Lectures on Kaehler Geometry at Academia Sinica

1998 August

Chin-Lung Wang
NTU

Chapter I

Hodge–Kodaira Theory

Chapter II

Hermitian–Yang–Mills and Kaehler–Einstein Geometry

Chapter III

Kodaira–Spencer–Kuranishi Theory

Chapter IV

Harmonic Maps

Ch. 1 Hodge-Kodaira Theory

Čech coh.
Derived funct?

X cpx mfd (not nec. cpt)

T_X holomorphic tang. bundle

$\Omega_X^i = \Lambda^i(T_X^*)$ sheaf of holo i-form

$E \to X$ holo. vector bundle

$\Omega_X^i \otimes E$ sh. of i-form with value in E

Basic invariants: (holo. inv)

$H^q(X, \Omega^p \otimes E)$

Using analytic method to get coh.

Dolbeault thm:

$$0 \to \Omega^p \to \Lambda^{p,0} \xrightarrow{\bar\partial} \Lambda^{p,1} \xrightarrow{\bar\partial} \Lambda^{p,2} \to \cdots$$

is a fine resolution of Ω_X^p

$$\Rightarrow \quad H^q(X, \Omega^p \otimes E) \simeq \underline{H^q(\Gamma(X, \Lambda^{p,\cdot}), \bar\partial)}$$

How to get unique repr?

Using metric on X and E, let X cpt

Hodge $*$ operator: (pointwise)

$$*: \Lambda^{p,q} \otimes E \longrightarrow \Lambda^{n-p,n-q}(E^*) \quad \text{st} \quad *^2 = id$$

st. $\alpha, \beta \in \Lambda^{p,q} \otimes E \Rightarrow \langle \alpha, \beta \rangle = \alpha \wedge (*\beta)$

$$\bar\partial^* := - * \bar\partial * : \Lambda^{p,q}(E) \to \Lambda^{p,q-1}(E)$$

Laplace: $\Delta_{\bar\partial} := \bar\partial \bar\partial^* + \bar\partial^* \bar\partial$

st $(\bar\partial \alpha, \beta) = (\alpha, \bar\partial^* \beta)$ (global)

$$H^{p,q}_{\bar\partial}(E) := \ker \Delta_{\bar\partial}$$

$$\equiv \ker \bar\partial \cap \underline{\ker \bar\partial^*}$$

Gauge condition

Hodge thm:

$$I = H + \Delta G, \quad \dim H < \infty.$$

Formal decomposition :

$$V = H \oplus \Delta V$$

for V : pre-Hilbert space

In general :

$$P : V \longrightarrow V, \quad P^* : V \longleftarrow V$$

$$\langle Pv, w \rangle = \langle v, P^* w \rangle$$

Self adjoint operator $\Delta : V \to V$

Spectral theory $\Rightarrow$

$$\Delta \sim \begin{bmatrix} 0 & \\ & \lambda_1 \\ & & \ddots \end{bmatrix} \begin{array}{l} \leftarrow H \text{ part} \\ \leftarrow G = \Delta^{-1} \text{part} \end{array}$$

$$V = \ker \Delta \oplus \operatorname{Im} \Delta$$

$\langle \Delta v, w \rangle = \langle v, \Delta w \rangle = 0$

if $w \in \ker \Delta$ then

$w \in (\operatorname{Im} \Delta)^\perp$

conversely, $w \in (\operatorname{Im} \Delta)^\perp$

$\Rightarrow$ $\ker \Delta = (\operatorname{Im} \Delta)^\perp$

$$w \in \ker P^* \Leftrightarrow w \in (\operatorname{Im} P)^\perp$$

$$\ker P^* = (\operatorname{Im} P)^\perp$$

Prob : $V = \operatorname{Im} P \oplus (\operatorname{Im} P)^\perp$?

need that $\operatorname{Im} P$ is closed , eg. finite codim.

true if $\ker P^*$ finite dim

for $P = P^*$, get : If P is Fredholm ($\Leftarrow$ elliptic op. on cpt Riem. mfd)

$$V = \ker P \oplus \operatorname{Im} P = H \oplus PV$$

$$v = h_1 + Pv_1 = h_2 + Pv_2$$

$\Rightarrow$ $h_1 - h_2 = P(v_2 - v_1)$ must $= 0$ so $h_1 = h_2$

v_2 is unique up to H. pick ! st in $H^\perp$

$$\boxed{v = H(v) + PGv}$$

Case of $\bar\partial$ Laplace : $\Delta = \bar\partial \bar\partial^* + \bar\partial^* \bar\partial$

$$H = \ker \Delta = \ker \bar\partial \cap \ker \bar\partial^*$$
$$= (\ker \bar\partial) \cap (\operatorname{Im} \bar\partial)^\perp$$

$V = H + \bar\partial(\cdots) + \bar\partial^*(\cdots)$ orthog. decomp.

Important : $[\bar\partial, G] = 0 = [\bar\partial^*, G]$

follows from simply $[\bar\partial, \Delta] = 0 = [\bar\partial^*, \Delta]$ ※

Applications of Hodge theorem :

$$H^q(X, \Omega_X^p(E)) = \mathbb{H}_{\bar\partial}^{p,q}(E)$$

via harmonic theory

since $\quad * \Delta = \Delta * \qquad (\bar\partial^* = - * \partial *)$

$$\left(\begin{array}{l} *\Delta = *(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial) = - * \bar\partial * \partial * \ \underline{+ -(*)^2\partial * \bar\partial} \\ \Delta * = (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)* = - \underline{\bar\partial * \partial (*)^2} - * \bar\partial * \bar\partial * \end{array}\right)$$

get

$$\begin{array}{ccc} \mathbb{H}_{\bar\partial}^{p,q}(E) & \xrightarrow[\sim]{*} & \mathbb{H}_{\bar\partial}^{n-p,n-q}(E^*) \\ \| & & \wr\| \\ H^q(X, \Omega_X^p(E)) & \cong & H^{n-q}(X, \Omega_X^{n-p}(E^*)) \end{array}$$

This is the Kodaira - Serre duality thm

Rmk : (1). Serre for C-M alg. v. coh. sh. via Ext

only for $\mathcal{F}$ v.b. in the form $H^i(X, \mathcal{F}) \cong H^{n-i}(X, K_X \otimes \mathcal{F}^*)$

(2) Kodaira for v.b. E. but for all cpx mfds.

Serre - Grothendieck : $\mathrm{Ext}^i(\mathcal{F}, \omega_X) \xrightarrow{\sim} H^{n-i}(X, \mathcal{F})^\vee$

Q : In the case $E =$ trivial $(\mathcal{O}_X)$

$\omega \in \mathbb{H}_{\bar\partial}^{p,q}(X) = H^q(X, \Omega_X^p)$

is that true that $\underline{d\omega = 0\ ?}$ Answer : NO !

look at $H^0(X, \Omega_X^p)$ sp. of holomorphic p-forms

Iwasawa manifold :

$\mathbb{C}^3 \xrightarrow{\varphi} X = \mathbb{C}^3/G \quad : \quad \begin{bmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{bmatrix} \in GL(3, \mathbb{C})$

check : dz_1, dz_3, $\phi = dz_2 - z_3 dz_1$ are inv. hol. 1-forms

so $\exists$ hol. 1-form λ st $\varphi^*\lambda = \phi$, $\quad \lambda \in H^0(X, \Omega_X^1)$.

$d\phi = dz_1 \wedge dz_3 \neq 0 \quad$ (not strange, $\mathbb{C}^3$ is not cpt)

$\Rightarrow d\lambda \neq 0$ on X.

$\underline{\lambda \text{ is not a top class !}}$

CONNECTIONS AND CURVATURE :

$$\begin{array}{cl} E & \text{hol. v.b.} \\ \downarrow & \\ X & \text{cpx. mfd} \end{array}$$

$\nabla : \Gamma(E) \longrightarrow \Gamma(E \otimes \Lambda^1_X)$ st. $\mathbb{R}$-linear (or $\mathbb{C}$-linear)

$\nabla(fs) = df \otimes s + f \cdot \nabla s$ Lebnitz rule.

Let h be a hermitian metric on E

Def: ∇ is comp. with h if $d\langle \alpha, \beta \rangle = \langle \nabla\alpha, \beta \rangle + \langle \alpha, \nabla\beta \rangle$
(or called metric conn)

Def: ∇ is called holomorphic if in $\nabla = \underset{\Lambda^{1,0}_X}{\nabla'} + \underset{\Lambda^{0,1}_X}{\nabla''}$, $\nabla'' \equiv \bar\partial$

Fundametal Thm in hermitian geom.

$\exists!$ holo conn ∇ comp. with the given h.

Lemma: $R := \nabla^2$ is function-linear, called the curvature

if ∇ metric. hol. then in a unitary frame s_i

$\nabla s_i = \theta_{ij} s_j$ θ_{ij} are skew hermitian, so is R_{ij}

But $R = R^{2,0} + R^{1,1} + R^{0,2}$ and $R^{0,2} = \bar\partial^2 = 0$ so $R^{2,0} = 0$

ie. $\boxed{R = R^{1,1} \in \Lambda^{1,1}_X(\mathrm{End}(E))}$ and $d^{\nabla} R = 0$
(Bianchi identity)

From here can define Chern classes. Wait!

Let $h_{ij} := h(s_i, s_j)$:

$$dh_{ij} = \underset{\Lambda^{1,0}_X}{\theta_{ik}} h_{kj} + \underset{\Lambda^{0,1}_X}{\bar\theta_{jk}} h_{ik} \Rightarrow \underline{\theta_{ik} = (\partial h_{ij})(h_{kj})^{-1}}$$

$$\| \quad \partial h_{ij} + \bar\partial h_{ij}$$

$$R s_i = \nabla^2 s_i = \nabla(\theta_{ij} s_j) = d\theta_{ij} \otimes s_j - \theta_{ij} \nabla s_j$$
$$= (d\theta_{ij} - \theta_{ik} \wedge \theta_{kj}) s_j \Rightarrow \underline{R_{ij} = d\theta_{ij} - \theta_{ik} \wedge \theta_{kj}}$$

$= \bar\partial \theta_{ij}$!
simple enough

special case : $E = L$: line bundle

$s_i = s$. one section $h(s) = |s|^2$

$\theta = \partial h \cdot h^{-1} = \partial \log h$

$R = d\theta = \bar\partial \partial \log h = -\partial\bar\partial \log h$

$\mathbb{Z}$ in fact!

Thm: $\frac{i}{2\pi} R = -\frac{i}{2\pi} \partial\bar\partial \log h$ repr $c_1(L) \in H^2_{DR}(X, \mathbb{R})$

NO Kahler condition is needed

Kahler mfd = "real-like" cpx mfd :

- X almost cpx : $J : T_{\mathbb{R}}X \to T_{\mathbb{R}}X$ st $J^2 = -Id$

  $g \in Sym^2_+ T^*X$ Riem. metric

  $\Rightarrow$ $\hat{g}(X,Y) := g(X,Y) + g(JX,JY)$ is hermitian

  ie $h(JX,JY) = h(X,Y)$

- X cpx, h hermitian metric

  $\Rightarrow$ $ds^2 = \sum h_{i\bar{j}}(z)\, dz^i \otimes d\bar{z}^j$

  associated 2-form $\omega(X,Y) := h(X, JY)$ (real) skew-sym.

  $\Rightarrow$ $\omega = \frac{i}{2} \sum h_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$ (1,1) form

- **Def:** X is Kahler if $\exists\, h$ st. $d\omega = 0$

  Cor: $h^{2i}(X) \neq 0$ $\forall\, i = 0 \dots n$ (since $\omega^n \sim$ Vol. form)

  Cor: Any cpx sub mfd $Y \hookrightarrow X$ is also Kahler

  in fact, Y is area minimizing in $[Y] \in H^*(X)$.

  MOST IMPORTANT EQUIVALENT CONDITION :

  $d\omega = 0 \Leftrightarrow$ $\forall p \in X$ $\exists\, z$ near p st. $ds^2 = \sum \left(\delta_{i\bar{j}} + [2]\right) dz^i \otimes d\bar{z}^j$

  Einstein's move with it at that moment !

  cor: an identity involves order ≤ 1 can be checked in the euclidean metric.

  (order 0 case even not use the kahler condi)

  Examples:

  I. $[\Lambda, \partial] = -i\, \bar{\partial}^*$ $\Rightarrow$ Hodge decomp

  II. $[\Lambda, L] = n - (p+q)$ on $\Lambda^{p,q}(X)$ $\Rightarrow$ Lefschetz decomp

- Fubini-Study metric on $\mathbb{P}^n = (\mathbb{C}^{n+1} - 0)/\mathbb{C}^\times$

  $Z \in \mathbb{C}^{n+1} - 0$ section of $\mathcal{O}(-1)$. ie. $Z : U \to \mathbb{C}^{n+1} - 0$, $U \subset \mathbb{P}^n$

  $\boxed{\omega = \frac{i}{2\pi} \partial \bar{\partial} \log |Z|^2}$ for $Z = (1, w_1, \dots, w_n)$ get

  $$\omega = \frac{i}{2\pi}\left(\frac{\sum dw_i \wedge d\bar{w}_i}{1+|W|^2} - \frac{\sum \bar{w}_i dw_i \wedge w_j d\bar{w}_j}{(1+|W|^2)^2} \right) \sim \left(\delta_{i\bar{j}} + [2]\right) dw_i\, d\bar{w}_j$$

  at $(1, 0, 0 \cdots 0)$.

