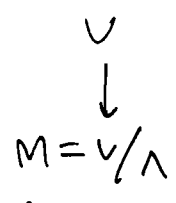


$M = V/\Lambda$  cpx tori  $V \cong \mathbb{C}^n$

her. product on  $V \leftrightarrow$  inv Kähler metric on  $M$

cpx structure under inv metric : harmonic forms  $\leftrightarrow$  inv. forms =  $\{dZ^I \wedge d\bar{Z}^J\}$



$\Rightarrow H_1(M, \mathbb{Z}) = \pi_1(M, \mathbb{Z}) = \Lambda$

$\Lambda = \mathbb{Z}\lambda_1 \oplus \dots \oplus \mathbb{Z}\lambda_{2n}$  int. basis  
real cov.  $x_1, \dots, x_{2n}$

Integral structure

$\Rightarrow H^1(M, \mathbb{Z}) = \text{Hom}(\Lambda, \mathbb{Z}) = \Lambda^*$

$H^k(M, \mathbb{Z}) = \Lambda^k(\Lambda^*)$

$= \mathbb{Z}(dX^I)_{|I|=k}$

$\Lambda^* \otimes \mathbb{R} = \mathbb{R}(dx_1, \dots, dx_{2n})$   
even /  $\mathbb{Z}$

Q: want a integral positive (1,1) form

$\omega = \frac{i}{2} \sum h_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$  (may assume, inv)

let  $\Pi$  be the  $2n \times n$  matrix st. (ie.  $\tilde{\Pi} = (\Pi, \bar{\Pi})_{2n \times 2n}$ )

$dx = \Pi dz + \bar{\Pi} d\bar{z}$

if  $\omega = \sum g_{ij} dx^i \wedge dx^j$  is an inv. int. 2-form (real)

$= g_{ij} (\pi_{i\alpha} dz^\alpha + \bar{\pi}_{i\alpha} d\bar{z}^\alpha) \wedge (\pi_{j\beta} dz^\beta + \bar{\pi}_{j\beta} d\bar{z}^\beta)$

$= \frac{g_{ij} \pi_{i\alpha} \pi_{j\beta}}{2} dz^\alpha \wedge dz^\beta$

$+ g_{ij} \pi_{i\alpha} \bar{\pi}_{j\beta} dz^\alpha \wedge d\bar{z}^\beta + g_{ij} \pi_{j\beta} \bar{\pi}_{i\alpha} d\bar{z}^\alpha \wedge dz^\beta$

$+ \frac{g_{ij} \bar{\pi}_{i\alpha} \bar{\pi}_{j\beta}}{2} d\bar{z}^\alpha \wedge d\bar{z}^\beta$   $i \leftrightarrow j \quad \alpha \leftrightarrow \beta$

(1,1) type  $\Leftrightarrow \Pi^t Q \Pi = 0$

then,  $\omega$  positive  $\Leftrightarrow 2 \cdot \frac{2}{i} \Pi^t Q \bar{\Pi}$  is her. pos. def.

This is the Riemann condition (relation)

Dual Form: (Usual Form)

p. 2

recall:  $\lambda_1, \dots, \lambda_{2n} \in \mathbb{Z}$  basis of  $\Lambda = H_1(M, \mathbb{Z})$

$e_1, \dots, e_n \in \mathbb{C}$  basis of  $V \cong \mathbb{C}^n$

so  $dz^1, \dots, dz^n$  basis of  $H^0(X, \Omega_M^1) = H^{1,0}(M)$

$\Omega :=$  period matrix  $\begin{bmatrix} \int_{\lambda_1} \vec{dz} & \int_{\lambda_2} \vec{dz} & \dots & \int_{\lambda_{2n}} \vec{dz} \end{bmatrix}$   
 $n \times 2n$

$$dz = \Omega dx = [\omega_{\alpha i}] \quad \text{st.} \quad \boxed{dz^\alpha = \omega_{\alpha i} dx^i} \Leftrightarrow \lambda_i = \omega_{\alpha i} e_\alpha$$

(or.  $\Lambda^t = \Omega^t E^t$ ) ( $d\bar{z}^\alpha = \bar{\omega}_{\alpha i} dx^i$ )

so  $\tilde{\Omega} := \begin{bmatrix} \Omega \\ \bar{\Omega} \end{bmatrix}$  is the inverse of  $(\Pi, \bar{\Pi})$

$$\left( \text{i.e. } \begin{bmatrix} \Omega \\ \bar{\Omega} \end{bmatrix} (\Pi, \bar{\Pi}) = \begin{bmatrix} \Omega \Pi & \Omega \bar{\Pi} \\ \bar{\Omega} \Pi & \bar{\Omega} \bar{\Pi} \end{bmatrix} = I_{2n \times 2n} \text{ omit} \right)$$

$$\Leftrightarrow \underline{\underline{\Omega \Pi = I_n, \quad \Omega \bar{\Pi} = 0}}$$

