Lectures on Abelian Varieties

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Reference

Griffiths—Harris: Principles of Algebraic Geometry

Mumford: Abelian Varieties

Contents

- $M = V/\Lambda$ cpx tori $V \cong \mathbb{C}^n$

her. product on $V \longleftrightarrow$ inv Kähler metric on M

cpx structure

under inv metric :

harmonic form $\longleftrightarrow$ inv. forms $= \{ dz^I \wedge d\bar{z}^J \}$

$$\begin{array}{c} V \\ \downarrow \\ M = V/\Lambda \end{array} \Rightarrow H_1(M, \underset{\mathbb{Z}}{\not\mathbb{A}}) = \pi_1(M, \mathbb{Z}) = \Lambda$$

$\Lambda = \mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_{2n}$ int. basis

integral structure

real coor. $x_1, \cdots, x_{2n}$

$$\Rightarrow H^1(M, \mathbb{Z}) = \mathrm{Hom}(\Lambda, \mathbb{Z}) = \Lambda^*$$

$$H^k(M, \mathbb{Z}) = \Lambda^k(\Lambda^*)$$

$$= \mathbb{Z}(dx^I)_{|I|=k}$$

$\Lambda^* \otimes \mathbb{R} = \mathbb{R}(dx_1, \cdots, dx_{2n})$ even$/\mathbb{Z}$

Q: want a <u>integral</u> <u>positive</u> $(1,1)$ form

$$\omega = \frac{i}{2} \sum h_{\alpha\bar{\beta}} \, dz^\alpha \wedge d\bar{z}^\beta \qquad \text{(may assume inv)}$$

let Π be the $2n \times n$ matrix st. (i.e. $\tilde{\Pi} = (\Pi, \bar{\Pi})$ $2n \times 2n$)

$$\boxed{dx = \Pi \, dz + \bar{\Pi} \, d\bar{z}}$$

if <u>$\omega = \sum q_{ij} \, dx^i \wedge dx^j$</u> is an inv. int. 2-form (real)

$$= q_{ij} (\pi_{i\alpha} dz^\alpha + \bar{\pi}_{i\alpha} d\bar{z}^\alpha) \wedge (\pi_{j\beta} dz^\beta + \bar{\pi}_{j\beta} d\bar{z}^\beta)$$

$$= \underline{q_{ij} \pi_{i\alpha} \pi_{j\beta}} \, dz^\alpha \wedge dz^\beta$$

$$+ q_{ij} \pi_{i\alpha} \bar{\pi}_{j\beta} \, dz^\alpha \wedge d\bar{z}^\beta + q_{ij} \pi_{j\beta} \bar{\pi}_{i\alpha} \, d\bar{z}^\alpha \wedge dz^\beta$$

$i \leftrightarrow j \quad \alpha \leftrightarrow \beta$

$$+ \underline{q_{ij} \bar{\pi}_{i\alpha} \bar{\pi}_{j\beta}} \, d\bar{z}^\alpha \wedge d\bar{z}^\beta$$

$(1,1)$ type $\Leftrightarrow$ $\Pi^t Q \Pi = 0$

then, ω positive $\Leftrightarrow 2 \cdot \frac{2}{i} \Pi^t Q \bar{\Pi}$ is her. pos. def.

This is the <u>Riemann condition</u> (relation)

· Dual Form: (Usual Form)

recall: $\lambda_1 \dots \lambda_{2n}$ $\mathbb{Z}$ basis of $\Lambda = H_1(M,\mathbb{Z})$

$e_1, \dots, e_n$ $\mathbb{C}$ basis of $V \cong \mathbb{C}^n$

so $dz^1, \dots, dz^n$ basis of $H^0(X, \Omega^1_M) = H^{1,0}(M)$

$\Omega :=$ period matrix $n \times 2n$ $\left[\int_{\lambda_1} \vec{dz}, \int_{\lambda_2} \vec{dz} \dots, \int_{\lambda_{2n}} \vec{dz} \right]$

$dz = \Omega\, dX$ $= [\omega_{\alpha i}]$ st. $\boxed{dz^\alpha = \omega_{\alpha i}\, dx^i} \leftrightarrow \lambda_i = \omega_{\alpha i} e_\alpha$

(or. $\Lambda^t = \Omega^t E^t$) $\quad (d\bar{z}^\alpha = \bar{\omega}_{\alpha i}\, dx^i)$

so $\tilde{\Omega} := \begin{bmatrix} \Omega \\ \bar{\Omega} \end{bmatrix}$ is the inverse of $(\Pi, \bar{\Pi})$

(ie. $\begin{bmatrix} \Omega \\ \bar{\Omega} \end{bmatrix} (\Pi, \bar{\Pi}) = \begin{bmatrix} \Omega\Pi & \Omega\bar{\Pi} \\ \bar{\Omega}\Pi & \bar{\Omega}\bar{\Pi} \end{bmatrix} = I_{2n \times 2n}$ omit

$\Leftrightarrow$ $\underline{\Omega\Pi = I_n, \ \Omega\bar{\Pi}} = 0$)

R.C. $\tilde{\Pi}^t Q \bar{\tilde{\Pi}} = \begin{pmatrix} \Pi^t \\ \bar{\Pi}^t \end{pmatrix} Q (\bar{\Pi}, \Pi) = \begin{pmatrix} \Pi^t Q \bar{\Pi} & \Pi^t Q \Pi \\ \bar{\Pi}^t Q \bar{\Pi} & \bar{\Pi}^t Q \Pi \end{pmatrix}$

$$\boxed{\frac{1}{i} \tilde{\Pi}^t Q \bar{\tilde{\Pi}} = \begin{pmatrix} H & 0 \\ 0 & -H^t \end{pmatrix}} \quad H > 0 \text{ (pos. her)}.$$

$\Leftrightarrow$ $i \bar{\tilde{\Omega}} Q^{-1} \tilde{\Omega}^t = \begin{pmatrix} H^{-1} & \\ & -H^{t-1} \end{pmatrix}$ $\quad H > 0$

$\Leftrightarrow$ $\underline{\underline{\Omega Q^{-1} \Omega^t = 0 \ \ \& \ \ i \bar{\Omega} Q^{-1} \Omega^t > 0}}$

- If Q is only a $\mathbb{R}$-form (ω). then this condition is simply the condition that ω being Kähler

this is the (Hodge)-Riemann bilinear relations •

<u>Lemma (Algebra)</u>: Let $Q(\ ,\ )$ be an integral skew-sym form on Λ

then $\exists$ $\mathbb{Z}$ basis $\lambda_1, \dots, \lambda_{2n}$ of Λ st.

$$Q = \begin{bmatrix} 0 & \Delta_\delta \\ -\Delta_\delta & 0 \end{bmatrix} ; \ \Delta_\delta = \begin{bmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{bmatrix} \quad \delta_i \mid \delta_{i+1}, \ \in \mathbb{Z}$$

In term of the new basis

$$\omega = \sum_{i=1}^{n} \delta_i \, dx^i \wedge dx^{n+i}$$

ω is non-deg. $\Leftrightarrow$ $\delta_i \neq 0$

take the cpx basis of V by $\underline{e_\alpha = \frac{\lambda_\alpha}{\delta_\alpha}}$, $\alpha = 1 \dots n$

then

$$\Omega = (\Delta_\delta, Z)$$

the Riemann conditions become

$$\Omega Q^{-1} \Omega^t = 0 \Leftrightarrow [\Delta_\delta, Z] \begin{bmatrix} 0 & -\Delta_\delta^{-1} \\ \Delta_\delta^{-1} & 0 \end{bmatrix} \begin{pmatrix} \Delta_\delta \\ Z \end{pmatrix} = Z - Z^t = 0$$

ie. $\underline{Z \text{ is symmetric}}$

$$i \bar{\Omega} Q^{-1} \Omega^t > 0 \Leftrightarrow i [\Delta_\delta, \bar{Z}] \begin{bmatrix} 0 & -\Delta_\delta^{-1} \\ \Delta_\delta^{-1} & 0 \end{bmatrix} \begin{pmatrix} \Delta_\delta \\ Z \end{pmatrix}$$

$$= -i (Z - \bar{Z}^t) = -i (Z - \bar{Z}) = 2 \, \underline{\mathrm{Im} Z > 0}$$

Theorem (Riemann)

$M = V/\Lambda$ is an abelian variety $\Leftrightarrow$ $\exists$ $\mathbb{Z}$ basis of Λ and $\mathbb{C}$ basis of V st. for $dZ = \Omega \, dx$, one have

$$\Omega = (\Delta_\delta, Z)$$

st. Z is symmetric & $\mathrm{Im} Z$ is positive definite.

Rmk: in this form. $i \bar{\Omega} Q^{-1} \Omega^t = 2 \, \mathrm{Im} Z$ is actually $\underline{\text{real matrix}}$

Later will see that $[\omega] \in H^2(M, \mathbb{Z}) \cap H^{1,1}(M)$ is the 1st chern class of a line bundle $L_\delta \to M$

in fact: (*) $\underline{h^0(M, L_\delta) = \delta_1 \cdots \delta_n}$ moreover

<u>Lefschetz embedding thm:</u>

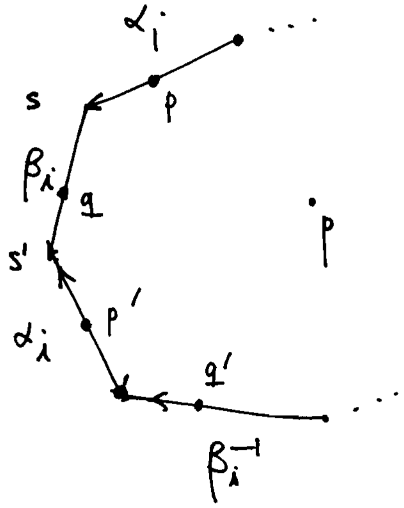
$$L \text{ ample} \Rightarrow \begin{array}{ll} L^k \text{ is b.p.f} & \forall k > 2 \\ L^k \text{ is v.a.} & \forall k \geq 3 \end{array}$$

This follows from the theory of theta functions. (later)

(*) follows from the R-R formula since $c_1(L)^n = \int_M \omega^n = n! \cdot \prod \delta_\alpha$

and K-V $\Rightarrow$ $\underline{h^0(L) = \chi(L) = \frac{L^n}{n!}}$ since M is flat.

X : R.S genus $= g$. Abel-Jacobi $X \to J(X) = \mathbb{C}^g/\Gamma$



ω holo. 1-form

Δ simply conn.

$\omega = d\varphi$ with $\varphi = \int_{p_0}^{z} \omega$ holo. fun.

$$\underline{\varphi(p') - \varphi(p)} = \int_p^{p'} d\varphi = \int_p^{p'} \omega$$

$$= \cancel{\int_p^s \omega} + \int_s^{s'} \omega + \cancel{\int_{s'}^{p'} \omega}$$

$$= \underline{\int_{\beta_i} \omega} \quad \text{indep of the position of } p!$$

Same reason : $\underline{\varphi(q') - \varphi(q) = -\int_{\alpha_i} \omega}$

Basic integration identity :

$$\int_X \omega \wedge \eta = \int_{\partial\Delta} \varphi\eta = \sum_{i=1}^{g} \int_{\alpha_i + \alpha_i^{-1}} \varphi\eta + \int_{\beta_i + \beta_i^{-1}} \varphi\eta$$

$$= \sum_{i=1}^{g} \left(-\int_{\beta_i} \omega \cdot \int_{\alpha_i} \eta + \int_{\alpha_i} \omega \int_{\beta_i} \eta \right)$$

for ω holo. η any 1-form.

we may normalize $\omega_1 \cdots \omega_g$ s.t. $\int_{\alpha_i} \omega_j = \delta_{ij}$

then the period matrix $\underline{\Omega = (I, Z)}$ $\quad Z_{ij} = \int_{\beta_i} \omega_j$

$$0 = \int_X \omega_i \wedge \omega_j = \sum_k \left(-\int_{\beta_k} \omega_i \int_{\alpha_k} \omega_j + \int_{\alpha_k} \omega_i \int_{\beta_k} \omega_j \right)$$

$$= -Z_{ji} + Z_{ij} \Rightarrow \underline{Z \text{ is symmetric}}$$

$$\text{"}0 <\text{"}\ i \int \omega_i \wedge \overline{\omega_j} = i \cdot \sum_k \left(-Z_{ji} + \overline{Z_{ij}} \right) = 2\,\mathrm{Im}\, Z_{ij}$$

$\Rightarrow$ <u>$\mathrm{Im}\, Z$ is positive definite</u>.

hence : $\mathbb{C}^g/\Gamma$ is an principally polarized abelian variety.

the case $n=1$: $\mathbb{C}/\Lambda$ elliptic curves :

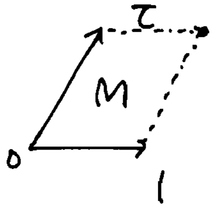
Λ has a basis $\mathbb{Z} \oplus \mathbb{Z}\tau$. $\operatorname{Im}\tau > 0$

i.e. $\lambda_1 = 1$, $\lambda_2 = \tau$ $\quad e_\alpha = e_1 = 1$

hence $\lambda_i = \omega_{\alpha i}\, e_\alpha \Rightarrow \omega_1 = 1,\ \omega_2 = \tau$

i.e. $\Omega = (\omega_{\alpha i}) = \underline{(1, \tau)}$ $\quad$ 1×2 matrix

i.e. $\underline{\Delta_\delta = 1.\ \ Z = \tau.}$



moduli space of elliptic curves

$\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ and $\Lambda' = \mathbb{Z} \oplus \mathbb{Z}\tau'$

determine the same torus $\Leftrightarrow$ $\Lambda' = \mu \cdot \Lambda$ for some $\mu \in \mathbb{C}$

i.e. $\exists \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$ s.t.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \tau \\ 1 \end{bmatrix} = \begin{bmatrix} \mu\tau' \\ \mu \end{bmatrix}$$

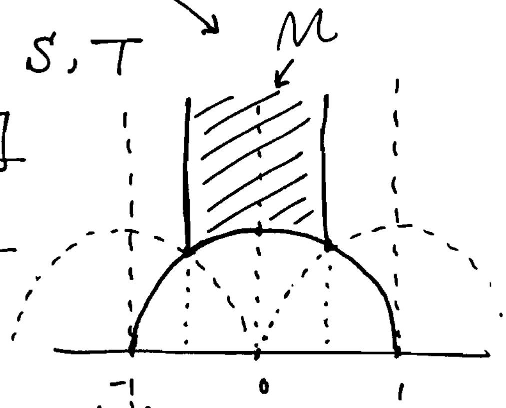
Define $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \dfrac{az+b}{cz+d}$ möbius transform

get Moduli of $\mathcal{M} = \mathbb{H}/SL(2,\mathbb{Z})$

<u>Lemma</u> : $SL(2,\mathbb{Z})$ is generated by S, T

$S := z \mapsto z+1 = \underline{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}$

$T := z \mapsto \frac{-1}{z} = \underline{\begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}}$



General n :

for principally polarized abelian varieties :

$\Omega = (\Delta, Z)$, $\delta \equiv 1$

$\mathcal{H} = \{ Z \subset \mathbb{C}^{\frac{n(n+1)}{2}}$: with $\operatorname{Im} Z > 0 \}$

this is the ~~Segal~~ Siegal upper half space

$\underline{\mathcal{M}_g = \mathcal{H}_g / Sp(2g, \mathbb{Z})}$ symmetric space. (待續).