X, ω Kahler, $L : \Lambda^{p,q} \to \Lambda^{p+1,q+1}$; $L\alpha = \alpha \wedge \omega$

$\Lambda : \Lambda^{p,q} \to \Lambda^{p-1,q-1}$ adjoint of L

Fundamental Kahler identity (due to Hodge)

$$\boxed{[\Lambda, \partial] = -i\bar\partial^*}$$

take $\bar{\ } \Rightarrow [\Lambda, \bar\partial] = i\partial^*$

$$\Delta_d = dd^* + d^*d = (\partial+\bar\partial)(\partial^*+\bar\partial^*) + (\partial^*+\bar\partial^*)(\partial+\bar\partial)$$

$$= (\partial\partial^* + \partial^*\partial) + (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)$$

$$+ \cancel{\bar\partial\partial^*} + \partial\bar\partial^* + \bar\partial^*\partial + \cancel{\partial^*\bar\partial} = \Delta_\partial + \Delta_{\bar\partial}$$

$$\Delta_\partial = \partial\partial^* + \partial^*\partial = -i\left(\partial(\Lambda\bar\partial - \bar\partial\Lambda) + (\Lambda\bar\partial - \bar\partial\Lambda)\partial\right)$$

$$= -i\left(\partial\Lambda\bar\partial - \partial\bar\partial\Lambda + \Lambda\bar\partial\partial - \bar\partial\Lambda\partial\right)$$

$$\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial = i\left(\bar\partial(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\bar\partial\right)$$

$$= i\left(\bar\partial\Lambda\partial - \bar\partial\partial\Lambda + \Lambda\partial\bar\partial - \partial\Lambda\bar\partial\right)$$

So, $\boxed{\Delta_\partial = \Delta_{\bar\partial} \quad \text{and} \quad = \tfrac{1}{2}\Delta_d \ !}$

get Hodge decomposition of cpx cohomologies.

$$H^k_{DR}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^q(X, \Omega^p_X)$$

// - Dolbeault thm

via $H^k_{DR}(X, \mathbb{C}) = \mathbb{H}^{p,q}_{\bar\partial}(X)$

Hodge thm — harmonic space wrt to $\bar\partial$

CONSEQUENCES :

- $H^{p,q} = \overline{H^{q,p}}$ since d is real

  $\Rightarrow$ h^i is even for i odd. eg. h^1 is even !

- Serre duality $\longrightarrow$ Hodge duality $H^{p,q} \underset{*}{\cong} H^{n-p,n-q}$

rational Hodge structure of $H^k(X, \mathbb{Q}) \otimes \mathbb{C}$.

1-1 correspondence between hermitian metric h on L and $c_1(L)$

h L $c_1(L) \in H^2_{DR}(M)$ line bundle

$-\frac{i}{2\pi}\partial\bar\partial \log h$ $= [\omega]$

real closed

$-\frac{i}{2\pi}\partial\bar\partial\log(e^f h) = -\frac{i}{2\pi}\partial\bar\partial f - \frac{i}{2\pi}\partial\bar\partial\log h \underset{\text{want}}{=} \omega$

ie. α, β (1,1) form

$[\alpha]=[\beta] \underset{?}{\Rightarrow} \alpha-\beta = \partial\bar\partial f$

($\partial\bar\partial$-lemma): Let η be $\bar\partial$ closed + ∂ exact then $\eta = \partial\bar\partial\xi$
or ∂ closed + $\bar\partial$ exact

pf: $id = H + G\Delta$

$\eta = H(\eta) + G\Delta\eta$ if η is $\bar\partial$ exact then η is closed in ∂ too? (No!) But $H_{\bar\partial}(\eta) = 0$ so $H_\partial(\eta) = 0$ too

$= G_\partial\Delta_\partial\eta$

$= G_\partial(\partial\partial^* + \partial^*\partial)\eta$

$= G_\partial\partial\partial^*\bar\partial\xi$ $([\partial^*,\bar\partial]=0)$

$= \partial\bar\partial(G_\partial\partial^*\xi)$

$$H^q(M, \Omega^p \otimes L) \cong \mathbb{H}^{p,q}_{\bar\partial}(L)$$

h metric on L, ample, has pos. curvature
$\Rightarrow$ M is Kahler

$R = Ric(h) = 2\pi/i \ \omega$ i.e. $\frac{i}{2\pi} Ric(h) = \omega$

$$R = \nabla^2 = (\nabla' + \bar\partial)^2 = \nabla'\bar\partial + \bar\partial\nabla'$$

Let $\eta \in \mathbb{H}^{p,q}(L)$

$$R\eta = \bar\partial\nabla'\eta + \nabla'\bar\partial\eta$$

1st. identity

$$\boxed{\text{"Kahler identity"} \quad [\Lambda, \bar\partial] = -i\,\nabla'^*}$$

$$\langle \Lambda R\eta, \eta\rangle = \langle \Lambda\bar\partial\nabla'\eta, \eta\rangle$$
$$= \langle ([\Lambda,\bar\partial] + \bar\partial\Lambda)\nabla'\eta, \eta\rangle$$
$$= -i\langle \nabla'^*\nabla'\eta, \eta\rangle + \langle \bar\partial\Lambda\nabla'\eta, \eta\rangle$$
$$= -i\langle \nabla'\eta, \nabla'\eta\rangle + \langle \Lambda\nabla'\eta, \bar\partial^*\eta\rangle$$

$$\langle R\Lambda\eta, \eta\rangle = \langle \bar\partial\nabla'\Lambda\eta, \eta\rangle + \langle \nabla'\bar\partial\Lambda\eta, \eta\rangle$$
$$= \langle \nabla'(-[\Lambda,\bar\partial] + \Lambda\bar\partial)\eta, \eta\rangle$$
$$= i\langle \nabla'\nabla'^*\eta, \eta\rangle = i\langle \nabla'^*\eta, \nabla'^*\eta\rangle$$

$$\langle [\Lambda, \tfrac{i}{2\pi}R]\eta, \eta\rangle = \frac{1}{4\pi}\left(|\nabla'\eta|^2 + |\nabla'^*\eta|^2\right)$$

$\parallel$

$$\boxed{[\Lambda, L] = \text{number operator } n-(p+q)}$$

2nd identity

so $p+q > n \Rightarrow \eta \equiv 0$, i.e.

Kodaira vanishing thm:

$H^q(X, \Omega^p_X \otimes L) = 0$ for $p+q > n$

CONSEQUENCES :

1. Hard Lefschetz : $H^{n-i}(X) \xrightarrow[\sim]{L^i} H^{n+i}(X)$

this is the consequence of Lefschetz decomp. sl_2 repr theory.

2. Weak Lefschetz : $i: H \subset X$ st $\mathcal{L}_H$ is positive, then

$$H^k(X) \xrightarrow[i^*]{} H^k(H) \quad \begin{array}{l} \text{isom for } k \leq n-2 \\ \text{injective } k = n-1 \end{array}$$

pf : look at $H^q(X, \Omega^p_X) \xrightarrow[i^*]{} H^q(H, \Omega^p_H)$ need $\Omega^p_X \to \Omega^p_X|_H \to \Omega^p_H$

(i) $0 \to \Omega^p_X(-H) \to \Omega^p_X \to \Omega^p_X|_H \to 0$

$$0 \to \underline{H^q(X, \Omega^p_X) \to H^q(X, \Omega^p_X|_H)} \to H^{q+1}(X, \Omega^p_X(-H))$$

for $p+q \leq n-1$

(ii) $0 \to \Omega^{p-1}_H(N^*_H) \to \Omega^p_X|_H \to \Omega^p_H \to 0$

$\Omega^{p-1}_H(N^*_H) = \Omega^{p-1}_H(-H)$

$0 \to T_H \to T_X|_H \to N_H \to 0$

$0 \to N^*_H \to T^*_X|_H \to T^*_H \to 0$

line. take Λ^p.

$$0 \to \underline{H^q(H, \Omega^p_X|_H) \to H^q(H, \Omega^p_H)} \to H^{q+1}(H, \Omega^{p-1}_H(-H))$$

for $p-1+q \leq (n-1)-1$

Q.E.D. i.e. $p+q \leq n-1$

3. Hodge conjecture for (1,1). (Lefschetz's thm.) for proj. v.

$H^{p,p}(X) \cap H^{2p}(X, \mathbb{Z})$ repr by $\underline{\text{alg. cycles}}$.

analytic cycles.

pf. $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\times \to 0$ get

$$H^1(X, \mathcal{O}^\times) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O})$$

$c_1 \qquad \alpha \qquad$ proj map $\qquad H^{0,2}_{\bar\partial}(X)$

$\alpha \mapsto 0$

$\Rightarrow \alpha = c_1(L)$ for a line bundle

now every L has mero. section : or use $L + kH$. $k \gg 0$.

induction on $\dim X$ and see

$0 \to \mathcal{O}_X(L+(k-1)H) \to \mathcal{O}_X(L+kH) \to \mathcal{O}_H(L+kH) \to 0$ Q.E.D.

Kodaira embedding thm:

A line bundle L on a kahler X is positive $\Leftrightarrow$ L is ample

i.e. X alg. $\Leftrightarrow$ $\exists$ closed positive $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$.

CHERN CLASSES:

$$\begin{array}{ccc} E & \text{cpx v.b. } rk = r & & S \text{ universal subbundle} \\ \downarrow & & & \downarrow \\ X & \text{top. space} & f: X \to & G(r,N) \quad N \text{ large} \end{array}$$

pointwise-linear mapping $E \to \mathbb{C}^N$, $E = f^*S$

$E_1 \cong E_2 \Leftrightarrow f_1 \sim f_2$ as homotopy classes in $[X, G(r,N)]$

$H^*(G(r,N), \mathbb{Z})$ is generated by the "universal chern classes" as a ring $\mathbb{Z}[c_1, c_2, c_3, \dots, c_r]$ $\deg c_i = 2i$

eg. $H^2(G(r,N)) = \mathbb{Z}c_1$

$H^4(G(r,N)) = \mathbb{Z}c_1^2 \oplus \mathbb{Z}c_2$

$H^6(G(r,N)) = \mathbb{Z}c_1^3 \oplus \mathbb{Z}c_1c_2 \oplus \mathbb{Z}c_3$

$H^8(G(r,N)) = \mathbb{Z}c_1^4 \oplus \mathbb{Z}c_1^2c_2 \oplus \mathbb{Z}c_1c_3 \oplus \mathbb{Z}c_2^2$ etc ...

(This can be calculated by writing down the CW cpx decomp of $G(r,N)$ using the "schubert cycles")

Call $c_i(S)$ = this c_i and

$c_i(E) = f^*c_i(S) \in H^{2i}(X, \mathbb{Z})$ for $i = 1 \dots r$

basic properties:

- functorial if $X \xrightarrow{g} Y$ and $E \to Y$ then
  $$g^* c_i(E) = c_i(g^*E)$$
- whitney summation formula
  $$c_k(E \oplus F) = \sum_{i+j=k} c_i(E) \cup c_j(F)$$

(How to prove this?) How about $c_k(E \otimes F)$??