R.C.  $\tilde{\Pi}^t Q \tilde{\Pi} = \begin{pmatrix} \Pi^t \\ \bar{\Pi}^t \end{pmatrix} Q \begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix} = \begin{pmatrix} \Pi^t Q \Pi & \Pi^t Q \bar{\Pi} \\ \bar{\Pi}^t Q \Pi & \bar{\Pi}^t Q \bar{\Pi} \end{pmatrix}$

$$\boxed{\frac{1}{i} \tilde{\Pi}^t Q \tilde{\Pi} = \begin{pmatrix} H & 0 \\ 0 & -H^t \end{pmatrix}} \quad H > 0 \text{ (pos. her.)}$$

$$\Leftrightarrow i \bar{\Omega} Q^{-1} \Omega^t = \begin{pmatrix} H^{-1} & \\ & -H^{t-1} \end{pmatrix} \quad H > 0$$

$$\Leftrightarrow \underline{\underline{\Omega Q^{-1} \Omega^t = 0 \quad \& \quad i \bar{\Omega} Q^{-1} \Omega^t > 0}}$$

• If  $Q$  is only a  $\mathbb{R}$ -form ( $\omega$ ). then this condition is simply the condition that  $\omega$  being Kähler

this is the (Hodge)-Riemann bilinear relations •

Lemma (Algebra): let  $Q(\cdot)$  be an integral skew-sym form on  $\Lambda$   
then  $\exists \mathbb{Z}$  basis  $\lambda_1, \dots, \lambda_{2n}$  of  $\Lambda$  st.

$$Q = \begin{bmatrix} 0 & \Delta_\delta \\ -\Delta_\delta & 0 \end{bmatrix}; \quad \Delta_\delta = \begin{bmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{bmatrix} \quad \delta_i | \delta_{i+1}, \in \mathbb{Z}$$

In terms of the new basis

P. 3

$$\omega = \sum_{i=1}^n \delta_i dx^i \wedge dx^{n+i}$$

$\omega$  is non-deg.  $\Leftrightarrow \delta_i \neq 0$

take the cpx basis of  $\mathbb{A}^n$  by  $e_\alpha = \frac{\lambda_\alpha}{\delta_\alpha}$ ,  $\alpha = 1 \dots n$

then

$$\Omega = (\Delta_\delta, \mathbb{Z})$$

the Riemann conditions become

$$\Omega \varrho^{-1} \Omega^t = 0 \Leftrightarrow [\Delta_\delta, \mathbb{Z}] \begin{bmatrix} 0 & -\Delta_\delta^{-1} \\ \Delta_\delta^{-1} & 0 \end{bmatrix} \begin{pmatrix} \Delta_\delta \\ \mathbb{Z} \end{pmatrix} = \mathbb{Z} - \mathbb{Z}^t = 0$$

ie.  $\mathbb{Z}$  is symmetric

$$i \bar{\Omega} \varrho^{-1} \Omega^t > 0 \Leftrightarrow i [\Delta_\delta, \bar{\mathbb{Z}}] \begin{bmatrix} 0 & -\Delta_\delta^{-1} \\ \Delta_\delta^{-1} & 0 \end{bmatrix} \begin{pmatrix} \Delta_\delta \\ \mathbb{Z} \end{pmatrix} \\ = -i (\mathbb{Z} - \bar{\mathbb{Z}}^t) = -i (\mathbb{Z} - \bar{\mathbb{Z}}) = 2 \text{Im } \mathbb{Z} > 0$$

Theorem (Riemann)

$M = V/\Lambda$  is an abelian variety  $\Leftrightarrow \exists \mathbb{Z}$  basis of  $\Lambda$  and  $\mathbb{C}$  basis of  $V$  st. for  $d\mathbb{Z} = \Omega dx$ , one have

$$\Omega = (\Delta_\delta, \mathbb{Z})$$

st.  $\mathbb{Z}$  is symmetric &  $\text{Im } \mathbb{Z}$  is positive definite.

Rmk: in this form.  $i \bar{\Omega} \varrho^{-1} \Omega^t = 2 \text{Im } \mathbb{Z}$  is actually real matrix

Later will see that  $[\omega] \in H^2(M, \mathbb{Z}) \cap H^{1,1}(M)$

is the 1st chern class of a line bundle  $L_\delta \rightarrow M$

in fact: (\*)  $h^0(M, L_\delta) = \delta_1 \dots \delta_n$  moreover

Lefschetz embedding thm:

$$L \text{ ample} \Rightarrow \begin{array}{l} L^k \text{ is b.p.f. } \forall k > 2 \\ L^k \text{ is v.a. } \forall k \geq 3 \end{array}$$

This follows from the theory of theta functions. (later)

