Sol: $\frac{q-1}{q^{r/s+1}-1} = \frac{w^s-1}{w^{r+s}-1} = \frac{s}{r+s} = \frac{1}{\alpha+1}$

$w = q^{1/s}$

$\alpha = \frac{r}{s} \in \mathbb{Q}$, $s \in \mathbb{N}$

$\alpha + 1 = \frac{r+s}{s} \neq 0$.

Ex 3. $f(x) = \log x$.



$$\overset{\shortparallel}{e^0} \qquad \overset{\shortparallel}{e^{\frac{kb}{n}}} \qquad \overset{\shortparallel}{e^b} = e^{\log a}$$

$$\int_1^a \log x \, dx = x\log x - x \Big|_1^a = a\log a - a + 1$$

$$F_n = \sum_{k=1}^{n} \underbrace{\left(e^{\frac{kb}{n}} - e^{\frac{(k-1)b}{n}}\right)}_{\Delta x_k} \underbrace{\log e^{\frac{kb}{n}}}_{f(x_k)}$$

Again, subdivison by geometric series.

$$F_n = \frac{b}{n}\left[\left(e^{\frac{1\cdot b}{n}} - e^{\frac{0b}{n}}\right)\cdot 1 + \left(e^{\frac{2b}{n}} - e^{\frac{1b}{n}}\right)\cdot 2 + \cdots + \left(e^{\frac{nb}{n}} - e^{\frac{(n-1)b}{n}}\right)\cdot n\right]$$

$$= \frac{b}{n}\left(1 - e^{\frac{b}{n}} - e^{\frac{2b}{n}} - \cdots - e^{\frac{(n-1)b}{n}} + ne^b\right)$$

$$= be^b - \frac{b}{n} \cdot \frac{1-e^{\frac{n}{n}b}}{1-e^{b/n}}.$$

when $n \to \infty$, $e^{\frac{b}{n}} \to 1$.

Lemma: $\lim_{x\to 0} \frac{e^x-1}{x} = 1$.

(Hint: Write $x = \log(1+y)$)

Ex 4. $f(x) = \sin x$.

$$S_n := \sin\theta + \sin 2\theta + \cdots + \sin n\theta$$

$$\sin\theta \cdot S_n = \frac{1}{2}\left(\cos [illegible] - \cos 2\theta + \cos\theta - \cos 3\theta + \cdots + \cos(n-1)\theta - \cos(n+1)\theta\right)$$

$$= \frac{1}{2}\left(1 + \cos\theta - \cos n\theta - \cos(n+1)\theta\right)$$

For $\int_0^b \sin x \, dx = \lim_{n\to\infty} F_n$ $\quad x_k = \frac{kb}{n}$, $\Delta x_k = \frac{b}{n}$ (equal)

let $\theta = \frac{b}{n}$

$$F_n = \frac{b}{n} \cdot \frac{1}{2} \frac{1}{\sin\frac{b}{n}}\left(1 - \cos b + \cos\frac{b}{n} - \cos\frac{n+1}{n}b\right)$$

$$\xrightarrow[n\to\infty]{} 1 - \cos b = -\cos x \Big|_0^b.$$

2.3 – 2.4 Rules of integration:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_i)\,\Delta x_i \quad (a \le b) \quad f: p\text{-conti.}$$

$\Rightarrow \int_a^b + \int_b^c = \int_a^c$, hence we define $\int_a^b = -\int_b^a$ if $a > b$.

Linearity: $\int_a^b$: p-conti. fun. on $[a,b] \to \mathbb{R}$ is $\mathbb{R}$-linear.

comparison: $f(x) \geq g(x)$ on $[a,b] \Rightarrow \int_a^b f \geq \int_a^b g$.

Cor. $m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$

M = maximum
m = minimum
they exist only requires f : p-conti

Mean Value Theorem: If f is conti.

then $\int_a^b f(x)\,dx = f(\xi)(b-a)$

for some $\xi \in [a,b]$. More generally, for $p \geqslant 0$, p-conti

then $\int_a^b f(x)\,p(x)\,dx = f(\xi)\int_a^b p(x)\,dx$ for some $\xi \in [a,b]$.

Theorem: The indefinite integral

$\phi(x) := \int_\alpha^x f(u)\,du$ is conti. on $x \in [a,b]$, & Lipschitz.

Pf: $|\phi(x) - \phi(y)| = \left|\int_y^x f(u)\,du\right| \le M \cdot |x-y|$. □

2.5 – 2.7 Log defined by integral.

This is taken as a simple example, can be skipped mostly.

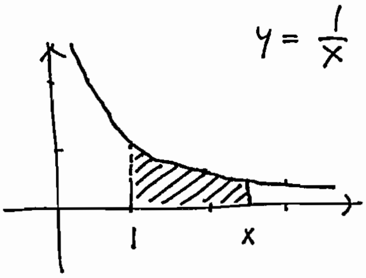
$$\int_a^b x^\alpha\,dx = \frac{1}{\alpha+1}x^{\alpha+1}\Big|_a^b \quad \text{for } \alpha \neq -1$$

How about if $\alpha = -1$? New functions appear!

$\log x := \int_1^x \frac{1}{u}\,du$ for all $x > 0$.

Addition Theorem: $\log xy = \log x + \log y$

(Hint: Show $\int_x^{xy}\frac{1}{u}\,du = \int_1^y \frac{1}{u}\,du$ by definition.)

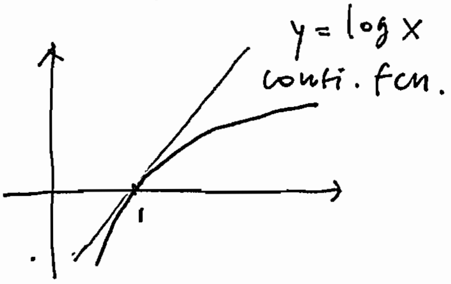


$\Downarrow$

Cor. $\log x^\alpha = \alpha \log x$ $\forall \alpha \in \mathbb{Q}$, hence $\mathbb{R}$.

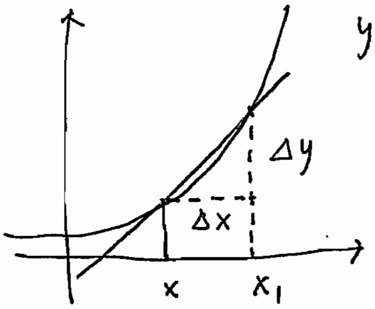
Theorem: $\log e = 1$ (Naturality of e).

Pf: $\lim_{n\to\infty}\log\left(1+\frac{1}{n}\right)^n = n\log\left(1+\frac{1}{n}\right) = n\cdot\frac{1}{\xi}\,\frac{1}{n} \to 1$.



Calculus 10/5

2.8 Derivative / differentiation



$y = f(x)$

$$f'(x) = \lim_{x_1\to x}\frac{f(x_1)-f(x)}{x_1-x} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\left(\frac{dy}{dx} = Df(x) = \dot{y} = \frac{df}{dx}(x) \text{ etc.}\right)$$

$$= \text{slope of tangent} = \tan \underset{\substack{\uparrow\\ [0,\pi)}}{\alpha}$$

require $x, x_1 \in \operatorname{Dom} f$ and $x_1 \to x$ makes sense.

Ex 1. $y = f(x) = x^{\alpha}$, $\alpha = p/q \in \mathbb{Q}$, $q \in \mathbb{N}$

$$\frac{f(x_1)-f(x)}{x_1-x} = \frac{x_1^{p/q}-x^{1/q}}{x_1-x} = \frac{\xi_1^{p}-\xi^{p}}{\xi_1^{q}-\xi^{q}} \underset{\substack{\uparrow\\ \text{if } p>0}}{=} \frac{\xi_1^{p-1}+\cdots+\xi^{p-1}}{\xi_1^{q-1}+\cdots+\xi^{q-1}}$$

Q: $\alpha \in \mathbb{R}$?

Ex 2. $$(\sin x)' = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$

$$= \lim_{h\to 0}\left(\sin x\,\frac{\cos h-1}{h} + \cos x\,\frac{\sin h}{h}\right) = \cos x.$$

Similary, $(\cos x)' = -\sin x$.

Ex 3: $$(e^x)' = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = e^x\lim_{h\to 0}\frac{e^h-1}{h} = e^x \;!$$

$$(\log x)' = \lim_{h\to 0}\frac{1}{h}\left(\log(x+h)-\log x\right) = \lim_{h\to 0}\frac{1}{x}\cdot\log\left(1+\frac{h}{x}\right)^{\frac{x}{h}} = \frac{1}{x}.$$

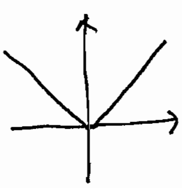
This explains the role of nat'l base e

and def'n of log x via $\int_1^x \frac{du}{u}$.

Differentiable ⇒ conti, but NOT conversely.

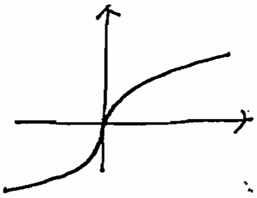
$$\lim_{x_1\to x}\left(f(x_1)-f(x)\right) = \lim_{x_1\to x}\frac{f(x_1)-f(x)}{x_1-x}\cdot(x_1-x) = f'(x)\cdot 0 = 0.$$

eg.



$y=|x|$ or $y = x^{1/3}$

not diff'ble at $x=0$.



$$\frac{f(h)-f(0)}{h} = \frac{1}{h^{2/3}} \to \infty \text{ as } h\to 0.$$

The mean value theorem MVT .

If f is conti on $[x_1, x_2]$, diff'ble on (x_1, x_2)

then $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi)$ for some $\xi \in (x_1, x_2)$.

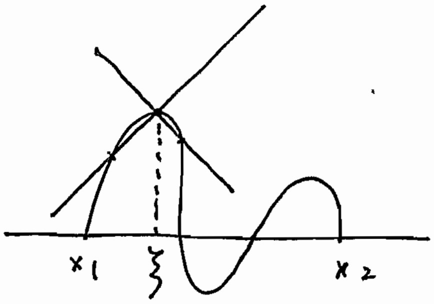
Rolles' Thm : Same condition, with $f(x_1) = 0 = f(x_2)$.

Then $\exists\ \xi \in (x_1, x_2)$ st. $f'(\xi) = 0$.

Pf: May assume that $f \not\equiv 0$.

let $f(\xi)$ = maximum , $\xi \in (x_1, x_2)$

$$\lim_{h \to 0} \frac{f(\xi + h) - f(\xi)}{h} \quad \begin{array}{ll} \le 0 & h \to 0^+ \\ \ge 0 & h \to 0^- \end{array}$$



Since $f'(\xi)$ exists, so it must be 0 . □

Pf of MVT : Simply consider $\phi(x) = f(x) - g(x)$

where $g(x) = f(x_1) + \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$. □

Basic Applications : let f' exists on (a, b) .

① f monotonic increasing (decreasing) $\Leftrightarrow$ $f' \ge 0$ ($f' \le 0$)

② f = constant $\Leftrightarrow$ $f' \equiv 0$.

Remark : $f'(x)$, if exists, is NOT nec. conti or even bounded.

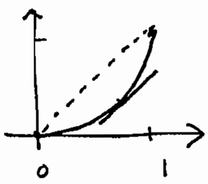
eg. $f(x) = x^2 \sin \frac{1}{x}$; $f(x) = x^2 \sin \frac{1}{x^2}$ (cf. ch3. A.2) .

Need Lebnitz rule $(fg)' = f'g + fg'$ and

chain rule $f(g(x))' = f'(g(x)) \cdot g'(x)$ to calculate it.

③ f' exists on $[0, 1]$, $f(0) = 0$, $f(1) = 1$, $f'(0) = 0$

then $\forall\ \xi \in (0, 1)$ st. $f'(\xi) = 1/2$.



④ Lipschitz conti vs. differentiable .

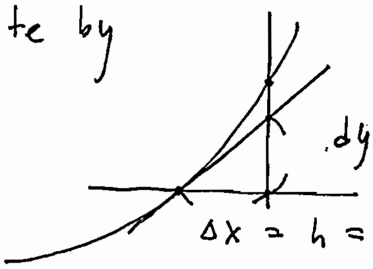
$$\begin{tikzcd}
 & \text{Lipschitz} \arrow[dl] & \\
\text{conti.} & & C^1([a,b]) \arrow[ul] \arrow[dl] \\
 & \text{Differentiable} \arrow[ul] &
\end{tikzcd}$$

.

Differentials and linear approximation.

By def'n, $\underset{f(x+h)-f(x)}{\underset{\shortparallel}{\Delta y}} = f'(x)\,\underset{h}{\underset{\shortparallel}{\Delta x}} + \varepsilon(\Delta x)\,\Delta x$ and $\lim_{h\to 0} \varepsilon(h) = 0$

Denote by



$dy := f'(x)\,dx$, the differential

$\Delta x = h = dx$ when x is the indep variable

i.e. $\Delta y = dy +$ "$\varepsilon \cdot dx$"

Rough estimate of $\varepsilon(h)$: Assume $f'' = (f')'$ exists on $[x, x+h]$,

$$\varepsilon = \frac{f(x+h)-f(x)}{h} - f'(x) = f'(\xi) - f'(x) = f''(\eta)\cdot(\xi - x)$$

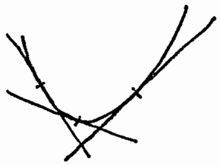
$\Rightarrow$ $f(x+h) = f(x) + f'(x)\,h + \varepsilon\cdot h$

with $|\varepsilon \cdot h| \leq \mathrm{Max}\,|f''(\eta)| \cdot h^2$.

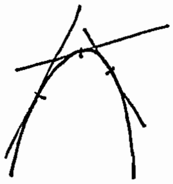
Higher derivatives and differentials.

$$y'' = f''(x) := \frac{d}{dx}\big(f'(x)\big) \underset{?}{=\!=} \frac{d^2y}{dx^2} \quad ; \quad y''' = f'''(x) = \frac{d^3y}{dx^3} \quad \text{etc.}$$

Usage: $f'' > 0 \Rightarrow f' \nearrow \Rightarrow$ the graph is part of



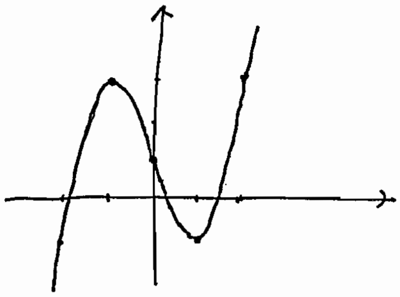
$f'' < 0 \Rightarrow f' \searrow \Rightarrow \cdots$



e.g. Graph of $f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3$, $f'(x) = 0 \iff x = \pm 1$

$f''(x) = 6x$. Sketch:



Theorem: If $f''(x)$ exists and conti. in a neighborhood of $x = a$, then

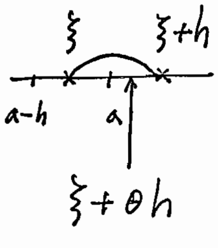
$$f''(a) = \lim_{h\to 0} \frac{\big(f(a+h) - f(a)\big) - \big(f(a) - f(a-h)\big)}{h^2}.$$

pf: Let $g(x) = f(x+h) - f(x)$ be the 1st. difference.

Then the 2nd diff $= g(a) - g(a-h) = g'(\xi)\,h$

$= \big(f'(\xi+h) - f'(\xi)\big)\,h = f''(\xi + \theta h)\,h^2$. □

$0 < \theta < 1$.



Calculus 10/7

2.9 The Fund. Thm. of Calculus

FTC part I: let f be conti. $\phi(x) := \int_a^x f(u)\,du$,

then $\phi'(x)$ exists and $\phi'(x) = f(x)$.

pf: $\phi'(x) = \lim_{h\to 0} \dfrac{\phi(x+h)-\phi(x)}{h} = \lim_{h\to 0} \dfrac{1}{h}\int_x^{x+h} f(u)\,du$

$= \lim_{h\to 0} f(\xi) = f(x)$, since f is conti. □

Reverse the process of diff: Given f, find F st. $F' = f$.
called primitive fcn of f.