Line bundles on $V/\Lambda = X$:

(1) any line bundle on $\mathbb{C}^n$ is trivial :

$$H^1(\mathbb{C}^n, \mathcal{O}) \to H^1(\mathbb{C}^n, \mathcal{O}^\times) \to H^2(\mathbb{C}^n, \mathbb{Z})$$

$H^1(\mathbb{C}^n,\mathcal{O}) = 0$ (Dolbeault lemma), $H^2(\mathbb{C}^n,\mathbb{Z}) = 0$

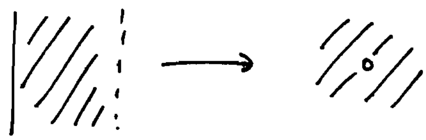
(2) any line bundle on $\mathbb{C}^{\times n}$ is determined by c_1 :

$$H^1(\mathbb{C}^{\times n}, \mathcal{O}) \to H^1(\mathbb{C}^{\times n}, \mathcal{O}^\times) \xrightarrow{c_1} H^2(\mathbb{C}^{\times n}; \mathbb{Z}) \to H^2(\mathbb{C}^{\times n}, \mathcal{O})$$

$H^1(\mathbb{C}^{\times n},\mathcal{O}) = 0$, $H^2(\mathbb{C}^{\times n},\mathcal{O}) = 0$

(bec. original pf of Dolbeault lemma also applies to the case $\mathbb{C}^\times$. Ex 15.
or use the fact that $\mathbb{C}^{\times n}$ is an <u>affine variety</u>)

now : $\mathbb{C}/\mathbb{Z} \xrightarrow[e^{2\pi i z}]{} \mathbb{C}^\times$



hence $\mathbb{C}^n \cong V \longrightarrow V/\mathbb{Z}\langle \lambda_1, \dots, \lambda_n \rangle \cong \mathbb{C}^{\times n}$ bec. $e_\alpha = \frac{\lambda_\alpha}{\delta_\alpha}$ $\alpha = 1 \dots n$

$e^{2\pi i x} \cdot e^{-2\pi y}$

$\pi : V \to X$, $\pi_1 : V/\mathbb{Z}\langle \lambda_1, \dots, \lambda_n \rangle \to X$

We want to consider line bundle $L \downarrow X$ with $\underline{c_1(L) = \omega}$
$\underline{= \sum \delta_\alpha \, dX^\alpha \wedge dX^{n+\alpha}}$:

Step 1. (1) $\Rightarrow$ $\pi^* L \to \mathbb{C}^n$ is trivial

fix a trivialization $\varphi : \pi^* L \longrightarrow V \times \mathbb{C}$

$\varphi_z \longrightarrow \varphi_{z+\lambda}$ differ by linear auto.

$\Rightarrow$ $\exists$ holomorphic functions $e_\lambda(z) \in \mathcal{O}^\times(\mathbb{C}^n)$, $\forall \lambda \in \Lambda$

st. $\underline{e_{\lambda+\lambda'}(z) = e_{\lambda'}(z+\lambda) \cdot e_\lambda(z)}$ (*)

which is determined by $e_{\lambda_i}(z)$ $i = 1 \dots 2n$

which is equiv to (*)': for basis λ_α : $e_{\lambda_\alpha}(z+\lambda_\beta) \cdot e_{\lambda_\beta}(z) = e_{\lambda_\beta}(z+\lambda_\alpha) \cdot e_{\lambda_\alpha}(z)$

Step 2. (2) $\Rightarrow$ $\pi_1^*(\delta_\alpha dx_\alpha) = \pi_1^*(dz_\alpha) = 0$

hence $\pi_1^* \omega = 0$. ie. if φ respects π_1 then may let

$$\underline{e_{\lambda_\alpha}(z) \equiv 1, \quad \alpha = 1 \dots n}$$

step 3. Recall $Z = Z^t$, $\operatorname{Im} Z > 0$, $\Omega = (\Delta_\delta, Z)$

Let $Z = X + iY$, $dz = \Omega\, dX$

the main observation is:

$$\underline{\omega = \sum \delta_\alpha\, dx^\alpha \wedge dx^{n+\alpha} = \frac{i}{2} \sum Y^{\alpha\beta}\, dz^\alpha \wedge d\bar{z}^\beta}$$

pf: $\sum Y^{\alpha\beta}\, dz^\alpha \wedge d\bar{z}^\beta$

$= \sum Y^{\alpha\beta} (\delta_\alpha dx^\alpha + Z_{\alpha p}\, dx^{n+p}) \wedge (\delta_\beta dx^\beta + \bar{Z}_{\beta q}\, dx^{n+q})$

$= \sum Y^{\alpha\beta} \cancel{\delta_\alpha \delta_\beta}\, dx^\alpha \wedge dx^\beta$

$+ \sum Y^{\alpha\beta} (Z_{\alpha p} \delta_\beta\, dx^{n+p} \wedge dx^\beta + \delta_\alpha \bar{Z}_{\beta q}\, dx^\alpha \wedge dx^{n+q})$

$+ \sum Y^{\alpha\beta} Z_{\alpha p} \bar{Z}_{\beta q}\, dx^{n+p} \wedge dx^{n+q}$

eq. $Y^{qp} Z_{qp} \cdot \bar{Z}_{pq} = Y^{pq} Z_{pq} \cdot \bar{Z}_{qp}$

$= \sum Y^{\alpha\beta} (\bar{Z}_{\beta q} - Z_{\beta q})\, \delta_\alpha\, dx^\alpha \wedge dx^{n+q}$

$= -2i \sum \underline{Y^{\alpha\beta} Y_{\beta q}}\, \delta_\alpha\, dx^\alpha \wedge dx^{n+q} = -2i\,\omega$ #

step 4. Want to find positive function $h(z)$ st

$\frac{i}{2\pi} [-\partial\bar{\partial} \log h] = \omega$; Let $K = \log h$. ie.

$$\partial\bar{\partial} K = -\frac{2\pi}{i} \cdot \frac{i}{2} \sum Y^{\alpha\beta}\, dz^\alpha \wedge d\bar{z}^\beta$$

the natural choice is then $K(z) = -\pi \sum Y^{\alpha\beta} z^\alpha \bar{z}^\beta$

but this function should be inv under $z \mapsto z + \lambda_\alpha$, $\alpha = 1 \dots n$

ie. inv. under $z^\alpha \mapsto z^\alpha + 1$. so use

$$\underline{\log h(z)} = K(z) = \frac{1}{2}\pi \sum Y^{\alpha\beta} (z^\alpha - \bar{z}^\alpha)(z^\beta - \bar{z}^\beta) \quad (\text{内有 } 2\text{ 項})$$

$$= \underline{-2\pi \sum Y^{\alpha\beta} \operatorname{Im} z^\alpha \cdot \operatorname{Im} z^\beta}$$

step 5. Find $e_{\lambda_{n+\gamma}}$:

Wish h defines metric on L. hence h satisfies

$$** \quad \underline{h(z + \lambda_{n+\gamma}) = |e^{-1}_{\lambda_{n+\gamma}}(z)|^2\, h(z)}$$

$$K(z + \lambda_{n+\gamma}) = -2\pi \sum Y^{\alpha\beta} \operatorname{Im}(z^\alpha + Z_{\alpha\gamma}) \operatorname{Im}(z^\beta + Z_{\beta\gamma})$$

$$= K(z) - 4\pi \sum Y^{\alpha\beta} Y_{\alpha\gamma} \operatorname{Im} z^\beta - 2\pi \sum Y^{\alpha\beta} Y_{\alpha\gamma} Y_{\beta\gamma}$$

$$= K(z) - 4\pi \operatorname{Im} z^\gamma - 2\pi Y_{\gamma\gamma}$$

$\left(** : \underline{\|\tilde{\theta}(z)\|^2} = h(z) |\theta(z)|^2 = \underline{\|\tilde{\theta}(z+\lambda)\|^2} = h(z+\lambda) |\theta(z+\lambda)|^2 \right)$

Let $e_{\lambda_{n+\gamma}}(z) = e^{a_\gamma(z)}$

then $K(z + \lambda_{n+\gamma}) = -2\,\mathrm{Re}\, a_\gamma(z) + K(z)$

$\Rightarrow$ $\mathrm{Re}\, a_\gamma(z) = +2\pi\, \mathrm{Im}\, z^\gamma + \pi Y_{\gamma\gamma}$

may take $\underline{a_\gamma(z) = -2\pi i z^\gamma + \pi Y_{\gamma\gamma}}$ ↑ const. c^γ

In fact it's easy to see that

$$\begin{cases} e_{\lambda_\alpha} \equiv 1 & : \quad \alpha = 1 \dots n \\ e_{\lambda_{n+\alpha}}(z) = e^{-2\pi i z^\alpha + c^\alpha} & : \quad {}^\alpha \text{ any const.} \end{cases}$$

satisfies the compatibility condition $(*)$:

$$\left(\begin{aligned} & e_{\lambda_{n+\alpha}}(z+\lambda_{n+\beta}) \cdot e_{\lambda_{n+\beta}}(z) \\ = \;& e^{-2\pi i (z^\alpha + Z_{\alpha\beta}) + c^\alpha} \cdot e^{-2\pi i z^\beta + c^\beta} \\ = \;& e^{-2\pi i ((z^\alpha + z^\beta) + Z_{\alpha\beta}) + (c^\alpha + c^\beta)} \qquad \text{sym. in } \alpha, \beta \end{aligned} \right)$$

Step 6. meaning of the const. c^α:

$\equiv$ line bundles with the same c_1 :

let $\mu \in X$ and $\tau_\mu : X \to X$ $\quad z \mapsto z + \mu$ translation by μ

then τ_μ homotopic to id_X hence $c_1(\tau_\mu^* L) = c_1(L)$

the new $\underline{e'_\lambda(z) := e_\lambda(z+\mu)}$ ($\tau_\mu^* L$ top. isom to L but not holomorphically)

$$\begin{cases} e'_{\lambda_\alpha}(z) = e_{\lambda_\alpha}(z+\mu) = 1 & \alpha = 1 \dots n \\ e'_{\lambda_{n+\alpha}}(z) = e_{\lambda_{n+\alpha}}(z+\mu) = e^{-2\pi i (z+\mu)^\alpha + c^\alpha} \\ \qquad = e^{-2\pi i z^\alpha + \underbrace{(c^\alpha - 2\pi i \mu^\alpha)}_{\text{const.}}} \end{cases}$$

Fact: $c_1(L) = 0$

exp sequence of top vs holo.

$\Leftrightarrow$ flat line bundle

$\Leftrightarrow$ constant transition function

$\Leftrightarrow$ constant trivialization $e_\lambda(0) = \begin{cases} 1 & \lambda = \lambda_\alpha \\ e^{c^\alpha} & \lambda = \lambda_{n+\alpha} \end{cases}$

this can also be seen from

$$0 \to \mathrm{Pic}^0(X) \to H^1(X, \mathcal{O}^\times) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$ principal homog. space.

$\mathrm{Pic}^0(X) = H^1(X, \mathcal{O}) / H^1(X, \mathbb{Z})$ dual a.v. of $X = \check{X}$ acting on X ~~by translation~~?

$$\boxed{\begin{array}{l}\text{Dimension Formula (R-R):} \\ \text{if } \omega(L) = \sum \delta_\alpha \, dx^\alpha \wedge dx^{n+\alpha} \\ \text{then } \quad h^0(X, \mathcal{O}(L)) = \delta_1 \cdots \delta_n .\end{array}}$$

pf: $h^0(\mathcal{O}(L))$ is inv. under translation, may consider L

st. $\quad \theta(z+\lambda_\alpha) = \theta(z) \qquad \alpha = 1 \dots n$

$$\theta(z+\lambda_{n+\alpha}) = e^{-2\pi i z^\alpha - \pi i Z_{\alpha\alpha}} \, \theta(z)$$

(corr. to transl. by $\underline{\mu = \frac{1}{2} \sum Z_{\alpha\alpha} e_\alpha}$)

Q: what's the meaning of this choice μ ?

so θ is periodic in $z^1, \dots, z^n$ with periods $\delta_1, \dots, \delta_n$

$$\Rightarrow \quad \theta(z) = \sum_{l \in \mathbb{Z}^n} a_l \, e^{\frac{2\pi i}{\delta_1} l_1 z^1} \cdots e^{\frac{2\pi i}{\delta_n} l_n z^n}$$

$$= \sum_{l \in \mathbb{Z}^n} a_l \, e^{2\pi i \langle l, \Delta_\delta^{-1} z \rangle}$$

$$\theta(z+\lambda_{n+\alpha}) = \sum_{l \in \mathbb{Z}^n} \underline{a_l} \, e^{2\pi i \langle l, \Delta_\delta^{-1} z \rangle} \cdot \underline{e^{2\pi i \langle l, \Delta_\delta^{-1} \lambda_{n+\alpha} \rangle}}$$

$$\parallel$$

$$e^{-2\pi i z^\alpha - \pi i Z_{\alpha\alpha}} \cdot \sum_{l \in \mathbb{Z}^n} a_l \, e^{2\pi i \langle l, \Delta_\delta^{-1} z \rangle}$$

$$= \sum_{l \in \mathbb{Z}^n} \underline{e^{-\pi i Z_{\alpha\alpha}} \cdot a_l} \, e^{2\pi i \langle \underline{l - \Delta_\delta e_\alpha}, \, \Delta_\delta^{-1} z \rangle}$$

$$\Rightarrow \quad a_{l+\lambda_\alpha} = e^{2\pi i \langle l, \Delta_\delta^{-1} \lambda_{n+\alpha} \rangle + \pi i Z_{\alpha\alpha}} a_l \quad (***)$$

hence θ is completely determined by

$$a_l \ ; \ l \in \mathbb{Z}^n / \langle \lambda_1, \dots, \lambda_n \rangle \ \Rightarrow \ \dim \le \prod_{\alpha=1}^n \delta_\alpha$$

Now

$$\theta(z) = \sum_{\substack{\xi \\ 0 \le \xi_\alpha < \delta_\alpha}} \sum_{l \in \mathbb{Z}^n} a_{\xi + \Delta_\delta l} \, e^{2\pi i \langle \xi + \Delta_\delta l, \, \Delta_\delta^{-1} z \rangle}$$

$$= \sum_\xi e^{2\pi i \langle \xi, \Delta_\delta^{-1} z \rangle} \cdot \underbrace{\sum_{l \in \mathbb{Z}^n} a_{\xi+\Delta_\delta l} \, e^{2\pi i \langle l, z \rangle}}_{\substack{\parallel \\ \theta_\xi(z) \\ \parallel \\ \sum_l b_l e^{2\pi i \langle l, z \rangle}}}$$

Solve relation $***$ gives

$$b_l = a_{\xi + \Delta_\delta l} = e^{2\pi i \langle \xi, \Delta_\delta^{-1} Z l \rangle + \pi i \langle l, Z l \rangle}$$

(basically: $2(1+2+\dots+(n-1)) + \underbrace{(1+\dots+1)}_{n}) = n^2$)

This is the reson to put μ.

Need to verify that $\theta_\xi(z)$ converges :

but $|b_\ell| = e^{-2\pi \langle \xi, \Delta_\delta^{-1} Y \ell \rangle - \pi \langle \ell, Y\ell \rangle}$ ↘ $2\mathcal{R}$.

$\leq e^{-c\|\ell\|^2}$ as $\|\ell\| \nearrow$.

and only has $\leq \ell^n$ terms with size $\leq \ell$. hence OK.

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when $\Delta_\delta = I_n$, $\Omega = (I_n, Z)$

X is called principally polarized a.V.

then $h^0(X, \mathcal{O}(L)) = 1$. called the Riemann theta function:

$$\vartheta(z) := \sum_{\ell \in \mathbb{Z}^n} e^{2\pi i \langle \ell, z \rangle + \pi i \langle \ell, Z\ell \rangle}$$

$$\text{st.} \begin{cases} \vartheta(z + \underline{\underline{e_\alpha}}) = \vartheta(z) & \text{periodic} \\ \vartheta(z + \underbrace{\lambda_{n+\alpha}}_{= Z_\alpha}) = e^{-2\pi i z^\alpha - \pi i Z_{\alpha\alpha}} \vartheta(z). \end{cases}$$

all are periods Ω_i in Ω.