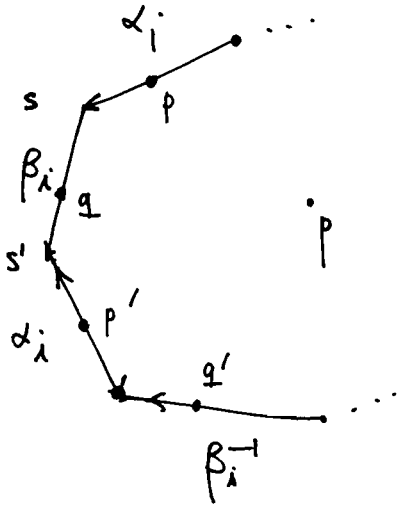
(\*) follows from the R-R formula since  $\chi(L)^n = \int_M \omega^n = n! \cdot \pi \delta_n$   
and K-V  $\Rightarrow$   $h^0(L) = \chi(L) = \frac{L^n}{n!}$  since  $M$  is flat.

$X: R.S \text{ genus} = g. \text{ Abel-Jacobi } X \rightarrow J(X) = \mathbb{C}^g / \Gamma \quad p.4$

$\omega$  holo. 1-form

$\Delta$  simply conn.

$\omega = d\varphi$  with  $\varphi = \int_{p_0}^z \omega$  holo. fun.



$$\frac{\varphi(p') - \varphi(p)}{\quad} = \int_p^{p'} d\varphi = \int_p^{p'} \omega$$

$$= \int_p^s \omega + \int_s^{s'} \omega + \int_{s'}^{p'} \omega$$

$$= \int_{\beta_i} \omega \quad \text{indep of the position of } p!$$

Same reason:  $\frac{\varphi(q') - \varphi(q)}{\quad} = - \int_{\alpha_i} \omega$

Basic integration

identity:

$$\int_X \omega \wedge \eta = \int_{\partial \Delta} \varphi \eta = \sum_{i=1}^g \int_{\alpha_i + \alpha_i^{-1}} \varphi \eta + \int_{\beta_i + \beta_i^{-1}} \varphi \eta$$

$$= \sum_{i=1}^g \left( - \int_{\beta_i} \omega \cdot \int_{\alpha_i} \eta + \int_{\alpha_i} \omega \cdot \int_{\beta_i} \eta \right)$$

for  $\omega$  holo.  $\eta$  any 1-form.

we may normalize  $\omega_1, \dots, \omega_g$  st  $\int_{\alpha_i} \omega_j = \delta_{ij}$

then the period matrix  $\Omega = (I, Z) \quad Z_{ij} = \int_{\beta_i} \omega_j$

$$0 = \int_X \omega_i \wedge \omega_j = \sum_k \left( - \int_{\beta_k} \omega_i \int_{\alpha_k} \omega_j + \int_{\alpha_k} \omega_i \int_{\beta_k} \omega_j \right)$$

$$= -Z_{ji} + Z_{ij} \Rightarrow \underline{Z \text{ is symmetric}}$$

$$0 < \int \omega_i \wedge \overline{\omega_j} = i \sum_k \left( -Z_{ji} + \overline{Z_{ij}} \right) = 2 \operatorname{Im} Z_{ij}$$

$\Rightarrow \underline{\operatorname{Im} Z \text{ is positive definite.}}$

hence:  $\mathbb{C}^g / \Gamma$  is an principally polarized abelian variety.

the case  $n=1$ :  $\mathbb{C}/\Lambda$  elliptic curves:

P. 5

$\Lambda$  has a basis  $\mathbb{Z} \oplus \mathbb{Z}\tau$ .  $\text{Im}\tau > 0$

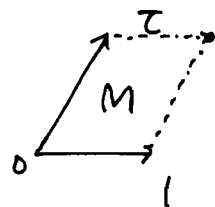
ie.  $\lambda_1 = 1, \lambda_2 = \tau$   $e_\alpha = e_1 = 1$

hence  $\lambda_i = \omega_{\alpha_i} e_\alpha \Rightarrow \omega_1 = 1, \omega_2 = \tau$

ie.  $\Omega = (\omega_{\alpha_i}) = \underline{(1, \tau)}$   
 $1 \times 2$  matrix

ie.  $\Delta_g = 1, \mathbb{Z} = \tau$ .

moduli space of elliptic curves



$\Lambda = \mathbb{Z} + \mathbb{Z}\tau$  and  $\Lambda' = \mathbb{Z} \oplus \mathbb{Z}\tau'$

determine the same torus  $\Leftrightarrow \Lambda' = \mu \cdot \Lambda$  for some  $\mu \in \mathbb{C}$

ie.  $\exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$  st.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} \mu\tau' \\ \mu \end{pmatrix}$$