FTC part II:

Any 2 primitives differ by a const.

pf: $(F_1 - F_2)' = f - f = 0 \Rightarrow F_1 - F_2 = c$ (MVT). □

Notation: (Indefinite integral)

$F(x) = \int f(x)\,dx$ means $F(x) = c + \int_a^x f(u)\,du$ for some a, c.

Thus $\int_a^b \underset{\substack{\| \\ F'(x)\,dx \,=\, dF}}{f(x)\,dx} = F(b) - F(a) = F(x)\Big|_a^b$

Caution: NOT TRUE for general F with F' exists. Need C^1.
Also, need $[a,b] \subset \mathrm{Dom}\, f$.

Ex 1. $\int_a^b x^\alpha\,dx = \dfrac{1}{\alpha+1}\left(b^{\alpha+1} - a^{\alpha+1}\right)$; $\alpha \neq -1$.

Ex 2. $\int_a^b \cos x\,dx = \sin b - \sin a$, $\int_a^b \sin x\,dx = -\cos x\Big|_a^b$.

Ex 3. $\int_a^b e^x\,dx = e^b - e^a$,

But $\int_a^b \log x\,dx = ?$ How to get $F(x)$, $F'(x) = \log x$?

Ans: $(x\log x - x)' = \log x + x\cdot\frac{1}{x} - 1 = \log x$.

In general, need to develop techniques of diff / int.

Thm : $f \in C([a,b]) \Rightarrow \int_a^b f(x)\,dx$ exists.

Recall Unif. conti. Thm.

For any given $\varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon) > 0$ st.

$|f(\xi) - f(\eta)| < \varepsilon$ for any $\xi, \eta \in [a,b]$ with $|\xi - \eta| < \delta$.

For a subdivison S_n : $\underset{\substack{\shortparallel \\ a}}{x_0} < x_1 < \cdots < \underset{\substack{\shortparallel \\ b}}{x_n}$

the span (size)

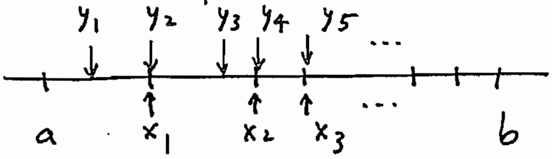
of S_n is the longest of Δx_i : Span $S_n := \max\{\Delta x_i\}$.

if Span $S_n < \delta$, then

$$|F_n - F_n'| = \left|\sum_{i=1}^n \left(f(\xi_i) - f(\xi_i')\right)\Delta x_i\right| < \varepsilon\cdot(b-a).$$

To handle different subdivison, need "refinement" :

$\Rightarrow\ |F_N - F_n| < \varepsilon\cdot(b-a)$.



- For any sequence F_n with Span $S_n \longrightarrow 0$.

Will show that $\{F_n\}$ is a Cauchy sequence hence $\lim_{n\to\infty} F_n$ exists.

Pf: $\forall\ \varepsilon > 0$, $\exists$ unif. modulus $\delta = \delta(\varepsilon)$, fix δ

$\exists\ n_1$ st. Span $S_n < \delta$ for all $n > n_1$

Now let $n, m > n_1$; F_N' any refinement of both,

$\Rightarrow\ |F_n - F_m| \le |F_n - F_N'| + |F_m - F_N'| \le 2\varepsilon\cdot(b-a)$. □

- $\lim_{n\to\infty} F_n$ is independent of choices of S_n and ξ_i etc.

Pf: Let $\lim_{n\to\infty} \underset{\text{wrt. } S_n}{F_n} = F$, $\lim_{n\to\infty} \underset{\text{wrt } S_n'}{F_n'} = F'$

Then $|F_n - F_n'| \le |F_n - F_N''| + |F_n' - F_N''| < 2\varepsilon\cdot(b-a)$

by way of refinement F_N'', as long as

both Span S_n, Span $S_n' < \delta$.

$\lim_{n\to\infty}$ get $|F - F'| \le 2\varepsilon\cdot(b-a)$. This holds $\forall\ \varepsilon > 0$, hence $F = F'$.

End

Calculus 10/12

3.1 – 3.3 Rules for Differentiation

Thm. $(af+bg)' = af' + bg'$

(4 Rules) $(fg)' = f'g + fg'$ (Lebnitz rule)

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'g - fg'}{g(x)^2}$$ at x with $g(x) \neq 0$

Ex1. $(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x$

$(\csc x)' = \left(\frac{1}{\sin x}\right)' = \frac{-\cos x}{\sin^2 x} = -\csc x \cdot \cot x$

$(\sec x)' = \sec x \cdot \tan x$

$(\cot x)' = -\csc^2 x$.

Thm. If $y = f(x)$ has inverse $x = \phi(y)$ and f' exists,
(inverse) then $\phi'(y)$ exists for x s.t. $f(x) \neq 0$. Also, $\phi'(y) \cdot f'(x) = 1$

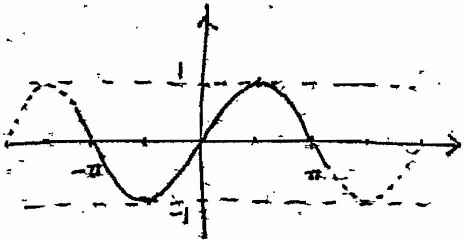
Ex 2. $y = e^x$, $x = \log y$ i.e. $\frac{dx}{dy} = \frac{1}{dy/dx}$

So, $\frac{de^x}{dx} = \frac{1}{dx/dy} = \frac{1}{1/y} = y = e^x$.

Ex 3. Inverse $\triangle$ functions : Branches

(1) $y = \sin x$, $x = \sin^{-1} y$

$\equiv \arcsin y$



$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{\cos x} = \frac{\pm 1}{\sqrt{1-\sin^2 x}}$

$= \pm 1/\sqrt{1-y^2}$ Sign depends on branch

principal branch $[-1,1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$, $(\sin^{-1})' > 0$.

(2) $(\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}}$ on p.b. $(0, \pi)$

(3) $(\tan^{-1} x)' = \frac{1}{(\tan y)'} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

$(-\infty, \infty) \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ the p.b.

On p.6. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$, also $-\cos^{-1} x$ why?

Also $\tan^{-1} x = \int_0^u \frac{du}{1+u^2}$, new def'n of Δ-fcns.

Thm: Chain Rule. $f(g(x))' = f'(g(x)) \cdot g'(x)$.

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta \phi} \cdot \frac{\Delta \phi}{\Delta x}$$ with $\phi = g(x)$. let $\Delta x \to 0$ done?

Pf (Rigorous): $\Delta y = f'(\phi)\Delta\phi + \varepsilon \Delta\phi$, $\varepsilon := 0$ if $\Delta\phi = 0$

$\Delta\phi = g'(x)\Delta x + \eta \Delta x$ ↑ key!

$\Rightarrow \Delta y = (f'(\phi) + \varepsilon).(g'(x) + \eta).\Delta x$, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ done. □

Ex 4. For any $x > 0$, $\alpha \in \mathbb{R}$.

$$(x^\alpha)' = (e^{\alpha \log x})' = e^{\alpha \log x} \cdot \frac{\alpha}{x} = \alpha \cdot x^{\alpha - 1}$$

$$(a^x)' = (e^{x \log a})' = a^x \cdot \log a.$$

$$(x^x) = (e^{x \log x})' = x^x.(\log x + 1).$$

$(\log|x|)' = 1/x$ no matter $x > 0$ or $x < 0$.

Thm: Generalized MVT. Let F, G conti on $[a,b]$

If F', G' exists on (a,b) and $G' > 0$, then

$$\frac{F(b) - F(a)}{G(b) - G(a)} = \frac{F'(\xi)}{G'(\xi)}$$ for some $\xi \in (a,b)$.

Pf: Let $u = G(x)$ has inverse $x = g(u)$, let $f = F \circ g$. □

Alt. pf: Set $\phi(x) = (F(b) - F(a))(G(x) - G(a)) - (G(b) - G(a))(F(x) - F(a))$

3.4 Applications of exp.

Thm: Diff eq'n $y' = \alpha y$ has sol. $y = f(x) = c e^{\alpha x}$

Pf: $(y e^{-\alpha x})' = y' e^{-\alpha x} + y e^{-\alpha x}(-\alpha) \equiv 0 \Rightarrow y e^{-\alpha x} = c$. □

eg. Newton's Law of cooling; chemical reaction etc.

$y' = -ky$.

Calculus 10/14

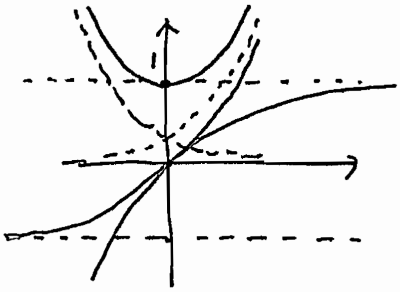
3.5 Hyperbolic functions.

$\sinh x = \frac{e^x - e^{-x}}{2}$, odd. $\quad \cosh x = \frac{e^x + e^{-x}}{2} > 0$, even, $\tanh x = \frac{\sinh x}{\cosh x}$. etc.

$\cosh^2 x - \sinh^2 x = 1$

parametrizes $x^2 - y^2 = 1$

Q: why not using $\sec^2 t - \tan^2 t = 1$?



$\sinh' x = \cosh x \quad , \quad \cosh' x = \underset{\text{sign}}{\sinh x}$

Also, $\tanh' x = \operatorname{sech}^2 x$.

Inverse function : $y(t) = \sinh t = \frac{1}{2}(e^t - e^{-t})$

$\Rightarrow e^{2t} - 2y e^t - 1 = 0 \Rightarrow e^t = y \pm \sqrt{y^2+1} > 0$ pick +.

i.e. $t = \operatorname{ar}\sinh y = \sinh^{-1} y = \log(y + \sqrt{y^2+1})$.

Ex. $x = \cosh t \Rightarrow t = \cosh^{-1} x = \log(x \pm \sqrt{x^2-1})$.

2 branches

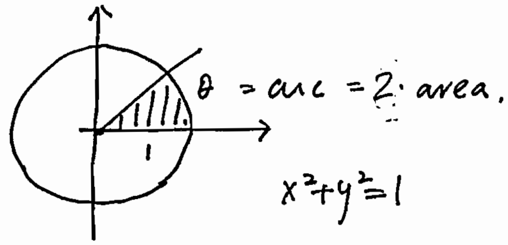
Let $y = \sinh^{-1} x$,

- $\frac{d}{dx}\sinh^{-1} x = \frac{1}{\frac{d}{dy}\sinh y} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$

FTC $\Rightarrow \int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + c = \log(x + \sqrt{x^2+1})$!

- $\frac{d}{dx}\cosh^{-1} x = \frac{\pm 1}{\sqrt{x^2-1}}$. $\quad \int \frac{dx}{\sqrt{x^2-1}} = \log(x + \sqrt{x^2-1})$. check.

Analogies with $\sin\theta$, $\cos\theta$:

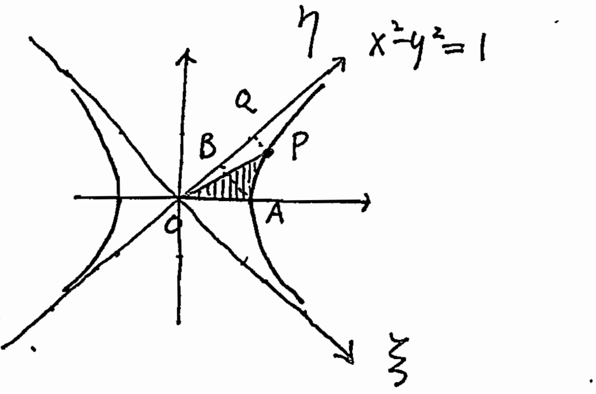


change axes :

$x = \frac{1}{\sqrt{2}}(\xi + \eta)$, $y = \frac{1}{\sqrt{2}}(\xi - \eta)$

eq'n : $\xi\eta = 1/2$.

$|OAP| = |ABQP|$



$$= \int_{1/\sqrt{2}}^{\eta} \frac{du}{2u} = \frac{1}{2}\log u \Big|_{1/\sqrt{2}}^{(x+y)/\sqrt{2}}$$

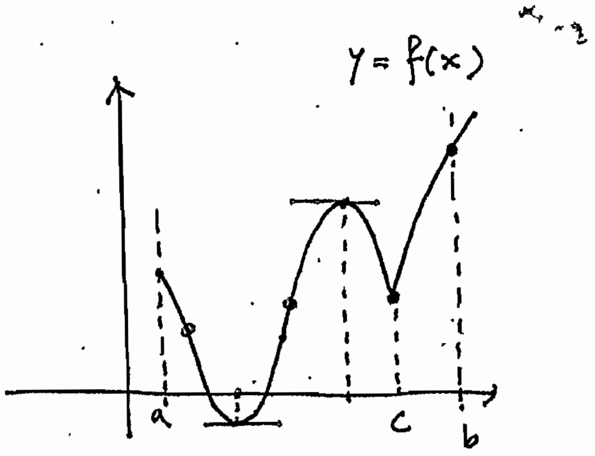
$$= \frac{1}{2}\log(y + \sqrt{y^2+1}) = \frac{1}{2}\sinh^{-1} y = t.$$

3.6 Maxima & Minima

$f : [a,b] \to \mathbb{R}$ conti.

f' exists except at finite pts

local extremal (candidates)
or relative



- Stationary pts (= critical pts.)
  i.e. $\xi \in (a,b)$ with $f'(\xi) = 0$
- End pts + non-diff pts. (cf. $f'(b)$ may not $= 0$)

1st derivative test : $f' \geq 0$ in RHS of ξ $\Rightarrow$ $f(x)$ has rel. min at ξ.
$f' \leq 0$ in LHS

2nd derivative test : $f'' > 0 \Rightarrow f' \nearrow$ i.e. convex.
$f'' < 0 \Rightarrow f' \searrow$ i.e. concave.

inflection pt : $f'' = 0$.

Ex 1. $y = f(x) = x^6 - 3x^2 - 1$; $[-2, 2] \to \mathbb{R}$

$f'(x) = 6x^5 - 6x = 6x(x^2+1)(x+1)(x-1)$

x	-2	-1	0	1	2
$f(x)$	51	-3	-1	-3	51
$f'(x)$		+	−	+	

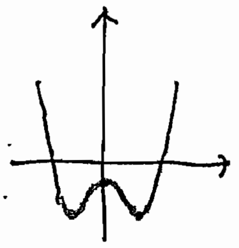
f'	−		+		−		+
		-1		0		1	
f	↘		↗		↘		↗

$f''(x) = 6(5x^4 - 1) = 30\left(x^2 + \frac{1}{\sqrt{5}}\right)\left(x + \frac{1}{\sqrt[4]{5}}\right)\left(x - \frac{1}{\sqrt[4]{5}}\right)$.

Ex 2. Nearest pt from a pt to ellipse.

simple case, pt $= (c, 0)$. $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

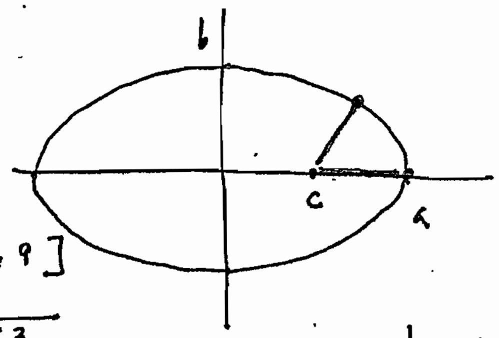
$b < a$.



$f(x) = d^2(x) = (x-c)^2 + b^2\left(1 - \frac{x^2}{a^2}\right)$

domain of d : $x \in [-a, a]$. f = deg 2 poly function.

$f'(x) = 2\left[\left(1 - b^2/a^2\right)x - c\right]$



$f''(x) = 2\left(1 - b^2/a^2\right) > 0$ convex

$f'(x) = 0 \Leftrightarrow x = c/\left(1 - b^2/a^2\right), \overset{?}{\in} [-a, a]$

If $|c| \leq a\left(1 - \frac{b^2}{a^2}\right)$, $d_{min} = b\sqrt{1 - \frac{c^2}{a^2 - b^2}}$, otherwise $d = a - |c|$

3.7 The notion of order of magnitude of functions

$p(x) = a_n x^n + \dots + a_0$

$a^x \quad (a > 1)$ all $\to \infty$ as $x \to \infty$, which is faster?