$\Theta = (\vartheta)$ is called the <u>theta divisor</u>

In fact, for general $\Omega = (\Delta_\delta, Z)$

Let $\Lambda' = \mathbb{Z}(e_1, \cdots, e_n, \lambda_{n+1}, \cdots \lambda_{2n})$, $X' = V/\Lambda'$

then get covering map

$$\pi' : X \longrightarrow X'$$

then $\Omega' = (I_n, Z)$, X' is prin. polarized :

$$\omega = \sum \delta_\alpha \, dx^\alpha \wedge dx^{n+\alpha} \equiv \sum dx'^\alpha \wedge dx'^{n+\alpha}$$

so, $\exists$ line bundle $L' \to X'$ st. $\underline{\pi'^* L' = L}$

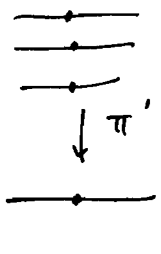
$$\theta(z) = \sum_{\xi \in \Lambda'/\Lambda} e^{2\pi i \langle \xi, \Delta_\delta^{-1} z \rangle} \cdot \theta_\xi(z)$$

notice that $\underline{\theta_0(z) = \vartheta(z)}$ and $\underline{\theta_\xi \text{ "=" } T_\xi^* \vartheta(z)}$ "translates".

??

Principle: General results about abelian v.

can be reduced to the case of principally polarized abelian v. .



$X = V/\Lambda$

π'

$X' = V/\Lambda'$

$\boxed{\text{Lefschetz embedding thm: } H^0(X, L^{\otimes k}) \text{ gives proj. embed. for } k \geq 3}$

pf: only need to do the case $\Delta_\delta = I_n$:

- base point free: ($k \geq 2$ is enough)
  $z_0 \in X$, need to find $\varphi \in H^0(L^{\otimes 2})$ st. $\varphi(z_0) \neq 0$
  consider $\varphi(z) = \vartheta(z+\mu) \cdot \vartheta(z-\mu) \in H^0(L^{\otimes 2})$
  for some μ st. $\vartheta(z_0+\mu) \neq 0 \neq \vartheta(z_0-\mu)$. (not individual)
- seperate points (1-1) :
  If $\theta_0, \theta_1, \cdots, \theta_N$ be a basis of $H^0(X, L^{\otimes 3})$ st.
  $$\theta_i(z_1) = \rho\, \theta_i(z_2) \quad \forall i$$
  In particular, $\dfrac{\vartheta(z_1+\mu)\,\vartheta(z_1+\nu)\,\vartheta(z_1-\mu-\nu)}{\vartheta(z_2+\mu)\,\vartheta(z_2+\nu)\,\vartheta(z_2-\mu-\nu)} = \rho$
  for all μ, ν. (they are all in $H^0(X, L^{\otimes 3})$)
  claim $\dfrac{\vartheta(z_1+z)}{\vartheta(z_2+z)}$ is non zero holo (entire)
  trivial: for any $z = \mu$. find ν st. $\frac{a}{b}\frac{c}{d}\frac{e}{f} = \rho$
  $cedf \neq 0$. ※

But then $\psi(z) := \log \dfrac{\vartheta(z_1+z)}{\vartheta(z_2+z)}$ is entire. st.

$$\begin{cases} \psi(z+e_\alpha) = \psi(z) \\ \psi(z+\lambda_{n+\alpha}) = \psi(z) + \left(-2\pi i\left[(z_1+z)^\alpha - (z_2+z)^\alpha\right]\right) \\ \qquad\qquad\quad = \psi(z) - 2\pi i \cdot \underbrace{(z_1-z_2)^\alpha}_{\text{const.}} \end{cases}$$

$\Rightarrow \dfrac{\partial \psi}{\partial z^i} =$ constant (bec. periodic in all directions)

$\Rightarrow \psi(z) =$ linear $\Rightarrow$ ※ can't be periodic in $z^1 \ldots z^n$.

Exercise 15. Complete the proof by showing that

$$J(z) := \begin{pmatrix} \theta_0(z) & \cdots & \theta_N(z) \\ \partial_1\theta_0(z) & \cdots & \partial_1\theta_N(z) \\ \vdots & \ddots & \vdots \\ \partial_N\theta_N(z) & \cdots & \partial_N\theta_N(z) \end{pmatrix}$$

has full rank. hence $H^0(X, L^{\otimes 3}) : X \longrightarrow \mathbb{P}^N$
is an immersion. Q.E.D.

D. Mumford. Abelian varieties.

$V \xrightarrow{\pi} X = V/\Lambda$ line bundle $L \to X$, $\pi^* L \to V$ is trivial p.7

Λ acts on $V \times \mathbb{C}$ holomorphically

$u \in \Lambda$, $\phi_u : (z, \alpha) \mapsto (z+u, e_u(z) \cdot \alpha)$

compatibility: $e_u \in \mathcal{O}^\times(V)$

$$\phi_u \circ \phi_{u'}(z, \alpha) = \phi_{u+u'}(z, \alpha) = (z+u+u', \underline{e_{u+u'}(z) \cdot \alpha})$$

$$\phi_u(z+u', e_{u'}(z) \cdot \alpha)$$

$$(z+u'+u, \underline{e_u(z+u') \cdot e_{u'}(z) \cdot \alpha})$$

let $e_u(z) = e^{2\pi i f_u(z)}$

$$\Rightarrow f_u(z+u') + f_{u'}(z) = f_{u+u'}(z) \pmod{\mathbb{Z}}$$

$c_1(L)$ is given by the integral 2 form (for any $z \in V$):

$$E(u,v) = f_v(z+u) + f_u(z) - f_u(z+v) - f_v(z)$$

$$\boxed{\begin{array}{c} 0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{e^{2\pi i}} \mathcal{O}^\times \to 0 \\ H^1(X, \mathcal{O}^\times) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \end{array}}$$

since $E(iu, iv) = E(u,v)$ (type (1,1))

may set Her. form $\underline{H(x,y) = E(ix, y) + i E(x,y)}$

In fact e_u has better representation:

Let $\alpha : \Lambda \to \mathbb{C}^\times$ st

$\underline{\alpha(u+v) = e^{i\pi E(u,v)} \alpha(u) \cdot \alpha(v)}$ $\forall u, v \in \Lambda$

Then, $\underline{e_u = \alpha(u) e^{\pi H(z,u) + \frac{1}{2}\pi H(u,u)}}$

define the line bundle L with $c_1(L) = E \in H^2(X, \mathbb{Z})$)

Algebraic Theory of Abelian Varieties

Def'n: An ab. var X is a proper alg. gp. $/k$
when $k = \mathbb{C}$, analytic method $\Rightarrow$ $X \cong \mathbb{C}^n/\Lambda$
st Λ has the Riemann condition.

In general we ask:

Q1: X as an abstract gp.

- X is comm. and divisible
- $X_n := \ker n_X : X \to X \quad a \mapsto \underbrace{a + \cdots + a}_{n} = na$

$\begin{cases} X_n \cong (\mathbb{Z}/n\mathbb{Z})^{2g} & \text{if char } k \nmid n \\ X_{p^m} \cong (\mathbb{Z}/p^m\mathbb{Z})^{i} & \text{if } p = \text{char } k,\ m > 0. \end{cases}$

Q2: Compute $H^q(X, \Omega^p)$?

Q3: Structure of $\mathrm{Pic}(X)$?

Q4: characterize ample line bundle, Riemann-Roch.
this part require Riemann form?

Recall: $f : X \to Y$ proper $\mathcal{F}$ coh. sheaf on X

Thm: $R^p f_* \mathcal{F}$ is coherent on Y $\forall p$.

$R^p f_* \mathcal{F}$ is the sh. asso. to $U \mapsto H^p(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)})$.

Main Q: Relation between $(R^p f_* \mathcal{F})_y$ and $H^p(X_y, \mathcal{F}|_{X_y})$?

* Theorem (Cohomology & Base change):

(a) the function $y \mapsto h^p(X_y, \mathcal{F}_y)$ is upper semi-cont.
(ie. can jump up!)

(b) $y \mapsto \chi(\mathcal{F}_y)$ is locally constant.

(c) TFAE: (for Y reduced & connected) (i) $h^p(X_y, \mathcal{F}_y)$ is constant in y.
(ii) $R^p f_* \mathcal{F}$ is locally free and
$\varphi_p : (R^p f_* \mathcal{F}) \otimes_{\mathcal{O}_Y} k(y) \xrightarrow{\sim} H^p(X_y, \mathcal{F}_y)$.

Moreover, then both $\Rightarrow$ φ^{p-1} is also an isom $\forall y$.

(d) Y not nec. reduced, $H^p(X_y, \mathcal{F}_y) = 0$ $\forall y$ $\Rightarrow$ φ_{p-1} isom $\forall y$.

Seesaw Theorem.

let $X \times T$ (X proper var, T any), L line bundle on $X \times T$, then $T_1 := \{ t \in T \mid L|_{X \times (t)}$ is trivial on $X \times (t) \}$ is <u>closed</u> in T.

trivial loci.

Moreover, $L|_{X \times T_1} \cong p_2^* M$ for some line bundle M on $X \times T_1$.

pf: A line bd $\mathcal{L}$ on proper X is trivial
$\iff \Gamma(X, \mathcal{L}) \neq 0$ and $\Gamma(X, \mathcal{L}^{-1}) \neq 0$

(if X is projective then it's trivial by int. with certain curve.)

bec. if $\sigma : \mathcal{O}_X \to \mathcal{L}$ a section

$\tau : \mathcal{O}_X \to \mathcal{L}^{-1} \Rightarrow \tilde{\tau} : \mathcal{L} \to \mathcal{O}_X$

then $\tilde{\tau} \circ \sigma : \mathcal{O}_X \to \mathcal{O}_X$ is a nowhere 0 section.

X proper $\Rightarrow \tilde{\tau} \circ \sigma$ is a const. $\Rightarrow \sigma, \tau$ both isom.

On T_1; the above $\Rightarrow T_1 = \{ x \in X, h^0(L|_{X \times (t)}) > 0$ & $h^0(L^{-1}|_{X \times (t)}) > 0 \}$

T_1 is closed is just the upper <u>semi-continuity</u> thm.

Replace T by T_1. So $L|_{X \times (t)}$ = trivial $\forall t \in T$.

so $h^0(X \times (t), L|_{X \times (t)}) = 1$

by Cor 2. (<u>coh & base change</u>), $\Rightarrow p_{2*} \mathcal{L} = M$

M is a locally free sheaf of rk = 1. i.e. line bd. on T.

Moreover,

$$M \otimes_{\mathcal{O}_T} k_{\text{at } t} \simeq H^0(X \times \{t\}, L|_{X \times \{t\}})$$

hence $p_2^* M \xrightarrow{\sim} L$ is an isom. ※

Theorem (of the cube):

X, Y, Z variety with X, Y proper

x_0, y_0, z_0 base points. L line bundle on $X \times Y \times Z$

if $L|_{x_0 \times Y \times Z}$, $L|_{X \times y_0 \times Z}$, $L|_{X \times Y \times z_0}$ trivial $\Rightarrow$ L is trivial.

Rmk: In char 0 this follows from Lefschetz (1,1) thm + Künneth formula.

thm is equiv. to

cor 1. $L \cong p_{12}^*(L_{12}) \otimes p_{13}^*(L_{13}) \otimes p_{23}^*(L_{23})$.

cor 2. $f, g, h : X \to Y$ - ab. var.
any variety

$\forall L \in \mathrm{Pic}(Y)$:

$$(f+g+h)^* L \cong (f+g)^* L \otimes (f+h)^* L \otimes (g+h)^* L \otimes f^* L^{-1} \otimes g^* L^{-1} \otimes h^* L^{-1}.$$

pf: $p_i : Y \times Y \times Y \to Y$

$m_{ij} := p_i + p_j : Y \times Y \times Y \to Y$

$m = p_1 + p_2 + p_3$:

$M := m^* L \otimes m_{12}^* L^{-1} \otimes m_{13}^* L^{-1} \otimes m_{23}^* L^{-1} \otimes p_1^* L \otimes p_2^* L \otimes p_3^* L$

let $q : Y \times Y \to Y \times Y \times Y$, $(y, y') \mapsto (0, y, y')$

then $q^* M = \underbrace{\eta^* L}_{\text{from } m} \otimes q_1^* L^{-1} \otimes q_2^* L^{-1} \otimes \underbrace{\eta^* L^{-1}}_{\text{from } m_{23}} \otimes 0^* L \otimes q_1^* L \otimes q_2^* L$

= trivial.

$= M$; view as on $(0) \times Y \times Y$.

Similarly, trivial on $Y \times (0) \times Y$, $Y \times Y \times (0)$.

$\Rightarrow M$ is trivial

Result follows from Pullback M via

$(f, g, h) : X \to Y \times Y \times Y$ $\#$.

<u>Cor 3.</u> X ab. var. $n \in \mathbb{Z}$, $L \in \mathrm{Pic}(X)$, then

$$n_X^* L \cong L^{\otimes \frac{1}{2}n(n+1)} \otimes (-1_X)^* L^{\otimes \frac{1}{2}n(n-1)}.$$

Pf: Let $f = (n+1)_X$, $g = 1_X$, $h = (-1)$ in Cor 2.

$\Rightarrow (n+1)_X^* L \cong (n+2)_X^* L \otimes n_X^* L \otimes 0_X^* L \otimes (n+1)_X^* L^{-1} \otimes 1_X^* L^{-1} \otimes (-1_X)^* L^{-1}$

i.e. $\underline{(n+2)_X^* L \otimes (n+1)_X^* L^{-2} \otimes n_X^* L} \cong \underline{L \otimes (-1_X)^* L}$.

= 階差數列 　　　　 indep. of n

eg. $a_{n+2} - 2a_{n+1} + a_n = c$

let $b_n = a_{n+1} - a_n$ then get $b_{n+1} - b_n = c$

$\vdots$

$\Rightarrow \underline{b_{n+1}} = (\underline{n+1})c + b_0$ 　　 $b_1 - b_0 = c$

so $a_n - a_{n-1} = nc + b_0$

$\vdots$

$a_1 - a_0 = c + b_0$ $\Rightarrow$ $a_n = \underbrace{\frac{n(n-1)}{2}}\, c + \underbrace{nb_0} + \underbrace{a_0}$

$\Rightarrow n_X^* L \cong (L \otimes (-1_X)^* L)^{\otimes \frac{1}{2}n(n-1)} \otimes M_1^n \otimes M_2$

$n=0 \Rightarrow M_2$ trivial, $n=1 \Rightarrow$

$M_1 \cong L$ #

<u>Cor 4.</u> (Thm of square) for $x \in X$ let $T_x : X \to X$, $a \mapsto a+x$

then $T_{x+y}^* L \otimes L \cong T_x^* L \otimes T_y^* L$. So,

let $\phi_L : X \to \mathrm{Pic}(X) : x \mapsto [T_x^* L \otimes L^{-1}]$

is a group homomorphism.

Pf: let $x = y$, f = const map to x, g = " to y, $h = id_X$.

in Cor 2. get result.

clearly then $\phi_L(x+y) = [T_{x+y}^* L \otimes L^{-1}]$

$\cong [T_x^* L \otimes L^{-1}] \otimes [T_y^* L \otimes L^{-1}]$

$= \phi_L(x) \otimes \phi_L(y)$ #

Rmk: better to write τ_x instead of T_x.

By definition:

(a) $\phi_{L_1 \otimes L_2} = \phi_{L_1} + \phi_{L_2}$

$\otimes$ law in $\operatorname{Pic}(X)$

(b) $\phi_{T_x^* L} = \phi_L$.

let $K(L) := \operatorname{Ker} \phi_L = \{ x \in X : T_x^* L \cong L \}$

it is a Zariski closed subgp of X.

pf: consider $m^* L \otimes p_2^* L^{-1}$ on $X \times X$

addition map. $p_1 + p_2$

then $\{ x \in X : m^* L \otimes p_2^* L^{-1} |_{\{x\} \times X} \text{ trivial} \}$

is Zariski closed. (by Coh. & Base change thm)

But $m^* L \otimes p_2^* L^{-1} |_{\{x\} \times X} \cong T_x^* L \otimes L^{-1}$ on X

$\Rightarrow$ this set is $K(L)$ $\#$.

Application I.

let $D \subset X$ - ab. var, eff. div., $L := L(D)$. TFAE:

① $H := \{ x \in X : T_x^* D = D \}$ is finite.

equality of divisors.