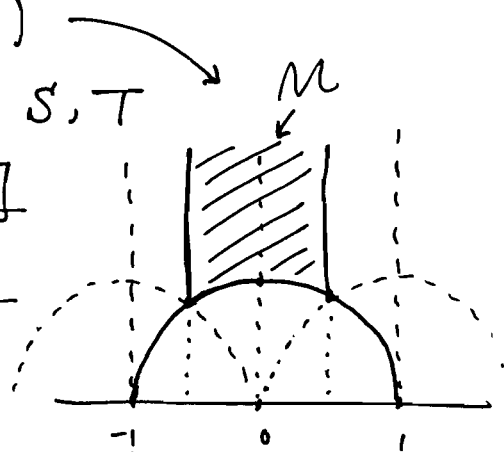
Define  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$  Möbius transform

get Moduli of  $M/\Gamma = \mathbb{H}/\text{SL}(2, \mathbb{Z})$

Lemma:  $\text{SL}(2, \mathbb{Z})$  is generated by  $S, T$

$$S := z \mapsto z + 1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$T := z \mapsto \frac{-1}{z} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



General  $n$ :

for principally polarized abelian varieties:

$$\Omega = (\Delta, \mathbb{Z}), \quad d \equiv 1$$

$$\mathcal{H} = \left\{ Z \in \mathbb{C} \mid \text{Im} Z > 0 \right\}$$

this is the Siegel upper half space  
Siegel

$$\mathcal{M}_g = \mathcal{H}_g / \text{Sp}(2g, \mathbb{Z}) \quad \text{symmetric space. (待續)}$$

Line bundles on  $V/\Lambda = X$  :

(1) any line bundle on  $\mathbb{C}^n$  is trivial :

$$\begin{array}{ccccc} H^1(\mathbb{C}^n, \mathcal{O}) & \rightarrow & H^1(\mathbb{C}^n, \mathcal{O}^*) & \rightarrow & H^2(\mathbb{C}^n, \mathbb{Z}) \\ \parallel & & & & \parallel \\ 0 & & & & 0 \end{array}$$

(Dolbeault lemma)

(2) any line bundle on  $\mathbb{C}^{*n}$  is determined by  $u$  :

$$\begin{array}{ccccccc} H^1(\mathbb{C}^{*n}, \mathcal{O}) & \rightarrow & H^1(\mathbb{C}^{*n}, \mathcal{O}^*) & \rightarrow & H^2(\mathbb{C}^{*n}, \mathbb{Z}) & \rightarrow & H^2(\mathbb{C}^{*n}, \mathcal{O}) \\ \parallel & & & & \parallel & & \parallel \\ 0 & & & & u & & 0 \end{array}$$

(bec. original pf of Dolbeault lemma also applies to the case  $\mathbb{C}^*$ . EX 15.)

or use the fact that  $\mathbb{C}^{*n}$  is an affine variety )

now:  $\mathbb{C}/\mathbb{Z} \xrightarrow{e^{2\pi i z}} \mathbb{C}^*$

hence  $\mathbb{C}^n \cong V \xrightarrow{\pi} V/\mathbb{Z}\langle \lambda_1, \dots, \lambda_n \rangle \xrightarrow{\pi_1} X$   $e^{2\pi i x} \cdot e^{-2\pi i y}$   
 $\mathbb{C}^{*n}$  bec.  $e_\alpha = \frac{\lambda_\alpha}{\delta_\alpha}$   $\alpha=1 \dots n$

We want to consider line bundle  $L \downarrow X$  with  $u(L) = \omega = \sum \delta_\alpha dx^\alpha \wedge dx^{n+\alpha}$  :

Step 1. (1)  $\Rightarrow \pi^*L \rightarrow \mathbb{C}^n$  is trivial

fix a trivialization  $\varphi: \pi^*L \rightarrow V \times \mathbb{C}$

$\varphi_z \rightarrow \varphi_{z+\lambda}$  differ by linear auto.

$\Rightarrow \exists$  holomorphic functions  $e_\lambda(z) \in \mathcal{O}^*(\mathbb{C}^n)$ ,  $\forall \lambda \in \Lambda$

st.  $\underline{e_{\lambda+\lambda'}(z) = e_{\lambda'}(z+\lambda) \cdot e_\lambda(z)}$  (\*)

which is determined by  $e_{\lambda_i}(z)$   $i=1 \dots 2n$

which is equiv to (\*)': for basic  $\lambda_\alpha$ :  
 $e_{\lambda_\alpha}(z+\lambda_\beta) = e_{\lambda_\beta}(z)$   
 $= e_{\lambda_\beta}(z+\lambda_\alpha) \cdot e_{\lambda_\alpha}(z)$

Step 2. (2)  $\Rightarrow \pi_1^*(\delta_\alpha dx^\alpha) = \pi_1^*(dz_\alpha) = 0$

hence  $\pi_1^*\omega = 0$ . ie. if  $\varphi$  respects  $\pi_1$ , then may let

$e_{\lambda_\alpha}(z) \equiv 1$ ,  $\alpha=1 \dots n$

step 3. Recall  $Z = Z^T$ ,  $\text{Im } Z > 0$ ,  $\Omega = (\Delta_\alpha, \bar{Z})$  p. 2