$\log_a x$

starting : $\frac{x}{a^x} = x a^{-x} = f(x)$, $f'(x) = a^{-x} - x \log a \; a^{-x}$

i.e. f ↘ so only need to take $x \in \mathbb{N}$. $= (1 - x \log a)\, a^{-x} < 0$ for x large

$$a^n = (1+h)^n = 1 + nh + \frac{n(n-1)}{2} h^2 + \cdots$$

hence $\frac{n}{a^n} \to 0$ as $n \to \infty$.

Corollary : $\frac{x^m}{a^x} = \left(\frac{x}{a^{x/m}} \right)^m = \left[\frac{x}{(a^{1/m})^x} \right]^m \to 0 \quad \forall m \in \mathbb{R}^+$.

$$\frac{\log_a x}{x^\varepsilon} = \frac{y}{a^{\varepsilon y}} \to 0 \quad (x = a^y) .$$

Landau's notation : $f(x) = O(g(x)) \iff \left| \frac{f(x)}{g(x)} \right| < M$ for x large

$f(x) = o(g(x)) \iff \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$.

may replace $x \to \infty$ by $x \to a$ if both $f(x), g(x) \to \infty$.

Order of smallness : when $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$

a could be finite ($\in \mathbb{R}$), or ∞.

eg. $e^{-1/x} = o(x^\alpha)$ as $x \to 0$ from $\mathbb{R}^+$.

eg. $f(x+h) - f(x) = f'(x) h + o(h) \iff f'(x)$ exists.

f'' exists and conti near $a \Rightarrow$

$f(a+h) - f(a) = f'(a) h + O(h^2)$, better estimate.

All these apply to the case of sequences a_n too.

eg. $\log n = o(n^\epsilon)$; $n^k = o(a^n)$; $a^n = o(n!)$; $n! = o(n^n)$.

in fact, $n! \sim \sqrt{2n\pi} \left(\frac{n}{e} \right)^n$ (Sterling).

Appendix of Ch. 3

Some special functions & differentiability.

1. $y = e^{-1/x^2}$ $x \neq 0$. Since $\lim_{x\to 0} e^{-1/x^2} = 0$

we define $y(0) = 0$ to make y a conti. function.

Thm : $y \in C^{\infty}(\mathbb{R})$, but $y^{(n)}(0) := \frac{d^n y}{dx^n}(0) = 0$ $\forall n$.

pf : $y'(x) = \frac{2}{x^3} e^{-1/x^2}$ for $x \neq 0$

$y'(0) = \lim_{h\to 0} \frac{1}{h} e^{-1/h^2} = 0$ Since $P(n) = o(e^{n^2})$ for $n \to \infty$.

Similarly $\lim_{x\to 0} y'(x) = 0 = y'(0)$.

the pf continues for $y''(x)$ etc. $\square$

2. $y = e^{-1/x}$ $x \neq 0$, $y(0)$ can't be conti. defined.

in fact $\lim_{x\to 0^-} y^{(n)}(x) = \infty$ & $\lim_{x\to 0+} y^{(n)}(x) = 0$ $\forall n$.

3. $y = \sin 1/x$ not conti at 0

$y = x \sin \frac{1}{x}$ conti, not diff at 0

$y = x^2 \sin \frac{1}{x}$ y' exists, but not conti. at 0, y' bounded

$y = x^2 \sin \frac{1}{x^2}$ y' exists, not conti at 0, Also y' not bounded.

This is NOT Lipshitz, why?

$y = x^3 \sin \frac{1}{x}$, y' conti.

$\Downarrow$ $y'(x)$ is not Riem. integrable.

Theorem (CVF)

for $f \in C([a,b])$, $\int_a^b f(x)\,dx = \int_\alpha^\beta f(u(t))\,u'(t)\,dt$

where $x = u(t)$, $C^1([\alpha,\beta])$, $a = u(\alpha)$, $b = u(\beta)$.

pf: let $\phi(s) = \int_\alpha^s f(u(t))\,u'(t)\,dt$, $F(w) = \int_a^w f(x)\,dx$

then $\phi'(t) = f(u(t))\,u'(t) = F'(u(t))\,u'(t) = \dfrac{d\,F\circ u}{dt}$

$\Rightarrow \quad \phi(t) = F(u(t)) + c = \int_a^{u(t)} f(x)\,dx + c$

let $t = \alpha$, get $c = 0$. $\square$

what happens really?

Key Point: u needs not to be 1-1, $u'(t)$ can change sign.

eg. $\int_0^1 \frac{\sqrt{x}\,dx}{\sqrt{2-x}} \underset{\sqrt{2-x}=u}{\overset{\uparrow}{=}} -\int_{\sqrt{2}}^1 \sqrt{2-u^2}\,du = -\cos^{-1}\sqrt{1-\frac{x}{2}} + \sqrt{\frac{x}{2}}\sqrt{1-\frac{x}{2}}\Big|_0^1$

$= \frac{1}{2} - \frac{\pi}{4}$.

careful about range. "limits"

eg. $F(t) := \int_{t^2}^{\sin t} e^{u^2}\,du = G(\sin t) - G(t^2)$

where $G(x) = \int_\alpha^x e^{u^2}\,du$, $\Rightarrow F'(t) = e^{\sin^2 t}\cos t - e^{t^4}\,2t$.

Theorem (Integration by parts)

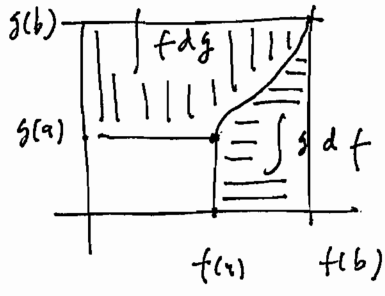
$f, g \in C^1([a,b]) \Rightarrow \int_a^b f(x)\,g'(x)\,dx = f(x)\,g(x)\Big|_a^b - \int_a^b g(x)\,f'(x)\,dx$

Pf: Simply $\int_a^b$ of $(fg)'(x) = f'(x)\,g(x) + f(x)\,g'(x)$. $\square$

eg. Wallis formula:

$$\lim_{m\to\infty} \frac{(m!)^2\,2^{2m}}{(2m)!\,\sqrt{m}} = \sqrt{\pi}$$

using $I_m = \int_0^{\pi/2} \sin^m x\,dx$.

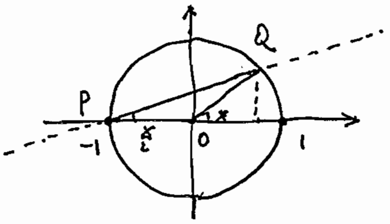


geometric meaning

- $\int R(\sin x, \cos x)\, dx$

In general, $t = \tan\frac{x}{2}$ i.e.

Δ-Substitution: $dt = \frac{1}{2}\frac{dx}{\cos^2\frac{x}{2}}(1+t^2)$



$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$= 2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cos^2\frac{x}{2}$

~~$\sin x =$~~ $\frac{2t}{t^2+1}$

$\cos x = \frac{1-t^2}{1+t^2}$

eg. $$\int \frac{dx}{2\sin x - \cos x + 5}$$

$$= \int \frac{dt}{3t^2+2t+2} = \frac{1}{3}\int \frac{dt}{(t+\frac{1}{3})^2+(\frac{\sqrt{5}}{3})^2}$$

$$= \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{1}{\sqrt{5}}\left(3\tan\frac{x}{2}+1\right)\right) + C.$$

eg. $$\int \frac{dx}{(2+\cos x)\sin x} = \int \frac{1+t^2}{t(3+t^2)}\, dt$$

$\cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

$= \cos^2(\frac{x}{2})(1-t^2)$

$$= \int\left(\frac{1}{3t} + \frac{1}{3}\frac{2t}{3+t^2}\right)dt = \frac{1}{3}\log|t(3+t^2)| + C.$$

special : (a) $R(-\sin x, \cos x) = -R(\sin x, \cos x)$ 令 $\cos x = t$.

(b) $R(-\sin x, -\cos x) = R(\sin x, \cos x)$ 令 $\tan x = t$.

---

eg. $$\int \frac{\sin^2 x}{1+\sin^2 x}\, dx = \int\left(1 - \frac{1}{1+\sin^2 x}\right)dx = x - \int \frac{\sec^2 x}{\sec^2 x + \tan^2 x}\, dx$$

$$= -\int \frac{d(\tan x)}{2\tan^2 x + 1} + x = x - \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) + C.$$

- All are based on integrability of rational functions in terms of elementary functions (via partial fractions):

$\frac{P(x)}{Q(x)}$ with $Q(x) = Q_1(x)(x-a)^k$, $P(a) \neq 0$, $Q(a) \neq 0$

then : $$\frac{P(x)}{Q(x)} - \frac{c}{(x-a)^k} = \frac{P(x) - q_1(x)c}{Q_1(x)(x-a)^k} = \frac{P_1(x)}{Q_1(x)(x-a)^{k_1}}$$

pick c st. $P(a) - Q_1(a)c = 0$, then $k_1 < k$.

By Gauss' fund. thm. of Algebra, get PF decomposition.

for $a \in \mathbb{C}$, merge $a, \bar{a}$ get $(ax+b)/(x^2+px+q)^k$.

$c\int\frac{dx}{(x-a)^k}$ easy. $\int\frac{(ax+b)\,dx}{(x^2+px+q)^k} \mapsto \int\frac{du}{u^k} + \int\frac{dx}{(x^2+1)^k}$ ← by int. by parts.

※

Calculus 11/2

3.15 Improper (Riemann) Integrals

f conti on (a,b) but not defined or conti at a, b.

Then $\int_a^b f(x)\,dx := \lim_{\epsilon \to 0} \int_{a_\epsilon}^{b_\epsilon} f(x)\,dx$

if the RHS exists and is indep. of all choices of $a_\epsilon \to a$, $b_\epsilon \to b$.

Theorem: If f is conti and bounded on (a,b) then $\int_a^b f(x)\,dx$ exists. (eg. $\int_0^1 \sin\frac{1}{x}\,dx$).

pf: WLOG, assume that f is conti at b, then for $F(\alpha) := \int_\alpha^b f(x)\,dx$, any sequence $\alpha_n \to a$ $(\alpha > a)$.

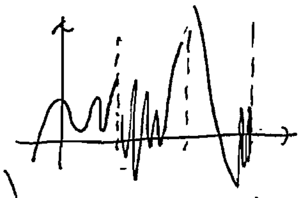
$$|F(\alpha_n) - F(\alpha_m)| \le \underset{\searrow \text{ bound for } |f|}{M} |\alpha_n - \alpha_m|$$

$\Rightarrow$ $F(\alpha_n)$ is a Cauchy sequence

$\Rightarrow$ $\int_a^b f(x)\,dx = \lim_{\alpha \to a} F(\alpha)$ exists. (Q: Why indep of choices of α_n?) $\square$

Cor. Riem. int. exists for piecewise conti. bounded functions.

Q: Discuss FTC for $\int f'(x)\,dx$ where $f(x) = x^2 \sin\frac{1}{x}$ or $x^2 \sin\frac{1}{x^2}$.



eg. $J = \int_0^1 \frac{dx}{x^\alpha} = \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{dx}{x^\alpha} = \lim_{\epsilon \to 0} \frac{1}{1-\alpha}\left(1 - \epsilon^{1-\alpha}\right)$ $(\alpha > 0)$

$\uparrow$ for $\alpha \neq 1$ ($\alpha = 1$ get $-\log\epsilon \to \infty$)

J exists $\Leftrightarrow$ $1 - \alpha > 0$, ie $\alpha < 1$ (or $-\alpha > -1$).

for $\alpha \geq 1$ the integral "diverges".

eg. $\int_0^{1-\epsilon} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(1-\epsilon)$ whose $\lim_{\epsilon \to 0}$ is clearly $\frac{\pi}{2}$

though it has ∞ discontinuity at $x = 1$

[since $(1-x^2)^{1/2} = (1+x)^{1/2}(1-x)^{1/2} \sim \epsilon^{1/2}$.]

Test for convergence: $f(x) = O\left(\frac{1}{(b-x)^\mu}\right)$ for $\mu < 1$ $\Rightarrow$ $\int^b f(x)\,dx$ exists.

eg. Elliptic integral ($|k| < 1$)

$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ at $x=1$ get $O\left(\frac{1}{(1-x)^{1/2}}\right)$, conv.

but if $k=1$ the int. diverges.

Infinite interval for integration

$$\int_a^\infty f(x)\,dx := \lim_{A\to\infty} \int_a^A f(x)\,dx \quad \text{if the RHS is convergent.}$$

eg. $\int_1^A \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}\left(A^{1-\alpha}-1\right)$ need $\alpha > 1$ to get conv.

($\alpha = 1$ get $\log A \to \infty$)

Test for conv:

if $\exists\ \nu > 1$ st. $f(x) = O\left(\frac{1}{x^\nu}\right)$ then $\int_a^\infty f(x)\,dx$ conv.

if $f(x) \geqslant x/N$ then the int. div. (sign of f is important)

eg. $\int_0^\infty \frac{dx}{1+x^2} = \lim_{A\to\infty}\left(\tan^{-1}A - \tan^{-1}0\right) = \frac{\pi}{2}$.

The actual usage is when the int. exists but the value is unknown.

① Gamma function.

$$\Gamma(n) := \int_0^\infty e^{-x}x^{n-1}\,dx. \quad \text{for } n > 0 \ (n \in \mathbb{R}^+)$$

$\int_0^1$ part conv. $\int_1^\infty$ part also conv. via $\frac{x^{n-1}}{e^x} < \frac{M}{x^2}$,

from $\int e^{-x}x^{n-1}dx = \int x^{n-1}\,d(-e^{-x}) = -e^{x}x^{n-1} + (n-1)\int e^{-x}x^{n-2}dx$

get $\Gamma(n) = (n-1)\Gamma(n-1)$ for $n > 1$.

$\Gamma(1) = \int_0^\infty e^{-x}dx = 1$, get $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$

eg. $\int_0^\infty e^{-x^2}dx = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$, $\int_0^\infty x^n e^{-x^2}dx = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)$, $n > -1$.

Remark: (Hard) $\Gamma(s)$ satisfies func. eq'n:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}$$, this extends to $s \in \mathbb{R}$ or even $\mathbb{C}$.

cf:- complex analysis.

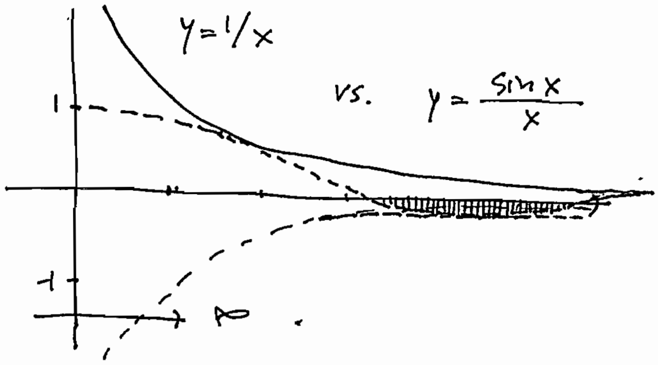
② The Dirichlet Integral

$I = \int_0^\infty \frac{\sin x}{x}\,dx$ conv. but $\int_0^\infty \frac{|\sin x|}{x}\,dx$ diverges.

in fact, on $[n\pi-\frac{\pi}{4}, n\pi+\frac{\pi}{4}]$

$|\sin x| \geq \frac{1}{\sqrt{2}}$

$\frac{1}{x} \geq \frac{1}{(n+\frac{1}{4})\pi}$ hence



$\int_0^\infty \frac{|\sin x|}{x}\,dx > \sum_{n=1}^{\infty} \frac{1}{\sqrt{2}\pi}\cdot\frac{1}{n+\frac{1}{4}} \to \infty$.

for changing sign, the area cancelle out. Later will see more systematic way to test alternating series $\sum a_n$, $|a_n| \searrow 0$.