Better formulation: Chern characters

the total chern class

$$c(E) := 1 + c_1(E) + c_2(E) + \cdots \in H^*(X, \mathbb{Z})$$

formally factorizes $= (1+\lambda_1)(1+\lambda_2)\cdots(1+\lambda_r)$

λ_i called chern roots

so $c_k(E)$ = elementary sym function of λ_i with $\deg = k$.

$$\mathrm{ch}(E) = \sum_{i=1}^{r} e^{\lambda_i} \quad \in H^*(X,\mathbb{Z}) \otimes \mathbb{Q}$$
$$= \sum_{i=1}^{r} \left(1 + \lambda_i + \frac{\lambda_i^2}{2} + \frac{\lambda_i^3}{3!} + \cdots\right)$$
$$= r + c_1 + \frac{c_1^2 - 2c_2}{2} + \frac{c_1^3 - 3c_1c_2 + 3c_3}{6} + \cdots$$

which does not dep on the existence of λ_i

Splitting principle:

any eq'n of c_i's can be reduced to check the line bundle case.

$$\left(\begin{aligned}\sum \lambda_i^3 &= \lambda_1^3 + \lambda_2^3 + \lambda_3^3 - 3\lambda_1\lambda_2\lambda_3 + 3c_3 \\ &= c_1(c_1^2 - 3c_2) + 3c_3\end{aligned}\right)$$

$$\begin{tikzcd} \varphi^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ \mathbb{P}(E) \arrow[r, "\varphi"] & X \end{tikzcd}$$

for each $y \in \mathbb{P}(E)$ y corr to a line $\cong \mathbb{C}$ in $E_{\pi(y)} = \varphi^*E_y$

hence there is a sub line bundle $\xi \subset \varphi^*E$

(in fact, universal one $\mathcal{O}(-1)$)

Repeat r times get

$$\begin{tikzcd} \varphi^*E \arrow[r] \arrow[d] & E \arrow[d] \\ X^{(r)} \arrow[r, "\varphi"] & X \end{tikzcd} \qquad \varphi^*E = \xi_1 \oplus \cdots \oplus \xi_r$$
$$\underline{c_1(\xi_i) \in H^2(X^{(r)},\mathbb{Z}) = \lambda_i}$$

Moreover, $H^i(X,\mathbb{Z}) \xrightarrow{\varphi^*} H^i(X^{(r)},\mathbb{Z})$ is injective

In fact Q: write down explicit relation between $H^*(\mathbb{P}(E),\mathbb{Z})$ and $H^*(X,\mathbb{Z})$ (Leray thm)

Now its clear that

$$\mathrm{ch}(E \oplus F) = \sum_{i=1}^{r_1+r_2} e^{\lambda_i} = \mathrm{ch}(E) + \mathrm{ch}(F)$$
$$\mathrm{ch}(E \otimes F) \quad \left(\sum_{i=1}^{r_1} e^{\lambda_1}\right)\left(\sum_{i=1}^{r_2} e^{\lambda_2}\right) = \sum_{i,j} e^{\lambda_i + \lambda_j}$$

$$E \otimes F \text{ "=" } \left(\sum \xi_i\right) \otimes \left(\sum \xi_j'\right) = \sum_{i,j} \xi_i \otimes \xi_j'$$

but for line bundles, $c_1(\xi_i \otimes \xi_j') = c_1(\xi_i) + c_1(\xi_j')$

$$= \mathrm{ch}(E) \cdot \mathrm{ch}(F).$$

Get $\mathrm{ch}: K(X) \longrightarrow H^{even}(X,\mathbb{Q})$ chern homomorphism

Example I: Let $\underset{\text{cpx. v.b.}}{E} \longrightarrow X$ cpt cpx mfd

$$\mathrm{End}(E) = E^* \otimes E \;; \quad \text{so} \quad c_1(\mathrm{End}\, E) = 0 \quad (\text{see also below})$$

Write $c_2(E^* \otimes E)$ in terms of $c_i(E)$:

$$ch(E^* \otimes E) = ch(E^*) \cdot ch(E) \qquad E: \text{semi-stable bundle}$$

$$r^2 + c_1' + \frac{c_1'^2 - 2c_2'}{2} = \left(r - c_1 + \frac{c_1^2 - 2c_2}{2}\right) \cdot \left(r + c_1 + \frac{c_1^2 - 2c_2}{2}\right) + \cdots$$

$$= r^2 + r \cdot (c_1^2 - 2c_2) - c_1^2 \cdots$$

$$\Rightarrow \begin{cases} c_1' = 0 \\ -c_2' = (r-1)\, c_1^2 - 2r\, c_2 \end{cases}$$

trivial formula

$$\boxed{c_2(\mathrm{End}\, E) = 2r\, c_2(E) - (r-1)\, c_1(E)^2}$$

This is useful later.

Example II: Adjunction Formula

Let X be a cpt cpx mfd, $j: D \hookrightarrow X$ a smooth divisor

the chern class of a cpx mfd X is defined to be the chern class of the holomorphic tangent bundle T_X. $c(X) := c(T_X)$.

then

$$0 \longrightarrow T_D \longrightarrow \underset{\substack{\| \\ j^* T_X}}{T_X|_D} \longrightarrow N_D \longrightarrow 0$$

$N_D \cong \mathcal{O}_D(D)$ is the "normal bundle" of D.

Notice also that $c_1(\mathcal{O}_X(D)) = [D] \in H^2(X, \mathbb{Z})$, $c_i(\mathcal{O}_X(D)) = 0 \quad \forall i \geq 2$

so $j^* c(X) = c(D) \cdot c(N_D)$, may also use

$$j^* ch(X) = ch(D) + ch(N_D)$$

$$\Rightarrow \quad c_1(X)|_D = c_1(D) + D|_D \quad (\text{ie. } K_D = (K_X + D)|_D)$$

$$c_2(X)|_D = c_2(D) + c_1(D) \cdot D|_D \quad \text{etc.}$$

Point of View from Diff. Geom :

How to construct $c_i(E)$ using diff forms in $H^*_{DR}(X,\mathbb{R})$?

$$\begin{array}{ll} E \text{ cpx v.b.} & \text{connection } \nabla : C^\infty(E) \longrightarrow C^\infty(E\otimes\Lambda^1) \\ \downarrow & \qquad\qquad\qquad\quad \overset{\shortparallel}{\Lambda^0(E)} \qquad\quad \overset{\shortparallel}{\Lambda^1(E)} \\ X \;\; C^\infty \text{ manifold} & \text{st. } \nabla(fs) = df\cdot s + f\,\nabla s \end{array}$$

$$\left(\begin{aligned} &\text{and } \nabla_X s = (\nabla s)(X) \text{ so} \\ &\nabla_X(fs) = df(X)\cdot s + f\,\nabla_X s \\ &\qquad\quad = Xf\cdot s + f\,\nabla_X s \text{ . etc ..} \end{aligned} \right)$$

Extension via Leibnitz rule : (Generalized de Rham cpx)

$$\Lambda^0(E) \xrightarrow{d^\nabla} \Lambda^1(E) \xrightarrow{d^\nabla} \Lambda^2(E) \xrightarrow{d^\nabla} \cdots$$

$$\text{st} \quad d^\nabla(\eta s) = d\eta\wedge s + (-1)^{\deg\eta}\,\eta\wedge d^\nabla s$$

Lemma : $R := d^\nabla \circ d^\nabla$ (usually ∇^2, $D^2\cdots$) $\in \Lambda^2(\mathrm{End}\,E)$

pf: only need to prove that R is "function linear", ie.

$$\begin{aligned} R(fs) &= d^\nabla d^\nabla(fs) = d^\nabla(df\cdot s + f\nabla s) \\ &= \cancel{d^2 f}\cdot s - \cancel{df\wedge\nabla s} + \cancel{df\wedge\nabla s} + f\,\nabla^2 s = f\,R(s) \quad \square \end{aligned}$$

R is called the curvature matrix 2-form of (E,∇)

Cartan Formula :

$$\begin{aligned} d^\nabla\omega\,(X_0,\dots,X_p) &= \sum (-1)^i\, d^\nabla_{X_i}\,\omega(X_0\cdots\hat{X_i}\cdots X_p) \\ &\quad + \sum (-1)^{i+j}\,\omega([X_i,X_j],\dots\hat{X_i},\dots\hat{X_j}\cdots X_p) \end{aligned}$$

Curvature Formula :

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} \qquad \left([\nabla_X,\nabla_Y] - \nabla_{[X,Y]} \right)$$

In case $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$ then R_{ij} (matrix) $:= R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$

$= [\nabla_i, \nabla_j]$ since $[\partial_i,\partial_j] = 0$.

measure the "non commutativity" of double derivative

Def : $c(E) := \det\left(I + \frac{i}{2\pi}R\right)$ total chern form !

characteristic poly : $\sum c_i(E)t^i = \det\left(tI + \frac{i}{2\pi}R\right)$

Lemma (Bianchi) : $d^\nabla R = 0$, hence $d\,c(E) = 0$. closed!

Local Formula: For simplicity, consider only the case

$E \downarrow X$: E hermitian holo. v.b. with inner product $\langle\, ,\, \rangle = h$

X cpx mfd with her. metric g,

$\boxed{\exists!\ \text{conn. } \nabla \text{ st in } \nabla = \nabla' + \nabla'' \text{ wrt } \Lambda^1 = \Lambda^{1,0} \oplus \Lambda^{0,1}, \text{ have } \nabla'' = \bar\partial}$

s_i holo. frame of E. ∇ any connection on E

$\boxed{\nabla s_i := \sum_j \omega_i^j s_j}$, then

$R s_i = \nabla^2 s_i = \nabla(\omega_i^j s_j) = d\omega_i^j s_j - \omega_i^j \wedge \omega_j^k s_k$

ie. $\Omega_i^j s_j = (d\omega_i^j - \omega_i^k \wedge \omega_k^j) s_j$

$$\boxed{(*)\quad \Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j}$$

Now $\det\left(I + \frac{i}{2\pi}\Omega\right) = 1 + c_1 + c_2 + \cdots$; $c_i \in \Lambda^{2i}(X)$, chern forms

eg. $c_1 = \frac{i}{2\pi} \sum_i \Omega_i^i$

$c_2 = -\frac{1}{8\pi^2} \sum_{j,k} \left(\Omega_j^j \wedge \Omega_k^k - \Omega_k^j \wedge \Omega_j^k\right)$

the 2nd one follows from the fact that

$\sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}\left[\left(\sum_i \lambda_i\right)^2 - \sum_i \lambda_i^2\right] = \frac{1}{2}\left[(\mathrm{tr} A)^2 - \mathrm{tr}(A^2)\right]$

with $A = \frac{i}{2\pi}\Omega$ and λ_i eigenvalues of A

We are interested in rewriting the chern forms c_1, c_2 using the notion of "Ricci tensor". Namely:

- since Ω_i^j are in fact $(1,1)$ forms (if use ∇ the unique conn. st $\nabla'' = \bar\partial$, then $R^{(0,2)}$ part $= \nabla''^2 = \bar\partial^2 = 0$, also R is skew-hermitian $\Rightarrow R^{(2,0)}$ part $= 0$, so R is $(1,1)$)

  Let $\Omega_i^j = R^j_{i\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta$

  and $R_{i\bar j\alpha\bar\beta} = R^k_{i\alpha\bar\beta} h_{k\bar j}$

  There are 2 Ricci tensors $P_{\alpha\bar\beta} = \sum_i R^i_{i\alpha\bar\beta}$ and $K_i^j = R^j_{i\alpha\bar\beta}\, g^{\alpha\bar\beta}$

Introduction to Hermitian-Yang-Mills / Kähler-Einstein Geometry

From : Kobayashi's book IV. §4. P.112 —

Given holo. v.b. rank $E = r$:

E, h : hermitian v.b. $h = \langle\ ,\ \rangle$

$\downarrow$ $\quad \exists!$ metric connection (Levi-Civita)

X cpx mfd $\quad \nabla : \Gamma(E) \to \Gamma(\Lambda^1 \otimes E)$

g : her. $\quad d\langle s, t\rangle = \langle \nabla s, t\rangle + \langle s, \nabla t\rangle$

st. in $\nabla = \nabla' + \nabla''$, have $\nabla'' = \bar\partial$

Fact : Curvature $R \in \Gamma(\Lambda^2 \otimes \operatorname{End} E) := R(s) = \nabla^2 s$ is (1,1) type

Def : total Chern form $c(E,\nabla) := \det\left(I + \frac{i}{2\pi} R\right) \in \Lambda^{even}(X)$.

$$= 1 + c_1(E,\nabla) + c_2(E,\nabla) + \cdots$$

on U, local unitary frame

$s_1, \ldots, s_r \in \Gamma(U, E)$

$\theta^1, \ldots, \theta^n \in \Gamma(U, T^*X)$

Write $\quad R(s_i) = \sum R^j_{i\alpha\bar\beta}\, s_j \otimes (\underset{\theta^\alpha}{dz^\alpha} \wedge \underset{\bar\theta^\beta}{d\bar z^\beta}) =: \sum \Omega^j_i\, s_j$

$$\Rightarrow \begin{cases} c_1(E,\nabla) = \frac{i}{2\pi} \operatorname{tr} R = \frac{\sqrt{-1}}{2\pi} \sum \Omega^i_i \quad \in \Lambda^{1,1} \\ c_2(E,\nabla) = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \cdot \frac{1}{2}\left[(\operatorname{tr} R)^2 - \operatorname{tr} R^2\right] = \frac{-1}{8\pi^2}\left(\Omega^i_i \Omega^j_j - \Omega^j_i \Omega^i_j\right) \end{cases}$$

let $\omega = \sqrt{-1} \sum \theta^\alpha \wedge \bar\theta^\alpha$ pre-Kähler form $\quad \in \Lambda^{2,2}$

Fact : $n(n-1)\, \theta^\alpha \wedge \bar\theta^\beta \wedge \theta^\gamma \wedge \bar\theta^\delta \wedge \omega^{n-2} = \begin{cases} \omega^n & \alpha = \delta \neq \beta = \gamma \\ -\omega^n & \alpha = \beta \neq \gamma = \delta \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow n(n-1)\, \Omega^i_i \wedge \Omega^j_j \wedge \omega^{n-2} = n(n-1) R^i_{i\alpha\bar\beta} R^j_{j\gamma\bar\delta}\, \theta^\alpha \wedge \bar\theta^\beta \wedge \theta^\gamma \wedge \bar\theta^\delta \wedge \omega^{n-2}$$

$$= -\left(R_{\alpha\bar\alpha} R_{\gamma\bar\gamma} - R_{\alpha\bar\gamma} R_{\gamma\bar\alpha}\right) \omega^n$$

$$=: -\left(R^2 - \|\rho\|^2\right) \omega^n \qquad \cdot\, R_{\alpha\bar\gamma} \overline{R_{\alpha\bar\gamma}} = \sum |R_{\alpha\bar\delta}|^2 =: \|\rho\|^2$$

$$n(n-1)\, \Omega^i_j \wedge \Omega^j_i \wedge \omega^{n-2} = n(n-1) R^i_{j\alpha\bar\beta} R^j_{i\gamma\bar\delta}\, \theta^\alpha \wedge \bar\theta^\beta \wedge \theta^\gamma \wedge \bar\theta^\delta \wedge \omega^{n-2}$$

$$= -\left(R^i_{j\alpha\bar\alpha} R^j_{i\gamma\bar\gamma} - R^i_{j\alpha\bar\gamma} R^j_{i\gamma\bar\alpha}\right) \omega^n$$

$$=: \left(\|R\|^2 - \|K\|^2\right) \omega^n \qquad K^i_j := \sum R^i_{j\alpha\bar\alpha} \text{ hermitian}$$

No.