Let  $Z = X + iY$ ,  $dZ = \Omega dx$

the main observation is:

$$\omega = \sum \int_\alpha dx^\alpha \wedge dx^{n+\alpha} = \frac{i}{2} \sum Y^{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta$$

pf:  $\sum Y^{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta$

$$= \sum Y^{\alpha\beta} (\delta_\alpha dx^\alpha + Z_{\alpha p} dx^{n+p}) \wedge (\delta_\beta dx^\beta + \bar{Z}_{\beta q} dx^{n+q})$$

$$= \sum Y^{\alpha\beta} \overrightarrow{d\alpha} \overleftarrow{d\beta} dx^\alpha \wedge dx^\beta + \sum Y^{\alpha\beta} (Z_{\alpha p} \delta_\beta dx^{n+p} \wedge dx^\beta + \delta_\alpha \bar{Z}_{\beta q} dx^\alpha \wedge dx^{n+q})$$

$$+ \sum Y^{\alpha\beta} \overrightarrow{Z_{\alpha p}} \overleftarrow{\bar{Z}_{\beta q}} dx^{n+p} \wedge dx^{n+q}$$

eq.  $Y^{\alpha p} Z_{p q} \cdot \bar{Z}_{p q} = Y^{p q} Z_{p q} \cdot \bar{Z}_{p q}$

$$= \sum Y^{\alpha\beta} (\bar{Z}_{\beta q} - Z_{\beta q}) \delta_\alpha dx^\alpha \wedge dx^{n+q}$$

$$= -2i \sum Y^{\alpha\beta} Y_{\beta q} \delta_\alpha dx^\alpha \wedge dx^{n+q} = -2i \omega \quad \#$$

step 4. Want to find positive function  $h(z)$  st

$$\frac{i}{2\pi} [-\partial\bar{\partial} \log h] = \omega \quad ; \quad \text{Let } K = \log h. \quad \text{ie.}$$

$$\partial\bar{\partial} K = -\frac{2\pi}{i} \cdot \frac{i}{2} \sum Y^{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta$$

the natural choice is then  $K(z) = -\pi \sum Y^{\alpha\beta} z^\alpha \bar{z}^\beta$

but this function should be inv under  $z \mapsto z + \lambda_\alpha$   
 $\alpha = 1, \dots, n$

ie. inv. under  $z^\alpha \mapsto z^\alpha + 1$ . so use

$$\log h(z) = K(z) = \frac{1}{2} \pi \sum Y^{\alpha\beta} (z^\alpha - \bar{z}^\alpha)(z^\beta - \bar{z}^\beta) \quad (\text{no } 2\pi k)$$

$$\text{step 5. Find } e_{\lambda_{n+\gamma}} = \frac{-2\pi \sum Y^{\alpha\beta} \text{Im } z^\alpha \cdot \text{Im } z^\beta}{2}$$

Wish  $h$  defines metric on  $L$ . hence  $h$  satisfies

$$** \quad h(z + \lambda_{n+\gamma}) = |e_{\lambda_{n+\gamma}}^{-1}(z)|^2 h(z)$$

$$K(z + \lambda_{n+\gamma}) = -2\pi \sum Y^{\alpha\beta} \text{Im}(z^\alpha + Z_{\alpha\gamma}) \text{Im}(z^\beta + Z_{\beta\gamma})$$

$$= K(z) - 4\pi \sum Y^{\alpha\beta} Y_{\alpha\gamma} \text{Im } z^\beta - 2\pi \sum Y^{\alpha\beta} Y_{\alpha\gamma} Y_{\beta\gamma}$$

$$= K(z) - 4\pi \text{Im } z^\gamma - 2\pi Y_{\gamma\gamma}$$

$$(** : \|\hat{\theta}(z)\|^2 = h(z) |\theta(z)|^2 = \|\hat{\theta}(z + \lambda)\|^2 = h(z + \lambda) |\theta(z + \lambda)|^2)$$

Let  $e_{\lambda_{n+\gamma}}(z) = e^{a_\gamma(z)}$

then  $K(z + \lambda_{n+\gamma}) = -2 \operatorname{Re} a_\gamma(z) + K(z)$

$\Rightarrow \operatorname{Re} a_\gamma(z) = +2\pi \operatorname{Im} z^\gamma + \pi Y_{\gamma\gamma}$

may take  $a_\gamma(z) = -2\pi i z^\gamma + \pi Y_{\gamma\gamma}$

const.  $c^\gamma$

In fact it's easy to see that

$$\begin{cases} e_{\lambda_\alpha} \equiv 1 : \alpha = 1 \dots n \\ e_{\lambda_{n+\alpha}}(z) = e^{-2\pi i z^\alpha + c^\alpha} : \alpha \text{ any const.} \end{cases}$$

satisfies the compatibility condition (\*):

$$\left( \begin{aligned} & e_{\lambda_{n+\alpha}}(z + \lambda_{n+\beta}) \cdot e_{\lambda_{n+\beta}}(z) \\ &= e^{-2\pi i (z^\alpha + Z_{\alpha\beta}) + c^\alpha} \cdot e^{-2\pi i z^\beta + c^\beta} \\ &= e^{-2\pi i ((z^\alpha + z^\beta) + Z_{\alpha\beta}) + (c^\alpha + c^\beta)} \quad \text{sym. in } \alpha, \beta \end{aligned} \right)$$