② $K(L)$ is finite.

③ ($|2D|$ is bpf. &) $\Phi_{L^{\otimes 2}} : X \to \mathbb{P}^N$ a finite morphism

④ L is ample on X.

pf: ① $\leftarrow$ ② $\leftarrow$ ④ $\leftarrow$ ③

③ $\Rightarrow$ ④ is a general fact (Nakai-Moishezon Criterion)

④ $\Rightarrow$ ②: If $K(L)$ not finite, then $\dim K(L) \geq 1$ since it is an alg. var.

let Y = conn. comp. of 0 of it.

i.e. Y is an ab. var. of $\dim Y \geq 1$.

now L ample $\Rightarrow$ $L_Y := L|_Y$ is ample on Y.

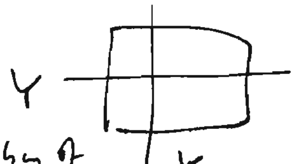
also $\tau_y^* L_Y \cong L_Y \quad \forall y \in Y$.

Consider $m^* L_Y \otimes p_1^* L_Y^{-1} \otimes p_2^* L_Y^{-1}$ on $Y \times Y$ $\qquad p_i : Y \times Y \to Y$

But then it is trivial on all slices $\qquad m = p_1 + p_2$.

?!

Notice this is $\forall$ slices. In thm of cube only "one slice".



$\underline{\text{Coh. \& Base change}}$ (See Saw thm) $\Rightarrow$ trivial. bdle.

Pull back by $Y \to Y \times Y$, $y \mapsto (y, -y)$ get

$L_Y \otimes (-1_Y)^* L_Y$ is trivial on Y

But L_Y ample $\Rightarrow$ $(-1_Y)^* L_Y$ also ample

bec. -1_Y is an automorphism on Y.

get $*$ since $\dim Y > 0$.

② $\Rightarrow$ ① trivial since $H \subset K(L)$.

① $\Rightarrow$ ③ : Same as in the pf of Lefschetz thm 1st part. in char $= 0$ case.

By cor 4. $\underbrace{\tau_x^* D}_{\text{ie. } D + x} + \underbrace{\tau_{-x}^* D}_{\text{ie. } D - x} \sim \tau_{x+(-x)}^* D + D = 2D$

for any $u \in X$, $\exists$ x st $u \pm x \notin \operatorname{Supp} D$

ie. $u \notin \operatorname{Supp}(\tau_x^* D \cup \tau_{-x}^* D) \Rightarrow Bs|2D| = \phi$.

thus a morphism $\phi : X \to \mathbb{P}^N$

To see ϕ is indeed finite, first :

$\underline{\text{lemma}}$: $C \hookrightarrow X$ curve, $E \hookrightarrow X$ divisor (irred) st. $C \cap E = \phi$

then E is invariant under $\tau_{x_1 - x_2}$ $\forall x_1, x_2 \in C$.

Pf : $L := L(E)$ is trivial on C.

$\Rightarrow$ $\tau_x^* L|_C$ has degree 0. $\forall x \in X$.

$\Rightarrow$ $\tau_x(c)$ either $\subset E$ or disjoint.

let $x_1, x_2 \in C$, $y \in E$, then

$\tau_{y-x_2}(C) \cap E \ni \{y\} \Rightarrow \tau_{y-x_2}(C) \subset E$

i.e. $y \underline{-x_2+x_1} \in E \quad \forall x_1, x_2$! ※.

Now if ϕ is not finite, $\exists$ irred. curve C st $\phi(C) = pt$. So $\forall E \in |2D|$, either $E \supset C$ or $E \cap C = \phi$. In particular, $\underline{\underline{\forall'}} x \in X$

$\tau_x^* D + \tau_{-x}^* D$ is disjoint ~~from~~ C

Zariski open condition.

let $D = \sum n_i D_i$, D_i irred.

by lemma, say $x + D_i$ is inv under translation by all pts $x_1 - x_2$, $x_i \in C$

i.e. $x_1 - x_2 + x + D_i = x + D_i$

$\Rightarrow x_1 - x_2 + D_i = D_i$. true $\forall D_i$

$\Rightarrow \underbrace{x_1 - x_2}_{\infty\text{-many choices}} + D = D$. ※ to $|H| < \infty$ ※.

$\underline{\underline{\text{Cor}}}$. Any abelian variety is projective.

pf: let $U \hookrightarrow X$, U any affine open.

let $D_1 \cdots D_t$ the complement $X - U$ reduced str

let $D := \sum_{i=1}^t D_i$

may assume $0 \in U$,

Since H is a closed subgp ($\subset K(L(D))$ say), in X (hence proper)

$:= \{x : \tau_x^* \underline{D = D}\}$

Def'n $\Rightarrow$ H stablizes U (fix the boundary), so $H + 0 \subset U$

but H proper, C affine U

$\Rightarrow |H| < \infty$, so $\Rightarrow$ D is ample ※.

Application 2.

X ab. var $\Rightarrow$ X is divisible & X_n is finite.

pf: let $n_X : X \to X$,

$\dim(\ker n_X) > 0 \iff \dim(\mathrm{Im}\, n_X) < \dim X$

so $X_n := \ker n_X$ is finite $\iff$ $\underline{n_X \text{ surjective}}$.

bec. $\dim \mathrm{Im} = \dim X$ $\Rightarrow$ surj since X proper.

let L be an ample line bd.

Cor 3 $\Rightarrow$ $n_X^* L \cong L^{\frac{1}{2}n(n+1)} \otimes (-1_X)^* L^{\frac{1}{2}n(n-1)}$ is still ample ($\frac{1}{2}n(n+1) > 0$, -1_X auto, $\frac{1}{2}n(n-1) \geq 0$).

$\Rightarrow$ $n_X^* L$ can't be trivial on any sub. var of $\dim > 0$

But $n_X^* L \,|_{\ker n_X}$ is trivial, so $\Rightarrow$ $\ker n_X$ is finite ※

Application 3. On $|X_n|$:

Recall for $f : X \twoheadrightarrow Y$ X, Y proper

then $f^* k(Y) \hookrightarrow k(X)$, finite alg. ext.

let d = degree $[k(X) : k(Y)]$

d_s = separable degree $[\cdots]_s$

d_i = insep degree $[\cdots]_i$

if f is separable, i.e. $d = d_s$, then

$d = \# \underline{f^{-1}(y)}$ $\forall' y \in Y$.

in general this is the d_s for general f.

Moreover,

$$(f^* D_1, \cdots, f^* D_n)_X = d\,(D_1, \cdots, D_n)_Y$$

for int. # of Cartier divisors $D_1, \cdots, D_n$ on Y.

Def'n: $f : X \to Y$ homo. between ab. var.

is called an isogeny if it is surj. with finite kernel. eg. $n_X : X \to X$.

clearly, $\# \ker f = \# f^{-1}(y) = d_s(f)$.
$\forall y \in Y$

Let $f = n_X$.

Let D be an ample symmetric divisor.
i.e. $(-1_X)^* D = D$.

eg. take $D' = D + (-1_X)^* D$.

Cor 3 $\Rightarrow$ $n_X^* D \sim \frac{1}{2} n(n+1) D + \frac{1}{2} n(n-1) D = n^2 D$

if $g = \dim X$, then

$$n_X^* D \cdots n_X^* D = d(n_X) \cdot D^n \text{ in } X$$
$$\shortparallel$$
$$n^2 D \cdot \cdots \cdot n^2 D = n^{2g} \cdot D^n .$$

$\Rightarrow$ $\underline{\text{degree}(n_X) = n^{2g}}$.

- If p is a prime $\nmid n$, then $p \nmid n^{2g} = \deg(n_X)$
  char k, X var./k. $\to$ so $k(Y)$, $k(X)$ all have char $= p$ ground field.

  $\Rightarrow$ n_X is separable

  ie. $|X_n| = n^{2g}$

  moreover $m|n \Rightarrow X_m \subset X_n$
  with $|X_m| = m^{2g}$

  finite abelian gp theory $\Rightarrow$ $\underline{X_n \cong (\mathbb{Z}/n\mathbb{Z})^{2g}}$.

- If $p|n$. then the tangent map

  $dn_X = 0 : T_{X,0} \to T_{X,0}$.
  simply = multiplication by n.

  ω inv. diff form on X $\Rightarrow$ $n_X^* \omega$ still inv.
  translation transl.
  and is 0 at $T_{X,0}$

  hence $n_X^* \omega = 0$.

  But inv. form on X gen Ω_X^1 over $\mathcal{O}_X$
  hence gen $k(X)/k$-differential
  (as $k(X)$-module)

$\Rightarrow$ n_X^* is 0 on rat'l differentials $\Omega^1_{k(x)/k}$

★ $\Rightarrow$ n_X^* on $k(x)$ maps $\underline{\text{into } k(x)^p}$

since each variable (transcen) has deg $\geq p$ — general field theory diff criterion cf. eg. Bourbaki Alg II. ch 6. § $\underline{\underline{13.2}}$ p. 103 Prop 6.

so the $d_i(\underset{=}{p}x) \geq p^g$. $\uparrow n = p$.

Now X_p is killed by p with $|X_p| = d_s(px)$.

so $\underline{X_p \cong (\mathbb{Z}/p\mathbb{Z})^i}$ for some $0 \leq i \leq g$.

Now $X_{p^2} \xrightarrow[p_X]{} \!\!\!\!\!\to X_p$ because X is divisible

conclude that $X_{p^2} \cong (\mathbb{Z}/p^2\mathbb{Z})^i$, inductively

get $X_{p^m} \cong (\mathbb{Z}/p^m\mathbb{Z})^i$ ✱. by str. of finite ab. gps.

Remark about field extensions:

Bourbaki Alg II. ch 6. § 13-2. (p. 103) Prop. 6

Let K be a field. char $K = p$. L ext of K

$\underline{\text{a) for } x \in L,\ d_{L/K}x = 0 \Leftrightarrow x \in K(L^p)}$.

b) $\Omega_K(L) = 0 \Leftrightarrow L = K(L^p)$.

In the above ★ case. $\underset{0 =}{}\ n^* da = d(n^*a)$ by a) $\Rightarrow$ $n^*a \in k(x)^p$

Apply to $K = k$, $L = k(x)$. $a \in k(x)$

(the ~~converse~~ part is trivial)

bec. $k^p = k$ (perfect fields) this is the case for eg. $p = 0$, finite, or alg. closed.

$Pic^0(X)$ of ab. var. Moduli Problem.
In case of $k = \mathbb{C}$: X general sm Kähler

$$H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to Pic(X) \to H^2(X,\mathbb{Z})$$

$$H^1(X,\mathcal{O})/H^1(X,\mathbb{Z}) =: Pic^0(X) \hookrightarrow Pic(X) \xrightarrow{c_1} H^2(X,\mathbb{Z})$$
$$Pic(X) \twoheadrightarrow NS(X) \hookrightarrow H^2(X,\mathbb{Z})$$
$$NS(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$$

Q: How to read out $Pic^0(X)$ in general case of k?

Answer: if char $k = p > 0$ may take $\ell \neq p$. use étale coh. $H^2_{et}(X, \mathbb{Z}_\ell)$. and define $c_1(\mathcal{L})$.

Special construction in ab. var.

$L \in Pic(X)$ defines a map $\phi_L : X \to Pic(X)$
$x \mapsto [T_x^* L \otimes L^{-1}]$.

$k = \mathbb{C}$ case: View

$$Pic^0(X) \hookrightarrow Pic(X) \xrightarrow{c_1} NS(X)$$

as a principal homogeneous space (each fiber $\simeq Pic^0(X)$)

Fact: $c_1(L) = 0$

$\overset{1}{\Leftrightarrow}$ L is topologically trivial since $H^i(X,\mathcal{C}) = 0$ $\forall i \geq 1$.

$0 \to H^1(X,\mathcal{C}^\times) \xrightarrow{\sim} H^2(X,\mathbb{Z}) \to 0$

$\uparrow \exp(2\pi i \cdot)$

$H^1(X,\mathcal{C})$

($\overset{2}{\Leftrightarrow}$ C^∞ flat line bundle in differential geom.)

$\overset{3?}{\Leftrightarrow}$ constant transition function

$\overset{4}{\Leftrightarrow}$ constant trivialization

$e_{\lambda_\alpha}(z) \equiv e_{\lambda_\alpha}(0)$

$L = \{ (U_{\alpha\beta}, \varphi_{\alpha\beta}) : \varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha} = 1 \}$

$\Rightarrow$ $\log \varphi_{\alpha\beta} + \log \varphi_{\beta\gamma} + \log \varphi_{\gamma\alpha} \equiv 0 \mod 2\pi i$
not well-defined!

$\Rightarrow$ $c_1(L)$ is the class
$U_{\alpha\beta\gamma} \mapsto$
$\frac{1}{2\pi i}(\log \varphi_{\alpha\beta} - \log \varphi_{\beta\gamma} + \log \varphi_{\gamma\alpha})$

Rmk: $\tilde{L}$ is trivial on $\mathbb{C}^g \times \mathbb{C}$.

$\overset{5}{\Leftrightarrow}$ L is translation invariant
i.e. $T_x^* L \cong L$ $\forall x \in X$.

$\overset{6}{\Leftrightarrow}$ $\phi_L \equiv 0$.

Rmk: this holds more generally for any $\tilde{X} \to X$ (univ. covering). up to step 4. ✱

Now for general k:

Def'n: Recall $\phi_L : x \mapsto [\tau_x^* L \otimes L^{-1}] \in \mathrm{Pic}(X)$

$\mathrm{Pic}^\circ(X) :=$ all L st $\phi_L \equiv 0$.

Notice $\phi_L(x) \in \mathrm{Pic}^\circ(X)$ auto. bec. of thm of square:

$$\tau_y^*(\tau_x^* L \otimes L^{-1}) \otimes (\tau_x^* L \otimes L^{-1})^{-1}$$
$$\cong \tau_{x+y}^* L \otimes \tau_y^* L^{-1} \otimes \tau_x^* L^{-1} \otimes L \cong \mathcal{O}_X$$

$$\Rightarrow \quad 0 \to \mathrm{Pic}^\circ(X) \to \mathrm{Pic}(X) \to \mathrm{Hom}(X, \underline{\underline{\mathrm{Pic}^\circ(X)}})$$
$$L \mapsto \phi_L$$

exact.

GOAL: Show $\mathrm{Pic}^\circ(X)$ has an structure of ab. $\underline{\underline{\text{var}}}$ too, denoted by $\hat{X}$ later.

properties:

① $L \in \mathrm{Pic}^\circ(X) \Leftrightarrow \tau_x^* L \cong L \;\; \forall x \in X \Leftrightarrow m^* L \cong p_1^* L \otimes p_2^* L$ on $X \times X$, with $p_1 : X \times X \to X$ and $p_2 : X \times X \to X$

by restriction to slices $X \times \{a\}$ or $\{a\} \times X$ then apply see saw thm.

② $L \in \mathrm{Pic}^\circ(X) \Rightarrow \forall f, g : S \to X$, $(f+g)^* L \cong f^* L \otimes g^* L$! any scheme — pullback ① to $S \to X \times X$

③ $L \in \mathrm{Pic}^\circ(X) \Rightarrow \underline{n_X^* L \cong L^n}$.

④ $L \in \mathrm{Pic}(X) \Rightarrow \underline{\underline{n_X}}^* L \cong L^{\underline{\underline{\otimes n^2}}} \otimes A$ with $A \in \mathrm{Pic}^\circ(X)$.

how to feel this non-trivial law geometrically ?! even $k = \mathbb{C}$

$\underline{\underline{\text{pf}}}$: in general,

$$n_X^* L \cong L^{\frac{1}{2}n(n+1)} \otimes (-1_X)^* L^{\frac{1}{2}n(n-1)}$$
$$= L^{\otimes n^2} \otimes [\underline{L \otimes (-1_X)^* L^{-1}}]^{-\frac{1}{2}n(n-1)}$$

need this in $\mathrm{Pic}^\circ(X)$

this is obvious in the c_1 consideration.