Now define "Einstein type tensor"

$$T^i_{j\alpha\bar\beta} := R^i_{j\alpha\bar\beta} - \frac{1}{r}\delta^i_j R_{\alpha\bar\beta}$$

$$\Rightarrow \|T\|^2 = \sum \left| R^i_{j\alpha\bar\beta} - \frac{1}{r}\delta^i_j R_{\alpha\bar\beta} \right|^2$$

$$= \sum |R^i_{j\alpha\bar\beta}|^2 - \frac{2}{r}\sum |R_{\alpha\bar\beta}|^2 + \frac{1}{r^2}\cdot r\sum |R_{\alpha\bar\beta}|^2$$

$$= \|R\|^2 - \frac{1}{r}\|\rho\|^2$$

≥ 0 and $=0 \Leftrightarrow R^i_{j\alpha\bar\beta} = \frac{1}{r}\delta^i_j R_{\alpha\bar\beta}$ (proj. flat)

Now consider $c_2(\operatorname{End} E) = 2r\, c_2(E) - (r-1)c_1^2(E)$

$$\wedge\,\omega^{n-2} := \frac{1}{4\pi^2 n(n-1)}\left(r R^2 - r\|\rho\|^2 + r\|R\|^2 - r\|K\|^2 - (r-1)R^2 + (r-1)\|\rho\|^2 \right)$$

$$= \frac{1}{4\pi^2 n(n-1)}\left(\underline{r\|T\|^2} + \left(\underline{R^2 - r\|K\|^2}\right)\right)$$

Hermitian-Yang-Mills eq'n : $\boxed{\Lambda R = \varphi I}$ ← $\Lambda := L^*$, $L\alpha := \omega\wedge\alpha$

ie. $K^i_j = \varphi\,\delta^i_j \Rightarrow \|K\|^2 = \varphi^2 r$ & $R = \varphi r$

$$\Rightarrow R^2 = \varphi^2 r^2 = r\|K\|^2$$

In this case : get

$$c_2(\operatorname{End} E).\omega^{n-2} = \left(2r\,c_2(E) - (r-1)c_1^2(E)\right).\omega^{n-2}$$

$$= \frac{r}{4\pi^2 n(n-1)}\|T\|^2 \geq 0.$$

even not necessary !

<u>Theorem</u> (Donaldson alg. case / Uhlenbeck-Yau '87 <u>Kähler case</u>)

E is ω-stable $\Longleftrightarrow$ $\exists$ metric h on E st. $\Lambda R = \varphi I$.

under this. then $\left(2r\,c_2(E) - (r-1)c_1^2(E)\right).\omega^{n-2} \geq 0$

<u>Definition</u>: ω Kähler form, E is ω-stable if $\forall$ torsion free subsheaf $\mathcal{F}$ of $\mathcal{E}$: $\dfrac{\int c_1(\mathcal{F})\,\omega^{n-1}}{rk\,\mathcal{F}} < \dfrac{\int c_1(\mathcal{E})\,\omega^{n-1}}{rk\,\mathcal{E}}$

Ex. <u>Special case</u>: X alg. surface, E rk 2.

Bogomolov instability thm :

$4c_2(E) - c_1^2(E) < 0 \Rightarrow \exists\ 0 \to M \to E \to N\otimes I_Z \to 0$ ← 0-dim'l

st. $(M-N)^2 > 0$

$(M-N).H > 0 \quad \forall$ ample H.

For the case $E = T_X$ tangent bundle $(r=n)$, $c_i(X) := c_i(T_X)$.
May consider the refined tensor, notice that $R_{i\bar j k\bar \ell} = R_{k\bar \ell i\bar j}$.

$$T_{i\bar j k\bar \ell} := R_{i\bar j k\bar \ell} - \frac{R}{n(n+1)}\left(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{jk}\right)$$

$T=0 \Leftrightarrow$ holo. sectional curvature $\Leftrightarrow \tilde{X} = \mathbb{P}^n, \mathbb{C}^n$ or D^n.

$$\|T\|^2 = \|R\|^2 - \frac{4R^2}{n(n+1)} + \frac{R^2}{n^2(n+1)^2}\left(n^2+2n+n^2\right)$$

Ex. 1) show first that $R = \text{const}$.
2) show $\Rightarrow$.

$$= \|R\|^2 - \frac{2}{n(n+1)}R^2$$

So $\left(2(n+1)c_2(X) - n\, c_1^2(X)\right).\,\omega^{n-2}$

$$= \frac{1}{4\pi^2 n(n-1)}\Big((n+1)R^2 - (n+1)\|\rho\|^2 + (n+1)\|R\|^2 - (n+1)\|K\|^2 - nR^2 + n\|\rho\|^2\Big)$$

notice $\rho \equiv K \equiv Ric$

$$= \frac{1}{4\pi^2 n(n-1)}\left(\left[(n+1)\|R\|^2 - \frac{2}{n}R^2\right] + \left(1+\frac{2}{n}\right)R^2 - (n+2)\|\rho\|^2\right)$$

$$= \frac{1}{4\pi^2 n(n-1)}\left[(n+1)\|T\|^2 + \frac{n+2}{n}\left(R^2 - n\|\rho\|^2\right)\right]$$

For Kähler-Einstein metrics $\boxed{R_{\alpha\bar\beta} = \frac{R}{n}\delta_{\alpha\beta}}$ ie. $Ric = \frac{R}{n}I$.

$\Rightarrow \|\rho\|^2 = \left(\frac{R}{n}\right)^2 \cdot n = \frac{1}{n}R^2$.

here we have to assume that ω is the K-E form.

hence K.E $\Rightarrow$ $\int_X \left(2(n+1)c_2 - n c_1^2\right)\omega^{n-2} \geq 0$

with "=" $\Leftrightarrow$ universal cover of $X = \mathbb{P}^n, \mathbb{C}^n$ or D^n.

This is Yau's uniformization theorem.

THEOREM (Yau: 1976)

X cpt Kähler mfd with

$c_1(X) = 0$ (ie K_X trivial) $\Rightarrow \exists!$ K-E metric ($Ric = 0$) in each Kähler class

$c_1(X) < 0$ (ie. K_X ample) $\Rightarrow \exists!$ K-E metric.

Rmk: $c_1 > 0$ (Fano case) $\longrightarrow$ Works of G. Tian (1990~).

Appendix:

(I) $\Lambda F_h = \varphi I$ H-E metric for $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ $rk = r$, holo. v.b, cpx mfd.

(II) $\begin{cases} DF = 0 \\ D^*F = 0 \end{cases}$ Y-M connection; $DF=0$ auto true (Bianchi). $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ cpx v.b. $rk = r$, real 4 mfd.

when X is a Kähler surface

$$0 = D^*F = *D*F \iff D(*F) = 0$$

special solution : SD or ASD solutions

SD/ ASD YM: $*F = \pm F$

since $* : \Lambda^2 \longrightarrow \Lambda^2$ maps $\Lambda^{2,0} \longrightarrow \Lambda^{0,2}$, $\Lambda^{1,1} \longrightarrow \Lambda^{1,1}$, $\Lambda^{0,2} \longrightarrow \Lambda^{2,0}$

$\Rightarrow$ F is of $(1,1)$ type ; ie.

in $D = D' + D''$ get $D''^2 = 0$

hence an unique holo. str. on E st. $D'' = \bar\partial$.

Ex. Eq'n $*F = F$ $F \in \Lambda^{1,1}(\mathrm{End}\,E)$

is equiv to $\Lambda F = \varphi I$ for some const. φ.

Bogomolov Thm :

$4C_2(E) - c_1^2(E) < 0 \implies \exists M, N$ line bundle, Z 0-cycle

$0 \to M \to E \to N \otimes I_Z \to 0$

st. $(M-N)^2 > 0$

$(M-N).H > 0 \quad \forall$ ample H

Obviously :

$c_1(E) = M + N$

$C_2(E) = MN + |Z|$

$4C_2(E) - c_1^2(E) = 4MN + 4|Z| - (M+N)^2$

$= -(M-N)^2 + 4|Z|$

Notice : Mumford's

H-in-stability means $\frac{M.H}{1} \geq \frac{(M+N).H}{2}$ ie. $(M-N).H \geq 0$

pf $\Rightarrow$: Donaldson/Uhlenbeck-Yau : H-(poly)stable $\Rightarrow \exists$ H-E $\Rightarrow 4C_2 - c_1^2 \geq 0$

hence $4C_2 - c_1^2 < 0 \Rightarrow$ not (even) (poly) stable $\forall H$ ample.

So $\exists M, N \ldots$ st. $(M-N).H \geq 0$ and $0 \to M \to E \to N \otimes I_Z \to 0$

but then $(M-N)^2 = -(4C_2 - c_1^2) + 4|Z| > 0$

Now $H^2 > 0$ (ample) ; Hodge index thm $\Rightarrow (M-N).H > 0$.

✱

Further directions :

- Simpson's theory of Higgs fields
  — general uniformization thm
- Harmonic metric / variation of Hodge structures
- General fiberwise - Ricci flat metric

Let X minimal, assume abundance conjecture :

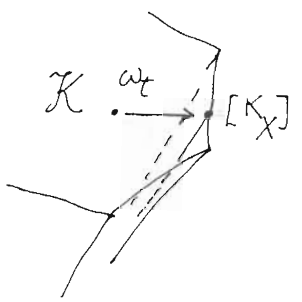
$$|rK| : X \xrightarrow[\varphi]{} Z \subset \mathbb{P}^N$$

$$[\tfrac{1}{r}\varphi^*\omega_{FS}] \in K = -c(X)$$

in any Kähler class $[\omega]$, $\exists!$ Kähler metric

ω st. $Ric(\omega) = -\frac{1}{r}\varphi^*\omega_{FS} = -\beta$

ie. $Ric(\omega)|_{X_z} = 0$ along any fiber X_z.



consider solu. ω_t

$[\omega_t] \longrightarrow [K_X]$ as $t \to \infty$

compare $\lim\limits_{t\to\infty} \omega_t$ with β

May also replace ω_{FS} by any metric

eg. K-E if $\exists$.

$$(1+\tfrac{2}{n})R^2 - (n+2)\|\rho\|^2$$

$$= \frac{n+2}{n}\left(R^2 - n\|\rho\|^2\right)$$

Ch.3 Kodaira – Spencer theory :

X almost cpx : $TX \xrightarrow{J} TX$ st $J^2 = -id$

$TX \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$

eigen space of i ; $-i$

"holomorphic part"

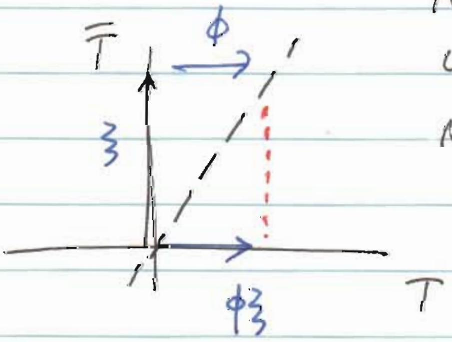
$T = T^{1,0}$ span by $v - iJv$ since $J(v - iJv) = i(v - iJv)$

$\bar{T} = T^{0,1}$ span by $v + iJv$. so $T^{0,1} = \overline{T^{1,0}}$.

Dual $T_X^* \otimes \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1}$

$\Lambda^0 \xrightarrow{d} \Lambda^1 = \Lambda^{1,0} \oplus \Lambda^{0,1}$; $d = \partial + \bar\partial$

Newlander – Nirenberg : J integrable $\Leftrightarrow$ $\bar\partial^2 = 0$



New (near by) almost cpx str comes from a map $\phi : \bar T \to T$

New $\bar{T}'$: $\xi + \phi\xi$, $\xi \in \bar T$

How New $\bar\partial$ operator looks like?

$\bar\partial_\phi = \bar\partial + \phi$ as operators : $\bar T = T^{0,1}$ spanned by $\frac{\partial}{\partial \bar z^\beta}$

$\Rightarrow$ $\bar T_\phi = T^{0,1}_\phi$ spanned by $\frac{\partial}{\partial \bar z^\beta} + \phi\left(\frac{\partial}{\partial \bar z^\beta}\right)$

$\phi = \sum \phi^\alpha_{\bar\beta}\, d\bar z^\beta \otimes \frac{\partial}{\partial z^\alpha}$; $\phi\left(\frac{\partial}{\partial \bar z^\beta}\right) = \sum_\alpha \phi^\alpha_{\bar\beta} \frac{\partial}{\partial z^\alpha}$

Formal reason :

Want $(\bar\partial + \phi)^2 = \bar\partial^2 + \bar\partial\phi + \phi\bar\partial + \phi^2 = 0$

actual form $\bar\partial\phi = \frac{1}{2}[\phi, \phi]$. A simple calculation !