Step 6. meaning of the const.  $c^\alpha$ :

$\equiv$  line bundles with the same  $u$ :

let  $\mu \in X$  and  $\tau_\mu: X \rightarrow X \quad z \mapsto z + \mu$  translation by  $\mu$

then  $\tau_\mu$  homotopic to  $\operatorname{id}_X$  hence  $u(\tau_\mu^* L) = u(L)$

the new  $e'_\lambda(z) := e_\lambda(z + \mu)$  ( $\tau_\mu^* L$  top. isom to  $L$  but not holomorphically)

$$\begin{cases} e'_{\lambda_\alpha}(z) = e_{\lambda_\alpha}(z + \mu) = 1 & \alpha = 1 \dots n \\ e'_{\lambda_{n+\alpha}}(z) = e_{\lambda_{n+\alpha}}(z + \mu) = e^{-2\pi i (z + \mu)^\alpha + c^\alpha} \\ = e^{-2\pi i z^\alpha + (c^\alpha - 2\pi i \mu^\alpha)} \end{cases}$$

$\uparrow$   
const.

Fact:  $u(L) = 0$   
 exp sequence of top vs holo.  $\begin{cases} \Leftrightarrow \text{flat line bundle} \\ \Leftrightarrow \text{constant transition function} \\ \Leftrightarrow \text{constant trivialization} \end{cases}$   
 $e_\lambda(0) = \begin{cases} 1 & \lambda = \lambda_\alpha \\ e^{c^\alpha} & \lambda = \lambda_{n+\alpha} \end{cases}$

this can also be seen from

$0 \rightarrow \operatorname{Pic}^0(X) \rightarrow H^1(X, \mathcal{O}^*) \xrightarrow{u} H^1(X, \mathbb{Z})$  principal homog. space.

$H^1(X, \mathcal{O}) / H^1(X, \mathbb{Z})$  dual a.v. of  $X = \check{X}$  acting on  $X$  by translation?



# Dimension Formula (R-R) :

if  $u(L) = \sum d_\alpha dx^\alpha \wedge dx^{n+\alpha}$

then  $h^0(X, \theta(L)) = \delta_1 \cdots \delta_n$

pf:  $h^0(\theta(L))$  is inv. under translation, may consider  $L$

st.  $\theta(z + \lambda_\alpha) = \theta(z) \quad \alpha = 1 \dots n$

$\theta(z + \lambda_{n+\alpha}) = e^{-2\pi i z^\alpha - \pi i Z_{\alpha\alpha}} \theta(z)$

(cov. to transl. by  $\mu = \frac{1}{2} \sum Z_{\alpha\alpha} e_\alpha$ )

$\alpha$ : what's the meaning of this choice  $\mu$ ?

so  $\theta$  is periodic in  $z^1, \dots, z^n$  with periods  $\delta_1, \dots, \delta_n$

$$\Rightarrow \theta(z) = \sum_{\ell \in \mathbb{Z}^n} a_\ell e^{\frac{2\pi i}{\delta_1} \ell_1 z^1 \dots e^{\frac{2\pi i}{\delta_n} \ell_n z^n}}$$

$$= \sum_{\ell \in \mathbb{Z}^n} a_\ell e^{2\pi i \langle \ell, \Delta_\delta^{-1} z \rangle}$$

$$\theta(z + \lambda_{n+\alpha}) = \sum_{\ell \in \mathbb{Z}^n} a_\ell e^{2\pi i \langle \ell, \Delta_\delta^{-1} z \rangle} \cdot \frac{e^{2\pi i \langle \ell, \Delta_\delta^{-1} \lambda_{n+\alpha} \rangle}}{e^{-2\pi i z^\alpha - \pi i Z_{\alpha\alpha}}}$$

$$= \sum_{\ell \in \mathbb{Z}^n} e^{-2\pi i z^\alpha - \pi i Z_{\alpha\alpha}} \cdot \sum_{\ell \in \mathbb{Z}^n} a_\ell e^{2\pi i \langle \ell, \Delta_\delta^{-1} z \rangle}$$

$$= \sum_{\ell \in \mathbb{Z}^n} e^{-\pi i Z_{\alpha\alpha}} \cdot a_\ell e^{2\pi i \langle \ell - \Delta_\delta e_\alpha, \Delta_\delta^{-1} z \rangle}$$

$$\Rightarrow a_{\ell + \lambda_\alpha} = e^{2\pi i \langle \ell, \Delta_\delta^{-1} \lambda_{n+\alpha} \rangle + \pi i Z_{\alpha\alpha}} a_\ell \quad (***)$$

hence  $\theta$  is completely determined by

$$a_\ell ; \ell \in \mathbb{Z}^n / \langle \lambda_1, \dots, \lambda_n \rangle \Rightarrow \dim \leq \prod_{\alpha=1}^n \delta_\alpha$$