Now $\tau_x^*(L \otimes (-1_X)^* L^{-1})$

$$\cong \tau_x^* L \otimes \underline{\underline{(-1_X)^*}} [\tau_{-x}^* L^{-1} \otimes \underline{(L^{-1})^{-1}}] \otimes \underline{(-1_X)^* L^{-1}}$$

commutation — "$\phi_{L^{-1}}(-x) \in \mathrm{Pic}^\circ(X)$"

$$\cong \underline{\tau_x^* L \otimes \tau_{-x}^* L \otimes L^{-1}} \otimes (-1_X)^* L^{-1} \quad \text{may apply ③}$$

$$\cong \underline{\underline{L}} \otimes (-1_X)^* L^{-1}, \quad \text{hence is } \tau_x^*\text{-inv } \forall x \in X$$

by thm of square

✱

⑤ $L \in \mathrm{Pic}(X)$ has finite order $\Rightarrow L \in \mathrm{Pic}^0(X)$.

again trivial for $q = 0$ ($H^2(X,\mathbb{Z})$ is free in the ab. var case)

pf: since X is divisible,

$$\phi_L(x) = \phi_L\left(n\cdot\left(\tfrac{1}{n}x\right)\right) = n\,\phi_L(y) = \phi_{L^{\otimes n}}(y) \equiv 0$$

say $L^{\otimes n} \cong \mathcal{O}_X$, some $y \in X$ #

⑥ $\begin{matrix} L \\ \downarrow \\ X \times S \end{matrix}$ any var $\Rightarrow \underline{L_{s_1} \otimes L_{s_0}^{-1}} \in \mathrm{Pic}^0(X)$; $\forall s_i \in S$.

this apriori means ~~only~~ alg. equiv. in general ~~seems does not~~ implies (numerical). hom equiv.

pf: By induction and transitivity, by rest to open set of S, we may assume $L|_{\{0\}\times S}$ trivial.

also L_{s_0} is trivial:
(replace L by $L \otimes p_1^* L_{s_0}^{-1}$)

remains to show $\underline{L_s \in \mathrm{Pic}^0(X)}$; $\forall s \in S$.

by ① enough to show that $m^*(L_s) \otimes p_1^*(L_s^{-1}) \otimes p_2^*(L_s^{-1}) = A$ trivial.

But for $M := \mu^* L \otimes p_{13}^* L^{-1} \otimes p_{23}^* L^{-1}$

$$\begin{matrix} M \\ \downarrow \\ X \times X \times S \end{matrix} \xrightarrow[p_{13},\, p_{23},\, \mu]{} X \times S \qquad \mu = (p_1 + p_2, \mathrm{id}_S)$$

M is trivial on $X \times (0) \times S$, $(0) \times X \times S$ and $X \times X \times (\underline{s_0})$

So M is trivial by thm of cube,

but then $M|_{X\times X\times(s)} = A$ #.

Proposition (Vanishing)

⑦ If $L \in \mathrm{Pic}^0(X) \setminus (0)$ then $\underline{H^i(X,L) = 0 \quad \forall i}$. vanishing

i.e. not trivial

pf: step 1: $H^0(X,L) = 0$:

if not, then $L \cong \mathcal{O}_X(D)$, $D \geq 0$, $\neq 0$.

but $\begin{cases} L^{-1} \cong \mathcal{O}_X(-D) & (<0) \\ L^{-1} \cong (-1_X)^* L \cong \mathcal{O}_X(\underbrace{(-1_X)^* D}_{\text{auto}}) & \geq 0 \end{cases}$

but cf X (-ef), so get ✗.

- step 1 is true very generally, say when X is projective.
- for Step 2, it is not simply a homological statement, need $\underline{X}$ ab.

※ there are much more line bds than divisors! from $\mathrm{Pic}^0(X)$.

Step 2: Now suppose $H^i(X, L) = 0$ for $i \leq k-1$, $(k \geq 1)$
will show $H^k(X, L) = 0$ too.

let $X \xrightarrow{i_1} X \times X \xrightarrow{m} X$ notice $m \circ i_1 = 1_X$

then $H^k(X, L) \xrightarrow{m^*} H^k(X \times X, m^* L) \xrightarrow{i_1^*} H^k(X, (\underline{\underline{m \circ i_1}})^* L)$

$m^*L = p_1^* L \otimes p_2^* L$ (bec in $\mathrm{Pic}^0(X)$), $(m \circ i_1)^* L = L$

Kunneth for: $H^k(X \times X, m^* L) = \bigoplus_{i+j=k} H^i(X, L) \otimes H^j(X, L)$

$= 0$, at least one i or $j < k$ since $k \geq 1$.

so $i_1^* \circ m^* = id_X^* = id$ factor thr 0 $\Rightarrow$ $H^k(X, L) = 0$

※

THEOREM : If L ample, then $\phi_L : X \twoheadrightarrow \mathrm{Pic}^0(X)$!

ie. $\forall M \in \mathrm{Pic}^0(X)$, $\exists x \in X$ st. $M \simeq T_x^* L \otimes L^{-1}$.

ie. the abstract gp $\mathrm{Pic}^0(X) \simeq X/K(L)$ has a str. of alg. var.
but does this structure depends on L ??

pf: consider $H^*(X \times X, A)$
with $A = m^* L \otimes p_1^* L^{-1} \otimes p_2^*(L^{-1} \otimes M^{-1})$
$M \in \mathrm{Pic}^0(X)$ any given one. Then

(1) $H^e(X, R^k p_{1*} A) \Rightarrow H^{k+e}(X \times X, A)$

(2) $H^e(X, R^k p_{2*} A) \Rightarrow H^{k+e}(X \times X, A)$

Notice that
$$\left.\begin{aligned} A|_{\{x\} \times X} &= T_x^* L \otimes L^{-1} \otimes M^{-1} \\ A|_{X \times \{x\}} &= T_x^* L \otimes L^{-1} \end{aligned}\right\} \in \mathrm{Pic}^0(X)$$

<u>if $\forall x \in X$, $T_x^* L \otimes L^{-1} \not\simeq M$</u>, then $A|_{\{x\} \times X}$ is not trivial, by vanishing thm, get $H^k(\{x\} \times X, A|_{\{x\} \times X}) = 0$ $\forall i$

by Coh & Base change,

$\Rightarrow R^k p_{1*} A \equiv 0$ $\forall k$, hence $H^*(X \times X, A) = 0$ by (1).

but by (2), for $x \notin K(L)$, which is finite since L ample

$A|_{X \times \{x\}}$ is not trivial & $\in \mathrm{Pic}^0(X)$, so

again has vanishing cohomology outside $K(L)$,
coh & base change $\Rightarrow$ $\mathrm{Supp}(R^k p_{2*} \mathcal{A}) \subset K(L)$
and (2) $\Rightarrow$ the spectral sequence degenerate to

$$\bigoplus_{x \in K(L)} (R^k p_{2*} \mathcal{A})_x \xrightarrow{\sim} H^k(X \times X, \mathcal{A}) \underset{\uparrow}{=} 0$$

if thm is false

so $R^k p_{2*} \mathcal{A} \equiv 0$ — in fact this part already only $k=0$ left.

But this $\Rightarrow$ $H^k(X \times \{x\}, \mathcal{A}|_{X \times \{x\}}) = 0 \quad \forall x \in X$

which is NOT true for $x = 0 \in K(L)$:

$A|_{X \times \{0\}}$ is trivial, its $H^0 \neq 0$ $\times$. QED.

CONSTRUCTING PICARD VARIETY / DUAL AB. VAR :

<u>Moduli Problem</u>:

we would like to put variety structure on $\mathrm{Pic}^0(X)$

call it $\hat{X}$, together with a universal line bundle

or Poincaré bundle $\begin{matrix} P \\ \downarrow \\ X \times \hat{X} \end{matrix}$ st.

(a) $P_\alpha := P|_{X \times \{\alpha\}}$ repr the element of $\alpha \in \hat{X} \cong \mathrm{Pic}^0(X)$

we also <u>adjust</u> P st $P|_{(0) \times \hat{X}}$ is trivial.

(when $\hat{X}$ given, st P is unique by (a))

(b) for any $\begin{matrix} A \\ \downarrow \\ X \times S \end{matrix}$ st $A_s := A|_{X \times \{s\}}$ for all $s \in S$ — S any normal var

$\mathrm{Pic}^0(X)$ (one s is enough) and $A|_{(0) \times S}$ trivial

the set map : $S \xrightarrow{f} \hat{X}$ st $A_s \cong P_{f(s)}$

is an $\underline{\text{algebraic morphism}}$ (C). & (F) $\underline{A \cong (1_X \times f)^* P}$.

((b) forces the uniqueness of alg. str of $\hat{X}$!)

<u>Define</u>: $\varphi_L : X \longrightarrow \mathrm{Pic}^0(X) \cong X/K(L) =: \hat{X}$

call the morphism $\pi : X \twoheadrightarrow \hat{X}$, apriori dep on L.

Since $X \times \hat{X} = (X \times X) / \underbrace{(0) \times K(L)}_{\text{finite gp}}$

construct P is equiv to construct $\tilde{P}$ on $X \times X$ inv. under it.

From (b) $\nearrow$(F): $\begin{array}{c}\tilde{P}\\ \downarrow \\ X \times (X) \\ S\end{array}$ with map $X \xrightarrow[(\pi)]{f} \hat{X}$

$\tilde{P}$ has to be $(1_X \times \pi)^* P$

but one such a choice of $\tilde{P}$ is

$\tilde{P} := m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$ (see (a))

it remains to construct the action of $(1_X) \times K(L)$ on $\tilde{P}$.

Card (0) $\quad \ker(1_X \times \pi)$ to get P.

But for $a \in K(L)$, $T^*_{(0,a)} \tilde{P} \cong \tilde{P}$ since $T_a^* L = L$.

so $\exists$ automorphism $\phi_a : \tilde{P} \xrightarrow{\sim} \tilde{P}$ lifts $T_{(0,a)} : X \times X \to X \times X$

if $(\circledast)$ $\phi_a \circ \phi_b = \phi_{a+b}$ $\forall a, b \in K(L)$ then done,

since X is proper, ϕ_a is det. up to a scalar.

So only need $\circledast$ to hold at one point.

Now the trivial line bundle

$$\tilde{P}|_{(0)\times X} \simeq m^* L|_{(0)\times X} \otimes p_1^* L^{-1}|_{(0)\times X} \otimes p_2^* L^{-1}|_{(0)\times X}$$

canonical $\simeq L \otimes (L^{-1}(0) \times X) \otimes L^{-1} = L^{-1}(0) \times X$.

fiber at 0 of X

We may adjust ϕ_a so that to "fix" this fiber $L^{-1}(0) \times X$

$\downarrow$

$(0) \times X$

$$L^{-1}(0) \times X \longrightarrow L^{-1}(0) \times X$$
$$(\lambda, x) \mapsto (\lambda, x + a)$$

We may do this because auto of trivial line bundle is only a constant function. . Done!

Theorem: $(\hat{X}, P)$ solves the universal problem.

Pf: only need to check in (b), $\begin{array}{c} A \\ \downarrow \\ X \times S\end{array}$

the set map $f : S \longrightarrow \hat{X}$

is a morphism. because then the

statement $A \simeq f^* P$ follows from the See Saw Thm:

" A is det. by its restrictions to $X \times \{s\}$ $\forall s$ & $(0) \times S$.

consider

$$E = p_{12}^* A \otimes p_{13}^* (P^{-1})$$
$$\downarrow$$
$$X \times S \times \hat{X}$$

then $E|_{X \times (s,\alpha)} = A_s \otimes P_\alpha^{-1}$ and

$\Gamma := \{ (s,\alpha) \in S \times \hat{X} \mid E|_{X \times (s,\alpha)}$ trivial $\}$

is Zariski closed, hence alg. var.

In fact, Γ = graph of $f : S \to \hat{X}$

since $E|_{X \times (s,\alpha)}$ trivial $\Leftrightarrow A_s \cong P_\alpha$

so $s \overset{f}{\longmapsto} \alpha$.

Example: Let $F : a \mapsto a^p$; if $a^p = b^p$ then $(a-b)^p = 0$ so $a = b$

$k = \bar{k}$, char $k = p > 0$; $\mathbb{A}^1 \xrightarrow{F} \mathbb{A}^1$

F is bijective morphism but is not nec. birational (not invertible) ※

$X = \{ x + y + z = 0 \}$; $Y = \{ x^p + y^p + z^p = 0 \}$

Frobenius map. $x_i \mapsto x_i^p$

in order to be able to invert F, the downstair space Y or $\mathbb{A}^1$, in general $\hat{X}$ should be carefully constructed!

$$\begin{tikzcd} & \Gamma \subset S \times \hat{X} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ S \arrow[rr, "f"'] & & \hat{X} \end{tikzcd}$$

in particular, $\Gamma \xrightarrow{\pi_1} S$ is a bijective morphism. Now if π_1 is birational which is true if char $k = 0$, then Zariski Main Thm $\Rightarrow$ π_1 is an isomorphism hence π_2 is a morphism $\Rightarrow$ f is an morphism. ※※

Unfortunately, when char $k = p > 0$, this π_1 is not nec. "invertible". see above example!!

Theorem: $P \to X \times \hat{X}$ realizes $X = \hat{\hat{X}}$ with same P.

Simplest gp schemes :

(1) $\mathbb{G}_a$ additive gp / k := Spec $k[T] = \mathbb{A}^1$

gp of S-valued pt $\underline{\mathbb{G}_a}(S) = \Gamma(S, \mathcal{O}_S)$.

when char $k = p > 0$,

α_{p^n} := sub gp scheme Spec$\left(k[T]/\underline{T}^{p^n}\right) = (\mathbb{G}_a)_n$

$\alpha_{p^n}(S) = \{ f \in \Gamma(S, \mathcal{O}_S) \mid f^{p^n} = 0 \}$ ie p^n-th nilpotent ele.

Notice Lie $\mathbb{G}_a$ = Lie$(\alpha_{p^n}) = k \cdot \frac{\partial}{\partial T}$ $(\forall n \geq 1)$.

other α_ℓ do not have nontrivial Lie. this is tangent vector for α_{p^n} since $\frac{\partial}{\partial T}(T^{p^n}) = p^n T^{p^n - 1} = 0$.

(2) $\mathbb{G}_m$ multiplicative gp / k := Spec$k[T, \frac{1}{T}] = \mathbb{A}^1 - (0)$.

gp of S-valued pt $\underline{\mathbb{G}_m}(S) = \Gamma(S, \mathcal{O}_S^\times)$.

when char $k = p > 0$,

μ_{p^n} := sub gp scheme Spec $k[T, T^{-1}] / (\underbrace{T - 1}_{= m_e})^{p^n} \equiv \underbrace{\text{Spec } k[T]/T^{p^n} - 1}_{= (\mathbb{G}_m)_n}$

$\mu_{p^n}(S) = \{ f \in \Gamma(S, \mathcal{O}_S) \mid f^{p^n} = 1 \}$ ie p^n-th roots of 1.

Notice Lie $\mathbb{G}_m$ = Lie $\mu_{p^n} = k \cdot T \frac{\partial}{\partial T}$

other μ_ℓ $((\ell, p) = 1)$ is just a reduced gp.

Characterize Scheme through its Functor of points :

X scheme / k, finite type / k, $k = \bar{k}$. $\underline{X}(S) := \text{Hom}_k(S, X)$ is a contravariant functor : $\underline{X} : \text{Sch}/k \to \text{Sets}$ or $\text{Alg}/k \to \text{Sets}$ by $\underline{X}(R) := \underline{X}(\text{Spec} R)$

- Theorem :

$$\begin{aligned} \text{Sch}/k &\longrightarrow \text{Functor}(\text{Sch}/k \text{ or } \text{Alg}/k, \text{Sets}) \\ X &\longmapsto \underline{X} \end{aligned}$$ is fully faithful.

ie. $\text{Hom}_{\text{Sch}}(X, Y) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(\underline{X}, \underline{Y})$, $\mathcal{C} := \text{Func}(\cdots)$.