For now:

$$I_{AB} = \int_A^B \frac{\sin x}{x}dx = \int_A^B \frac{d(1-\cos x)}{x} = \frac{1-\cos x}{x}\Big|_A^B + \int_A^B \frac{1-\cos x}{x^2}dx$$

the key point to insert "1" is to make $\lim_{A\to 0} = 0$.

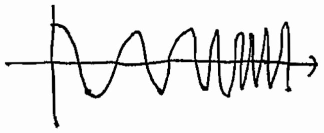
$\Rightarrow \int_0^\infty \frac{\sin x}{x}dx = \lim_{B\to\infty}\lim_{A\to 0} I_{AB} = \int_0^\infty \frac{1-\cos x}{x^2}dx$ which exists.

③ The Fresnel Integral

- we can change variable for conv. improper integals as we just did for Γ.

$F_1 = \int_0^\infty \sin(x^2)\,dx$ notice that $\lim_{x\to\infty}\sin(x^2) \neq 0$!

$= \frac{1}{2}\int_0^\infty \frac{\sin u}{\sqrt{u}}du$



$\int_A^B \frac{\sin u}{\sqrt{u}}du = \frac{1-\cos u}{\sqrt{u}}\Big|_A^B + \frac{1}{2}\int_A^B \frac{1-\cos u}{u^{3/2}}du$

what happens?

$\Rightarrow F_1$ exists, Similarly for $F_2 = \int_0^\infty \cos(x^2)\,dx$.

The fcn can be even unbounded!

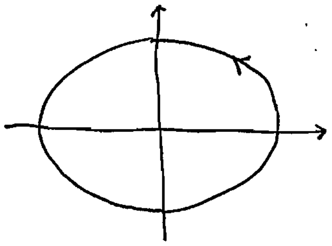
eg. $\int_0^\infty 2u\cos(u^4)\,du = \int_0^\infty \cos(x^2)\,dx$.
↑
$u^2 = x$.

Q: cf. the case $x^2\sin\frac{1}{x^2}$ at $x=0$.

※

Calculus 11/11 (We skip Physis Appli.)

4.1-4.2 Plane curves



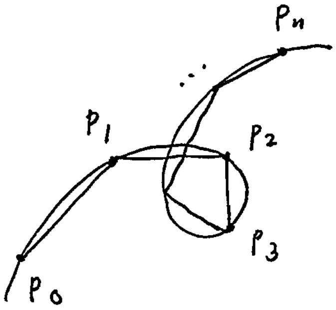
- implicit form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (non-parametric)
- parametric form $x = a\cos\theta$, $y = b\sin\theta$

- change of parameters $t = \tan\frac{\theta}{2}$ (ie. $\theta = 2\tan^{-1} t$)

$$x = a\,\frac{1-t^2}{1+t^2} \; ; \quad y = b\,\frac{2t}{1+t^2}$$

- orientation = direction = sense $\quad \theta'(t) = \frac{2}{1+t^2} > 0$ preserving "sense"

Natural parameter : arc length



$(x(t), y(t)) : [a,b] \longrightarrow \mathbb{R}^2$

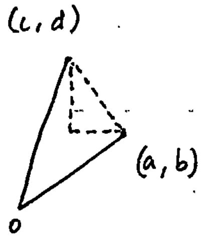
$a = t_0 < t_1 < \cdots < t_n = b$

$$S_n = \sum_{i=1}^{n} |\overline{P_{i-1}P_i}| = \sum_{i=1}^{n} \left(\Delta x_i^2 + \Delta y_i^2\right)^{1/2}$$
$$= \sum_{i=1}^{n} \left(\dot{x}(\xi_i)^2 + \dot{y}(\eta_i)^2\right)^{1/2} \Delta t_i$$

Theorem: For C^1 curves, $L = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2}\,dt$.

pf: $\left|\sqrt{a^2+b^2} - \sqrt{c^2+d^2}\right| \le |a-c| + |b-d|$

and use unif. continuity of $\dot{x}(t), \dot{y}(t)$ on $[a,b]$ *



Ex1. Graph of a function $(x, y = f(x))$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

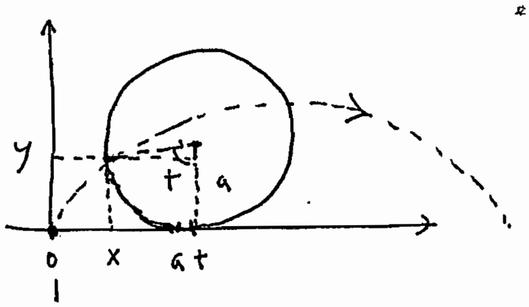
eg. Parabola $y = \frac{1}{2}x^2$, get $\int_a^b \sqrt{1+x^2}\,dx$

let $x = \sinh t$

$$= \int \cosh^2 t\,dt = \frac{1}{2}\left(t + \sinh t \cdot \cosh t\right)$$
$$= \frac{1}{2}\left(\sinh^{-1} x + x\sqrt{1+x^2}\right)\Big|_a^b$$

Ex 2. Cycloid.

$$x(t) = a(t - \sin t)$$
$$y(t) = a(1 - \cos t).$$



$$L = \int_0^\alpha a\left((1-\cos t)^2 + \sin^2 t\right)^{1/2} dt$$

$$= \sqrt{2}\,a \int_0^\alpha \sqrt{1-\cos t}\; dt = 2a \int_0^\alpha \sin\frac{t}{2}\, dt = -4a \cos\frac{t}{2}\Big|_0^\alpha$$

$$= 8a \sin^2\frac{\alpha}{4}.$$

implicit form : (say $a = 1$)

$\cos t = 1 - y$, $\Rightarrow$ $\sin t = \pm\sqrt{1-(1-y)^2} = \pm\sqrt{y(2-y)}$

so $x = \cos^{-1}(1-y) \mp \sqrt{y(2-y)}$ not any better.

Ex 3. Ellipse, say $b > a$.

$$L = \int_0^\alpha \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\; d\theta = b\int_0^\alpha \sqrt{1-\underset{=:\, k^2}{\left(1-\left(\frac{a}{b}\right)^2\right)}\sin^2\theta}\; d\theta$$

no explicit form, called elliptic integral. (2nd kind)

Ex 4. Polar coordinates : $r = r(\theta)$, $\theta \in [\alpha, \beta]$

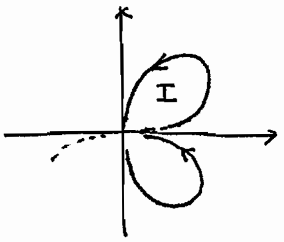
Formula : $\dot{x}(\theta)^2 + \dot{y}(\theta)^2 = (\dot{r}\cos\theta - r\sin\theta)^2 + (\dot{r}\sin\theta + r\cos\theta)^2$

$$= r^2 + \dot{r}^2.$$

eg. $r = \sin 2\theta$, part I $\leftrightarrow$ $\theta \in [0, \frac{\pi}{2}]$

$$L = \int_0^\alpha \left(\sin^2 2\theta + 4\cos^2 2\theta\right)^{1/2} d\theta$$

$\rightarrow$ elliptic integral.



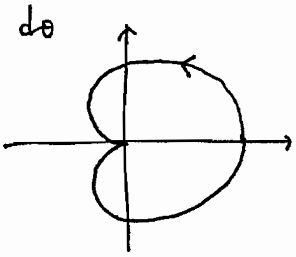
$x^2 + y^2 = x^3y^3$

Cardioid

eg. $r = 1 + \cos\theta$ ($\sim r = \sin^2 2\theta$)

$$L = \int_0^\alpha \sqrt{(1+\cos\theta)^2 + \sin^2\theta}\; d\theta = \sqrt{2}\int_0^\alpha \sqrt{1+\cos\theta}\; d\theta$$

$$= 4a \sin\frac{\theta}{2}\Big|_0^\alpha.$$



$(x^2+y^2-x)^2 = x^2+y^2$

Arc length as parameter :

$$s = s(t) := c + \int_{t_0}^{t} \sqrt{\dot{x}(u)^2 + \dot{y}(u)^2}\cdot du.$$

Calculus 11/16
4.1-4.2 continued.

speed formula : $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$

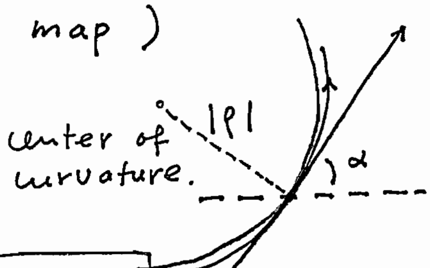
differential notation $ds = \sqrt{dx^2+dy^2}$!

- curvature : $K := \frac{d\alpha}{ds}$ (via Gauss map)

$$\dot\alpha = \frac{\ddot y\dot x - \dot y\ddot x/\dot x^2}{1+(\dot y/\dot x)^2} = \frac{\dot x\ddot y - \dot y\ddot x}{\dot x^2+\dot y^2}$$

$$\Rightarrow K = \frac{d\alpha/dt}{ds/dt} = \frac{\dot x\ddot y - \dot y\ddot x}{(\dot x^2+\dot y^2)^{3/2}}$$

$\boxed{\text{Q : sign of } K = ?}$



$\alpha = \tan^{-1}\frac{dy}{dx}$
$= \tan^{-1}\dot y/\dot x$
(CVF)
for some branch.

eg. $x^2+y^2=R^2 \Rightarrow K = 1/R$

so we called $\rho := 1/K$, radius of curv. $=: |\rho|$.

Signed -
Area within closed curves.

Def'n : simple closed (oriented) curve. piece-wise C^1.

$C : \mathbf{x} : [a,b] \to \mathbb{R}^2$ $\quad \mathbf{x}(t_1) \neq \mathbf{x}(t_2)$ for $t_1 \neq t_2$
$\quad\ \overset{\shortparallel}{(x(t), y(t))}$ $\quad$ except $\mathbf{x}(a) = \mathbf{x}(b)$

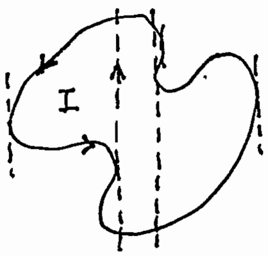
Def'n : Line integral along C : $\int_a^b F(x(t), y(t))\,dt$

or $\int_C p\,dx + q\,dy$

is indep of parameters of same orientation. (by CVF). get (-1) if changing orientation.

Theorem : Area inside $C = -\int_C y\,dx = \int_C x\,dy = \frac{1}{2}\int_C x\,dy - y\,dx$.

Pf:



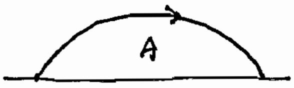
only need to prove for simple region, which is clear for $-\int_C y\,dx$, eg. in I ※.

Ex1. Ellipse : $A = \frac{1}{2}\int_0^{2\pi}(x\dot y - y\dot x)\,dt = \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab$.

$x = a\cos t$, $y = b\sin t$.

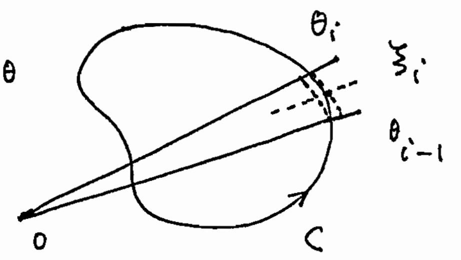
Ex2. Cycloid : $A = \int_0^{2\pi} y\,dx = a^2\int_0^{2\pi}(1-\cos t)^2\,dt = \cdots = 3a^2\pi$.

$x = a(t-\sin t)$, $y = a(1-\cos t)$.



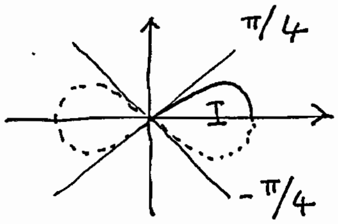
Ex 3. Area in polar coord. $r = r(\theta)$

R.S. $\frac{1}{2}\sum r^2(\xi_i)\,\Delta\theta_i \to \frac{1}{2}\int_C r^2\,d\theta$



eg. Lemniscate : $r^2 = \cos 2\theta$

Area in I = $\int_{-\pi/4}^{\pi/4} \cos 2\theta\, d\theta = 1$.



But $s = \int\sqrt{r^2+\dot{r}^2}\,d\theta = \int\left(\cos^2 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}\right)^{1/2} d\theta$

$= \int\frac{d\theta}{\sqrt{\cos 2\theta}} = \int\frac{d\theta}{\sqrt{1-2\sin^2\theta}}$ elliptic integral (1st kind)

More Applications : (Trivial)

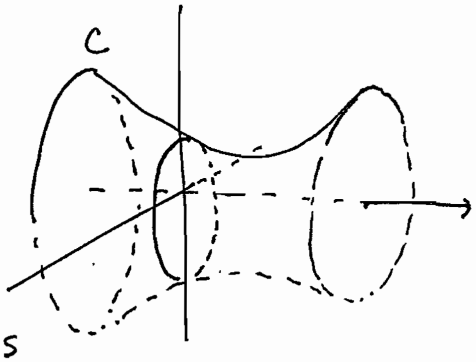
1. Moment x-component $T = M\int_{s_0}^{s_1} x\,ds$, center of mass $(\bar{x},\bar{y}) = \left(\frac{\int x\,ds}{s_1-s_0}, \frac{\int y\,ds}{s_1-s_0}\right)$

2. Surface of revolution

area $= 2\pi\int y\,ds = 2\pi\bar{y}.(s_1-s)$

why?

volume $= \pi\int y^2\,dx$



4.3 Diff & Integral of Vector functions

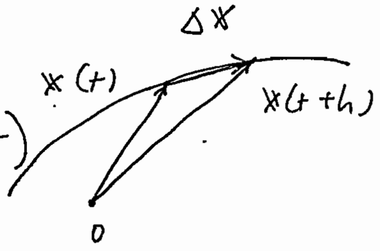
$\mathbb{X} : I \to \mathbb{R}^n$

$$\frac{d}{dt}\mathbb{X}(t) = \lim_{h\to 0}\frac{1}{h}\left(\mathbb{X}(t+h) - \mathbb{X}(t)\right)$$

$$= \left(\lim_{h\to 0}\frac{x(t+h)-x(t)}{h}, \lim_{h\to 0}\frac{y(t+h)-y(t)}{h}\right)$$

$$= (x'(t), y'(t)).$$

geom. meaning :



Similarly, $\int_a^b \mathbb{X}(t)\,dt = \left(\int x(t)dt, \int y(t)dt\right)$

Theorem :

(1) $(\mathbb{X}_1\cdot\mathbb{X}_2)' = \mathbb{X}_1'\cdot\mathbb{X}_2 + \mathbb{X}_1\cdot\mathbb{X}_2'$

(2) $(\mathbb{X}_1\times\mathbb{X}_2)' = \mathbb{X}_1'\times\mathbb{X}_2 + \mathbb{X}_1\times\mathbb{X}_2'$ (proof ?)

(3) $\frac{d}{dt}\int_a^t \mathbb{X}(u)\,du = \mathbb{X}(t)$.

notice in textbook
$\mathbb{X}_1\times\mathbb{X}_2$ is tempo.
only z-component.

Example 1. $L = \int_a^b |x'| \, dt$

signed area on plane $A = \frac{1}{2}\int_a^b (x \times x')\cdot e_3 \, dt$

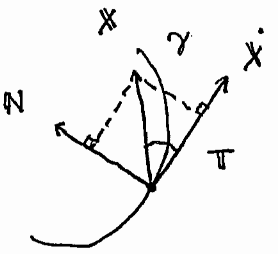
$K = \frac{x' \times x''}{|x'|^3}$. use temp. notation .

Example 2. Tangent / Normal comp of acceleration on $\mathbb{R}^2$:

$v^2 = \dot{x}\cdot\dot{x} \Rightarrow 2v\dot{v} = \ddot{x}\cdot\dot{x} + \dot{x}\cdot\ddot{x}$

$= 2\dot{x}\cdot\ddot{x}$

$= 2v|\ddot{x}|\cos\gamma$



So, $\dot{x} = V = vT$

$\ddot{x} = a$; $a_T = \dot{v}$

Also, $a_N = |\ddot{x}|\cdot\sin\gamma = K v^2$. So "$a = \dot{v}$" is not quite true .

Example 3. Frenet frame $\{T, N\}$

$$\begin{cases} T' = \frac{d}{dt}(\cos\alpha, \sin\alpha) = (-\sin\alpha, \cos\alpha)\frac{d\alpha}{dt} = KvN \\ N' = -KvT \end{cases}$$

. If $t = s$ arc length $\begin{pmatrix}T\\N\end{pmatrix}' = \begin{pmatrix}0&K\\-K&0\end{pmatrix}\begin{pmatrix}T\\N\end{pmatrix}$.

4.4 – 4.6 Some motions in Physics

Newton's Law : $F = ma$ works in any direction ξ can vary in t !!

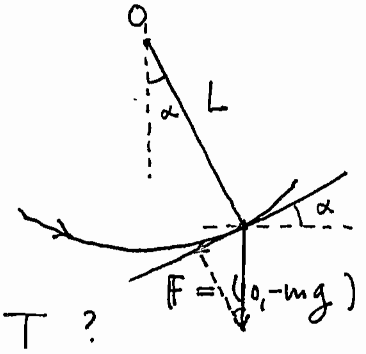
i.e. $F\cdot\xi = m a\cdot\xi$.

Ex 1. Simple pendulum. Take $\xi = T = \frac{dx}{ds}$

$\Rightarrow -mg\frac{dy}{ds} = m\frac{d^2s}{dt^2} = mL\frac{d^2\alpha}{dt^2}$

$y = -L\cos\alpha = -L\cos(s/L)$

$\Rightarrow \frac{d^2\alpha}{dt^2} = -\frac{g}{L}\sin\alpha$. Q1. Solve period T ?



Ex 2. Free fall of a body resisted by air . Take $\xi = e_2 = (0,1)$.

eg. $m\ddot{s} = -mg - r\dot{s}$ (since $\dot{s} < 0$)

$\Rightarrow \frac{dt}{dv} = -\frac{1}{g(1+k^2v)}$ ($k := \sqrt{\frac{r}{mg}}$) $\Rightarrow t = t_0 - \frac{1}{gk^2}\log(1+k^2v)$

i.e. $v = -\frac{1}{k^2}\left(1 - e^{-gk^2(t-t_0)}\right)$

Ex 3. Spring : $m\ddot{x} = -kx$. (Hooke's law) .

Sol. spanned by sin, cos. Q2. WHY ? Both Q's later. ※

Calculus 11/18

Q1: $\left((\dot\alpha)^2\right)' = 2\dot\alpha\ddot\alpha = -2\frac{g}{L}\sin\alpha\cdot\dot\alpha = \frac{2g}{L}(\cos\alpha)'$

$\Rightarrow \dot\alpha^2 = \frac{2g}{L}\cos\alpha + c$ At $t=0$, $\alpha=\alpha_0$, $\dot\alpha(0)=0$

$\Rightarrow \frac{d\alpha}{dt} = \sqrt{\frac{2g}{L}(\cos\alpha-\cos\alpha_0)}$

Period $T = 2\cdot\sqrt{\frac{L}{2g}}\int_{-\alpha_0}^{\alpha_0}\frac{d\alpha}{\sqrt{\cos\alpha-\cos\alpha_0}} = \sqrt{\frac{L}{g}}\int_{-\alpha_0}^{\alpha_0}\frac{d\alpha}{\sqrt{\sin^2\frac{\alpha_0}{2}-\sin^2\frac{\alpha}{2}}}$

let $u = \frac{\sin\alpha/2}{\sin\alpha_0/2}$ get $2\sqrt{\frac{L}{g}}\int_{-1}^{1}\frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}$, $k=\sin\frac{\alpha_0}{2}$.

Q2: (§3.16 Diff eq'n & Δ-fcn's)

All solutions of $u''(t)+u(t)=0$ are $a\cos t + b\sin t$.

- It is a vector space, contains $\sin t$, $\cos t$

claim: $\dim = 2$.

Idea: Energy and ini. cond.

$u''u' + uu' = 0$

$\Rightarrow \frac{1}{2}\left[(u')^2+u^2\right]' = 0$

i.e. $u'^2+u^2 = \text{const } c \geq 0$ ($c=0 \Leftrightarrow u(t)\equiv 0$)

2 sol. with same $u(0)$, $u'(0)$ are identical.

for $u(0)=a$, $u'(0)=b$, may sellect $a\cos t + b\sin t$.

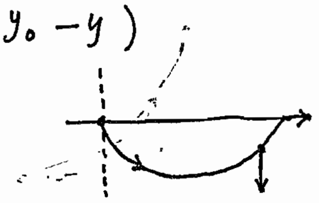
Hence, $m\ddot x = -kx$ has sol $x(t) = c_1\cos\omega t + c_2\sin\omega t$.
where $\omega = \sqrt{k/m}$.

4.7 Motion on a given curve

All these can be formalized: Energy $\longrightarrow$ 1st integral.

$$s'' = -g\frac{dy}{ds} \Rightarrow \frac{1}{2}(s')^2 = -gy + c = g(y_0-y)$$

$$\Rightarrow \frac{dt}{ds} = \pm\frac{1}{\sqrt{2g(y_0-y)}} \text{ i.e. } t = c_1 \pm \int\frac{ds}{\sqrt{2g(y_0-y)}}$$



$$= c_1 \pm \int\sqrt{\frac{\dot x^2+\dot y^2}{2g(y_0-y)}}\,d\theta$$

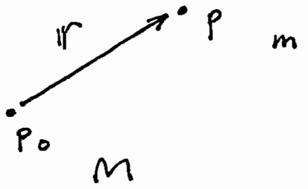
eg. Cycloid Pendulum

$$\begin{cases} x = a(\theta+\pi+\sin\theta) \\ y = -a(1+\cos\theta) \end{cases} \Rightarrow \frac{T}{2} = \frac{1}{\sqrt{2g}}\int_{-\theta_0}^{\theta_0}\frac{2\sqrt{a}\cdot\cos\theta/2}{\sqrt{\cos\theta-\cos\theta_0}}\,d\theta = 2\sqrt{\frac{a}{g}}\int_{-1}^{1}\frac{du}{\sqrt{1-u^2}} = 2\pi\sqrt{\frac{a}{g}}.$$

independent of θ_0!

14.8 Gravitational Field (Omit most discussions)

Newton : $\mathbb{F} = -\frac{GMm}{r^2}\hat{\mathbb{r}} = -\frac{GMm}{r^3}\vec{r}$



$$= ma = m\vec{r}''$$

Kepler's law : $(\vec{r} \times \vec{r}\,')' = \vec{r}\,' \times \vec{r}\,' + \vec{r} \times \vec{r}\,'' = 0$

i.e. $\vec{N} = \vec{r}(t) \times \vec{r}\,'(t)$ is constant in t, say $= \vec{N}(0)$

$\Rightarrow$ $\vec{r}(t) \perp \vec{N}(t) = \vec{N}(0)$, ie a plane motion

Also, $\vec{r} = r(\cos\theta, \sin\theta)$, $\vec{r}\,' = r'(\cos\theta, \sin\theta) + r\theta'(-\sin\theta, \cos\theta)$

$|\vec{r} \times \vec{r}\,'| = r^2\theta' |e_1 \times e_2| = r^2\theta'$ is a const.

This is the speed of area: $A(t) = \frac{1}{2}\int r^2 \frac{d\theta}{dt}\, dt$.

Q: Rule for conic? So for only use $\vec{F} \parallel \vec{r}$.

14.9 Work and Energy

law of motion in the π direction ($\xi = \pi$)

$\vec{F} = m\vec{a} \Rightarrow f = ms''$

$m s'' \cdot s' = f(s) \cdot s'$ ($s' = v$)

$\frac{1}{2} m v^2 \Big|_{t_0}^{t_1} = \int_{t_0}^{t_1} f(s) \frac{ds}{dt}\, dt = W$ (work done)

where $W = \int \vec{F} \cdot \vec{r}\,' dt = \int \underbrace{|F| \cos\theta}_{f} \cdot \underbrace{|\vec{r}\,'| dt}_{ds}$.

conservation field: If $W = V(x_0, y_0) - V(x_1, y_1)$

V: potential function

$\Rightarrow$ $\frac{1}{2} m v^2 + V = $ constant.

Q: When does V exist? (what kind of $\vec{F}$?) *

Calculus 11/23 Ch. 5.6.7 → One chapter

5.1 - 5.5 Taylor's Expansion

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^{n-1} + r_{n-1}(x)$$

with $r_{n-1}(x) = \frac{x^n}{1-x} \xrightarrow[n \to \infty]{} 0$ if $|x| < 1$.

$$-\log(1-x) = \int_0^x \frac{dt}{1-t} = x + \frac{x^2}{2} + \cdots + \frac{x^n}{n} + \int_0^x r_{n-1}(t)\, dt$$

with remainder $:= R_n(x) = \int_0^x \frac{t^n}{1-t} dt$

So far this is valid for all $x < 1$.

- $-1 \leq x \leq 0 \Rightarrow |R_n(x)| \leq \left| \int_0^x t^n dt \right| = \frac{|x|^{n+1}}{n+1} \xrightarrow[n \to \infty]{} 0$ even at $x = -1$.
- $0 \leq x < 1 \Rightarrow |R_n(x)| \leq \frac{1}{1-x} \int_0^x t^n dt = \frac{1}{1-x} \cdot \frac{x^{n+1}}{n+1}$.

i.e. $\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) \quad \forall x \in [-1, 1)$.

eg. $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

replace x by $-x$: $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots (-1)^{n-1} \frac{x^n}{n} + \cdots$

$\Rightarrow \frac{1}{2} \log \frac{1+x}{1-x} \equiv \tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots \quad \forall |x| < 1$.

Now $\frac{1+x}{1-x}$ can take value of any $\mathbb{R}^+$.

Similarly, $\frac{1}{1+t^2} = 1 - t^2 + t^4 + \cdots + (-1)^{n-1} t^{2n-2} + r_{2n-1}(t)$

$r_{2n-1}(t) = (-1)^n t^{2n}/1+t^2$

$\Rightarrow \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + R_{2n}(x)$

for $|x| \leq 1$, $|R_{2n}(x)| \leq \int_0^{|x|} t^{2n} dt = \frac{|x|^{2n+1}}{2n+1} \longrightarrow 0$ as $n \to \infty$.

eg. $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \cdots$ (Rmk: John Machin 1706: $\frac{\pi}{4} = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}$ gets 100 decimals.)

cf. Wiki

Taylor Polynomial:

$f(x)$ poly of degree $n \Rightarrow$

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Q: How $T_n(h) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} h^k$

approximats $f(a+h)$? Let $x=a+h$, $f_n(x) = T_n(x-a)$.

Theorem If $f^{(n+1)}$ exists, then

$$f(x) = f_n(x) + R_n(x) \quad \text{with} \quad R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

called Lagrange's form, for some $\xi \in (a, x)$.

pf: MVT ⇒ $\frac{R_n(x)}{(x-a)^{n+1}} = \frac{R_n'(\xi_1)}{(n+1)(\xi_1-a)^n} = \cdots = \frac{R_n^{(n+1)}(\xi_{n+1})}{(n+1)!}$

since $R_n^{(k)}(a) = 0$ for $k=0,1,\cdots,n$. □

Alt. pf: Int. by parts.

$$f(b)-f(a) = \int_a^b f'(x)\,d(x-b) = f'(x)(x-b)\Big|_a^b - \int_a^b (x-b)\, f''(x)\,dx$$

$$= f'(a)(b-a) - \int_a^b f''(x)\, d\,\frac{(x-b)^2}{2!}$$

$$= f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \int_a^b \frac{(x-b)^2}{2!} f'''(x)\,dx$$

inductively, get $f(b) = f_n(b) + R_n$

$R_n = \int_a^b \frac{(b-x)^n}{n!} f^{(n+1)}(x)\,dx$ # Cauchy's form: MVT ⇒ $\frac{(b-a)^{n+1}}{n!}(1-\theta)^n f^{(n+1)}(a+\theta(b-a))$.

HW. weighted MVT ⇒ Lag. form.

Elementary functions:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n$$

$$R_n = \frac{e^{\xi}}{(n+1)!}\, x^{n+1} \xrightarrow[n\to\infty]{} 0 \quad \forall x \in \mathbb{R}$$

Similarly, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ valid $\forall x \in \mathbb{R}$.

Newton's binomial series: $\alpha \in \mathbb{R}$:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots C_n^\alpha x^n + R_n$$

If $|x|<1$, then $\lim_{n\to\infty} R_n = 0$.

For the pf: Lag's form doesn't work! (later) #

Calculus 11/25

Proof of binomial theorem : $a = 0$, $b = x$

Cauchy's form $R_n = \frac{x^{n+1}}{n!}(1-\theta)^n \alpha(\alpha-1)\cdots(\alpha-n)(1+\theta x)^{\alpha-n-1}$

key : $0 \leq \frac{1-\theta}{1+\theta x} \leq 1$ (since $0 \leq \theta \leq 1$, direct check)

$\Rightarrow |R_n| \leq (1+\theta x)^{\alpha-1} |x\alpha| \cdot |x(\alpha-1)| \cdot |x(\frac{\alpha}{2}-1)| \cdots |x(\frac{\alpha}{n}-1)|$

---

pick $\varepsilon > 0$ st $|x| < 1-\varepsilon$, then $\exists N$ st $|x(\frac{\alpha}{n}-1)| < 1-\varepsilon$ $\forall n \geq N$.

$$\Rightarrow |R_n| \leq \underbrace{(1+\theta x)^{\alpha-1}(1\text{st } n \text{ terms})}_{\text{fixed number}}(1-\varepsilon)^{n-N} \xrightarrow[\text{as } n\to\infty]{} 0 . \quad \square$$

This can be proved [illegible]

The pf can be generalized to prove

Theorem (Appendix. I.4)

If $f \in C^\infty(a,b)$ and $f^{(k)}(x) \geq 0$ $\forall k$ large , then $\lim_{n\to\infty} R_n = 0$.

5.6 Applications

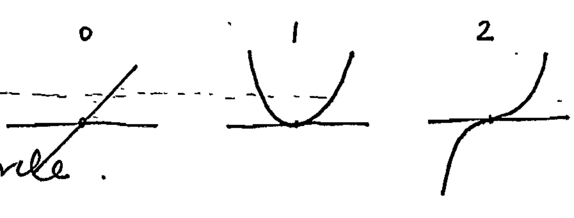
1) contact order : If $f^{(k)}(a) = g^{(k)}(a)$, $k = 0, 1, \cdots, n$ but not $n+1$,

$$\Rightarrow f(a+h) - g(a+h) = \frac{h^{n+1}}{(n+1)!}\Big(\underbrace{f^{(n+1)}(a+\theta_1 h) - g^{(n+1)}(a+\theta_2 h)}_{F(h)}\Big) \quad f \in C^{n+1}.$$

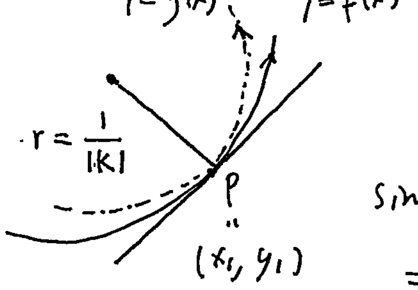
$\lim_{h\to 0} F(h) = F(0) \neq 0$, so $F(h) \neq 0$ for h small.

if n is even (odd) , then $f-g$ changes sign (does not) for $h \in (-\delta, \delta)$

Ex 1. $y = f(x) = x^n$ with $y = x$ (tangent line) at $x = 0$ has contact order $= n-1$.



Ex 2. Circle of curv. as contact circle.



the circle has curv $K(p)$ $\Rightarrow$

$$\frac{f''(x_1)}{(1+f'(x_1)^2)^{3/2}} = \frac{g''(x_1)}{(1+g'(x_1)^2)^{3/2}} .$$

since $f(x_1) = g(x_1)$, $f'(x_1) = g'(x_1)$ (slope) ,

$\Rightarrow f''(x_1) = g''(x_1) \Rightarrow$ order ≥ 2 .

2) Relative maxima / minima

Let $f(x) = f(a) + 0 + \cdots + 0 + \frac{f^{(n)}(a+\theta(x-a))}{n!}(x-a)^n \quad (n \geqslant 2)$

with $f \in C^n$ and $f^{(n)}(a) \neq 0$

for n even, get max ($f^{(n)}(a) < 0$) or min ($f^{(n)}(a) > 0$).

for n odd, inflection point (why?).

3) L'Hospital rule

$\frac{0}{0}$ form at $x=a$, a diff view: $\frac{f(x)}{g(x)} = \frac{\frac{f^{(n)}(\xi_1)}{n!}(x-a)^n}{\frac{g^{(n)}(\xi_2)}{n!}(x-a)^n}$

with n the 1st index st $f^{(n)}(a), g^{(n)}(a)$ not both 0.

4) Taylor series for e^{-x^2} ? $x^3 \sin x$? $\tan^{-1} x$?

Need uniqueness thm for power series (easy) and differentiability (hard).

Remark: It may happen $\lim_{n\to\infty} f_n(x)$ conv. but $R_n \not\to 0$.

eg. $f(x) = e^{-1/x^2}$ or $e^{-1/x^2}\sin\frac{1}{x}$, All $f^{(n)}(0) = 0$ so $f \neq$ Taylor.

Appendix. Other ways to approx by poly. Interpolation.

Lagrange / Chinese Remainder Theorem

Want $\phi(x_i) = f_i$, $i = 0, 1, \cdots, n$; $\deg \phi = n$

Sol. $\phi(x) = \sum_{j=0}^{n} \frac{\prod_{i\neq j}(x-x_i)}{\prod_{i\neq j}(x_j-x_i)} f_i$.

Newton's interpolation:

$\phi(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)\cdots(x-x_{n-1})$

then $\phi(x_i) = f_i$ $i = 0, \cdots, n$ determines $a_0, a_1, \cdots$ recursively.

If $\Delta x_i := x_{i+1} - x_i \equiv h$ is fixed, then using

$\Delta f_i := f_{i+1} - f_i$, $\Delta^2 f_i = \Delta f_{i+1} - \Delta f_i = f_{i+2} - 2f_{i+1} + f_i$ etc.

with $\Delta^n f_k = f_{k+n} - C^n_1 f_{k+n-1} + C^n_2 f_{k+n-2} - \cdots + (-1)^n f_k$,

We solve $a_k = \frac{1}{k!} \cdot \frac{\Delta^k f_0}{h^k}$ (pf by induction).

Notice, $h \to 0$ the Newton poly $\longrightarrow$ Taylor polynomial. *
if $f \in C^{n+1}$.

ch.6 Numerical Methods.

6.1 Integrals

$$J = \int_a^b f(x)\,dx = \sum_{\nu=1}^n J_\nu \quad , \quad J_\nu = \int_{x_{\nu-1}}^{x_\nu} f(x)\,dx$$

① Rectangle approx. Let $|f'| \le M_1$. $x_\nu = a + \nu h$, $h = \frac{b-a}{n}$

$$| J_\nu - h \underset{= f(x_{\nu-1})}{f_\nu} | = \left| \int_{x_{\nu-1}}^{x_\nu} \left(f(x) - f(x_\nu)\right) dx \right| \le \frac{M_1}{2} h^2$$

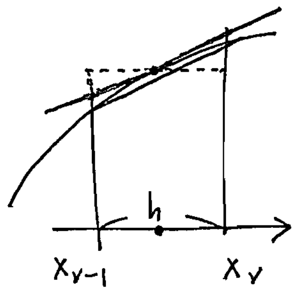
$$\Rightarrow \left| J - h \sum_{\nu=1}^{n} f_\nu \right| \le \frac{M_1}{2}(b-a)\, h .$$

② Trapezoid approx.

- secant : $\frac{1}{2} h \sum_{\nu=1}^{n} (f_\nu + f_{\nu-1})$

$$= h(f_1 + \cdots + f_{n-1}) + \frac{h}{2}(f_0 + f_n)$$

- tangent : (area = mid pt. rectangle)



$$f(x) = \underbrace{f(a_\nu) + f'(a_\nu)(x-a_\nu)}_{\phi_\nu(x)} + \frac{1}{2} f''(\xi)(x-a_\nu)^2$$

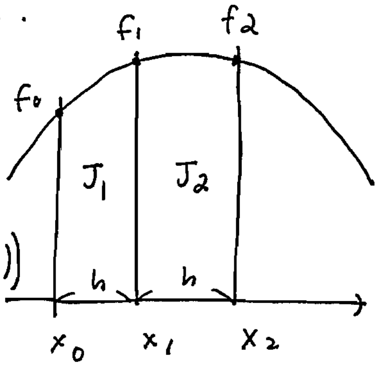
$$a_\nu = \frac{1}{2}(x_{\nu-1} + x_\nu)$$

$$| J_\nu - h f(a_\nu) | \le \frac{M_2}{2} \int_{x_{\nu-1}}^{x_\nu} (x-a_\nu)^2 dx = \frac{M_2}{24} h^3, \quad |f''| \le M_2$$

$$\Rightarrow \left| J - h \sum_{\nu=1}^{n} f(a_\nu) \right| \le \frac{M_2}{24}(b-a)\, h^2 .$$

③ Simpson's method : quadratic approx.

Newton interpolation ⇒



$$y = f_0 + a_1(x - x_0) + a_2\overset{h}{(x-x_0)}\overset{2h}{(x-x_1)}$$

$$a_1 = \frac{f_1 - f_0}{h} \qquad a_2 = \frac{1}{2h^2}\left(f_2 - f_0 - 2(f_1 - f_0)\right)$$

$$J_1 + J_2 \sim \int_{x_0}^{x_2} y\, dx = \frac{h}{3}\left(f_0 + 4f_1 + f_2\right)$$

partition $n = 2m$: $J \sim \frac{h}{3}\left(f_0 + \underline{4f_1} + \underline{2f_2} + \underline{4f_3} + \underline{2f_4} + \cdots + \underline{4f_{2m-1}} + f_{2m}\right)$

Error estimate : If $|f'''| \le M_3$, get $\frac{M_3}{3}(b-a)h^3$ as in ②, (?)

In fact, much better result holds :

Theorem : If $|f^{(4)}| \le M_4$, $n = 2m$ $\Rightarrow$ $|J - S_{2m}| \le \frac{M_4}{180}(b-a)\, h^4$.

pf: cubic interpolation $\phi(x) = g(x) + a_3(x-x_0)(x-x_1)(x-x_2)$

st. $\phi'(x_1) = f'(x_1)$.

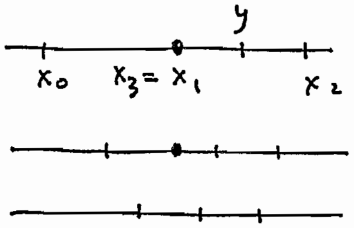
the cubic part $a_3 \int_{-h}^{h} u(u^2-h^2)\,du = 0$

- remainder estimate $R(x) := f(x) - \phi(x) = \frac{\prod_{i=0}^{n}(x-x_i)}{(n+1)!} f^{(n+1)}(\xi)$

in the current case, $n=3$ with $x_3 = x_1$.

pf: let $K(x) = R(x) - c\prod_{i=0}^{3}(x-x_i)$

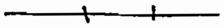
fix $y \neq x_i$, then $\exists\, c$ st $K(y) = 0$.



$K = 0$

$K' = 0$

$K'' = 0$



$K^{(3)} = 0$

$$\Rightarrow K^{(4)}(\xi) = 0$$
$$\shortparallel$$
$$R^{(4)}(\xi) - 4!\cdot c$$
$$\shortparallel$$
$$f^{(4)}(\xi) \quad ※$$

Simpson's error: $m M_4 \frac{1}{4!}\int_{-h}^{h} u^2(u^2-h^2)\,du = \frac{m M_4}{90} h^5 = \frac{M_4(b-a)}{180} h^4$.

6.2 Examples: (other than integrals)

(1) $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} \Rightarrow \tan^{-1}u + \tan^{-1}v = \tan^{-1}\frac{u+v}{1-uv}$

to get $\frac{\pi}{4} = \tan^{-1}1$, need $\frac{u+v}{1-uv} = 1$, ie. $(u+1)(v+1) = 2$

$\Rightarrow \frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$ Also $\frac{1}{2} = \frac{\frac{1}{3}+\frac{1}{7}}{1-\frac{1}{21}}$; $\frac{1}{3} = \frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{40}}$

$= 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = 2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{8}$.

(This is as good as $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$)

eg. to get π up to 10^{-6}; $\frac{(\frac{1}{5})^{2n+1}}{2n+1}\cdot 20 < 10^{-6}$

ie. $5^{2n+2}(2n+1) > 10^8 \Rightarrow n \geq 5$ in fact $n=4$ OK.

(2) linear approx in sciences:

Period of pendulum $T(\ell) \doteq 2\pi\sqrt{\frac{\ell}{g}}$ (not quite true)

$\log(\cdot)' \Rightarrow \frac{\Delta T}{T} = \frac{d\ell}{2\ell} \doteq \frac{\alpha}{2}\Delta t \Rightarrow$ time loss per day

if $\ell = \ell_0(1+\alpha(t-t_0))$ $= 86400\cdot\frac{\Delta T}{T}$

temperature

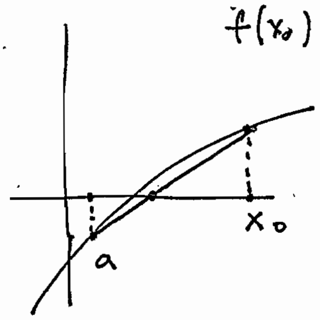
$\sim 43200\,\alpha\,\Delta t$. ※

Calculus 12/2

6.3 Numerical sol. of eq'ns.

Newton's iteration

f(x₀) > 0



$f(a) < 0$

works even without this.

- Secant (rule of false position)

$$\frac{y-f(a)}{x-a} = \frac{f(x_0)-f(a)}{x_0-a}$$

$$y=0 \quad \Rightarrow \quad x = a - \frac{f(\cancel{x}a)}{\left(\frac{f(x_0)-f(a)}{x_0-a}\right)}$$

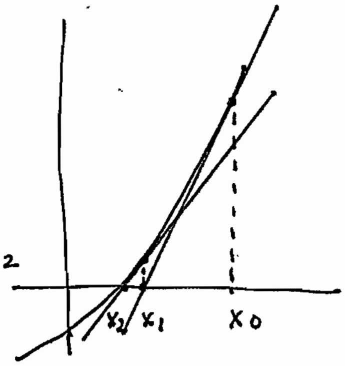
error = ?

- tangent : $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$
  $(a \to x_0)$

If $f(\xi)=0$ and $f'(\xi) \neq 0$, $f \in C^2$

$$x_{n+1} - \xi = x_n - \xi - \frac{f(x_n)}{f'(x_n)}$$

$$0 = f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(\eta)}{2!}(\xi - x_n)^2$$



$$\Rightarrow \quad x_{n+1} - \xi = -\frac{f''(\eta)}{2f'(x_n)}(x_n - \xi)^2$$

error estimate : pick δ small s.t. $\overset{\mu}{\overset{\shortparallel}{\left(\frac{M_2}{2m_1}\right)}}\delta < 1$ $(|f'| \geq m_1)$

then $|x_{n+1} - \xi| \leq (\mu\delta)^{n+1}|x_0 - \xi| \longrightarrow 0$ (*)

and $|x_{n+1} - \xi| \leq \mu\, |x_n - \xi|^2$ (quadratic conv.)

Q : why is this really better than the bi-section method $\left(\frac{1}{2}\right)^n$?

Ans. the power in (*) is actually $(\mu\delta)^{2^n - 1}$.

Example : $\sqrt{2}$

① Taylor : $(1.4)^2 = 2.56$ $\Rightarrow$ $\sqrt{2} = \sqrt{(1.4)^2 - 0.54} = 1.4(1-x)^{1/2}$

with $x = \frac{0.54}{2.56}$

② Newton : $f(x) = x^2 - 2$, $x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}$

for $x_0 = 1$, $x_1 = \frac{3}{2}$ $= 1.5$, $x_2 = \frac{3}{4} + \frac{2}{3} = \frac{17}{12}$ $\sim \underline{1.41}6$ $x_3 = \frac{17}{24} + \frac{12}{17} = \frac{577}{408}$ $\sim \underline{1.414215}$

$\sqrt{2} = \underline{1.414213}56\ldots$

General iteration

$x_{n+1} = \phi(x_n)$ solving fixed pt $a = \phi(a)$

$\phi \in C^1$ if $\lim_{n\to\infty} x_n$ exists.

Thm : If $a = \phi(a)$ with $|\phi'(a)| < 1$ (contraction map)
then $\exists\, \delta > 0$ st any iteration with $x_0 \in (a-\delta, a+\delta) \longrightarrow a$.

pf: pick $\delta > 0$ st $|\phi'(x)| \leq q < 1$ $\forall x \in (a-\delta, a+\delta)$,

then $|x_{n+1} - a| = |\phi(x_n) - \phi(a)| = |\phi'(\xi)(x_n - a)| \leq q\,|x_n - a|$ ※

Ex 1. Newton : $\phi(x) = x - \frac{f(x)}{f'(x)} \Rightarrow \phi'(x) = \cancel{1} - \cancel{1} + \frac{f(x) f''(x)}{f'(x)^2}$

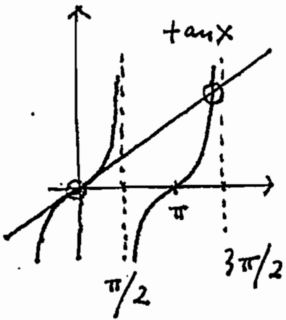
now use $f(a) = 0$, but $f'(a) \neq 0$.

Ex 2. A fixed pt may be repelling ($|\phi'(a)| > 1$)

eg. $\phi(x) = \tan x$, $\phi'(x) = \sec^2 x > 1$ at $x = a = \tan a$.

but then $\phi(x) := \tan^{-1} x$

works with the same fixed pts.



Appendix : Stirling's formula

Thm : $\sqrt{2n\pi}\left(\frac{n}{e}\right)^n < n! < \sqrt{2n\pi}\left(\frac{n}{e}\right)^n \cdot \left(1 + \frac{1}{4n}\right)$

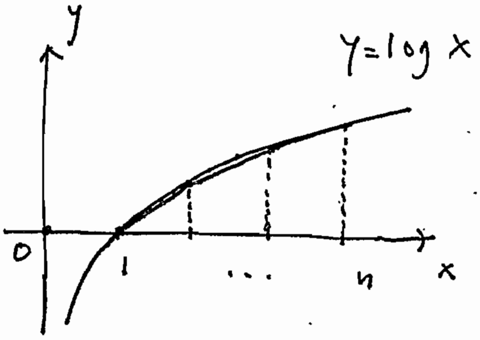
pf:
(part) $A_n = \int_1^n \log x \, dx = n \log n - n + 1$

$T_n = \log 2 + \cdots + \log(n-1) + \frac{1}{2}\log n$

$\nearrow$ Trapezoid $= \log(n!) - \frac{1}{2}\log n$

HW. claim : $a_n := A_n - T_n \nearrow$ & bounded.

$\Rightarrow n! = \alpha_n \sqrt{n}\left(\frac{n}{e}\right)^n$, $\alpha_n = e^{1-a_n} \searrow \alpha$.



$(2n)! = \alpha_{2n}\sqrt{2n}\left(\frac{2n}{e}\right)^{2n}$

$\Rightarrow \sqrt{n} \cdot \frac{(2n)!}{(n!)^2}\frac{1}{2^{2n}} = \frac{\alpha_{2n}}{\alpha_n^2}\sqrt{2} \xrightarrow{n\to\infty} \frac{\alpha\sqrt{2}}{\alpha^2} = \frac{\sqrt{2}}{\alpha}$

$\downarrow n \to \infty$

$1/\sqrt{\pi}$

i.e. $\alpha = \sqrt{2\pi}$ ※

Calculus 12/7

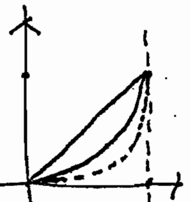
7.1 - 7.2 Absolute / Condi. convergence of series.

Calculus 12/9

7.3-7.4 $\boxed{\text{Unif. conv. of functions}}$ sequence/series

Ex 1. $f_n(x) = x^n$ on $[0,1]$ conti.

$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$$



The limit, even exists, may fail to be conti.

Def'n: $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly if on $[a,b]$ or $I \subset \mathbb{R}$

$\forall \varepsilon > 0$, $\exists N$ st $|f_n(x) - f(x)| < \varepsilon$

for any $n \geq N$, and $\forall x \in I$. (i.e. N is indp. of x)

$\boxed{\text{Thm}}$: $\boxed{\text{Unif. limit of conti. fcn is conti.}}$

pf (3-ε argument): let $f_n \xrightarrow{\text{unif}} f$ on I.

let $x_0 \in I$. given $\varepsilon > 0$, $\exists N$ st. $n \geq N \ni |f_n(x) - f(x)| < \frac{\varepsilon}{3}$

Pick one such n (say $= N$) $\forall x \in I$.

f_N is conti. at $x_0 \ni \exists \delta > 0$, $|f_N(x_0+h) - f_N(x_0)| < \frac{\varepsilon}{3}$

$\forall |h| < \delta$.

$\ni |f(x_0+h) - f(x_0)|$

$$\leq |f(x_0+h) - f_N(x_0+h)| + |f_N(x_0+h) - f_N(x_0)| + |f_N(x_0) - f(x_0)|$$

$< \varepsilon$. □

Now we apply to series:

given $g_1(x), g_2(x), \cdots g_n(x), \cdots$ $n = 1, 2, \cdots$

let $f_n(x) = \Sigma_{i=1}^{n} g_i(x)$ partial sum

(*) $\boxed{\text{Simple test}}$: If $|g_i| \leq a_i$ with $\Sigma_{i=1}^{\infty} a_i$ conv.

then $\Sigma_{i=1}^{\infty} g_i(x)$ conv. unif. & absolutely.

Reason: (Need to use $\boxed{\text{Cauchy's test}}$, why?)

$$|\Sigma_{i=m}^{n} g_i(x)| \leq \Sigma_{i=m}^{n} |g_i(x)| \leq \Sigma_{i=m}^{n} a_i$$

So $\Sigma_{i=1}^{n} g_i(x) \xrightarrow[n\to\infty]{} f(x)$ conv.

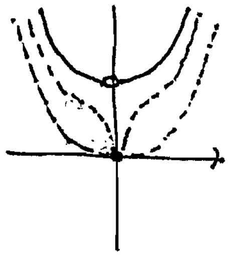
Why unif? $|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \varepsilon$ ※

Corollary: If $g_i(x)$ are conti. then $f(x)$ is also conti.

Ex 2. $g_k(x) = \frac{x^2}{(1+x^2)^k}$, $f(x) = g_0(x) + g_1(x) + g_2(x) + \cdots$

$f(0) = 0$, but for $x \neq 0$, $f(x) = \frac{x^2}{1 - 1/(1+x^2)} = 1 + x^2$

not conti at 0. This does not fit condi (★).



★: $\boxed{\text{Thm}}$ f_n conti & $\lim_{n\to\infty} f_n = f$ unif.

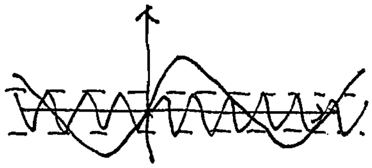
$$\Rightarrow \int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx .$$

pf: $\left| \int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx \right| \leq \int_a^b \left| f(x) - f_n(x) \right| dx \leq (b-a)\cdot\varepsilon$. □

Ex 3. Differentiation is harder : $f_n(x) := \frac{\sin(n^2 x)}{n} \xrightarrow[n\to\infty]{} 0$

but $f_n'(x) = n\cos(nx)$ has no limit.

We consider only a very simple case:



★' $\boxed{\text{Thm}}$ If $f_n \in C^1$, $f_n \longrightarrow f$ point-wise

$f_n' \longrightarrow h$ unif.

then $f' = h$. (hence $f \in C^1$ too).

pf: By ★, $\lim_{n\to\infty} \int_a^x f_n'(u)\,du = \int_a^x h(u)\,du$ (h is conti.)

$$\overset{\shortparallel}{\lim_{n\to\infty}} \left(f_n(x) - f_n(a) \right) = f(x) - f(a) \Rightarrow f' = h . \ \square$$

- when apply to series $f_n(x) = g_1(x) + \cdots + g_n(x)$,

  we get int. / diff. term by term.

Calculus 12/14

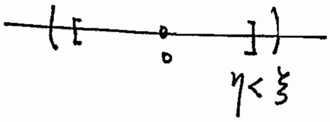
7.5 − 7.7 power series

$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$, WLOG set $a = 0$

Theorem If f conv. at $x = \xi$ then it conv. absolutely for $|x| < |\xi|$, and uniformly in $|x| \le \eta$ for any $\eta < |\xi|$.

Pf : $c_k \xi^k \to 0 \Rightarrow |c_k \xi^k| \le M \ \forall k$

for any $\zeta \in (0,1)$, on $|x| \le \zeta|\xi|$ $(= \eta)$



$|c_k x^k| \le |c_k \xi^k| \cdot \zeta^k \le M \zeta^k$, a conv. geom. series

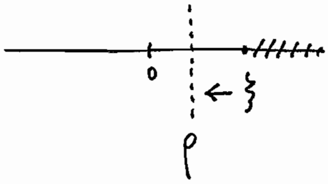
By simple test (last time) $\Rightarrow \sum_{k=0}^{\infty} c_k x^k$ conv. abs. & unif. □

Radius of convergence

If f conv $\forall x$, set $\rho = \infty$

otherwise, if f does not conv at $x = \xi$, then it div. on $|x| > |\xi|$ by above thm.

Now set $\rho =$ g.l.b of all such ξ .

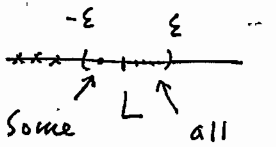


Theorem : $\rho = \dfrac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|c_n|}}$.

Pf : This follows from the root test for $\overline{\lim}$ version. □

Rmk : $L = \overline{\lim\limits_{n\to\infty}}\, a_n \overset{def}{=} \left(\begin{array}{l} \forall \varepsilon > 0 ,\ \exists N ,\ n \ge N \Rightarrow a_n < L + \varepsilon , \\ \forall \varepsilon > 0 , m \in \mathbb{N} ,\ \exists\, n \ge m \text{ st } a_n > L - \varepsilon . \end{array} \right.$

eg. $\overline{\lim}\,(-1)^n = 1$.



Thm. (Integration of power series)

$f = \sum c_k x^k$ is cont. in $|x| < \rho$ and

$$\int f(x)\,dx = \sum c_k \frac{x^{k+1}}{k+1} + c .$$

Notice : the radius of conv. is the same since $\sqrt[n]{\frac{1}{n+1}} \to 1$.

Thm (diff of power series)

$$f = \sum c_k x^k \text{ is } C^\infty \text{ on } |x|<\rho \;\; \& \;\; f'(x) = \sum_{k=1}^{\infty} k \cdot c_k x^{k-1}.$$

Pf: Since all partial sum $f_n = \sum_{k=0}^{n} c_k x^k \in C^1$ (polynomial) and $f_n'(x) = \sum_{K=1}^{n} k c_k x^{k-1}$, only need to prove it conv. unif. This holds for $|x|<\rho$ since $\sqrt[n]{|n|} \overset{n\to\infty}{=} 1$. $\square$

Corollary : (Uniqueness) $a_n = \frac{f^{(n)}(0)}{n!}$, $\forall n = 0, 1, 2 \cdots$

Calculus 12/21

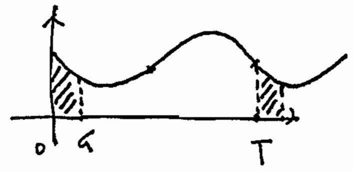
8.1 – 8.4 Trigonometric (Fourier) series

periodic functions : $f(x+T) = f(x)$

periodic extension f on $[a,b]$, use $\frac{1}{2}\left(f(a)+f(b)\right)$

in fact, replace all ξ by $\frac{1}{2}\left(f(\xi+0)+f(\xi-0)\right)$.

Fact : $\int_0^T f(x)\,dx = \int_a^{T+a} f(x)\,dx$



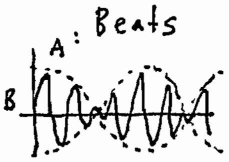
Major examples : Harmonic vibrations

$y = a \sin \omega(x-\xi)$

amplitude, angular frequency, $T = 2\pi/\omega$ period.

Superposition : $y = a \sin \omega_1 x + b \sin \omega_2 x$

eg. $a=b=1$, $y = \underbrace{2\cos\left[\frac{1}{2}(\omega_1-\omega_2)x\right]}_{A} \underbrace{\sin\left[\frac{1}{2}(\omega_1+\omega_2)x\right]}_{B}$



Δ-fcn's in cpx form : Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$

Lemma : $\int e^{inx}\,dx = \frac{1}{in} e^{inx}$ (in fact e^{ax}, $a \in \mathbb{C}$)

$(e^{ax})' = a e^{ax}$ for any $a \in \mathbb{C}$

Corollary : In particular, $\int_{-\pi}^{\pi} e^{inx} e^{-imx}\,dx = \begin{cases} 0 & n \neq m \\ 2\pi & n = m \end{cases}$

(orthogonal relation of Δ-fcn's)

In general we consider " Δ-polynomials "

$$S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}\left(a_k \cos kx + b_k \sin kx\right)$$

$$= \sum_{k=-n}^{n} \alpha_k e^{ikx} \quad \text{then} \quad \begin{aligned} a_k &= \alpha_k + \alpha_{-k}, \\ b_k &= i(\alpha_k - \alpha_{-k}) \end{aligned} \quad k = 0,1,2,\cdots$$

$S_n(x)$ is a real fcn $\Leftrightarrow$ $\alpha_{-k} = \overline{\alpha_k}$.

Lemma : $\sigma_n(x) := \frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx = \dfrac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}}$

conti in x, true even for $x=0$, get $\sigma_n(0) = n + \frac{1}{2}$.

Alt. pf : $\sigma_n(x) = \frac{1}{2}\sum_{k=-n}^{n} e^{ikx} = \frac{1}{2} e^{-inx} \cdot \dfrac{1-e^{(2n+1)ix}}{1-e^{ix}}$ #

Corollary : $\int_0^{\pi} \dfrac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt = \frac{1}{2}\pi$, indep of $n \in \mathbb{Z}$.

Remarkable formula:

for $f(x) = S_n(x)$, $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx$, $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$

in cpx form: $\alpha_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-ikx}\,dx$. $\left(e^{-ikx} \equiv \overline{e^{ikx}}\right)$

Definition: $f(x)$ is **piece-wise** ($\equiv$ **sectionally**) $\boxed{C^2}$ on $[a,b]$ if $[a,b] = \bigcup_{\text{finite}} [a_i, \underset{= a_{i+1}}{b_i}]$ st $f \in C^2(a_i, b_i)$ and f, f', f'' has limits at each end points.

Lemma: (**Weak Riemann-Lebesgue**) if $k(x) \in PC'([a,b])$,

then $K_\lambda := \int_a^b k(x)\sin(\lambda x)\,dx \longrightarrow 0$ as $\lambda \to \infty$.

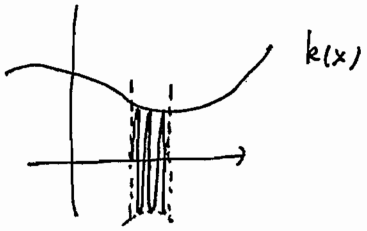
pf: On each C' interval $[a_i, a_{i+1}]$,

$$\int_{a_i}^{a_{i+1}} k(x)\sin\lambda x\,dx = \frac{-1}{\lambda}\int_{a_i}^{a_{i+1}} k(x)\,d(\cos\lambda x)$$

$$= \frac{-1}{\lambda}\left(k(x)\cos\lambda x\Big|_{a_i}^{a_{i+1}} - \int_{a_i}^{a_{i+1}} k'(x)\cos\lambda x\,dx \right)$$

$\longrightarrow 0$ as $\lambda \to \infty$ ※

idea:



Thm: The **Dirichlet integral** $I := \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

pf: Let $I_M = \int_0^M \frac{\sin x}{x}\,dx$, recall the conv. pf: $N > M > 0$

$$\left| I_N - I_M \right| = \left| \frac{-\cos x}{x}\Big]_M^N + \int_M^N \frac{\cos x}{x^2}\,dx \right| \leq \frac{1}{M} + \cancel{\frac{1}{N}} + \cancel{\left(-\frac{1}{N}\right)} + \frac{1}{M} = \frac{2}{M}.$$

idea: Rewrite into **finite** integral.

let $M = \lambda p$; $I_{\lambda p} = \int_0^p \frac{\sin\lambda x}{x}\,dx \xrightarrow[\text{as } \lambda\to\infty]{} I$

in fact, $|I - I_{\lambda p}| < \frac{1}{\lambda p}$, $\boxed{\text{conv. is unif in } p \geq \underline{p} > 0}$

We can't apply weak R-L since $\frac{1}{x}$ is NOT PC' at $x = 0$.

But we can do so for $k(x) := \frac{1}{x} - \frac{1}{2\sin\frac{x}{2}} \in PC'([0,\pi])$

when we set $k(0) = 0$. (**Exercise**)

$\Rightarrow \int_0^p \left(\frac{1}{x} - \frac{1}{2\sin\frac{x}{2}}\right)\sin\lambda x \to 0$

check $k(x)$, $k'(x)$ are bounded in $[0,\pi]$.

i.e. $I = \lim_{\lambda\to\infty} \int_0^p \frac{\sin\lambda x}{2\sin\frac{x}{2}}\,dx$

Now let $p = \pi$ and for $\lambda = n + \frac{1}{2}$ we get $I = \frac{\pi}{2}$ ※

Rmk: Will use these to prove **Fourier expansion** for $f \in PC^2$.

Calculus 12/23

Main Theorem. Let $f \in PC^2([0,2\pi])$

Then the Fourier poly $S_n(x) \xrightarrow[n\to\infty]{} \frac{1}{2}\left(f(x+)+f(x-)\right) =: f(x)$.

pf: The idea is similar to: $\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{\lambda\to\infty}\int_0^\pi \frac{\sin \lambda x}{x}\,dx = \frac{\pi}{2}$.

$$S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}\left(a_k \cos kx + b_k \sin kx\right)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\Big[\frac{1}{2} + \sum_{k=1}^{n}\underset{\cos k(u-x)}{\underset{\parallel}{\left(\cos ku \cos kx + \sin ku \sin kx\right)}}\Big]du$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\,\frac{\sin(n+\frac{1}{2})(u-x)}{2\sin\frac{1}{2}(u-x)}\,du$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt$$

Q: How to apply R-L ?

$$S_n(x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{0}\frac{f(x+t)-f(x-)}{2\sin\frac{t}{2}}\sin(n+\tfrac{1}{2})t\,dt + \frac{1}{\pi}\int_{0}^{\pi}\frac{f(x+t)-f(x+)}{2\sin\frac{t}{2}}\sin(n+\tfrac{1}{2})t\,dt$$

Now $\frac{f(x+t)-f(x+)}{2\sin\frac{t}{2}} = \underset{g(t)}{\underset{\parallel}{\frac{f(x+t)-f(x+)}{t}}}\cdot\frac{t}{2\sin\frac{t}{2}}$ ← this is C^∞ on $[0^-,\pi]$.

↖ need to show $g(0+)$, $g'(0+)$ exists.

$t>0$:

$$g(t) = f'(x+\theta t) \xrightarrow[t\to 0]{} f'(x+).$$

$$g'(t) = \frac{f'(x+t)t-(f(x+t)-f(x+))}{t^2} = \frac{f'(x+t)t - f'(x+\theta t)t}{t^2} = f''(x+\theta_1 t)$$

$0<\theta_1<\theta<1$.

$$\xrightarrow[t\to 0]{} f''(x+).$$

Similarly for $f(x+t)-f(x-)$ side on $[-\pi,0]$.

Weak R-L. $\Rightarrow \lim_{n\to\infty} S_n(x) = f(x)$. □

8.5 Examples

(1) $x^2 = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right)$

$x=0 \Rightarrow \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \Rightarrow \zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

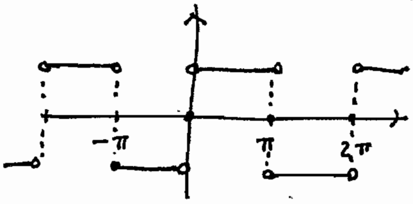
(or use $x=\pi$.)

even fcn $\Rightarrow$ cos.

(2) $\operatorname{sgn}(x) = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots \right)$

odd ⇒ sin. 2 jumping disconti.

to get just 1 pt see Ex (4)



(3) $f(x) = \cos \mu x$, $\mu \notin \mathbb{Z}$, $x \in [-\pi, \pi]$

$$a_k = \frac{2}{\pi} \int_0^{\pi} \cos\mu x \cdot \cos kx \, dx = \frac{1}{\pi} \int_0^{\pi} \left[\cos(\mu+k)x + \cos(\mu-k)x \right] dx$$

$$= \frac{1}{\pi}\left(\frac{\sin(\mu+k)\pi}{\mu+k} + \frac{\sin(\mu-k)\pi}{\mu-k} \right) = \frac{2\mu(-1)^k}{\pi(\mu^2-k^2)} \sin\mu\pi$$

$$\Rightarrow \cos\mu x = \frac{2\mu}{\pi} \sin\mu\pi \left(\frac{1}{2\mu^2} - \frac{\cos x}{\mu^2-1} + \frac{\cos 2x}{\mu^2-2^2} - + \cdots \right)$$

Notice the periodic ext is conti. at $x = \pi$.

$$\Rightarrow \cot \pi\mu = \frac{2\mu}{\pi}\left(\frac{1}{2\mu^2} + \frac{1}{\mu^2-1} + \frac{1}{\mu^2-2^2} + \cdots \right)$$

this is the partial fraction decomposition.

Reset $\mu = x \in [0, q]$, $q < 1$ ⇒ uniform convergence

$$\Rightarrow \int_0^x \left(\pi \cot \pi t - \frac{1}{t} \right) dt = \underset{\text{by term}}{\text{int. term}} = \lim_{n\to\infty} \sum_{k=1}^{n} \log\left(1 - \frac{x^2}{k^2}\right)$$

$$\parallel$$

$$\left(\log \sin \pi t - \log t \right)\Big|_0^x = \log \frac{\sin \pi t}{t} \Big]_0^x = \log \frac{\sin \pi x}{\pi x}.$$

$$\Rightarrow \frac{\sin \pi x}{\pi} = x \cdot \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2} \right).$$

HW. get $\zeta(2n)$.

8.6 Convergence / diff & int.

Bessel's inequality: $\forall n = 0, 1, 2, \cdots$, $\forall f \in PC([-\pi, \pi])$?,

$$2 \sum_{k=-n}^{n} |\alpha_k|^2 = \frac{1}{2} a_0^2 + \sum_{k=1}^{n} (a_k^2 + b_k^2) \leq M^2 \quad \left(M^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx \right).$$

Pf: Expand $\frac{1}{\pi} \int_{-\pi}^{\pi} \left(f(x) - \left(\frac{1}{2} a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx \right) \right)^2 dx \geq 0$

Rmk: Will see later that get "=" for $n \to \infty$ (Parseval, Hardy).

Main Theorem*:

In fact, $S_n(x) \longrightarrow f(x)$ if $f \in PC^1([-\pi, \pi])$,

the convergence is absolute & unif if f is also conti.

in general; it is unif. on every closed interval where f is conti.

Pf: Notice that $f'(x)$ has Fourier coefficients

(*) $$a_k' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} k\,f(x)\sin kx\,dx = k\cdot b_k$$

$\therefore\ b_k' = -k\,a_k$. Similarly, this step need $f \in C$

$$\Rightarrow |a_k\cos kx + b_k\sin kx| \le \sqrt{a_k^2+b_k^2} \le \frac{1}{2}\left(\frac{1}{k^2} + k^2(a_k^2+b_k^2)\right)$$

a conv. majorant

$$\Rightarrow S_n(x) \xrightarrow[\text{unif.}]{n\to\infty} S(x)$$ ← conti.

But why $S(X) = f(x)$?

Since we assume $f \in PC^1$ only, to apply Main Thm we consider $\int f$.

ie. Let $F(x) = \int_{-\pi}^{x}\left(f(t) - \frac{1}{2}a_0\right)dt \in PC^2$, & $F(-\pi) = 0 = F(\pi)$.

$$\Rightarrow F(x) = \text{its Fourier series} = \frac{1}{2}A_0 + \sum_{k=1}^{\infty}\Big(\underset{-b_k/k}{\underset{\|}{A_k}}\cos kx + \underset{a_k/k}{\underset{\|}{B_k}}\sin kx\Big)$$

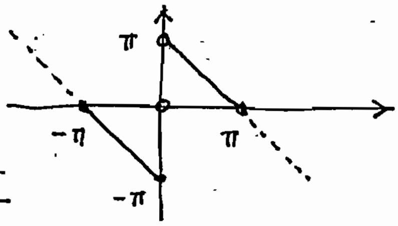
same as (*).

since the term by term differentiation converges uniformly, we get

$$F'(x) = f(x) - \frac{1}{2}a_0 = \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx).$$

To deal with jump discontinuity, need a "local model"

example (4) $$\chi(x) = \begin{cases} \pi - x & x > 0 \\ 0 & x = 0 \\ -\pi - x & x < 0 \end{cases}$$



From $\sigma_n(t) = \frac{1}{2} + \cos t + \cdots + \cos nt = \dfrac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}$

$\int_0^{x}$ and let $n\to\infty$ get $\frac{x}{2} + \sin x + \frac{\sin 2x}{2} + \cdots = \frac{\pi}{2}$ (indep of x!)

ie $\chi(x) = 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right)$, unif on $0 < \varepsilon \le |x| \le \pi$.

Now let $f^*(x) = f(x) - \frac{1}{2\pi}\sum_{i=1}^{m}\beta_i\,\chi(x-\xi_i) \in PC^2$

when f has jump β_i at $x = \xi_i$

which is also conti.

Then can apply the prev. case. □

Rmk: Carleson 1966: If $f \in L^2(I)$ (Lebesgue)

then $S_n(x) \longrightarrow f(x)$ a.e. in I. (Real analysis).

Easy results on diff/int

(1) If $f \in C^{k-1}$ and $f \in PC^k$, then $|a_n|, |b_n| \leq \frac{B}{n^k}$ for some B.

pf: $$\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx = \frac{1}{2\pi}\left(\frac{-1}{in}\right)\int_{-\pi}^{\pi} f(x)\, d\, e^{-inx}$$
$$\overset{*}{=} \frac{1}{2\pi}\left(\frac{-i}{n}\right)\int_{-\pi}^{\pi} f'(x)\, e^{-inx}\, dx = \cdots = \frac{1}{2\pi}\left(\frac{-i}{n}\right)^k \int_{-\pi}^{\pi} f^{(k)}(x)\, e^{-inx}\, dx.$$

(2) If $k \geq 2$, then can differentiate the Fourier series of $f(x)$ up to $k-2$ times (term by term), which equals $f^{(k-2)}(x)$.

If $|\alpha_n| \leq \frac{B}{n^{k+\varepsilon}}$ for some $\varepsilon > 0$ and $k \geq 1$

Then $\Sigma_{-\infty}^{\infty} \alpha_n e^{inx}$ sums to $f(x) \in PC^k$ and gets $f^{(k-1)}(x)$ by term by term differentiation. ($\boxed{\text{No conclusion for } \varepsilon = 0}$)

(3) Int can be done term-wise if $S_n(x) \xrightarrow[\text{unif.}]{} f(x)$, say $k \geq 1$, $\varepsilon > 0$.

But if fact much more is true

$\boxed{\text{(A.I.3) Thm}}$: If $f \in PC([-\pi, \pi])$, $\boxed{\text{without assuming any conv.}}$

$$\Rightarrow \int_a^X f(t)\, dt = \frac{1}{2} a_0 (X - a) + \sum_{k=1}^{\infty} \int_a^X \left(a_k \cos kx + b_k \sin kx\right) dx.$$

pf: $F(x) := \int_{-\pi}^{X} \left(f(t) - \frac{1}{2} a_0\right) dt \in C, PC'$ & $F(\pi) = 0 = F(-\pi)$

$$\text{Main Thm}^* \Rightarrow \frac{1}{2} A_0 + \Sigma_{k=1}^{\infty} \left(\underset{-b_k/k}{\underset{\parallel}{A_k}} \cos kx + \underset{a_k/k}{\underset{\parallel}{B_k}} \sin kx\right) \xrightarrow[\text{unif.}]{} F(x)$$

as in (*), using $F' = f$ and int by parts. □

Application : Weierstrass approximation theorem.

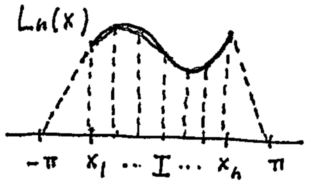
(a) $f \in C(I)$, $I \subset [-\pi, \pi]$, $\Rightarrow f$ is unif. approx. by a Δ-poly.

(b) f is unif. approx. by a poly $P(x)$ on I.

(c) If $f \in C'$, then $P_n(x)$ can be chosen st. $P_n'(x) \xrightarrow[\text{unif.}]{} f'(x)$.

pf: (a) By unif. conti. f is approx by p.L. function within $\varepsilon/2$ for $h = \Delta x_i$ small. Then $L_n(x) \in PC^2, C$.

$\Rightarrow L_n(x)$ unif & absolut. approx. by its Fourier poly $S_m(x)$ within $\varepsilon/2$.



(b) By Taylor series for $\sin kx$, $\cos kx$, $k = 1, \ldots, m$,

get poly $P_N(x)$ st. $|P_N(x) - S_m(x)| < \varepsilon$. (c) Approx. f' by Q_n then $P_n = \int Q_n$.

□

Calculus 12/30, 2011 (Final lecture)

8.7 **Approximations**

Theorem (Fejer's): $F_n(x) := \dfrac{S_0(x) + \cdots + S_n(x)}{n+1} \xrightarrow[\text{unif}]{n\to\infty} f(x)$

for f conti & periodic on $[-\pi, \pi]$.

Notice that this does not provide ANY series exp. of $f(x)$.

Pf: Recall $S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,\sigma_n(t)\,dt$

$$\Rightarrow F_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,\frac{\sigma_0(t) + \cdots + \sigma_n(t)}{n+1}\,dt$$

Now $\sigma_n(t) = \dfrac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}} = \dfrac{\sin\frac{t}{2}\sin(n+\frac{1}{2})t}{2\sin^2\frac{t}{2}} = \dfrac{1}{2}\,\dfrac{\cos(nt) - \cos(n+1)t}{1-\cos t}$

$$\Rightarrow \frac{1}{n+1}\sum_{k=0}^{n}\sigma_k(t) = \frac{1}{2(n+1)}\cdot\frac{1-\cos(n+1)t}{1-\cos t} = \frac{1}{2(n+1)}\left(\frac{\sin\frac{n+1}{2}t}{\sin\frac{t}{2}}\right)^2 =: \overset{\overset{0}{\wedge}}{S_n(t)}$$

Given any $\delta > 0$, (choose the one st. $|f(x) - f(x+t)| < \varepsilon/3$ $\forall\ |t| < \delta$, $x \in [-\pi, \pi)$)

let $|f| \le M$ on $[-\pi, \pi]$. then

$$f(x) - F_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi}\big(f(x) - f(x+t)\big)\,S_n(t)\,dt,$$
(All $\frac{1}{\pi}\int_{-\pi}^{\pi}\sigma_k(t)\,dt = 1$,

$$= \int_{-\delta}^{\delta} + \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}$$
here $\frac{1}{\pi}\int_{-\pi}^{\pi} S_n(t) = 1$.)

$$\Rightarrow |f(x) - F_n(x)| \le \frac{\varepsilon}{3\pi}\int_{-\delta}^{\delta} S_n(t)\,dt + \frac{2M}{\pi}\int_{-\pi}^{-\delta} S_n(t)\,dt + \frac{2M}{\pi}\int_{\delta}^{\pi} S_n(t)\,dt$$

$$< \frac{\varepsilon}{3\pi} + \frac{2M}{n+1}\cdot\frac{1}{\sin^2\delta/2}.$$

Now choose $n \gg 0$ (since δ is fixed), the result follows. $\square$

In contrast to sup norm approx.

Also useful to consider **L^2 norm approx.** $\to$ **inner prod. space**

On $PC([0,\ell])$, define $\langle f, g\rangle = \frac{1}{\ell}\int_0^{\ell} f(x)\,g(x)\,dx$, $\|f\|^2 := \langle f, f\rangle$.

$\|f\|$ is a distance function since $\|f\| = 0 \Leftrightarrow f \equiv 0$.

$\langle f, g\rangle \le \|f\|\cdot\|g\|$ (using $\|f + tg\|^2 \ge 0$ $\forall t \in \mathbb{R}$)

$\Rightarrow \|f + g\| \le \|f\| + \|g\|$

Theorem: (**Least square approx.**)

If $e_1, \cdots, e_m$ ONB of $V \subset PC([0,\ell])$, then $\|f - g\|$, $g \in V$

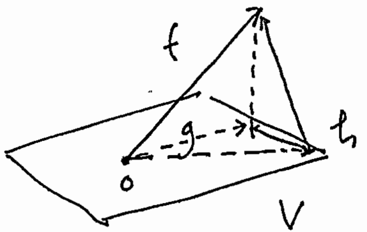
attains minima $\Leftrightarrow g = \mathrm{Proj}_V f = \sum_{i=1}^{m}\langle f, e_i\rangle e_i$.

pf: (Phytagoras) $\forall h \in V$

$$\|f-h\|^2 = \|(f-g)+(g-h)\|^2 = \|f-g\|^2 + \|g-h\|^2$$

since $f-g \perp V$ and $g-h \in V$

so min. $\Leftrightarrow$ $g = h$. $\square$



- Now e^{inx} is ON for $\langle f,g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\bar{g}\,dx$

equivalently, → need to use $\mathbb{C}^m$

- $\left\{ \frac{1}{\sqrt{2}}, \cos nx, \sin nx \right\}$ is ON for $\frac{1}{\pi}\int_{-\pi}^{\pi} fg\,dx$

so $h = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$ has

$\overset{\shortparallel}{\frac{a_0}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}}$ $\quad \|h\|^2 = \left(\frac{a_0}{\sqrt{2}}\right)^2 + \sum_{k=1}^{n}(a_k^2+b_k^2) \equiv \sum_{k=-n}^{n}|\alpha_k|^2$.

Theorem (Parseval's equality). If f is periodic and conti of period 2π, then $\frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2+b_k^2) = \|f\|^2$.

pf: $\exists\, T_n(x)$: Δ-poly st $f(x) - T_n(x) \longrightarrow 0$ unif.

of degree n

But then $\|f - S_n(x)\|^2 \leq \|f - T_n\|^2 \xrightarrow[n\to\infty]{} 0$.

$\overset{\shortparallel}{\|f\|^2 - \|S_n\|^2}$ (again since $f - S_n \perp S_n$.) $\square$

Rmk: Thm holds if f has finite discontinuities (why?) and in fact holds as long as $\int f^2$ makes sense.

A more natural framework is L^2 space & Lebesgue integral

example: $\operatorname{sgn}(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$

let $x = \frac{\pi}{2}$, get $2 = \frac{4^2}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$

$\Rightarrow \quad 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \pi^2/8 \quad \not\Rightarrow \quad \zeta(2) = \pi^2/6$ as well.

End.