Now

$$\theta(z) = \sum_{\substack{\zeta \\ 0 \leq \zeta_\alpha < \delta_\alpha}} \sum_{\ell \in \mathbb{Z}^n} a_{\zeta + \Delta_\delta \ell} e^{2\pi i \langle \zeta + \Delta_\delta \ell, \Delta_\delta^{-1} z \rangle}$$

$$= \sum_{\zeta} e^{2\pi i \langle \zeta, \Delta_\delta^{-1} z \rangle} \cdot \underbrace{\sum_{\ell \in \mathbb{Z}^n} a_{\zeta + \Delta_\delta \ell} e^{2\pi i \langle \ell, z \rangle}}_{\theta_\zeta(z)}$$

Solve relation \*\*\* gives

$$b_\ell = a_{\zeta + \Delta_\delta \ell} = e^{2\pi i \langle \zeta, \Delta_\delta^{-1} z \ell \rangle + \pi i \langle \ell, z \ell \rangle} \cdot \sum_{\ell} b_\ell e^{2\pi i \langle \ell, z \rangle}$$

(basically:  $2(1+2+\dots+(n-1)) + (1+\dots+1) = n^2$ )

This is the reason to put  $\mu$ .

Need to verify that  $\theta_{\xi}(z)$  converges :

$$\text{but } |b_{\ell}| = e^{-2\pi \langle \xi, \Delta_{\delta}^{-1} Y_{\ell} \rangle - \pi \langle \ell, Y_{\ell} \rangle} \leq e^{-c \|\ell\|^2} \text{ as } \|\ell\| \uparrow \quad \sim 2\mathcal{R}$$

and only has  $\leq \ell^n$  terms with size  $\leq \ell$ . hence OK.

when  $\Delta_{\delta} = I_n, \Omega = (I_n, \mathbb{Z})$

$X$  is called principally polarized a.v.

then  $h^0(X, \mathcal{O}(L)) = 1$ . called the Riemann theta function:

$$\theta(z) := \sum_{\ell \in \mathbb{Z}^n} e^{2\pi i \langle \ell, z \rangle + \pi i \langle \ell, \mathbb{Z} \ell \rangle}$$

$$\text{st. } \begin{cases} \theta(z + \underline{e}_d) = \theta(z) & \text{periodic} \\ \theta(z + \frac{\lambda_n + \alpha}{\mathbb{Z} \alpha}) = e^{-2\pi i z \alpha - \pi i \mathbb{Z} \alpha} \theta(z) \end{cases}$$

$\Theta = (\theta)$  is called the theta divisor

In fact, for general  $\Omega = (\Delta_{\delta}, \mathbb{Z})$

Let  $\Lambda' = \mathbb{Z}(e_1, \dots, e_n, \lambda_{n+1}, \dots, \lambda_{2n})$ ,  $X' = V/\Lambda'$

then get covering map

$$\pi' : X \longrightarrow X'$$

then  $\Omega' = (I_n, \mathbb{Z})$ ,  $X'$  is prin. polarized :

$$\omega = \sum \delta_{\alpha} dx^{\alpha} \wedge dx^{n+\alpha} \equiv \sum dx'^{\alpha} \wedge dx'^{n+\alpha}$$

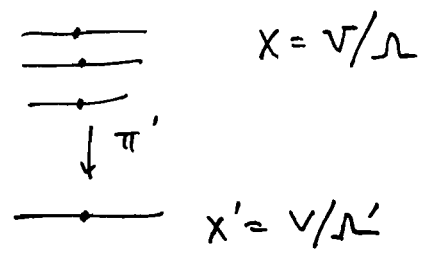
so,  $\exists$  line bundle  $L' \rightarrow X'$  st.  $\pi'^* L' = L$

$$\theta(z) = \sum_{\xi \in \Lambda' / \Lambda} e^{2\pi i \langle \xi, \Delta_{\delta}^{-1} z \rangle} \cdot \theta_{\xi}(z)$$

notice that  $\theta_0(z) = \theta(z)$  and  $\theta_{\xi} = \tau_{\xi}^* \theta(z)$  translates. ??

Principle: General results about abelian v.

can be reduced to the case of principally polarized abelian v.