- Corollary : A scheme G is a gp scheme $\Leftrightarrow$ $\underline{G}(S)$ is a gp $\forall S$ and $\underline{G}(S) \to \underline{G}(S')$ is a gp homo $\forall S' \to S$ of schemes.

Fact: A gp scheme / k with char $k = 0$ is always smooth (hence reduced).

Dual Abelian Variety in Any Characteristic.

X ab. var $\quad L$ line bundle

$K(L)$ closed subgp $= \{ x \in X \mid \tau_x^* L \simeq L \}$

GOAL: define subscheme structure on $K(L)$.

$M = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$

$$\begin{tikzcd} & X \times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd}$$

★ $\underline{K(L)} :=$ maximal <u>subscheme</u> of X st.

$M|_{\underline{K(L)} \times X}$ is trivial. Is it a subgroup scheme

<u>lemma</u>: $f \in \underline{K(L)}(S)$ (ie. S-valued pt of $\underline{K(L)}$)

$\Leftrightarrow \tau_f^* L_S \simeq L_S \otimes p_2^* N$ <u>(*)</u> some $N \in \mathrm{Pic}(S)$.

where $L_S = p_1^* L$ on $X_S = X \times S$, $\tau_f(x,s) = (x + f(s), s)$.

pf: composite $X \times S \xrightarrow{\tau_f} X \times S \xrightarrow{p_1} X$ equals

$X \times S \xrightarrow{1_X \times f} X \times X \xrightarrow{m} X$

So $\tau_f^* L_S \simeq (1_X \times f)^* m^* L$

since $L_S|_{(0) \times S}$ trivial and $\tau_f^* L_S|_{(0) \times S} \simeq f^* L$

<u>(*)</u> holds will $\Rightarrow$ $f^* L \simeq (L_S \otimes p_2^* N)|_{(0)\times S} \simeq N$

So <u>(*)</u> holds $\Leftrightarrow$ $(1_X \times f)^* m^* L \simeq p_1^* L \otimes p_2^* f^* L$

But $(1_X \times f)^* m^* L \otimes p_1^* L^{-1} \otimes p_2^* f^* L^{-1} \simeq (1_X \times f)^* M$

so <u>(*)</u> holds $\Leftrightarrow$ $(1_X \times f)^* M$ trivial

ie. f factors thru $\underline{K(L)}$ by def ★

or, equiv. $f \in \underline{K(L)}(S)$ ✱

since <u>(*)</u> satisfies a gp law for f, this $\Rightarrow$

<u>Corollary</u>: $\underline{K(L)}(S)$ is a subgp of $\underline{X}(S)$, $\forall S$

hence $\underline{K(L)}$ is a <u>subgroup scheme</u> of X.

Quotients by finite gp schemes :

Fundamental Theorem:

(A). $G \curvearrowright X$ scheme s.t. any orbit is contained in an affine open set of X.
G finite gp scheme
then $\exists$ pair (Y, π), $\pi : X \to Y$ s.t.
Y scheme, π morphism

(i) As ~~top. space~~ (Y, π) = quotient of X by G_{red}.

(ii) π is G-equivariant, and $\mathcal{O}_Y \xrightarrow{\sim} (\pi_* \mathcal{O}_X)^G$ see back side for some def.

π is finite surj and (Y, π) has univ. property :

$\forall$ G-equivariant morphism $f : X \to Z$
$\exists !$ $g : Y \to Z$ s.t. $f = g \circ \pi$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & Z \\ Y \arrow[ur] & \end{tikzcd} \quad Y =: X/G.$$

(B). If further the G-action is free
with $G = \operatorname{Spec} R$, $\dim_k R = n$ (not nec. $= 0$! even $G_{red} = pt$)
then $\pi_* \mathcal{O}_X$ is locally free $\mathcal{O}_Y$-module of rank $= n$.
(i.e. π is flat finite of deg n)
and the closed immersion

$$(\mu, p_2) : G \times X \longrightarrow X \times X$$

coincide with the subscheme $X \times_Y X \hookrightarrow X \times X$.

Finally, $\mathcal{F} \longmapsto \pi^* \mathcal{F}$
is an equivalence of cat. between
(coh. $\mathcal{O}_X$-mod) $\longleftrightarrow$ (coh. $\mathcal{O}_Y$-mod with G-action)

(loc. free $\mathcal{O}_X$-mod of finite rk) $\longleftrightarrow$ (loc. free $\mathcal{O}_Y$-mod of finite rk, with G-action)

Corollary: X ab. var. L ample, $K(L)$ finite gp scheme
then $\pi : X \twoheadrightarrow X/K(L)$ an ab. var.

Pf: Since $\mathcal{O}_X$ is reduced (X is an var),
$\mathcal{O}_Y = (\pi_* \mathcal{O}_X)^G$ must also be reduced, hence
$X/K(L)$ is an var. (homeo to prev. def'n, but diff scheme str.) ∎

Now let $\hat{X} = X/K(L)$

Want to construct Poincaré bundle $\begin{matrix} P \\ \downarrow \\ \hat{X} \times X \end{matrix}$

in the <u>variety case</u> before, $\begin{matrix} M \\ \downarrow \\ X \times X \end{matrix} = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$ with $K(L)$-action

then P = quotient line bundle of M by $K(L)$.

In the <u>present case</u>, $K(L)$ is a finite gp scheme

need to perform all construction for S-points. $\forall$ scheme S.

let $*_S$ = $*$ base change $S \to X$. here do not use this since p_i has diff meaning

$\underline{K(L)}(S)$ = sub gp of $f \in \underline{X}(S)$ st. $\tau_f^* L_S \simeq L_S \otimes \underbrace{p_2^* N}_{L_0}$ $\star$

on $X_S \times_S X_S = (X \times X)_S = X \times X \times S$

$M_S \hat{=} m_S^*(L_S) \otimes p_1^* L_S^{-1} \otimes p_2^* L_S^{-1}$

so $\tau^*_{(f,0)} M_S \hat{=} m_S^* \tau_f^* L_S \otimes p_1^* \tau_f^* L_S^{-1} \otimes p_2^* \; L_S^{-1}$

by $\star$ $\hat{=}$ $M_S \otimes p_1^* L_0^{-1} \otimes m_S^* L_0 \simeq M_S$

cancel, both come from $S \leftarrow N$.

$\Rightarrow \tau^*_{(f,0)} M_S \cong M_S$

it remains to fix this isomorphism consistently so that to get a $\underline{K(L)}(S)$-action.

As before, it is enough to fix the isomorphism on the subscheme $\underline{X_S \times_S 0_S}$.

because diff isom. diff by a unit, but

$H^0((X \times X)_S, \mathcal{O}^\times) \cong \underline{H^0(S, \mathcal{O}_S^\times)} \cong H^0(X_S \times_S 0_S, \mathcal{O}^\times)$

by the properness of X.

Now let V = dual of fiber of L at 0 on X

$i : X_S = X_S \times_S 0_S \longrightarrow X_S \times_S X_S$ the closed imm.

$i^* M_S \simeq i^* m_S^* L_S \otimes i^* p_1^* L_S^{-1} \otimes i^* p_2^* L^{-1}$

$\simeq L_S \otimes L_S^{-1} \otimes_k V \simeq V \times X_S$ (trivial line bundle)

canonically, require the action on M_S to be

$1_V \times \tau_f$ on $\overset{V \times X_S}{X_S \times_S 0_S}$, this is a $\underline{K(L)}(S)$ action. done $\divideontimes$

Universal Property for Moduli Problem:

THEOREM: S scheme. $L \to S \times X$ line bundle st. $L|_{S\times\{0\}}$ trivial and $L|_{\{s\}\times X} \in \mathrm{Pic}^0(X)$, $\forall s \in S$ then $\exists!$ morphism $\phi: S \to \hat{X}$ st $L \cong (\phi \times 1_X)^* P$.

Rmk: The pattern of pf is completely analogous to char = 0 case:

pf: $M := p_{23}^* P \otimes p_{13}^* L^{-1}$ on $S \times \hat{X} \times X$

$\Gamma_S \hookrightarrow S \times \hat{X}$ maximal subscheme st. M trivial

let $\pi : \Gamma_S \to S$ be projection $S \times X \to X$

~~Main point:~~ π is an isomorphism.

for then $S \xrightarrow{\pi^{-1}} \Gamma_S \longrightarrow \hat{X}$ is the desired morphism.

it is enough to show that for any closed subscheme $S_0 \hookrightarrow S$ with $\mathrm{Supp}\, S_0 = $ one pt,

$\Gamma_{S_0} = S_0 \times_S \Gamma_S \longrightarrow S_0$ is an isom.

wrt $L|_{S_0 \times X}$, the max. sub scheme st M trivial

ie. may assume $S = \mathrm{Spec}\, B$. B a f.d. local k-alg.

may also assume $L|_{\{s\}\times X}$ trivial

the unique point of S.

(by replacing L by $L \otimes p_2^*(L|_{\{s\}\times X})$.)

Part I: epimorphism.

Now $\exists$ only finite (s, x) st. $M|_{\{s\}\times\hat{X}\times\{x\}}$ trivial

(it is just the case $S = $ pt, reduced str. reduce to

$m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}|_{X\times(x)} = T_x^* L \otimes L^{-1}$ trivial.)

$\Rightarrow$ $R^p p_{13*} M$ has discrete support

$$\Rightarrow H^p(S\times\hat{X}\times X, M) \overset{s.s}{=} H^0(S\times X, R^p p_{13*} M)$$

$$\cong H^0(S\times X, (R^p p_{13*} \underline{p_{23}^* P}) \otimes \underline{L^{-1}})$$

take away since coh dep only on the space with reduced str. then L trivial on it.

reverse s.s.

$$\cong H^p(S\times\hat{X}\times X, p_{23}^* P)$$

$$\cong B \otimes_k \underline{H^p(\hat{X}\times X, P)}$$

since $p_{23}^* P$ is trivial in the factor S.

ie. coh. gps are all free B-modules. ($\hat{X}\times X$ is a variety $/k$)

Another direction : $R^p p_{12*} M$.

$M|_{(s)\times(\hat{x})\times X} \in \mathrm{Pic}^\circ X$ and trivial $\Leftrightarrow \hat{x} = 0$, hence

by coh. vanishing $\Rightarrow R^p p_{12*} M$ concentrate at $(s,0)$ hence a fi dim v.s. over the residue field $k((s,0))$.

Let $0 \to K_0 \to K_1 \to \cdots \to K_g \to 0$

be the cpx of finite free $A = B \otimes \mathcal{O}_{\hat{X},0}$ -module

st. $\left(R^i p_{12*} M \right)_{(s,0)} = H^i(K.)$ (length finite A-module)

hence also length finite over $\mathcal{O}_{\hat{X},0}$ (since B is finite-dim k).

But this $\Rightarrow$ $\underline{R^i p_{12*} M = 0 \quad \forall\ 0 \le i < g}$ $(g = \dim X)$

by : Lemma : Let $\mathcal{O}$ be a g-dim'l regular local ring, $0 \to K_0 \to K_1 \to \cdots \to K_g \to 0$ a cpx of finite free $\mathcal{O}$-module. If $H^i(K.)$ artinian, then $H^i(K.) = 0$ for $0 \le i < g$.

pf : $g=0$ ok. Assume $g>0$ and lemma holds for $\dim < g$.

choose $x \in m \setminus m^2$, so $\bar{\mathcal{O}} = \mathcal{O}/\mathcal{O}x$ reg of dim $g-1$.

let $\bar{K}. = \bar{\mathcal{O}} \otimes_{\mathcal{O}} K.$ then $0 \to K. \xrightarrow{x} K. \to \bar{K}. \to 0$

$\Rightarrow H^p(K.) \xrightarrow[x]{} H^p(K.) \to \underline{H^p(\bar{K}.)} \to H^{p+1}(K.) \xrightarrow[x]{} H^{p+1}(K.)$

this $\Rightarrow H^p(\bar{K}.)$ artinian, so $=0$ for $p < g-1$

$\Rightarrow H^{p+1}(K.) \xhookrightarrow[x]{} H^{p+1}(K.)$ for $p < g-1$

inductively $H^{p+1}(K.) \xhookrightarrow[x^n]{} H^{p+1}(K.)$

but artinian module was killed by x^n for $n \gg 0$

hence $\Rightarrow H^{p+1}(K.) = 0$ for $p < g-1$ i.e. $p+1 < g$ $\ast$.

Rmk: only need $\mathcal{O}$ to be CM.

Write exact sequence $0 \to K_0 \to K_1 \to \cdots \to K_g \to N \to 0$

then $N = R^g p_{12*} M_{(s,0)} \simeq H^g(S \times \hat{X} \times X, M)$

s.s. and since X is a complete var / k

had seen this is a free B-module.

consider the dual cpx of free A-module

$0 \to \hat{K}_g \to \cdots \to \hat{K}_1 \to \hat{K}_0 \to 0$

which also has artinian hol. module, hence by lemma $\Rightarrow$ exact sequence — artin A-module

$$0 \to \hat{K_g} \to \hat{K}_{g-1} \to \cdots \to \hat{K_0} \to K \to 0$$

From thm of Coh & Base change (p.46), get exact

$$0 \to \underbrace{H^0((s)\times(0)\times X, \mathcal{P}|_{(s)\times(0)\times X})}_{\text{trivial}} \to \underbrace{K_0 \otimes_A k \to K_1 \otimes_A k \to \cdots}_{\text{this cpx compute coh of fiber at }(s,0)\text{ (over fields)}}$$

$\cong k$

notice that $\otimes$ does not preserve left exact (injectivity).

$\Rightarrow$ coker of $\hat{K_1} \otimes_A k \to \hat{K_0} \otimes_A k$ is 1-dim'l

i.e. $K \otimes_A k = K/m_A K$ is 1-dim'l.

$\Rightarrow$ $\underline{K \simeq A/\mathfrak{a}}$ for ideal $\mathfrak{a} \lhd A$ and $(\hat{K_i})$ resolves $A/\mathfrak{a}$.

this will $\Rightarrow$ homologies of (K_i) are killed by $\mathfrak{a}$

hence $\underline{\mathfrak{a}\cdot N = 0}$. (Exercise)

In $A := B \otimes \mathcal{O}_{\hat{X},0}$, then get $\mathfrak{a} \cap (B \otimes 1) = (0)$

by the B-freeness of N.

equivalently, $B \hookrightarrow A/\mathfrak{a} = K$.

for any m_A-primary $\mathfrak{b} \lhd A$, let $V(\mathfrak{b}) \subset S \times \hat{X}$, then

a closed subscheme (supp $= (s)\times(0)$).

$$H^0(V(\mathfrak{b})\times X, M|_{V(\mathfrak{b})\times X}) \cong \mathrm{Ker}(K_0/\mathfrak{b}K_0 \to K_1/\mathfrak{b}K_1)$$

thm of coh. & base change $\quad \cong$

Since $\cdots \to \hat{K_1} \to \hat{K_0} \to A/\mathfrak{a} \to 0$ $\qquad \mathrm{Hom}_A(A/\mathfrak{a}, A/\mathfrak{b})$

get $0 \to \mathrm{Hom}_A(A/\mathfrak{a}, A/\mathfrak{b}) \to \mathrm{Hom}_A(\hat{K_0}, A/\mathfrak{b}) \to \mathrm{Hom}_A(\hat{K_1}, A/\mathfrak{b})$

$\qquad \mathrm{Hom}_A(\hat{K_0}, A/\mathfrak{b}) = K_0 \otimes_A A/\mathfrak{b}$ etc.

$M|_{V(\mathfrak{b})\times X}$ trivial $\Rightarrow$ gp $\neq 0$ $\Leftrightarrow$ $\mathfrak{a} \subset \mathfrak{b}$ (i.e. $V(\mathfrak{a}) \supset V(\mathfrak{b})$.)

so $V(\mathfrak{a}) \supset$ the maximal subscheme Γ_s.