(see next page)

$$\bar\partial_\phi^{\,2} = (\bar\partial+\phi)^2 = \cancel{\bar\partial^2 f} + \bar\partial\phi f + \phi\,\bar\partial f + \phi\circ\phi f$$

$$= \cancel{-\phi\,\bar\partial f} + \underline{(\bar\partial\phi) f} + \cancel{\phi\,\bar\partial f} + \underline{\phi\circ\phi f}$$

A. $\phi\circ\phi f = \sum \phi^\alpha_\beta\, d\bar z^\beta \frac{\partial}{\partial z^\alpha}\left(\phi^\gamma_\delta\, d\bar z^\delta \frac{\partial}{\partial z^\gamma} f\right)$

$$= \sum \phi^\alpha_\beta \left(\frac{\partial}{\partial z^\alpha}\phi^\gamma_\delta\right)\frac{\partial f}{\partial z^\gamma}\, \underline{\underline{d\bar z^\beta\wedge d\bar z^\delta}} + \cancel{\sum \phi^\alpha_\beta\, \phi^\gamma_\delta\, d\bar z^\beta\wedge d\bar z^\delta\, \frac{\partial}{\partial z^\alpha}\frac{\partial}{\partial z^\gamma} f}$$

B. $[\phi,\phi] = \sum \left[\phi^\alpha_\beta \frac{\partial}{\partial z^\alpha}, \phi^\gamma_\delta \frac{\partial}{\partial z^\gamma}\right] d\bar z^\beta\wedge d\bar z^\delta$

$$= \sum \left[\phi^\alpha_\beta\left(\frac{\partial}{\partial z^\alpha}\phi^\gamma_\delta\right)\frac{\partial}{\partial z^\gamma} - \phi^\gamma_\delta\left(\frac{\partial}{\partial z^\gamma}\phi^\alpha_\beta\right)\frac{\partial}{\partial z^\alpha}\right] d\bar z^\beta\wedge d\bar z^\delta$$

$$= 2\cdot\phi\circ\phi f$$

the equation $\bar\partial_\phi^{\,2} = 0 \iff (\bar\partial\phi + \phi\circ\phi) f = 0$

$$\iff \underline{\bar\partial\phi + \tfrac{1}{2}[\phi,\phi] = 0}$$

change sign : $\bar\partial\phi = \frac{1}{2}[\phi,\phi]$

change scale : $\boxed{\bar\partial\phi = [\phi,\phi]}$ integrable a.c. structure.

X cpx mfd

* Lie algebra str. on $\oplus \Lambda^{0,q}(T)$; $\mathbb{Z}_2$ graded

$\alpha \in \Lambda^{0,r}(T)$, $\beta \in \Lambda^{0,s}(T)$

α, β are linear combinations of the form

$$X \otimes \phi \quad \text{st. } \partial \phi = 0, \quad X \in C^\infty(T).$$

in local coor. ϕ looks like $d\bar{z}^I$.

consider $\alpha = X \otimes \phi$, $\beta = Y \otimes \psi$, then

$[\alpha, \beta] := [X, Y] \otimes \phi \wedge \psi$

Well-definedness:

$$[X_I \otimes d\bar{z}^I, Y_J \otimes d\bar{z}^J] = [X_I, Y_J]\, d\bar{z}^I \wedge d\bar{z}^J$$

$$X_I d\bar{z}^I = X_I \frac{\partial \bar{z}^I}{\partial \bar{w}^{I'}} d\bar{w}^{I'}$$ change coor $z \leftrightarrow w$

$$Y_J d\bar{z}^J = Y_J \frac{\partial \bar{z}^J}{\partial \bar{w}^{J'}} d\bar{w}^{J'}$$

$$\left[X_I \frac{\partial \bar{z}^I}{\partial \bar{w}^{I'}}, Y_J \frac{\partial \bar{z}^J}{\partial \bar{w}^{J'}}\right] = \frac{\partial \bar{z}^I}{\partial \bar{w}^{I'}} \cdot \frac{\partial \bar{z}^J}{\partial \bar{w}^{J'}} [X_I, Y_J]\, d\bar{w}^{I'} \wedge d\bar{w}^{J'}$$

$$+ \frac{\partial \bar{z}^I}{\partial \bar{w}^{I'}} \left(\cancel{X_I \frac{\partial \bar{z}^J}{\partial \bar{w}^{J'}}}\right) Y_J - \frac{\partial \bar{z}^J}{\partial \bar{w}^{J'}} \left(\cancel{Y_J \frac{\partial \bar{z}^I}{\partial \bar{w}^{I'}}}\right) X_I$$

this is zero! zero

$$\begin{aligned} [fv, gw] &= fv(gw) - gw(fv) \\ &= f(vg)w + fgvw - g(wf)v - gfwv \\ &= f(vg)w - g(wf)v + fg[v,w] \end{aligned}$$

Rmk: not true for $\Lambda^{p,q}(T)$!

Basic properties for $[\alpha, \beta]$

- $[\alpha, \beta] = -(-1)^{rs} [\beta, \alpha]$
- $\bar{\partial}[\alpha, \beta] = [\bar{\partial}\alpha, \beta] + (-1)^r [\alpha, \bar{\partial}\beta]$ Leibnitz rule
- for $\alpha \in \Lambda^{0,r}(T)$, $\beta \in \Lambda^{0,s}(T)$, $\gamma \in \Lambda^{0,t}(T)$ Jacobi identi

$$(-1)^{rt} [[\alpha, \beta], \gamma] + (-1)^{sr} [[\beta, \gamma], \alpha] + (-1)^{ts} [[\gamma, \alpha], \beta] = 0$$

Kuranishi theorem Part I

Eq. $\bar\partial\alpha = [\alpha,\alpha]$
has solution given by the power series
$\alpha = \alpha_1 + \bar\partial^* G[\alpha,\alpha] \iff H[\alpha,\alpha]=0$
(α in this form satisfies $\bar\partial^*\alpha=0$ automatically)

$G\bar\partial^*\bar\partial\alpha = G\bar\partial^*[\alpha,\alpha]$ still need $\bar\partial^*\alpha=0$
$\alpha - H\alpha$

pf: $\bar\partial\alpha = \bar\partial\bar\partial^* G[\alpha,\alpha]$
$= \Delta G[\alpha,\alpha] - \bar\partial^*\bar\partial G[\alpha,\alpha]$
$= [\alpha,\alpha] - \cancel{H[\alpha,\alpha]} - \bar\partial^* G\,\bar\partial[\alpha,\alpha]$

$\beta = \bar\partial\alpha - [\alpha,\alpha] = -2\bar\partial^* G[\bar\partial\alpha,\alpha]$
$= -2\bar\partial^* G[\beta + [\alpha,\alpha],\alpha]$
$= -2\bar\partial^* G[\beta,\alpha]$ Jacobi identity $[[\alpha,\alpha],\alpha]=0$
$\|\beta\|_{k+\alpha} \le C_{k,\alpha}\|\beta\|\cdot\|\alpha\|$
$k+\alpha+1-2+1$
$[\ ]\ G\ \bar\partial^*$
pick t small $*$
so must $\beta\equiv 0$

Here we have used the basic estimates:

I. $\|[\varphi,\psi]\|_{k+\alpha} \le C\|\varphi\|_{k+1+\alpha}\|\psi\|_{k+1+\alpha}$
C indep. of φ,ψ (EASY)

II. $\|G\varphi\|_{k+\alpha} \le C\|\varphi\|_{k-2+\alpha}$, $k\ge 2$
C dep only on k,α not on φ (HARD)

III. We have used $\bar\partial^* G = G\bar\partial^*$, $\bar\partial G = G\bar\partial$
which follows easily from $[\bar\partial,\Delta]=0=[\bar\partial^*,\Delta]$

IV. We are motivated by the cond; $\bar\partial^*\alpha=0$

Formal $\Rightarrow$ Convergence (Kodaira-Spencer-Nirenberg)

$\alpha = \alpha(t) = \alpha_1 t + \alpha_2 t^2 + \cdots$ constructed by
$= \alpha_1(t) + \bar\partial^* G[\alpha(t), \alpha(t)]$ $\qquad \alpha_1(t) = \sum \eta_i t_i$
i.e. $\alpha_2(t) = \bar\partial^* G[\alpha_1(t), \alpha_1(t)]$ $\qquad \eta_i$ basis of $H^{0,1}_{\bar\partial}(T)$
$\alpha_3(t) = \bar\partial^* G([\alpha_1(t), \alpha_2(t)] + [\alpha_2(t), \alpha_1(t)]) \ldots$
$\alpha_\ell(t) = \bar\partial^* G \sum_{i=1}^{\ell-1} [\alpha_i(t), \alpha_{\ell-i}(t)]$

for $|t|$ small, $\alpha(t)$ converges
pf: Let $\alpha^{(\ell)} = \alpha_1(t) + \cdots + \alpha_\ell(t)$
then $\alpha^{(\ell)} \cdot \bar\partial^* G[\alpha^{(\ell-1)}, \alpha^{(\ell-1)}]$ mod $t^{\ell+1}$
(*) $\Rightarrow \|\alpha^{(\ell)}\|_{k+\alpha} \leq C_{k,\alpha} \|\alpha^{(\ell-1)}\|^2_{k+\alpha}$
since k, α are fixed, pick $|t|$ small such that
($C_{k,\alpha} \cdot \|\alpha^{(1)}(t)\|_{k,\alpha} < 1$ get the result ?)
or use induction to show that
$\|\alpha^{(\ell)}(t)\|_{k+\alpha} \leq A(t) := \frac{\beta}{16\gamma} \sum_{\mu=1}^{\infty} \gamma^\mu (t_1 + \cdots + t_m)^\mu$
if $|\alpha^{(\ell-1)}|_{k+\alpha} \leq A(t)$ $\qquad$ componentwise
(*) $\Rightarrow \|\alpha^\ell\|_{k+\alpha} \leq C_{k,\alpha} A(t)^2$ $\qquad$ (then can be
take $|t|$ small st. $\qquad$ specialize to
$C_{k,\alpha} \cdot A(t) < 1$. done. □ $\qquad$ specific value of t)

Alg-geom $\longleftrightarrow$ Artin approximation theorem

Kuranishi's thm Part II : $\{t \mid H[\alpha(t), \alpha(t)] = 0\}$
is an analytic space and defines complete family of
cpx structures.

Bogomolov-Tian-Todorov theorem (BTT)
X cpt Kahler with $c_1(X)=0$ then the Kuranishi space of X is smooth of (maximal) dim $= h^{n-1,1}(X) = h^1(T_X)$

Need to show that given $\alpha_1 \in H^1_{\bar\partial}(X, T_X)$
$\exists$ formal power series $\alpha = \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \cdots$
with $\alpha_i \in \Lambda^{0,1}(T_X)$ such that

$$(A) \quad \bar\partial \alpha = [\alpha, \alpha]$$
$$\bar\partial^* \alpha = 0$$

equivalently $\bar\partial \alpha_k = \sum_{i=1}^{k-1} [\alpha_i, \alpha_{k-i}]$ and $\bar\partial^* \alpha_k = 0$

Idea: $H^1_{\bar\partial}(X, T_X) \xrightarrow{\sim} H^{n-1,1}(X)$
$v \longmapsto i(v)\Omega$

Ω the global nonvanishing holomorphic $(n,0)$ form.
Try to solve eq in $H^{n-1,*}(X)$ using $\partial\bar\partial$ lemma
In fact, we need $I: \Lambda^{0,q}(T) \xrightarrow{\sim} \Lambda^{n-1,q}$
$v \longmapsto i(v)\Omega$

Tian's lemma: For $\alpha, \beta \in \Lambda^{0,1}(T)$

$$\iota([\alpha,\beta])\Omega = \partial(\iota(\alpha)\iota(\beta)\Omega) - \iota(\alpha)\,\partial(\iota(\beta)\Omega) + \iota(\beta)\,\partial(\iota(\alpha)\Omega)$$

Induction, can solve α_k st $\partial(\iota(\alpha_k)\Omega) = 0 \quad k = 1, 2, \cdots$
$\partial(\iota(\alpha_1)\Omega) = 0$ can be done since α_1 is harmonic
so $\iota([\alpha_1, \alpha_1])\Omega = \partial(\iota(\alpha_1)\iota(\alpha_1)\Omega)$ is ∂ exact
But $\bar\partial \sum_{i=1}^{k-1} [\alpha_i, \alpha_{k-i}] = 0$ always by Jacobi identity
by $\partial\bar\partial$ lemma $\quad \iota([\alpha_1, \alpha_1]\Omega) = \bar\partial\partial\xi. \quad \xi \in \Lambda^{n-2,1}$

take $\alpha_2 = I^{-1}(\partial\xi)$ then $\partial(\iota(\alpha_2)\Omega) = \partial(I\alpha_2) = \partial\partial\xi = 0$
Now use induction $\square$ Use K-E metric can get α_i also ~~har~~ $\bar\partial^*\alpha_i = 0$

Recall $i_X \omega = \omega(X, \cdots)$; $df(X) = Xf$

THE PROOF OF TIAN'S LEMMA

$[i_X, d] = L_X$

$[[i_X, d], i_Y]\varphi$

$= [i_X, d]\, i_Y \varphi - i_Y [i_X, d]\varphi$

$= i_X d\, \varphi(Y) + \cancel{d i_X \varphi(Y)} - i_Y (i_X d + \underline{d i_X})\varphi$

$= d\varphi(Y)(X) + i_Y i_X d\varphi - d\varphi(X)(Y)$

$= X\varphi(Y) - Y\varphi(X) - d\varphi(X, Y)$

By Cartan formula

$d\varphi(X,Y) = X\varphi(Y) - Y\varphi(X) - \varphi([X,Y])$

Any better reason? $[L_X, \iota(Y)] = \iota(L_X Y)$

get that

$\boxed{[[i_X, d], i_Y]\varphi = i_{[X,Y]}\varphi}$ on 1-forms.

Both sides are derivatives of forms, hence has unique extension to any C^∞ forms φ. ie.

$$\iota([X,Y])\Phi = (\iota(X) d + d\,\iota(X))\,\iota(Y)\Phi - \iota(Y)(\iota(X) d + d\,\iota(X))\Phi$$

$$= \iota(X) d(\iota(Y)\Phi) - \iota(Y) d(\iota(X)\Phi) + d(\iota(X)\iota(Y)\Phi) - \iota(Y)\iota(X) d\Phi$$

We are interested in the case that replace $[X,Y]$ by $[\alpha, \beta]$, $\alpha, \beta \in \Lambda^{0,*}(T)$.

Rmk: Sign: $[L_1, L_2] = L_1 L_2 - (-1)^{s+t} L_2 L_1$

$s =$ ord L_1 as shifting of $\Lambda^* \to \Lambda^{*+s}$

eg. ord of i_X, d $= -1, +1$, ord $L_X = 0$

Rmk: $i_Y i_X d\varphi = i_Y(i_X d\varphi) = (i_X d\varphi)(Y)$

$= d\varphi(X, \cdot)(Y) = d\varphi(X, Y)$

$\boxed{i_X d\varphi(Z) = d\varphi(X, Z)}$

So, If $\alpha = X \otimes \varphi \quad \partial\varphi = 0 \quad \varphi \in \Lambda^{0,s}$ (In Tian's case, $s=t=1$)

$\beta = Y \otimes \psi \quad \partial\psi = 0 \quad \psi \in \Lambda^{0,t}$

$[\alpha, \beta] = [X, Y] \otimes \varphi \wedge \psi$

and $\iota([\alpha,\beta])\Phi$ means as usual

(for $\gamma = Z \otimes \xi$, $\quad Z \in C^\infty(\Lambda^p T)$

$\iota(\gamma)\Phi := \xi \wedge (\iota(Z)\Phi)$. $\quad \xi$ any q form.)

Get $\iota([\alpha,\beta])\Phi = \varphi\wedge\psi\wedge\iota([X,Y])\Phi$

~~Since X, Y all in T direction (i.e. $T^{1,0}$) and φ, ψ all in anti-holo direction, so we can commute them.~~

$$\begin{aligned}
&= \varphi\wedge\psi\wedge \iota(X)\, d(\iota(Y)\Phi) \to \varphi\wedge\psi\wedge\iota(X)\,\partial(\iota(Y)\Phi)\\
&\pm \varphi\wedge\psi\, d(\iota(X)\iota(Y)\Phi) \to \varphi\wedge\psi\wedge\partial(\iota(X)\iota(Y)\Phi)\\
&- \varphi\wedge\psi\wedge \iota(Y)\iota(X)d\Phi \to -\varphi\wedge\psi\wedge(\iota(Y)\iota(X)\partial\Phi)\\
&- \varphi\wedge\psi\wedge\iota(Y)\, d(\iota(X)\Phi) \to -\varphi\wedge\psi\wedge\iota(Y)(\partial\,\iota(X)\Phi)
\end{aligned}$$

When $\Phi \in C^\infty(\Lambda^{p,q})$, by comparing with type, the right hand can only apply ∂ but not $\bar\partial$. (type of LHS is $\Lambda^{p-1,q+s+t}$)

Now combine with $\boxed{\partial\varphi = 0 = \partial\psi}$ get

$$\begin{aligned}
\iota([\alpha,\beta])\Phi &= (-1)^{s+t}\,\partial(\iota(\alpha)\iota(\beta)\Phi) \leftarrow \text{good term}\\
&+ (-1)^t\,\iota(\alpha)\,\partial(\iota(\beta)\Phi)\\
&- (-1)^{st}(-1)^s\,\iota(\beta)\,\partial(\iota(\alpha)\Phi)\\
&- (-1)^{st}\,\iota(\beta)\iota(\alpha)\,\partial\Phi
\end{aligned}$$

For $s=t=1$, we get Tian's lemma

Variations of path, geodesics and Bonnet's theorem

The consequence we need is:	which follows from Bonnet's theorem:
pos. Ric $\Rightarrow$ π_1 finite with const. lower bound >0	positive Ricci $\Rightarrow$ compactness. with a const. lower bound >0

Let $\gamma:[0,\ell]\longrightarrow X$ be a curve with $|T|=|\frac{d\gamma}{dt}|=1$.

For Any variation $F:\underset{t}{[0,\ell]}\times\underset{s}{[-1,1]}\longrightarrow X$ of γ with $F|_{t=0}\equiv\gamma(0)$, $F|_{t=\ell}\equiv\gamma(\ell)$

Length of $F|_s = L(s) := \int_0^\ell \langle T,T\rangle^{\frac12}dt$

$$L'(s)=\int_0^\ell \frac{d}{ds}\langle T,T\rangle^{\frac12}dt=\int_0^\ell \frac{2\langle\nabla_N T,T\rangle}{2\cdot\langle T,T\rangle^{1/2}}\,dt \qquad (*)$$

For $\gamma\equiv F|_{s=0}$ to be a geodesic, this has to be 0 at $s=0$.

> Bonnet–Meyer's Thm:
> $\mathrm{diam}(X)\le\frac{\pi}{\sqrt{K}}$
> if $Ric\ge(n-1)K>0$

Note: $\nabla_N T-\nabla_T N=[N,T]=0$ since N,T are coordinates vectors.

so $\langle\nabla_N T,T\rangle=\langle\nabla_T N,T\rangle$
$=T\langle N,T\rangle-\langle N,\nabla_T T\rangle$

Get $L'(0)=\int_0^\ell T\langle N,T\rangle dt-\int_0^\ell\langle N,\nabla_T T\rangle\,dt \underset{\text{want}}{\overset{\uparrow}{=}} 0$

$\int_0^\ell T\langle N,T\rangle dt = \langle N,T\rangle|_0^\ell=0$

since $N(0)=0=N(\ell)$

Since this is true $\forall F$, take $N=\nabla_T T$ get $\int_0^\ell\|\nabla_T T\|^2dt=0\Leftrightarrow\boxed{\nabla_T T=0}$

Get Equation for geodesics:

If in local coor. system $\gamma(t)=(x^1(t),\cdots,x^n(t))$ then

$\frac{d}{dt}=T=\sum_i\frac{dx^i}{dt}\frac{\partial}{\partial x^i}$, so (recall $\nabla_{\partial_i}\partial_j=\Gamma_{ij}^k\partial_k$)

$$\nabla_T T=\sum_j\nabla_{\partial/\partial t}\left[\frac{dx^j}{dt}\partial_j\right]$$
$$=\sum_j\frac{d^2x^j}{dt^2}\frac{\partial}{\partial x^j}+\sum_j\frac{dx^j}{dt}\nabla_{\partial/\partial t}\left(\frac{\partial}{\partial x^j}\right)$$
$$=\sum_j\frac{d^2x^j}{dt^2}\frac{\partial}{\partial x^j}+\sum_{i,j}\frac{dx^j}{dt}\frac{dx^i}{dt}\Gamma_{ij}^k\frac{\partial}{\partial x^k}$$
$$=\sum_k\left(\frac{d^2x^k}{dt^2}+\sum_{i,j}\Gamma_{ij}^k\frac{dx^i}{dt}\frac{dx^j}{dt}\right)\frac{\partial}{\partial x^k}=0$$

Now assume that γ is a geodesic :

$$L'(s) = \int_0^\ell \frac{\langle \nabla_T N, T\rangle}{\langle T, T\rangle^{1/2}}\, dt$$

$$L''(s) = \int_0^\ell \left(\frac{\langle \nabla_N \nabla_T N, T\rangle + \|\nabla_T N\|^2}{\langle T,T\rangle^{1/2}} - \frac{\langle \nabla_T N, T\rangle \cdot \langle \nabla_T N, T\rangle}{\langle T, T\rangle^{3/2}} \right) dt$$

$$L''(0) = \int_0^\ell \|\nabla_T N\|^2 \boxed{+\langle \nabla_T N, \cancel{N}\rangle\Big|_0^\ell \; + \langle \nabla_N \nabla_T N, T\rangle\, dt} - \int_0^\ell \left|T\langle N,T\rangle - \langle N, \cancel{\nabla_T T}\rangle\right|^2 dt$$

goes away if we take the normal variation $\langle N, T\rangle = 0$ at $s=0$.

because :

$\langle \nabla_N \nabla_T N, T\rangle$

$= R(N,T,T,N) + \langle \nabla_T \nabla_N N, T\rangle$

$= -R(N,T,N,T) + T\langle \nabla_N N, T\rangle - \langle \nabla_N N, \cancel{\nabla_T T}\rangle$

so for end-pts fixed normal variations : Get 2nd variation formula:

$$L''(0) = \int_0^\ell \left(\|\nabla_T N\|^2 - R(N,T,N,T)\right) dt + \left(\langle \nabla_N N, T\rangle\Big|_0^\ell \text{ since } \underset{N\langle \cancel{N}, T\rangle - \langle N, \cancel{\nabla_N T}\rangle}{\shortparallel} \right)$$

Another form $\displaystyle = -\int_0^\ell \langle \nabla_T^2 N + R(N,T)T, N\rangle\, dt$

- pf of Bonnet-Meyer thm:

  if $\operatorname{diam}(X) > \frac{\pi}{\sqrt{k}}$ then $\exists\, p, q \in X$, $d(p,q) > \frac{\pi}{\sqrt{k}}$

  and shortest geodesic γ joins p, q $\quad d(p,q) = \ell$

  Hence must $L''(0) \geq 0$

  $$\sum_{i=1}^{2} \|\nabla_T N_i\|^2 - R(N_i, T, N_i, T)$$
  $$= \frac{\pi^2}{\ell^2}(n-1)\cos^2\left(\frac{\pi}{\ell}t\right) - Ric(T,T)\sin^2\left(\frac{\pi}{\ell}t\right)$$

take integration get $\frac{1}{2}\left[\frac{\pi^2}{\ell^2}(n-1) - k\right] < 0$ —×— Q.E.D.

- Here $e_1 = T$, $e_2, \cdots, e_n$ parallel. basis Let $\boxed{N_i = e_i \text{ for } i \geq 2 \quad \sin\left(\frac{\pi}{\ell}t\right)}$

  then $\|\nabla_T N\|^2 = \frac{\pi^2}{\ell^2}\cos\left(\frac{\pi}{\ell}t\right)^2$

  $R(N_i, T, N_i, T) = \sin\left(\frac{\pi}{\ell}t\right)^2 R_{NTNT}$

parallel means $\nabla_T e = 0$, parallel translation preserves inner product :

$\frac{d}{dt}\langle e_i, e_j\rangle = \langle \nabla_T e_i, e_j\rangle + \langle e_i, \nabla_T e_j\rangle \equiv 0$ along γ. Moreover,

$\nabla_T(f(t)e) = f'(t)e + f(t)\nabla_T e = f'(t)e$.

Harmonic maps : Let N, M cpt Riem. mfds.

$f: N \longrightarrow M$ locally $(t_1, \cdots, t_n) \mapsto (x_1, \cdots, x_m)$

$g \quad h$ $\qquad x_i = f^i(t_1, \cdots, t_n)$, C^∞

Riemannian metrics

Let $T_i = f_* \frac{\partial}{\partial t_i} \in \Gamma(f^* T_M)$ with induced conn. $f^*\nabla$

coor. vector fields $\qquad \nabla$ conn. of h

then the d-energy

$$E(f) = \int_N |df|^2 = \int_N g^{ij} \langle T_i, T_j \rangle_h$$

since $T_i = f_* \frac{\partial}{\partial t_i} = \sum_\alpha \frac{\partial f^\alpha}{\partial t_i} \cdot \frac{\partial}{\partial x^\alpha} = \sum_\alpha f_i^\alpha \partial_\alpha$

$$E(f) = \int_N g^{ij} f_i^\alpha f_j^\beta h_{\alpha\beta}$$

Harmonic map = critical pt of energy functional :

Let $F: \underset{t}{N} \times \underset{s}{(-1,1)} \longrightarrow M$ be a 1-parameter variation of f

$S := F_* \frac{\partial}{\partial s} \in \Gamma(F^* T_M)$ is also a coor. v.f on $N \times (-1,1)$.

$$\begin{aligned}
E'(s) &= \int_N g^{ij}\, S \langle T_i, T_j \rangle \\
&= 2\int_N g^{ij} \left(\langle \nabla_S T_i, T_j \rangle + \cancel{\langle T_i, \nabla_S T_j \rangle} \right) \\
&= 2\int_N g^{ij} \left(\langle \nabla_{T_i} S, T_j \rangle + \cancel{\langle T_i, \nabla_{T_j} S \rangle} \right) \\
&= 2\int_N g^{ij}\, T_i \langle S, T_j \rangle - g^{ij} \langle S, \nabla_{T_i} T_j \rangle \\
&= 2\int_N \underbrace{\cancel{\operatorname{div} \langle S, \cdot \rangle}}_{=0} - 2\int_N \langle S, g^{ij} \nabla_{T_i} T_j \rangle
\end{aligned}$$

So $E'(0) = 0 \iff \sum_{i,j} g^{ij} \nabla_{T_i} T_j = 0 \quad$ (since $\forall S$)

In local coordinates, this is, $\forall \alpha$

$$\boxed{\sum_{i,j} g^{ij} \left(\frac{\partial^2 f^\alpha}{\partial t^i \partial t^j} + {}^M\Gamma^\alpha_{\beta\gamma} \frac{\partial f^\beta}{\partial t^i} \frac{\partial f^\gamma}{\partial t^j} \right) = 0}$$

this generalizes the geodesic equation.

Local Formulas in Kähler Geometry

notations

real index $i, j, k \dots$

cpx index $\alpha, \beta, \gamma, \dots \quad \bar\alpha, \bar\beta, \bar\gamma, \dots$

- Kahler condition : Let $g_{\alpha\bar\beta}\, dz^\alpha \otimes d\bar z^\beta$ be a hermitian metric.

$$\omega = g_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta \qquad (g_{\bar\beta\alpha} = g_{\alpha\bar\beta} \ ; \ \overline{g_{\alpha\bar\beta}} = g_{\bar\alpha\beta})$$

$$d\omega = \partial_\gamma g_{\alpha\bar\beta}\, dz^\gamma \wedge dz^\alpha \wedge d\bar z^\beta + \partial_{\bar\gamma} g_{\alpha\bar\beta}\, d\bar z^\gamma \wedge dz^\alpha \wedge d\bar z^\beta$$

so $d\omega = 0 \iff \underline{\partial_\gamma g_{\alpha\bar\beta} = \partial_\alpha g_{\gamma\bar\beta}}$ & $\underline{\partial_{\bar\gamma} g_{\alpha\bar\beta} = \partial_{\bar\beta} g_{\alpha\bar\gamma}}$

In fact, locally $g_{\alpha\bar\beta} = \partial_\alpha \partial_{\bar\beta} K$ for some C^∞ function K (*)

- Levi-Civita connection : This is the unique connection st.

(1) metrical : $d\langle u, v\rangle = \langle \nabla u, v\rangle + \langle u, \nabla v\rangle$

(2) torsion free : $\nabla_u v - \nabla_v u - [u, v] = 0$

in the cpx case, (2) is equivalent to that for $\nabla = \nabla' + \nabla''$ wrt $\Lambda^1 = \Lambda^{1,0} \oplus \Lambda^{0,1}$, one has $\nabla'' \equiv \bar\partial$

It is well known in elementary diff. geom. that if

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k$$

then (1) + (2) $\Rightarrow \Gamma_{ij}^k = \Gamma_{ji}^k$ and

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \{ \partial_i g_{jl} + \partial_j g_{kl} - \partial_l g_{ij} \}$$

In the Kahler case,

$$\underline{\Gamma_{\alpha\beta}^\gamma} = \frac{1}{2} g^{\gamma\bar\delta} \{ \partial_\alpha g_{\beta\bar\delta} + \partial_\beta g_{\alpha\bar\delta} - \cancel{\partial_{\bar\delta} g_{\alpha\beta}} \} = \underline{g^{\gamma\bar\delta} \partial_\alpha g_{\beta\bar\delta}}$$

$\Gamma_{\alpha\beta}^{\bar\gamma} = 0 = \Gamma_{\bar\alpha\bar\beta}^{\gamma}$ clearly, also $\Gamma_{\alpha\bar\beta}^{\gamma} = 0 = \Gamma_{\alpha\bar\beta}^{\bar\gamma}$:

$$\Gamma_{\alpha\bar\beta}^{\gamma} = \frac{1}{2} g^{\gamma\bar\delta} \{ \cancel{\partial_\alpha g_{\bar\beta\bar\delta}} + \cancel{\partial_{\bar\beta} g_{\alpha\bar\delta}} - \cancel{\partial_{\bar\delta} g_{\alpha\bar\beta}} \} = 0$$

So the only nontrivial terms are of pure type

$\underline{\Gamma_{\alpha\beta}^\gamma = \Gamma_{\beta\alpha}^\gamma = g^{\gamma\bar\delta} \partial_\alpha g_{\beta\bar\delta}}$ and $\Gamma_{\bar\alpha\bar\beta}^{\bar\gamma} = \overline{\Gamma_{\alpha\beta}^{\gamma}}$.

At any point p, $\exists$ local coor. st $\Gamma_{\alpha\beta}^\gamma(p) \equiv 0$ (via (*))

For connection 1-forms ω_i^j, ω_α^β etc.

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k \quad \text{ie.}$$

$$\nabla \partial_j = (\Gamma_{ij}^k dx^i)\partial_k =: \omega_j^k \partial_k, \text{ so } \underline{\omega_j^k = \Gamma_{ij}^k dx^i}$$

In the Kähler case,

$$\underline{\omega_\alpha^\beta} = \Gamma_{\gamma\alpha}^\beta dz^\gamma ; \quad \omega_\alpha^{\bar\beta} = 0 \quad \text{(no mixed term)}$$

$$= g^{\beta\bar\delta}\partial_\gamma g_{\alpha\bar\delta}\, dz^\gamma = \underline{g^{\beta\bar\delta}\cdot \partial g_{\alpha\bar\delta}}$$

- Curvature form:

$$\Omega_i^j := d\omega_i^j - \omega_i^k \wedge \omega_k^j$$

In the Kähler case, since Ω_α^β is of type (1,1), so

$\nabla''^2 = \bar\partial^2 = 0 \Rightarrow$ no (0,2)
(1) $\Rightarrow \Omega$ skew hermitian
$\Rightarrow$ no (2,0) part.

$$\Omega_\alpha^\beta = \bar\partial \omega_\alpha^\beta = \bar\partial(g^{\beta\bar\nu}\cdot\partial g_{\alpha\bar\nu})$$

$$= \partial_{\bar\delta}(g^{\beta\bar\nu}\cdot\partial_\gamma g_{\alpha\bar\nu})\, d\bar z^\delta \wedge dz^\gamma$$

$$= \left[-g^{\beta\bar\nu}\cdot\partial_\gamma\partial_{\bar\delta} g_{\alpha\bar\nu} + g^{\beta\bar\lambda}(\partial_{\bar\delta} g_{\bar\lambda\mu})g^{\mu\bar\nu}\cdot\partial_\gamma g_{\alpha\bar\nu}\right] dz^\gamma \wedge d\bar z^\delta$$

ie.

$$\underline{\Omega_\alpha^\beta = \left(-g^{\beta\bar\lambda}\frac{\partial^2 g_{\alpha\bar\lambda}}{\partial z^\gamma \partial\bar z^\delta} + g^{\mu\bar\nu}\frac{\partial g_{\alpha\bar\nu}}{\partial z^\gamma}\cdot g^{\beta\bar\lambda}\frac{\partial g_{\mu\bar\lambda}}{\partial \bar z^\delta}\right) dz^\gamma\wedge d\bar z^\delta}$$

Write $\Omega_\alpha^\beta = R^\beta_{\alpha\gamma\bar\delta}$ and $R_{\alpha\bar\beta\gamma\bar\delta} := g_{\lambda\bar\beta}R^\lambda_{\alpha\gamma\bar\delta}$, get

$$\boxed{R_{\alpha\bar\beta\gamma\bar\delta} = -\partial_\gamma\partial_{\bar\delta} g_{\alpha\bar\beta} + g^{\mu\bar\nu}\partial_\gamma g_{\alpha\bar\nu}\cdot\partial_{\bar\delta} g_{\mu\bar\beta}}$$

This is in fact a very easy formula, since there is only one way to write down such an expression!

BASIC PROPERTIES:

(1) $R_{\alpha\bar\beta\gamma\bar\delta} = R_{\gamma\bar\delta\alpha\bar\beta}$ (symmetric)

(2) $R_{\alpha\bar\beta\gamma\bar\delta} = -R_{\alpha\bar\beta\bar\delta\gamma}$ (anti-symmetric)

(3) $R_{\alpha\beta\cdot\cdot} = 0 = R_{\bar\alpha\bar\beta\cdot\cdot}$ (no mixed terms)

(4) $R_{\alpha\bar\beta\gamma\bar\delta}$ $\alpha\leftrightarrow\gamma$, $\beta\leftrightarrow\delta$ (Bianchi symmetry + (3) or (*))

$R_{\alpha\bar\beta\gamma\bar\delta,\lambda}$; $R_{\alpha\bar\beta\gamma\bar\delta,\bar\mu}$ $\alpha\leftrightarrow\gamma\leftrightarrow\lambda$; $\beta\leftrightarrow\delta\leftrightarrow\mu$

HARMONIC MAPS IN KÄHLER GEOMETRY

$ds_N^2 = g_{i\bar{j}}\, dw^i \otimes d\bar{w}^j$; $ds_M^2 = h_{\alpha\bar{\beta}}\, dz^\alpha \otimes d\bar{z}^\beta$

Given $f : N \to M$ C^∞ map of Hermitian mfds :

Want to consider $\bar{\partial}$ - energy :

$$|\bar{\partial} f|^2 := g^{\bar{i}j} f^\alpha_{\bar{i}} \overline{f^\beta_{\bar{j}}} h_{\alpha\bar{\beta}} \quad ; \quad f^\alpha_{\bar{i}} := \frac{\partial f^\alpha}{\partial \bar{w}^i} .$$

- More intrinsically : for general mapping $f : N \to M$, C^∞

$$|df|^2 = \mathrm{Tr}_{ds_N^2} (f^* ds_M^2)$$

x^i : real coordinates
i, j, α, β run through all real index.

$$f^* ds_M^2 = h_{\alpha\beta} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} dx^i \otimes dx^j$$

Take trace $= g^{ij} h_{\alpha\beta} f^\alpha_i f^\beta_j$

- In the cpx case, since $g, h \neq 0$ only for mixed type

if define $|\partial f|^2 = g^{\bar{i}j} f^\alpha_j \overline{f^\beta_i} h_{\alpha\bar{\beta}}$

then $|df|^2 = |\partial f|^2 + |\bar{\partial} f|^2$

i.e. $\int_N |df|^2 = \int_N |\partial f|^2 + \int_N |\bar{\partial} f|^2$ energy ①

- Let $\dim_{\mathbb{C}} N = 1$. then,

$$f^*(h_{\alpha\bar{\beta}}\, dz^\alpha \wedge d\bar{z}^\beta) = h_{\alpha\bar{\beta}} \left(\frac{\partial f^\alpha}{\partial w} dw + \frac{\partial f^\alpha}{\partial \bar{w}} d\bar{w} \right) \wedge \left(\overline{\frac{\partial f^\beta}{\partial w}}\, dw + \overline{\frac{\partial f^\beta}{\partial \bar{w}}}\, d\bar{w} \right)$$

$$= h_{\alpha\bar{\beta}} \left(\frac{\partial f^\alpha}{\partial w} \overline{\frac{\partial f^\beta}{\partial w}} - \frac{\partial f^\alpha}{\partial \bar{w}} \overline{\frac{\partial f^\beta}{\partial \bar{w}}} \right) dw \wedge d\bar{w}$$

add g^{-1} add g factor

i.e. $\int_N |\partial f|^2 - \int_N |\bar{\partial} f|^2 = f^* \omega_M [N] = \omega_M([f(N)])$ ②

- so if $d\omega_M = 0$ (M is Kähler) then $\rightarrow$ is topological #.

$$\Rightarrow \quad E_{\bar{\partial}}(f) = \tfrac{1}{2} E_d(f) - \tfrac{1}{2} \omega_M [f(N)]$$
$$E_{\partial}(f) = \tfrac{1}{2} E_d(f) - \tfrac{1}{2} \omega_M [f(N)]$$

Hence they have the same critical points (harmonic maps)

1st variation of $\bar\partial$-energy: Let $f(s): N \to M$

$|s| < \varepsilon$, in $\mathbb{C}$

$$\frac{\partial}{\partial s}\int_N |\bar\partial f|^2 = \int_N \frac{\partial}{\partial s}\left(g^{\bar i j} f^{\alpha}_{\bar i}\, \overline{f^{\beta}_{\bar j}}\, h_{\alpha\bar\beta}\right)$$

$$= g^{\bar i j}\, \partial_s f^{\alpha}_{\bar i}\cdot \overline{f^{\beta}_{\bar j}}\, h_{\alpha\bar\beta} + g^{\bar i j} f^{\alpha}_{\bar i}\, \overline{\partial_{\bar s} f^{\beta}_{\bar j}}\, h_{\alpha\bar\beta}$$

A B

$$+ g^{\bar i j} f^{\alpha}_{\bar i}\, \overline{f^{\beta}_{\bar j}}\, \underline{\partial_\mu h_{\alpha\bar\beta}}\, f^{\mu}_s + g^{\bar i j} f^{\alpha}_{\bar i}\, \overline{f^{\beta}_{\bar j}}\, \underline{\partial_{\bar\nu} h_{\alpha\bar\beta}}\, f^{\bar\nu}_s$$

Let $p \in N \mapsto f(p) = q \in M$, will pick coor. st $dg_{i\bar j} = 0 = dh_{\alpha\bar\beta}$, so

$$\partial_s f^{\alpha}_{\bar i}\cdot \overline{f^{\beta}_{\bar j}} = \partial_{\bar i} f^{\alpha}_s\cdot \overline{f^{\beta}_{\bar j}} = \partial_{\bar i}\left(f^{\alpha}_s\, \overline{f^{\beta}_{\bar j}}\right) - f^{\alpha}_s\, \overline{\partial_i f^{\beta}_{\bar j}}$$

the relevant part is (for part A):

$$g^{\bar i j}\, \partial_{\bar i}\left(f^{\alpha}_s\, \overline{f^{\beta}_{\bar j}}\, h_{\alpha\bar\beta}\right) - \left(g^{\bar i j}\, \overline{\partial_i f^{\beta}_{\bar j}}\right) f^{\alpha}_s\, h_{\alpha\bar\beta}$$

$\int_N = 0$ by div. thm

Get $\bar\partial$-harmonic map equation:

$$\underline{g^{i\bar j}\, \nabla_{\partial_i} f^{\alpha}_{\bar j} = 0} \quad \forall \alpha, \text{ ie.}$$

$$\sum_{i,j} g^{i\bar j}\left(\frac{\partial^2 f^\alpha}{\partial t^i \partial \bar t^j} + {}^{M}\Gamma^{\alpha}_{\beta\gamma}\, \frac{\partial f^\beta}{\partial t^i}\cdot \frac{\partial f^\gamma}{\partial \bar t^j}\right) = 0 \quad \forall \alpha.$$

Rmk: For part (B):

by div. thm

$$g^{\bar i j} f^{\alpha}_{\bar i}\, \overline{\partial_{\bar j} f^{\beta}_{\bar s}}\, h_{\alpha\bar\beta} = g^{\bar i j}\, \partial_j\left(f^{\alpha}_{\bar i}\, \overline{f^{\beta}_{\bar s}}\, h_{\alpha\bar\beta}\right) - \left(g^{\bar i j}\, \partial_j f^{\alpha}_{\bar i}\right) \overline{f^{\beta}_{\bar s}}\, h_{\alpha\bar\beta}$$

get the same contribution.

2nd variation of $\bar\partial$-energy :

$$\frac{\partial}{\partial\bar s\,\partial s}\int_N |\bar\partial f|^2 = \int_N \text{ of } 6 \text{ terms :}$$

$$(1)\ g^{\bar i j}\,\partial_{\bar s}\partial_s f^\alpha_{\bar i}\cdot\overline{f^\beta_{\bar j}}\,h_{\alpha\bar\beta} + (2)\ g^{\bar i j}\,\partial_s f^\alpha_{\bar i}\cdot\overline{\partial_s f^\beta_{\bar j}}\,h_{\alpha\bar\beta}$$

$$(3) + g^{\bar i j}\,\partial_{\bar s} f^\alpha_{\bar i}\cdot\overline{\partial_{\bar s} f^\beta_{\bar j}}\,h_{\alpha\bar\beta} + (4)\ g^{\bar i j}\,f^\alpha_{\bar i}\,\overline{\partial_s\partial_{\bar s} f^\beta_{\bar j}}\,h_{\alpha\bar\beta}$$

$$(5) + g^{\bar i j}\,f^\alpha_{\bar i}\,\overline{f^\beta_{\bar j}}\,\partial_{\bar\nu}\partial_\mu h_{\alpha\bar\beta}\,f^\mu_s\,f^{\bar\nu}_{\bar s}$$

$$(6) + g^{\bar i j}\,f^\alpha_{\bar i}\,\overline{f^\beta_{\bar j}}\,\partial_\mu\partial_{\bar\nu} h_{\alpha\bar\beta}\,f^{\bar\nu}_s\,f^\mu_{\bar s}$$

$$\boxed{\begin{aligned} f^{\bar\nu}_s &= \frac{\partial \overline{f^\nu}}{\partial s} = \overline{f^\nu_{\bar s}} \\ f^{\bar\nu}_{\bar s} &= \frac{\partial\overline{f^\nu}}{\partial\bar s} = \overline{f^\nu_s}\end{aligned}}$$

where $\bar z^\nu = \bar f^\nu(\cdots)$

(2), (3), (5), (6) are good terms,

(1) + (4) go to

$$2\,\mathrm{Re}\ g^{\bar i j}\left(\frac{\partial^2}{\partial s\partial\bar s} f^\alpha_{\bar i}\right)\overline{f^\beta_{\bar j}}\,h_{\alpha\bar\beta} \quad (**)$$

To compute it, recall the Hermitian bundle picture:

$$\begin{array}{l} f^*T_M \\ \ \ \downarrow \\ f: N\to M\end{array}$$

T_M is the holo. tangent bundle of M

f^*T_M is only a hermitian vector bundle

$\nabla := f^{*\,M}\nabla$ is a hermitian connection

then $f_{\bar s}$ is a global (C^∞, cpx section of f^*T_M), $f_{\bar j}$ is a local section

consider

$$g^{\bar i j}\,\partial_{\bar i}\langle \nabla_s f_{\bar s}, f_{\bar j}\rangle \quad \text{(divergence form)}$$

$$= g^{\bar i j}\langle \nabla_s f_{\bar s}, \nabla_{\bar i} f_{\bar j}\rangle + g^{\bar i j}\langle \nabla_{\bar i}\nabla_s f_{\bar s}, f_{\bar j}\rangle$$

by harmonicity

since
$$\nabla_s f_{\bar s} = \left(\frac{\partial^2 f^\alpha}{\partial s\partial\bar s} + {}^M\Gamma^\alpha_{\mu\nu}\frac{\partial f^\mu}{\partial s}\cdot\frac{\partial f^\nu}{\partial\bar s}\right)\frac{\partial}{\partial z^\alpha}$$

so
$$= (**) + \partial_{\bar i}\,{}^M\Gamma^\alpha_{\mu\nu}\,\frac{\partial f^\mu}{\partial s}\cdot\frac{\partial f^\nu}{\partial\bar s}\left(\overline{\frac{\partial f^\beta}{\partial \bar t_j}}\right) h_{\alpha\bar\beta}\,g^{\bar i j}$$

$$= (**) + \underline{\partial_{\bar\lambda}\,{}^M\Gamma^\alpha_{\mu\nu}}\,g^{\bar i j}\,\underline{h_{\alpha\bar\beta}}\,f^\mu_s f^\nu_{\bar s}\,f^{\bar\lambda}_{\bar i}\,\overline{f^\beta_{\bar j}}\ ; \quad \partial_{\bar i} = f^{\bar\lambda}_{\bar i}\frac{\partial}{\partial\bar z^\lambda} + f^\lambda_{\bar i}\frac{\partial}{\partial z^\lambda}$$

but $f^\lambda_{\bar i}\frac{\partial}{\partial z^\lambda}$ has no contribution in later calculation

$$\partial_{\bar\lambda}\left(h^{\alpha\bar\delta}\,\partial_\nu h_{\mu\bar\delta}\,h_{\alpha\bar\beta}\right) = \partial_{\bar\lambda}\partial_\nu h_{\mu\bar\beta}$$

ie. $(**) = \mathrm{div}\langle\nabla_s,\cdot\rangle + g^{\bar i j}\,R_{\alpha\bar\beta\mu\bar\nu}\,f^\alpha_s\,\overline{f^\beta_{\bar j}}\,f^\mu_{\bar s}\,\overline{f^\nu_{\bar i}}$

i.e. $\frac{\partial^2}{\partial s \partial \bar{s}} \int_N |\bar{\partial} f|^2$

$$= \int_N g^{\bar{i}j}\left(\langle \nabla_{\partial_{\bar{i}}} f_s, \nabla_{\partial_{\bar{j}}} f_s \rangle + \langle \nabla_{\partial_{\bar{i}}} f_{\bar{s}}, \nabla_{\partial_{\bar{j}}} f_{\bar{s}} \rangle \right)$$
$$- g^{\bar{i}j}\left(R(f_{\bar{i}}, \overline{f_{\bar{j}}}, f_s, \overline{f_s}) + R(f_{\bar{i}}, \overline{f_{\bar{j}}}, f_{\bar{s}}, \overline{f_{\bar{s}}}) \right)$$
$$+ 2\,\mathrm{Re}\, g^{\bar{i}j} R(f_s, \overline{f_{\bar{j}}}, f_{\bar{s}}, \overline{f_{\bar{i}}})$$

In the special case that $\dim_{\mathbb{C}} N = 1$ with coor. $w, \bar{w}$, get

2nd Variation Formula:

$$\frac{\partial^2}{\partial s \partial \bar{s}}\Big|_{s=0} \int_N |\bar{\partial} f|^2 = \int_N \|\nabla_{\bar{w}} f_s\|^2 + \|\nabla_{\bar{w}} f_{\bar{s}}\|^2$$
$$- \int_N R(f_{\bar{w}}, \overline{f_{\bar{w}}}, f_s, \overline{f_s}) + R(f_{\bar{w}}, \overline{f_{\bar{w}}}, f_{\bar{s}}, \overline{f_{\bar{s}}})$$
$$+ 2\,\mathrm{Re} \int_N R(f_s, \overline{f_{\bar{w}}}, f_{\bar{s}}, \overline{f_w}) \quad \cdot \frac{i}{2}\, dw \wedge d\bar{w}$$

In the even more special case that $N = \mathbb{P}^1$, we are able to create certain special variations:

(1) $f^* T_M$ is a holo. v.b over $\mathbb{P}^1$

this means that $\exists$ sections v_α $\alpha = 1, \dots, m$ locally st. $\nabla_{\bar{w}} v_\alpha = 0$ and v_α form a basis

(2) Now assume that $f : \mathbb{P}^1 \longrightarrow M$ is energy minimizing and $f^* c_1(M)$ is non-negative on $\mathbb{P}^1$

Question: when can we conclude that f is holomorphic?

(1) + Grothendieck $\Rightarrow$ $f^* T_M = L_1 \oplus \cdots \oplus L_m$ over $\mathbb{P}^1$

L_i holo. line bundle

(2) then $\Rightarrow$ some L_i has non-neg degree over $\mathbb{P}^1$

R-R thm on $\mathbb{P}^1 \Rightarrow \exists$ global hol. section v of $f^* T_M$

$$h^0(L) - h^1(L) = \deg L + 1 - g = \deg L + 1$$
$$h^1(L) = h^0(K - L)$$
$$\deg(K-L) < 0 \Rightarrow h^1(L) = 0$$

Now construct a C^∞ family of C^∞ maps

$f(s): \mathbb{P}^1 \to M$, $|s| < \varepsilon$, $s \in \mathbb{C}$ st. $f(0) = f$

and $\underline{f_s = v \;;\; f_{\bar{s}} = 0}$ at time $s = 0$

But then $\nabla_{\bar{w}} f_s = \nabla_{\bar{w}} v = 0$ at $s = 0$

so

$$\underbrace{\frac{\partial^2}{\partial s \partial \bar{s}}\Big|_{s=0} \int_{\mathbb{P}^1} |\bar{\partial} f|^2}_{\geq 0} = -\int_{\mathbb{P}^1} R(f_{\bar{w}}, \overline{f_{\bar{w}}}, v, \bar{v}) \leq 0$$

But $R(u, \bar{u}, v, \bar{v}) > 0$ if $u \neq 0 \neq v$, hence $\Rightarrow f_{\bar{w}} = 0$

this means that $f: \mathbb{P}^1 \to M$ is holomorphic.

Frenkel Conjecture (Siu-Yau 1980)

If X is a cpt Kahler mfd with positive bisectional curvature, then $X = \mathbb{P}^n$.

sketch of proof:

step 1: Bonnet-Meyer $\Rightarrow \pi_1$ finite
may assume in fact π_1 = trivial

step 2: Bishop - Goldberg - Kobayashi $\Rightarrow H^2(X, \mathbb{Z})$ rk $= 1$.

step 3: Hurwicz thm $\pi_2(X) = H^2(X, \mathbb{Z})$
hence the generator $\alpha \in H^2(X, \mathbb{Z})_{\text{free}}$ is repr
by $\sum f_i$; $f_i: S^2 \to X$ energy minimizing
and $\sum E(f_i) = E(f)$ where $[f] \in \pi_2(X)$ corr. to α.
by Sacks-Uhlenbeck (Meeks-Yau)

step 4: The above $\Rightarrow f_i$ holo. or anti holo

step 5: Top. argument $\Rightarrow$ if more than 2 maps. then holo. anti-holo both occurs

step 6: Tube argument $\Rightarrow$ get smaller energy. Q.E.D.