Lefschetz embedding thm:

P. 6

$H^0(X, L^{\otimes k})$  gives proj. embed. for  $k \geq 3$

Pf: only need to do the case  $\Delta_g = I_n$ :

• base point free: ( $k \geq 2$  is enough)

$z_0 \in X$ , need to find  $\varphi \in H^0(L^{\otimes 2})$  st.  $\varphi(z_0) \neq 0$

consider  $\varphi(z) = \vartheta(z+\mu) \cdot \vartheta(z-\mu) \in H^0(L^{\otimes 2})$

for some  $\mu$  st.  $\vartheta(z_0+\mu) \neq 0 \neq \vartheta(z_0-\mu)$ . (not individual)

• separate points (1-1):

If  $\theta_0, \theta_1, \dots, \theta_N$  be a basis of  $H^0(X, L^{\otimes 3})$  st.

$$\theta_i(z_1) = \rho \theta_i(z_2) \quad \forall i$$

In particular, 
$$\frac{\vartheta(z_1+\mu) \vartheta(z_1+\nu) \vartheta(z_1-\mu-\nu)}{\vartheta(z_2+\mu) \vartheta(z_2+\nu) \vartheta(z_2-\mu-\nu)} = \rho$$

for all  $\mu, \nu$ . (they are all in  $H^0(X, L^{\otimes 3})$ )

claim  $\frac{\vartheta(z_1+z)}{\vartheta(z_2+z)}$  is non zero holo (entire)

trivial: for any  $z = \mu$ . find  $\nu$  st.  $\frac{a}{b} \frac{c}{d} \frac{e}{f} = \rho$   
 $c, e, d, f \neq 0$  . \*

But then  $\psi(z) := \log \frac{\vartheta(z_1+z)}{\vartheta(z_2+z)}$  is entire. st.

$$\begin{cases} \psi(z + \rho\alpha) = \psi(z) \\ \psi(z + \lambda\eta + \alpha) = \psi(z) + (-2\pi i [ (z_1+z)^\alpha - (z_2+z)^\alpha ]) \\ = \psi(z) - 2\pi i \cdot (z_1 - z_2)^\alpha \end{cases}$$

$\Rightarrow \frac{\partial \psi}{\partial z_i} = \text{constant}$  (bec. periodic in all directions)

$\Rightarrow \psi(z) = \text{linear} \Rightarrow \times$  can't be periodic in  $z^1 \dots z^n$ .

Exercise 15. Complete the proof by showing that

$$J(z) := \begin{pmatrix} \theta_0(z) & \dots & \theta_N(z) \\ \partial_1 \theta_0(z) & \dots & \partial_1 \theta_N(z) \\ \vdots & \dots & \vdots \\ \partial_N \theta_0(z) & \dots & \partial_N \theta_N(z) \end{pmatrix}$$

has full rank. hence  $H^0(X, L^{\otimes 3}) : X \rightarrow \mathbb{P}^N$   
 is an emersion. Q.E.D.

D. Mumford. Abelian varieties.

$V \xrightarrow{\cdot \pi} X = V/\Lambda$  line bundle  $L \rightarrow X$ ,  $\pi^*L \rightarrow V$  is trivial p. 7

$\Lambda$  acts on  $V \times \mathbb{C}$  homomorphically

$$u \in \Lambda, \phi_u: (z, \alpha) \mapsto (z+u, e_u(z) \cdot \alpha)$$

compatibility:

$$e_u \in \mathcal{O}^X(V)$$

$$\phi_u \circ \phi_{u'}(z, \alpha) = \phi_{u+u'}(z, \alpha) = (z+u+u', \underline{e_{u+u'}(z) \cdot \alpha})$$

$$\phi_u(z+u', e_{u'}(z) \cdot \alpha)$$

$$(z+u'+u, \underline{e_u(z+u') \cdot e_{u'}(z) \cdot \alpha})$$

let  $e_u(z) = e^{2\pi i f_u(z)}$

$$\Rightarrow f_u(z+u') + f_{u'}(z) = f_{u+u'}(z) \pmod{\mathbb{Z}}$$

$\omega(L)$  is given by the integral 2 form (for any  $z \in V$ ):

$$E(u, v) = f_v(z+u) + f_u(z)$$

$$- f_u(z+v) - f_v(z)$$

since  $E(iu, iv) = E(u, v)$  (type (1,1))

may set Her. form  $H(x, y) = E(ix, y) + iE(x, y)$

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathcal{O} & \rightarrow & \mathcal{O}^X \rightarrow 0 \\ & & & & e^{2\pi i} & & \\ & & & & & & H^1(X, \mathcal{O}^X) \xrightarrow{\omega} H^2(X, \mathbb{Z}) \end{array}$$

In fact  $e_u$  has better representation:

Let  $\alpha: \Lambda \rightarrow \mathbb{C}^*$  s.t

$$\alpha(u+v) = e^{i\pi E(u, v)} \alpha(u) \cdot \alpha(v) \quad \forall u, v \in \Lambda$$

Then,  $e_u = \alpha(u) e^{\pi H(z, u) + \frac{1}{2}\pi H(u, u)}$

define the line bundle  $L$  with  $\omega(L) = E \in H^2(X, \mathbb{Z})$