Now $A/\mathfrak{a} \simeq \mathrm{Hom}_A(A/\mathfrak{a}, A/\mathfrak{a}) \simeq H^0(V(\mathfrak{a})\times X, M|_{V(\mathfrak{a})\times X})$

this $\Rightarrow$ $M|_{V(\mathfrak{a})\times X}$ trivial, hence $\underline{V(\mathfrak{a}) \equiv \Gamma_s}$.

In particular, $B \longrightarrow A/\mathfrak{a} = H^0(\Gamma_s, M|_{\Gamma_s})$ injective.

($S = \mathrm{Spec}\, B$)

ie. $\pi: \Gamma_s \to S$ is "surjective", even this is so non-trivial !!

On the other hand, for $\pi : P_S \longrightarrow S$

$\pi^{-1}(s) \hookrightarrow P_S \cap ((s) \times X)$ is a closed subscheme st.

$\underline{(s) \times P} \mid_{\pi^{-1}(s) \times X}$ trivial (this is M)

since ~~by def of $\hat{X}$~~, (0) = max subscheme st P trivial

$\Rightarrow$ $\pi^{-1}(s)$ = the reduced point $(s, 0)$.

i.e. $A/\mathfrak{a} + m_B A = B$, so $B \twoheadrightarrow A/\mathfrak{a}$

combine get $B \xrightarrow{\sim} A/\mathfrak{a}$, i.e. $\pi : P_S \xrightarrow{\sim} S$ #.

<u>COROLLARY 1</u>: $H^i(\hat{X} \times X, P) = \begin{cases} k & i = g \\ 0 & \text{otherwise} \end{cases}$.

<u>COROLLARY 2</u>: $\dim_k H^p(X, \mathcal{O}) = C^g_p$, $g = \dim X$.

<u>COROLLARY 3</u>: $T_{(0)}\hat{X} \cong H^1(X, \mathcal{O}_X)$.

pf: Let $S = \operatorname{Spec} k[\varepsilon]/\varepsilon^2$. Then $T_{(0)}\hat{X} \cong \operatorname{Hom}_0(S, \hat{X})$

means $s_0 \longrightarrow 0$ unique pt of S.

= { Line bundle on $S \times X$ which is trivial on $(s_0) \times X$ }

by thm

$= \ker [H^1(S \times X, \mathcal{O}^*_{S \times X}) \longrightarrow H^1((s_0) \times X, \mathcal{O}^*_X)]$

From $1 \to 1 + \epsilon\mathcal{O}_X \longrightarrow \mathcal{O}^*_{S \times X} \longrightarrow \mathcal{O}^*_X \to 1$ exact sequence of multi sheaves

mod ϵ

$\cong \mathcal{O}_X$ as sheaf of abelian gps,

$(1 + \epsilon a)(1 + \epsilon b) = 1 + \epsilon(a + b)$

$\Rightarrow$ $0 \to H^1(\mathcal{O}_X) \to H^1(S \times X, \mathcal{O}^*_{S \times X}) \longrightarrow H^1(X, \mathcal{O}^*)$ done #.

<u>COROLLARY 4</u>: $f : X \to Y$ isogeny $\Rightarrow \deg f = \deg \hat{f}$.

pf: thm $\Rightarrow \exists !$ map $\hat{f} : \hat{Y} \to \hat{X}$ st. $(1 \times f)^* P_Y = (\hat{f} \times 1)^* P_X = Q$

apply to $(1 \times f)^* P_Y \to P_Y$, $\hat{Y} \times X \to \hat{Y} \times Y$

$\Rightarrow \chi(Q) = \deg f \cdot \chi(P_Y) = \deg \hat{f} \cdot \chi(P_X)$ (may also use divisor to avoid cor 1 ?)

but $\chi(P_Y) = \chi(P_X)$ both $= (-1)^g$. done.

<u>COROLLARY 5: (~~Duality Hypothesis~~)</u> :

The canonical morphism defined by $P \to X \times \hat{X}$:

$i : X \to \hat{\hat{X}}$ is an isomorphism.

#

Abelian Varieties : π_1, T_ℓ and Riemann Hypothesis

~~Etale coverings~~ / Alg. Fund. Groups

$f: X \to Y$ / $k = \bar{k}$ etale $\Leftrightarrow$ $\hat{f}^*: \hat{\mathcal{O}}_{f(x)} \xrightarrow{\sim} \hat{\mathcal{O}}_x$ ("formal isomorphism")

$$\boxed{\text{Thm (Serre-Lang): } X \text{ ab. var } f: Y \to X \text{ etale covering} \Rightarrow Y \text{ ab. var and } f \text{ becomes a separable isogeny}}$$

(any var)

i.e. $K(Y)$ over $K(X)$ separable. A priori only etale for generic pt, but gp law $\Rightarrow$ all pts.

Rmk: $f: Y \to X$ isogeny then $\exists g: X \to Y$ st an isogeny
(not nec. separable)
$f \circ g = n_X$ for some $n > 0$:

$\ker f \subset Y$ is a finite gp scheme, so killed by some $n > 0$, hence $\ker f \subset \ker(n_Y)$

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "n_Y"] & X \simeq Y/\ker f \arrow[dl, dashed, "g"] \\ Y & \end{tikzcd} \Rightarrow \exists g: X \to Y \text{ st } n_Y = g \circ f$$

but then $f \circ g(x) = f g f(y) = f(ny) = n f(y) = nx$ ✵

where $x = f(y)$ for some y

i.e. g is both left & right inverse of f (up to $\otimes \mathbb{Q}$), right is the key point.

~~(Algebraic) fundamental group~~

$$\pi_1(X, x_0) := \varprojlim_{(Y, y_0, \pi)} G_Y \quad \text{where}$$

finite gp $G_Y \circlearrowright Y \xrightarrow{\pi} X$ st $y_0 \to x_0$ and $X \simeq Y/G_Y$

~~acts freely on Y~~. Y connected. (X sm $\Rightarrow$ Y sm.)

Given 2 such
$\exists$ at most one
$f: Y_1 \to Y_2$ st

$$\begin{tikzcd} Y_1 \arrow[r, "\pi_1"] \arrow[d, "f"] & X \\ Y_2 \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

~~$\pi_2 \circ f = \pi_1$~~ & ~~$f(y_1) = y_2$~~. In this case, get

$\rho: G_{Y_1} \to G_{Y_2}$ st. $f(\sigma y) = \rho(\sigma) f(y)$.

hence set $Y_1 > Y_2$ to form a inverse system.

- $\pi_2 \circ f = \pi_1 \Rightarrow f: Y_1 \to Y_2$ is also an étale covering
  G_{Y_1} acts on Y hence on Y_2, get ρ
  in fact $Y_2 = Y_1/K$ with $K = \ker \rho$.

For X abelian var. $x_0 = 0$.

Serre-Lang thm $\Rightarrow$ all $\begin{array}{c}(Y, y_0) = (\text{ab. var}, 0) \\ \pi \downarrow \\ (X, 0)\end{array}$ separable isogeny.

hence $G_Y = \ker \pi$ acting on Y via translations.

$\Rightarrow$ $\pi_1(X)$ is an ab. gp.

By remark, given étale cover $Y \xrightarrow{\pi} X$ with $|\ker \pi| = \ell^m$, then $\exists g$ "right inv." $g \uparrow$, $X \xrightarrow{\ell^m_X}$

i.e. All such π is cofinal to the system $X \xrightarrow{\ell^n_X} X$

For the condition of free action, this is étale $\Leftrightarrow (\ell, p) = 1$.

with $G_n = X_{\ell^n} = \ker(\ell^n_X)$, $\to X_{\ell^{n+1}} \longrightarrow X_{\ell^n} \to \cdots$ (torsion subgps)

Def'n: For ℓ prime $\neq p$.

$T_\ell(X) := \varprojlim_n X_{\ell^n}$ the ℓ-adic Tate module ($\mathbb{Z}_\ell$-module)

as the ℓ-adic component of $\pi_1(X)$.

In fact, $X_{\ell^n} \simeq (\mathbb{Z}/\ell^n\mathbb{Z})^{2g}$ hence $T_\ell(X) \simeq \mathbb{Z}_\ell^{2g}$.

For $\ell = p$. $\mathrm{Ker}(p^n_X) = \underset{\uparrow}{X^{lo}_{p^n}} \times \underset{\uparrow}{X^{re}_{p^n}}$ finite gp scheme

local part, i.e. space = pt. but with nontrivial tangent space (Lie algebra).

the reduced part $r = p$-rank ($r \le g$).

$$\begin{array}{ccc} & Y & \\ ? & \uparrow & \searrow^{\pi} \\ \exists & X & \xrightarrow{p^n_X} X \\ & \downarrow & \nearrow_{\pi_n} \\ & Y_n := X/X^{lo}_{p^n} & \end{array}$$

Given ?

this is not étale

π_n is étale with degree $p^\cdot$

if only for the gp consideration, then just replace π by π_n.

i.e. all such π is cofinal to system $Y_{n+1} \longrightarrow Y_n$.

⟨ For the map ?, this should follows from π is étale ⟩

Def'n: $T_p(X) = \varprojlim_n X^{re}_{p^n} \simeq (\mathbb{Z}_p)^r$ where $r = p$-rk of X

the p-adic Tate module, p-adic cmp. of $\pi_1(X)$.

Full Fund. gp

$$\pi_1(X) \simeq \prod_{\text{all primes } \ell} T_\ell(X).$$

Rmk: when $k = \mathbb{C}$, $\pi_1(X)$ is always the profinite completion of $\pi_1^{top}(X)$, i.e. $= \widehat{\pi_1^{top}}(X) := \varprojlim$ (finite quotients).

End(X) : Poincaré irreducibility thm

$\mathrm{Hom}^\circ(X,Y) := \mathbb{Q}\otimes_{\mathbb{Z}} \mathrm{Hom}(X,Y)$

$\mathrm{End}^\circ(X) := \mathbb{Q}\otimes\mathbb{Z}\ \mathrm{End}(X)$. X, Y, ab. var.

form a category "ab. var. up to isogeny"

since for any isogeny $f: Y \longrightarrow X$

$\exists\ g: Y \longleftarrow X$ isogeny st.

$f\circ g = n_X$, hence isogeny are isom in this cat.

write f^{-1} $\in \mathrm{Hom}^\circ(X,Y)$.

$\boxed{\begin{array}{l}\text{Thm (Poincaré): This cat. is semi-simple abelian.}\\ \text{i.e. } Y\subset X \text{ ab. var. } \exists\ Z \text{ ab. } \hookrightarrow X \text{ st.}\\ Y\cap Z \text{ finite \& } Y+Z=X, \text{ i.e. } X \text{ isogeny to } Y\times Z\end{array}}$

pf: $i: Y \longrightarrow X$ inclusion

$\hat{i}: \hat{X} \to \hat{Y}$ dual morphism (restriction of line bundle)

take L ample: $\phi_L: X \to \hat{X}$

then $Z :=$ conn. comp. of 0 of $\phi_L^{-1}(\ker\hat{i})$

$\dim Z = \dim\ker\hat{i} \geq \dim\hat{X} - \dim\hat{Y} = \dim X - \dim Y$.

for $z\in Y$, $z\in\phi_L^{-1}(\ker\hat{i})\cap Y \Leftrightarrow \phi_L(z)|_Y$ trivial

$\Leftrightarrow z\in K(L|_Y) \Rightarrow$ finite gp.
ample

$\Rightarrow Z\cap Y$ finite

and $Z\times Y\longrightarrow X$, $(z,y)\mapsto(z-y)$ has finite kernel

$\Rightarrow \underbrace{\dim Z+\dim Y}_{\substack{\geq \text{ by above}\\ \dim X}} \stackrel{!}{=} \dim(Z\times Y)$ surjective hence $\Rightarrow$ isogeny.

#

Cor 1. $X \underset{\text{isogeny}}{\sim} X_1^{n_1}\times\cdots\times X_k^{n_k}$ (X_i simple ab. var i.e. contains no ab. sub. var $\neq 0$ and X_i.)

the isogeny type X_i and n_i are uniquely determined.

Cor 2. X simple $\Rightarrow \mathrm{End}^\circ X$ is a division ring

for X in Cor 1. Let $D_i = \mathrm{End}^\circ X_i$ then

$$\mathrm{End}^\circ(X) \cong M_{n_1}(D_1)\times\cdots\times M_{n_k}(D_k).$$

Will see that $\mathrm{End}^\circ X$ is <u>finite dim'l</u> semi-simple alg $/\mathbb{Q}$.

~~Cohomology of Line bundles~~ / Riemann - Roch Thm

R-R Thm: $L \to X$. X ab. var. $L \cong \mathcal{O}(D)$ then
$$\chi(L) = \frac{D^g}{g!} \text{ ; also } \deg \phi_L = (\chi(L))^2 .$$
dim $= g$

Def$^{\text{in}}$: ϕ on V a poly function of degree n
if $\phi(xv + yw)$ = poly in x, y of degree n
$\forall v, w \in V$.

Theorem: $\phi \mapsto \deg \phi$ on $\mathrm{End}(X)$ extends to a homog. poly fcn of degree $2g$ on $\mathrm{End}^0 X$.

Pf: R-R $\Rightarrow$ $\chi((n\phi + \psi)^* L) = \dfrac{[(n\phi + \psi)^* D]^n}{g!}$

$\underset{\text{int. theory}}{=} \deg (n\phi + \psi) \cdot \dfrac{D^n}{g!}$

if show $\chi((n\phi + \psi)^* L)$ a poly function in n then
since $\deg(n\phi) = \underline{\deg n_X} \deg \phi = \underline{n^{2g}} \deg \phi$
then done. last time

Let $L(n) := (n\phi + \psi)^* L$
apply cor 2 of thm of cube to $n\phi + \psi$, ϕ, ϕ get
$$\underline{L(n+2)} \otimes \underline{L(n+1)^{-2}} \otimes (2\phi)^* L^{-1} \otimes \underline{L(n)} \otimes (\phi^* L)^2 = 1$$

$$\Rightarrow \quad L(n) = L_1^{\frac{1}{2} n(n-1)} \otimes L_2^n \otimes L_3$$
since $\chi(L)$ is poly in L (eg. R-R)
$\Rightarrow \chi(L(n))$ = poly in n $\divideontimes$.

- The main point of all the following is that the geometric number $\deg \phi$ can be calculated by homological number (linearized) thr $T_\ell(X)$. This is the "Lefschetz fixed point"-style formula.

Think: geometric $\leftrightarrow$ coherent coh. $\leftrightarrow$ topological

End(X): ℓ-adic repr via Tate module:

Given $f: X_1 \to X_2$ get $(X_1)_{\ell^n} \xrightarrow{f} (X_2)_{\ell^n}$

hence $T_\ell(f): T_\ell(X_1) \to T_\ell(X_2)$.

i.e. $T_\ell : \mathrm{Hom}_{\text{ab. var}}(X_1, X_2) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(X_1), T_\ell(X_2))$

"ℓ-adic repr."

Theorem: $\mathrm{Hom}(X,Y)$ (always torsion free) is f.g. free ab. gp. and eg. $\cong \mathbb{Z}^\rho$

$$\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathrm{Hom}(X,Y) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(X), T_\ell(Y))$$

in particular $\rho \le 4 \dim X \cdot \dim Y$.

pf: if $M \subset \mathrm{Hom}(X,Y)$ f.g. st.

$M = \mathbb{Q}M \cap \mathrm{Hom}(X,Y)$ then $\mathbb{Z}_\ell \otimes_{\mathbb{Z}} M \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(X), T_\ell(Y))$

$\uparrow$ $\mathbb{Z}_\ell$-free

let $f_1, \dots, f_p$ a $\mathbb{Z}$-basis of M. If not inj.

then $\exists\, \alpha_i \in \mathbb{Z}_\ell$, $f := \sum \alpha_i f_i \mapsto 0$ & some α_i is an unit.

(RHS $\mathbb{Z}_\ell$-free. may take away factors.)

$\Rightarrow$ $T_\ell(\sum \alpha_i f_i)$ maps $T_\ell(X)$ to 0.

in particular maps X_ℓ to 0. hence f factorizes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\ell\cdot"'] & Y \\ X \arrow[ur, "g"'] & \end{tikzcd}$$

with $g \in \mathbb{Q}M \cap \mathrm{Hom}(X,Y) = M$, $g = \sum m_i f_i$

$\Rightarrow$ $\sum \alpha_i f_i = \ell \sum m_i f_i$ so $\ell \mid \alpha_i$ $\forall i$ ※

claim: $M \subset \mathrm{Hom}(X,Y)$ f.g. subgp $\Rightarrow$ $\mathbb{Q}M \cap \mathrm{Hom}(X,Y)$ is also f.g.

it is enough to prove for X, Y simple.

Again may assume $X \underset{\text{isogeny}}{\sim} Y$, otherwise $\mathrm{Hom}(X,Y) = (0)$

via isogeny $Y \to X$ get $\mathrm{Hom}(X,Y) \hookrightarrow \mathrm{End}\, X$,

so may assume $Y = X$. (simple ab. var).

Now $\underset{\mathbb{Z}}{\deg \phi} = \underset{\geq 1}{P(\phi)}$ poly of degree $2g$

if $\phi \neq 0$.

Now $\mathbb{Q}M$ is finite dim v.s/$\mathbb{Q}$ and $|P(\phi)| < 1$

is a nbd U of 0 st $U \cap \mathrm{End}\, X = (0)$

$\Rightarrow$ $\mathrm{End} X \cap \mathbb{Q}M$ is discrete in $\mathbb{Q}M$,

hence f.g. ※

Now start with M, replace by $\mathbb{Q}M \cap \mathrm{Hom}(X,Y)$, then apply the 1st part of pf. ※.

Remarks on Trace & Norm :

A/Γ finite dim asso. alg $/\Gamma$. $|\Gamma| = \infty$

A <u>trace(form)</u>$/\Gamma$: $S : A \to \Gamma$ is $\begin{cases} \Gamma\text{-linear} \\ S(XY) = S(YX) \end{cases}$

A <u>norm(form)</u>$/\Gamma$: $N : A \to \Gamma$ is a poly function $(\not\equiv 0)$

i.e. if v_i a Γ-basis of A

then $N(a) = N(\sum a_i v_i) =$ poly in a_i

st. $N(XY) = N(X) \cdot N(Y)$ for $X, Y \in A$.

<u>Lemma</u>: If A/Γ is a simple algebra whose center Λ is separable ext. of Γ

then $\exists$ canonical norm N° and trace Tr° /Γ

st. Any norm is of the type: $(Nm_{\Lambda/\Gamma} \circ N^\circ)^k$ some $k > 0$

Any trace is of the type: $\phi \circ Tr^\circ$

for some Γ-linear map $\phi : \Lambda \to \Gamma$.

If $[A:\Lambda] = d^2$, then N° is homogeneous of degree d.

pf: when $\Lambda = \Gamma$, and also separably closed,

it is well known that $A \simeq M_d(\Gamma)$

then simply take $N^\circ = \det$, $Tr^\circ = tr$.

because :

$$\begin{tikzcd} M_d(\Gamma) \arrow[r, "S"] & \Gamma \\ M_d(\Gamma) \arrow[ur, "tr"'] & \end{tikzcd}$$

($S \equiv 0$ on the $\overset{\text{sub}}{\text{space}}$ W spanned by $XY - YX$ which is exactly all <u>trace 0</u> matrix.)

so S is det by its value on the 1-dim'l space $M_d(\Gamma)/W$, hence $S \sim tr$. not used!

For $N : M_d(\Gamma) \to \Gamma$ it induces a gp homo.

$$\begin{tikzcd} GL(d) \arrow[r, "\rho"] & \mathbb{G}_m /\Gamma. \\ GL(d) \arrow[ur, "\det"'] & \end{tikzcd}$$

both induces $GL(d)/K \longrightarrow \Gamma^\times$

hence differs only by a map $\Gamma^\times \to \Gamma^\times$, $a \mapsto a^k$. (character)

in particular, $\det = N^\circ$ is homogeneous of degree d ✱

the general case was postponed.

$\boxed{\begin{array}{l}\underline{\text{Theorem}}.\ \text{let } f \in \mathrm{End}(X) \text{ (ab. var.) with induced} \\ T_\ell(f) \in \mathrm{End}(T_\ell(X)),\ \ell \neq \mathrm{char}\, k = p,\ \text{then} \\ \underline{\deg f = \det T_\ell(f)} \text{ hence} \\ \deg(n.1_X - f) = P(n) = \det(n - T_\ell(f)) \\ \qquad \text{the characteristic polynomial of } T_\ell(f). \\ P \text{ is monic of degree } 2g,\ \mathbb{Z}\text{-coeff and } P(f) = 0.\end{array}}$

pf: both $\deg f$ & $\det T_\ell(f)$ are $\underline{\text{degree } 2g}$ $\underline{\text{norms}}$ on the semi-simple $\mathbb{Q}_\ell$-alg. $(\mathrm{End}\, X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$

step 1: $\underline{\text{enough}}$ $\underline{\text{to show } |N_1 \alpha| = |N_2 \alpha|}$ ($|\cdot|$: ℓ-adic absolute value)

$$N_1 = \deg(\cdot) \qquad N_2 = \det T_\ell(\cdot)$$

for $\alpha \in \mathrm{End}\, X \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$.

ie. $\ell^\nu \mid \deg f \iff \ell^\nu \mid \det T_\ell(f)$. which is true:

this is the order of ker of $X_{\ell^n} \xrightarrow{f} X_{\ell^n}$ for n large. ($\underline{\text{so that } f^{-1}(0) \text{ as a finite}}$ $\underline{\text{subgp of } X \text{ all occurs in}}$ $\underline{X_{\ell^n}}$.) $\equiv$ order of coker of the same map. take limits $n \to \infty$ get $|\mathrm{coker}\, T_\ell(f)| = \ell^\nu$ but this $\iff \ell^\nu \mid \det T_\ell(f)$.

step 2:

$\underline{\text{To see enough}}$:

let $\mathrm{End}\, X \otimes_{\mathbb{Z}} \mathbb{Q}_\ell = \prod_{j=1}^r A_j$, A_j simple algebra.

N_1, N_2 go to norms to the product,

ie. $N_1(\alpha_1, \cdots, \alpha_r) = \prod_{j=1}^r N_j^0(\alpha_j)^{\nu_{1j}}$ also $1 \leftarrow 2$.

take $\alpha_j = 1$ for $j \neq j_0$, get $|N_{j_0}^0(\alpha_{j_0})|^{\nu_{1j_0} - \nu_{2j_0}} = 1$ $\forall \alpha_{j_0} \in A_{j_0}$

but $N_{j_0}^0$ homog. of positive degree

$\Rightarrow \nu_{1j_0} = \nu_{2j_0}$, $\Rightarrow$ true $\forall j_0 \Rightarrow N_1 = N_2$.

This is the 1st statement

✻

Step 3. Remaining integral structures:

Now $T_\ell(n1_X - f) = n \cdot 1_{T_\ell(X)} - T_\ell(f)$

so $\underset{\mathbb{Z}^{?}\ \forall n}{\deg(n1_X - f)} = \det(n - T_\ell(f))$ call $p(n)$

$\Rightarrow p(n)$ has <u>rational</u> coefficients.

since End X is a finite $\mathbb{Z}$-module,

$\Rightarrow f$ is integral over $\mathbb{Z}$ i.e.

f (hence $T_\ell(f)$) satisfies monic eq^n $/\mathbb{Z}$.

$\Rightarrow$ all eigenvalues of $T_\ell(f)$ are alg. integers

$\Rightarrow p(n) = \prod(n - \lambda_i)$ char. poly has ~~alg integer~~ as coefficients.

$\mathbb{Q}$ + Alg. $\mathbb{Z} \Rightarrow \mathbb{Z}$.

thus $p(f)$ is in End X.

Finally $p(T_\ell(f)) = T_\ell(p(f))$ (Cayley-Hamilton's trivial theorem)

$\qquad\qquad\qquad \| \ \det(T_\ell f) - T_\ell f)) = 0$

by injectivity of $T_\ell \Rightarrow p(f) = 0$ #

<u>Def^n</u>: The above $p(t) \in \mathbb{Z}[t]$, indep of ℓ is called the characteristic polynomial of f

$\left\{ \begin{array}{l} \underline{\text{norm of } f} := \text{constant term } p(0). \\ \underline{\text{trace of } f} := -\text{coeff of } t^{g-1}. \end{array} \right.$

<u>Rosati Involution</u>:

L ample, $\phi_L : X \to \hat{X}$ an isogeny

for $\phi \in \text{End}^\circ \hat{X}$ (same up to isogeny), $\phi' := \phi_L^{-1} \circ \phi \circ \phi_L \in \text{End}^\circ X$.

Will see next time that $\exists$ Riemann form

$E^L : T_\ell(X) \otimes T_\ell(\hat{X}) \to \mu_{\ell^\infty}$ st $T_\ell(\phi')$ is <u>adjoint</u> of $T_\ell(\phi)$

i.e. $E^L(\phi x, y) = E^L(x, \phi y)$.

Most important is positivity theorem:

<u>Thm</u>. for $L = \mathcal{O}(H)$ ample. $\quad \mathrm{Tr}(\phi\phi') = \frac{2g}{H^g}(H^{g-1} \cdot \phi^* H) > 0.$

Rieman Hypothesis : (for Abelian Var.)

Prop: X ab. var. $\alpha \in \mathrm{End}\, X$ st. $\alpha'\alpha = a \in \mathbb{Z}$
let $\omega_1, \dots, \omega_{2g}$ be the cpx roots of P : the characteristic polynomial of α. Then
- the subalgebra $\mathbb{Q}[\alpha] \subset \mathrm{End}\, X$ is semi-simple
- $|\omega_i|^2 = a$ $\forall i$ and $\omega_i \longmapsto \frac{a}{\omega_i}$ is a permutation of ω_i's

pf: let Q = min poly of α (as in $\mathrm{End}(X)$)
<u>claim</u> : <u>P, Q has the same cpx roots</u>.
since $P \in \mathbb{Z}[x]$ and $P(\alpha) = 0$, this $\Rightarrow Q \mid P$.
But P is also char poly of $T_\ell(\alpha)$ in the repr

$$T_\ell : \mathrm{End}(X) \longrightarrow \mathrm{End}(T_\ell(X)) \quad /\mathbb{Q}_\ell$$

if $\omega \in \overline{\mathbb{Q}}_\ell$ with $P(\omega) = 0$ then
ω is an eigenvalue of $T_\ell(\alpha)$
$\Rightarrow Q(\omega)$ " of $T_\ell(Q(\alpha)) \equiv 0 \Rightarrow Q(\omega) = 0$
ie. roots of P in $\overline{\mathbb{Q}}_\ell$ are roots of Q
$\Rightarrow P \mid Q^n$ some n $\Rightarrow$ the claim. $*$

Now let $S = \mathrm{tr}|_{\mathbb{Q}[\alpha]}$, which has <u>$S(xx') > 0$</u> if $x \in \mathbb{Q}[\alpha] \setminus (0)$.

α invertible in $\mathrm{End}^0 X$
$\mathbb{Q}[\alpha]$ finite dim'l $\Rightarrow \alpha^{-1} \in \mathbb{Q}[\alpha]$
(for min. poly $Q(\alpha) = 0 = 1 + \alpha(\cdots)$ etc.)
$\Rightarrow \alpha' = a \cdot \alpha^{-1} \in \mathbb{Q}[\alpha]$, so $\mathbb{Q}[\alpha]$ stable under involution.

<u>$\mathbb{Q}[\alpha]$ is semi-simple</u> : (commutative algebra)
if $\mathfrak{a} \triangleleft \mathbb{Q}[\alpha]$ any ideal, $\mathfrak{b}$ orthogonal complement of $\mathfrak{a}$ in $\mathbb{Q}[\alpha]$ wrt. $S(xx')$.
$\mathfrak{b}$ is also an ideal
(general fact : $\mathrm{tr}(u(\alpha v)') = \mathrm{tr}(u v' \alpha') = \mathrm{tr}(\alpha' u v')$
$= \mathrm{tr}(\underset{\mathfrak{a}}{(\alpha' u)} v') = 0$.)
so $\mathfrak{a} \cap \mathfrak{b} = (0)$, $\mathfrak{a} \oplus \mathfrak{b} = \mathbb{Q}[\alpha]$.
ie. semi-simple.

$\mathbb{Q}[\alpha] \cong K_1 \times \cdots \times K_p$, K_i alg. number field since simple factor has to be division ring

the Rosati involution maps K_i to K_i since $s(XX') > 0$.

$\Rightarrow$ each K_i is either

$\begin{cases} \text{totally real with identity involution or} \\ \text{totally imaginary quadratic extension} \end{cases}$

of a totally real field, involution $=$ cpx conjugation.

Galois theory w

$\Rightarrow$ Roots of min poly of α

$\longleftrightarrow$ image of α under various imbedding

$\phi_j : K_j \hookrightarrow \mathbb{C}$.

since $\phi_j(x') = \overline{\phi_j(x)}$ $\forall x \in \mathbb{Q}[\alpha]$.

$\Rightarrow a = \phi_j(a) = \phi_j(\alpha'\alpha) = \phi_j(\alpha')\phi_j(\alpha) = |\phi_j(\alpha)|^2$

i.e. $|\omega_i|^2 = a$. $\forall i$,

Now $\omega_i \mapsto \frac{a}{\omega_i}$ is a permutation since $\frac{a}{\omega_i} = \overline{\omega_i}$

and the polynomial P has real (integer) coeff ※

Abelian Var over Finite Fields.

General setup: $X_0 / \mathbb{F} = \mathbb{F}_q$, $q = p^f$. scheme of finite type $/\mathbb{F}$.

$\pi_0 : X_0 \to X_0$ Frobenius $\begin{cases} = \text{id on space} \\ f \mapsto f^q \text{ on } \mathcal{O}_{X_0} \end{cases}$

$k := \overline{\mathbb{F}}_q$, $X = X_0 \otimes_{\mathbb{F}} k$

$\pi : X \to X$ $(x_1, \ldots, x_m) \mapsto (x_1^q, \ldots, x_m^q)$

For X ab. var

$N_n := |X(\mathbb{F}_{q^n})| = \# \ker(1 - \pi^n)$: Galois theory

$= \deg(1 - \pi^n)$

$P_n(t)$: char. poly of $\pi^n = \prod_{i=1}^{2g} (t - \omega_i^n)$

$N_n = \deg(1 - \pi^n) = \underset{\text{by thm}}{P_n(1)} \equiv \det(1 - T_\ell(\pi^n)) = \prod_{i=1}^{2g} (1 - \omega_i^n)$

wish to show that $|\omega_i| = q^{1/2}$.

It is enough to show that
$|\omega_i^m| = q^{m/2}$ so may replace $\mathbb{F}_q$ by $\mathbb{F}_{q^m}$
X_0 by $X_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$
π by π^m

$\Rightarrow$ may assume $\exists$ line bundle L_0 on X_0 st $L \otimes_{\mathbb{F}} k$ ample on X

Recall $\phi' := \phi_L^{-1} \circ \phi \circ \phi_L$ the Rosati involution.

In order to apply prop. Need to show $\pi' \circ \pi = q$.

i.e. $\quad \phi_L^{-1}\, \pi\, \phi_L\, \pi(x) = q\, x \quad$ or

$$\pi\, \phi_L(\pi(x)) = q\, \phi_L(x)$$

π_0 acts on $\mathcal{O}_{X_0}$ by $f \mapsto f^q \Rightarrow \pi_0^* L_0 \simeq L_0^q$
$\Rightarrow \pi^* L \simeq L^q$

and notice $\underline{\pi \text{ acts on } \hat{X} \text{ as pull back}}$.

so $\quad \pi^*(\tau_{\pi x}^* L \otimes L^{-1}) \simeq \tau_x^*\, \pi^* L \otimes (\pi^* L)^{-1}$
$\simeq (\tau_x^* \otimes L^{-1})^{\otimes q}$

i.e. LHS = RHS $\quad$ *

General Remark on Weil conjectures: