Fubini's thm. $f : A \times B \subset \mathbb{R}^{n+m} \to \mathbb{R}$ bounded

$$\underline{\int}_{A\times B} f \le \underline{\int}_A \left(\underline{\int}_B f(x,y)\,dy \right) dx \le \overline{\int}_A \overline{\int}_B f \le \overline{\int}_{A\times B} f$$

in particular, $f \in R(A\times B) \Rightarrow \int_{A\times B} f = \int_A \overline{\int}_B f$

can change to lower $\int$.

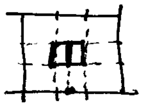
<u>pf:</u> $P_A \in \mathcal{P}(A)$, $P_B \in \mathcal{P}(B) \Rightarrow P_A \times P_B =: P \in \mathcal{P}(A\times B)$

subintervals $S_{ij} = S_i \times S_j'$

$$\underline{L(P,f)} = \sum_{ij} m_{S_{ij}}(f)\cdot \mu(S_{ij})$$

$$= \sum_i \Big(\sum_j \underbrace{m_{S_i \times S_j'}(f)}_{\le\, m_{S_j'}(f(x,\cdot))} \mu(S_j') \Big) \mu(S_i)$$

$\forall x_i \in S_i$ fixed



S_j', $x \in S_i$

$$\le \sum_i \Big(\underline{\int}_B f(x_i,\cdot) \Big) \mu(S_i) \le L(P_A, \mathcal{L}) \le U(P_A, \mathcal{L})$$

$$\underline{\int}_B f(x_i,\cdot) = \mathcal{L}(x_i)$$

$$L(P_A,\mathcal{U}) \le U(P_A,\mathcal{U}) \le \underline{U(P,f)}$$

※

Jordan measure $\equiv$ content. Let $S \subset I$ interval in $\mathbb{R}^n$

$\bar{c}(S)$ = Outer content $= \inf_{P\in\mathcal{P}(I)} \bar{J}(P,S)$, contains pt in $\bar{S}$

$\underline{c}(S)$ = inner content $= \sup \underline{J}(P,S)$ - contained in int S.

Lemma: $\bar{c}(\partial S) = \bar{c}(S) - \underline{c}(S)$, So S is Jordan measurable $\Leftrightarrow c(\partial S) = 0$

~~this is same as measure 0 for cpt set~~

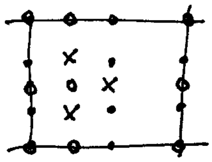
<u>pf:</u> $\bar{J}(P,\partial S) = \bar{J}(P,S) - \underline{J}(P,S) \ge \bar{c}(S) - \underline{c}(S)$

$\Rightarrow \bar{c}(\partial S) \ge \bar{c}(S) - \underline{c}(S)$.

for $\le$: Given $\varepsilon > 0$, $\exists P_1, P_2$ st. $\bar{J}(P_1,S) < \bar{c}(S) + \frac{\varepsilon}{2}$, $\underline{J}(P_2,S) > \underline{c}(S) - \frac{\varepsilon}{2}$ let <u>$P \supseteq$ "$P_1 \cup P_2$"</u>

$\Rightarrow \bar{c}(\partial S) \le \bar{J}(P,\partial S) = \bar{J}(P,S) - \underline{J}(P,S)$

$\le \bar{J}(P_1,S) - \underline{J}(P_2,S) < \bar{c}(S) - \underline{c}(S) + \varepsilon$ ※.



$P_1 \cup P_2$ is NOT in $\mathcal{P}(I)$?

Sol: the partition generated by $P_1 \cup P_2$.

Cor. Let S has content. $f \in \mathcal{B}(S)$.

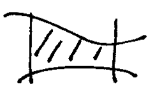
$\exists \int_S f := \int_I f \chi_S \Leftrightarrow D(f)$ has measure 0.

Cor. Bounded open set may not has Jordan content.

eg. A: open cover of $\mathbb{Q}\cap[0,1] \Rightarrow \partial A = [0,1] \setminus A$ not count (measure) 0.

but if $c(A)$ exists, then $c(A) = c(\bar{A}) = \int_{\bar{A}} 1 = \int_A 1$.

Cor. Fubini for basic shape between conti graphs:



since $c(\partial S) = 0$ (Ex.)

Thm: $f \in R(S)$, $S = A \cup B$, $\operatorname{int} A \cap \operatorname{int} B = \phi \Rightarrow \int_S f = \int_A f + \int_B f$.

<u>pf:</u> $S(P, f\chi_S) = S_A + S_B - S_C$ $\quad c(\partial A \cup \partial B) = 0 \Rightarrow S_C \to 0$ ※

$I_k \cap A \ne \phi$, $\cap B \ne \phi \Rightarrow \cap \partial A \ne \phi$, $\cap \partial B \ne \phi$

※ ~~only finitely~~ additive.

Lebesgue integral on $\mathbb{R}^n$ (Daniel int.)

$I \subset \mathbb{R}^n$ interval. $f \in S(I)$ if $\exists P \in \mathcal{P}(I)$ st $f = c_k$ on int I_k for each sub int.
cpt interval $\subset I$

$\int_I f := \sum_{k=1}^m c_k \mu(I_k)$

$f \in U(I)$ if $S(I) \ni s_n \nearrow f$ a.e. on I & $\lim_{n\to\infty} \int_I s_n$ exists.

$\int_I f := \lim_{n\to\infty} \int_I s_n$

$f \in L(I)$ if $f = u - v$, $u, v \in U(I)$. $\int_I f := \int_I u - \int_I v$.

Everything is true. eg. Levi thm. Lebesgue dominated conv. thm.

<u>Levi: $L(I) \ni f_n \nearrow$ a.e.</u> $\quad$ <u>$L(I) \ni f_n \longrightarrow f$ a.e. $|f_n| \leq_{a.e.} g \in L(I)$</u>

<u>$\int_I f_n < M$</u> $\Rightarrow f_n \nearrow f \in L(I)$ and $\int f = \lim_{n\to\infty} \int f_n$

$f \in M(I)$ if $S(I) \ni s_n \xrightarrow[a.e.]{} f$ $\quad$ Thm: $M(I) \ni f_n \xrightarrow[a.e.]{} f \Rightarrow f \in M(I)$.

$S \subset \mathbb{R}^n$ measurable if $\chi_S \in L(\mathbb{R}^n)$. $\mu(S) := \int_{\mathbb{R}^n} \chi_S \in \mathbb{R}_+ \cup \{\infty\}$

$f \in L(S)$ if $f\chi_S \in L(\mathbb{R}^n)$. $\int_S f := \int_{\mathbb{R}^n} f\chi_S$.

Thm (Countable additive property): $\mathcal{M}$ is a σ-algebra. Also,
$A_1, A_2, \dots$ measurable in $\mathbb{R}^n$, disjoint $\Rightarrow \mu(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$.

Thm: $S \subset \mathbb{R}^n$ open $\Rightarrow S = \bigcup I_k$ countable bdd intervals
Hence ~~open~~ & ~~closed~~ sets are all measurable (Lindelof is enough)

x pf: $F_m := \{$prod of $(\frac{k}{2^m}, \frac{k+1}{2^m}]\}$ count. disj. collect. union $= \mathbb{R}^n$

$F_{m+1} \equiv$ bisect. of F_m so $Q_{m+1} \subseteq Q_m$ or disj.

Subcollection: $G_0 = \phi$, $G_m = \{Q \in F_m \mid \bar{Q} \subset S, Q \subsetneq |G_{m-1}|\}$ $\frac{1}{2^m}$ Q_m

$\Rightarrow S \supseteq \coprod_{m=1}^\infty \coprod_{Q \in G_m} Q \supseteq S$: $p \in S \Rightarrow p \in Q \in F_m$ support $\frac{1}{2^m}$
cube smallest. ※

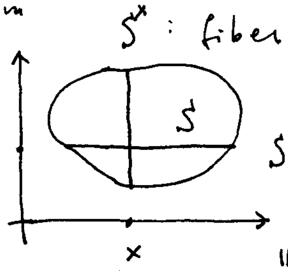
Structure of measure 0 sets:

Thm: $\mu(S) = 0 \overset{\circledast}{\Leftrightarrow} \exists$ count. J_k intervals $\sum_{k=1}^\infty \mu(J_k) < \infty$, st $S \ni s \in J_k$ for ∞ k.

pf. $\Rightarrow$ let $S \subset I_{m,1} \cup I_{m,2} \cup \cdots$ total $\mu < \frac{1}{2^m}$
then $A = \{I_{m,k}\}$ is countable, $\mu < 1$, every $s \in S$ appears in I_{m,k_m}.

$\Leftarrow$: trivial. $\forall \varepsilon > 0$, $\exists N$ st. $\sum_{k=N}^\infty \mu(J_k) < \varepsilon$. Still $\bigcup_{k=N}^\infty J_k \supset S$. ※

Thm μ: $S \subset \mathbb{R}^{n+m}$, $\mu(S) = 0 \Rightarrow \mu_{\mathbb{R}^m}(S_y) = 0$ $\forall' y$. $\quad$ S^x: fiber
(if S measurable, $\Leftarrow$ by "Fubini")



Pf. $\exists I_k = X_k \times Y_k$, $k \in \mathbb{N}$ has property $\circledast$.

$\infty > \sum \mu(I_k) = \sum \mu(X_k) \times \mu(Y_k) = \sum \int_{\mathbb{R}^m} \mu(X_k) \chi_{Y_k}$

~~Levi~~ $\Rightarrow$ $\sum \mu(X_k) \chi_{Y_k}(y) \to g(y)$ on $\mathbb{R}^m \setminus T$, $\mu(T) = 0$.

Claim: $\mu_{\mathbb{R}^n}(S_y) = 0$: $A(y) := \{X_k \mid y \in Y_k\}$ sum of measure $= g(y) < \infty$

$x \in S_y \Rightarrow (x, y) \in X_k \times Y_k$ ∞ many k $\Rightarrow x \in X_k$ ∞ many $k \Rightarrow \mu(S_y) = 0$ ※

Fubini's Thm. $f \in L(\mathbb{R}^{n+m}) \Rightarrow$

$G(y) := \int_{\mathbb{R}^n} f(x,y)\,dx \overset{(2).}{\in} L^1(\mathbb{R}^m)$

(1) exists for $y \in \mathbb{R}^m \setminus T$, $\mu_{\mathbb{R}^m}(T) = 0$

set $G(y) = 0$ for $y \in T$.

$\& \overset{(3).}{\int_{\mathbb{R}^m} G(y)\,dy} = \int_{\mathbb{R}^{n+m}} f$.

~~pf:~~ True if $f \in S(\mathbb{R}^{n+m})$. Also "$U(\mathbb{R}^{n+m})$" $\Rightarrow$ "$L(\mathbb{R}^{n+m})$".

So let $s_k \nearrow f$ on $\mathbb{R}^{n+m} \setminus S$, $\mu(S) = 0$.

$\Rightarrow s_k(x,y) \nearrow f(x,y)$ for $x \in \mathbb{R}^n \setminus S_y$

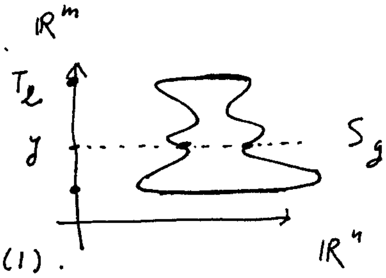
Let $t_k(y) := \int_{\mathbb{R}^n} s_k \nearrow$, then $\int_{\mathbb{R}^m} t_k = \int_{\mathbb{R}^{n+m}} s_k \le \int_{\mathbb{R}^{n+m}} f$

~~Levi~~ $\Rightarrow t_k \nearrow t \in U(\mathbb{R}^m)$ on $y \in \mathbb{R}^m \setminus T_1$, $\mu_{\mathbb{R}^m}(T_1) = 0$, $\int t_k \nearrow \int t$.

Fix such y, $\int_{\mathbb{R}^n} s_k(x,y)\,dx = t_k(y) \le t(y)$.

~~Levi~~$\Rightarrow$ $s_k(x,y) \nearrow g(x,y) \in U$ on $x \in \mathbb{R}^n \setminus A_y$, $\mu_{\mathbb{R}^n}(A_y) = 0$.

~~Thm μ~~$\Rightarrow$ $\mu_{\mathbb{R}^m}(S_y) = 0$ on $y \in \mathbb{R}^m \setminus T_2$, $\mu_{\mathbb{R}^m}(T_2) = 0$.



$y \in \mathbb{R}^m \setminus \underset{T_1 \cup T_2}{T} \Rightarrow f(x,y) = g(x,y)$ on $x \in \mathbb{R}^n \setminus (A_y \cup S_y)$

in particular $G(y) = \int f\,dx = \int g\,dx$

exists. $\Rightarrow$ (1).

$= \lim_{k\to\infty} \int_{\mathbb{R}^n} s_k(x,y)\,dx = \lim_{k\to\infty} t_k(y) = t(y) \Rightarrow G \in U(\mathbb{R}^m)$ i.e. (2).

$\underset{\text{a.e.}}{=} t$

Finally, $\int_{\mathbb{R}^m} G(y)\,dy = \lim_{k\to\infty} \int_{\mathbb{R}^m} t_k(y)\,dy$

$= \lim_{k\to\infty} \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^n} s_k(x,y)\,dx\right)dy = \lim_{k\to\infty} \int_{\mathbb{R}^{n+m}} s_k \overset{(3).}{=} \int_{\mathbb{R}^{n+m}} f$ ※

Cor/Thm: Tonelli-Hobson test for integrability:

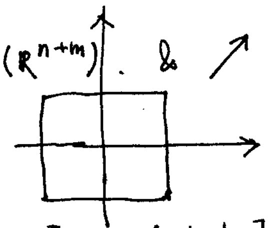
Let $f \in M(\mathbb{R}^{n+m})$. $\int_{\mathbb{R}^m}\int_{\mathbb{R}^n} |f|\,dx\,dy$ exists $\Rightarrow$ $|f|$, hence $f \in L(\mathbb{R}^{n+m})$.

this makes sense by thm μ.

~~pf:~~ consider $S(\mathbb{R}^{n+m}) \ni s_k := \chi_{I_k}$

$0 \le f_k := \min(s_k, |f|) \le s_k \Rightarrow f_k \in L(\mathbb{R}^{n+m})$. & $\nearrow$ in k.

$\int_{\mathbb{R}^{n+m}} f_k = \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} f_k \le \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} |f|$



Levi $\Rightarrow$ $f_k \nearrow_{\text{a.e.}} g \in L(\mathbb{R}^{n+m})$

$I_k = (-k,k]^{n+m}$

But $f_k(x,y) \longrightarrow |f(x,y)|$ as $k \to \infty$, $\Rightarrow |f| \underset{\text{a.e.}}{=} g \in L(\mathbb{R}^{n+m})$.

with $f \in M \Rightarrow f \in L$. ※.

- No such criterion for Riemann integrals.

G ~~primitive mapping~~ if $G(x) = (x_1, \dots x_{m-1}, g(x), x_{m+1} \cdots x_n)^t$
$\uparrow$ m-th $= x + (g(x) - x_m)\vec{e}_m$

$G' = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ D_1 g & \cdots & D_m g & \cdots & D_n g \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$; $J_G(a) = D_m g(a)$

Thm: $F \in C'(E, \mathbb{R}^n)$ $0 \in E \underset{\text{open}}{\hookrightarrow} \mathbb{R}^n$, $F(0)=0$, $F'(0)$ invertible

$\Rightarrow$ $F = \cancel{B_1 \cdots B_{n-1}}\, G_n \cdots G_1$; G_i C'-primitive, $0 \mapsto 0$, inverti. in a nbd of 0. B_i flip or identity.

~~Pf:~~ Denote $P_m x = (x_1, \dots, x_m, 0 \dots 0)^t$; $P_0 x = 0$.

$F_1 := F$, induction on m: for $1 \le m \le n-1$

$V_m \ni 0$, $F_m \in C'(V_m, \mathbb{R}^n)$, $0 \mapsto 0$, $F_m'(0)$ inv. $\underline{P_{m-1} F_m(x) = P_{m-1} x}$.

So, $F_m(x) = (x_1, \dots, x_{m-1}, \alpha_m(x), \dots, \alpha_n(x))^t$

$F_m'(0)\vec{e}_m = D_m F_m(0) = \sum_{i=m}^{n} D_m \alpha_i(0)\, \vec{e}_i \neq 0$, say at $i=k$.

let $\begin{cases} B_m \text{ flips the } k\text{-th } \& \ m\text{-th coor.} \\ G_m(x) := x + (\alpha_k(x) - x_m)\vec{e}_m \end{cases}$: $0 \in U_m \xrightarrow{\sim} V_{m+1} \ni 0$; $U_m \subset V_m$, $\exists\, G_m^{-1}$ by IFT.

$F_{m+1} := B_m F_m G_m^{-1}$

$P_m F_{m+1}(\underbrace{G_m(x)}_{y}) = P_m B_m F_m(x) = P_m \underbrace{G_m(x)}_{y \in V_{m+1}}$ Get F_{m+1}.

$\Rightarrow F = F_1 = B_1 F_2 G_1 = B_1(B_2 F_3 G_2)G_1 = B_1 \cdots B_{n-1} F_n G_{n-1} \cdots G_1$

by construction, $F_n =: G_n$ is also primitive. done $\square$.

Thm: CVF for Riem int. with ~~conti f~~. ~~cpt supp in I~~.

~~Pf:~~ step 1. CVF for g_1, g_2 $\Rightarrow$ CVF for $g_1 \circ g_2$

step 2. g primitive : Fubini + 1 variable case

step 3. Covering argument (finit)

let $f \equiv 0$ outside $S \underset{\text{cpt}}{\hookrightarrow} T$ (Also $S \underset{\text{cpt}}{\subset} I$)

$\forall t \in S$, $\exists$ open interval $I_t \subset T$ st

$g|_{I_t}$ = comp of primi. map. ($\Rightarrow$ CVF)

$S \subset \bigcup_{t \in S} (I_t \cdot \frac{1}{2})$ has a finite sub cover $I_{t_1}, \dots, I_{t_N}$
half size

$\Rightarrow \exists\, P \supseteq$ All division pts. $P \in \mathcal{P}(I)$ st. CVF holds $\forall I_k \in P$
finite many

Rmk: Also holds for ~~$f \in R(S)$~~, ~~S Jordan measurable~~, ~~$\bar{S} \subset T$~~. ⚹

3/8. Thm (CVF. ~~Transformation formula for Lebesgue int.~~)

$T \subset \mathbb{R}^n$ open $g: T \to \mathbb{R}^n$ C', 1-1 $\det g'(t) \neq 0$ $\forall t \in T$.

if $f \in L'(g(T))$ then $\int_{g(T)} f = \int_T f\circ g \, |\underbrace{\det g'}_{J_g}|$.

~~pf~~: ~~Step 1~~: $K \underset{\text{cpt cube}}{\hookrightarrow} g(T)$, replace T by $g^{-1}(K)$.

May assume $f \in U(K)$. $S(K) \ni s_k \nearrow f$ a.e. in K.

* Assumption : $\mu(I) = \int_{g^{-1}(I)} |J_g(t)|\,dt$ for cpt interval (cube)

(This is proved last time, in Rudin). This is Apostol p.428. Thm 15.16.

$$\Rightarrow \int_K s_k = \int_{g^{-1}(K)} (s_k\circ g)|J_g| = \int_{\mathbb{R}^n} f_k := (s_k\circ g)|J_g|\cdot \chi_{g^{-1}(K)}$$

$f_k \nearrow_{a.e.}$ and $\int_{\mathbb{R}^n} f_k = \int_K s_k \xrightarrow[k\to\infty]{} \int_K f$

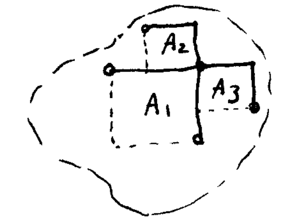
~~Levi~~ $\Rightarrow$ $f_k \nearrow_{a.e.} h \in L(\mathbb{R}^n)$. with $\int_{\mathbb{R}^n} h = \lim_{k\to\infty} \int_{\mathbb{R}} f_k = \int_K f$.

But $\lim_{k\to\infty} f_k(t) = f(g(t))\cdot |J_g(t)|\cdot \chi_{g^{-1}(K)}(t)$ a.e. in $\mathbb{R}^n$

Hence $h \underset{a.e.}{=} (f\circ g)\cdot|J_g|\cdot \chi_{g^{-1}(K)}$ ※

~~Step 2~~: $g(T)$ is open (by IFT),

$\Rightarrow g(T) = \coprod_{i=1}^{\infty} A_i$ with A_i interval of the form $\prod_{k=1}^n (a_k, b_k]$ and $K_i := \overline{A_i} \subset g(T)$.



** countable additivity $\Rightarrow$

$$\int_{g(T)} f = \sum_{i=1}^{\infty} \underbrace{\int_{K_i} f}_{\text{same as } A_i} = \sum_{i=1}^{\infty} \int_{g^{-1}(K_i)} (f\circ g)\cdot|J_g|$$

$$= \int_T (f\circ g)\cdot |J_g| \quad ※.$$

Rmk (*) the assumption is really a thm in Riem int & Jordan content.
as proved in Courant & John using ~~primitive decomp.~~ ($\Leftrightarrow$ IFT).

(**) the countable add. can be replaced by ~~partition of unity~~.
which is better since it works for Riem. int.

Thm (Partition of 1). Let $K \subset \mathbb{R}^n$ cpt, $K \subset \bigcup_\alpha V_\alpha$ open cover then $\exists\, \psi_1, \cdots, \psi_s \in C(\mathbb{R}^n)$, $0 \le \psi_i \le 1$, supported in some V_α, $\Sigma \psi_i = 1$.

~~pf~~: $x \in K \Rightarrow x \in V_{\alpha(x)} \Rightarrow \exists$ balls $\overline{B}_x \subset W_x \subset \overline{W}_x \subset V_{\alpha(x)}$

$\Rightarrow K \subset B_{x_1} \cup \cdots \cup B_{x_s}$. Let $\varphi_i|_{B_{x_i}} \equiv 1$ & $\varphi_i \equiv 0$ outside W_{x_i}.

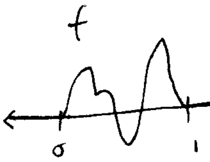
(~~Ex $\varphi_i \in C^\infty$~~)

Finally, $\psi_i := \varphi_i / \sum_{j=1}^{s} \varphi_j$. ※

This replaces the covering argument (step 3) of Fubini thm (Riem case).

Thm (Weierstrass). $f \in C[a,b] \Rightarrow \exists P_n(x)$ poly $\xrightarrow[\text{unif.}]{} f$

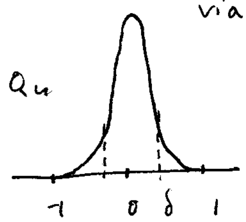
~~pf:~~ Assume $[a,b] = [0,1]$, $f(0) = f(1) = 0$ by subtr. linear poly.
and $f \equiv 0$ outside $[0,1]$.



let $Q_n(x) = c_n(1-x^2)^n$ st $\int_{-1}^{1} Q_n(x)\,dx = 1$

~~Legendre~~ (polynomial kernel) $\Rightarrow c_n < \sqrt{n}$

via $\int_{-1}^{1}(1-x^2)^n \geq 2\int_0^{1/\sqrt{n}}(1-nx^2) = \frac{2}{\sqrt{n}} - \frac{2}{3}n\frac{1}{\sqrt{n}^3} = \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}$



$\forall\ 0 < \delta < |x| \leq 1$, $\underline{Q_n(x) \leq \sqrt{n}(1-\delta^2)^n} \xrightarrow[\text{unif}]{} 0$

Now set $P_n(x) := \underline{\int_0^1 f(t)\,Q_n(t-x)\,dt}$ poly in x convolution

$x \in [0,1]$

$= \int_{-x}^{1-x} f(x+t)\,Q_n(t)\,dt = \int_{-1}^{1}$.

Notice f real (cpx) $\Rightarrow P_n$ real (cpx).

Then $|P_n(x) - f(x)| \leq \int_{-1}^{1}|f(x+t) - f(x)|\cdot Q_n(t)\,dt$

~~Given~~ $\varepsilon > 0$, $\exists\,\delta$ st. $\leq 2M\left(\int_{-1}^{-\delta} + \int_\delta^1 Q_n(t)\,dt\right) + \frac{\varepsilon}{2}\int_{-\delta}^{\delta} Q_n(t)\,dt$

$|y-x| < \delta \Rightarrow |f(y) - f(x)| < \frac{\varepsilon}{2}$

$\leq 4M\cdot\sqrt{n}(1-\delta^2)^n + \frac{\varepsilon}{2} < \varepsilon \quad \forall n \gg 0$.

Cor. for $|x|$ on $[-a,a]$, $\exists\, P_n(x)$ ~~$P_n(0)=0$~~ & $P_n(x) \to |x|$ unif.

~~pf:~~ Simply replace $P_n(x)$ by $P_n(x) - P_n(0)$ ※.

Def'n ① $\mathcal{A} \subset \mathrm{Map}(E, \mathbb{R})$ is an ~~$\mathbb{R}$-algebra~~ if $\mathcal{A}$ is a ring & $\mathbb{R}\mathcal{A} \subset \mathcal{A}$.
algebra simply means $\mathbb{C}$-algebra.

② $\bar{\mathcal{A}} := \{f \mid \exists f_n \in \mathcal{A} \xrightarrow[\text{unif}]{} f\}$ called the ~~unif closure~~ of $\mathcal{A}$.

③ $\mathcal{A}$ is ~~uniformly closed~~ if $\bar{\mathcal{A}}^{\text{unif}} = \mathcal{A}$.

So. $\mathcal{C}(S)$ is unif. closed. and $\mathbb{C}[x]$ has unif closure $C([a,b])$

~~Fact:~~ If $\mathcal{A}$ alg of ~~bdd fun's~~ then $\bar{\mathcal{A}}^{\text{unif}}$ is unif. closed.
used in $f_n g_n \to fg$

④ $\mathcal{A}$ ~~base-point-free~~ (vanishes at no pts on E): $x \in E \Rightarrow \exists g(x) \neq 0$

⑤ $\mathcal{A}$ ~~separates points~~ : $\forall x_1 \neq x_2$ in E, $\exists f$ st $f(x_1) \neq f(x_2)$

~~Eg.~~ $\mathbb{C}[x]$ satisfies ④. ⑤, but even poly does not satisfy ⑤.

Theorem (Stone). Let $\mathcal{A} \subset C(K, \mathbb{R})$, K cpt, $\mathbb{R}$-algebra

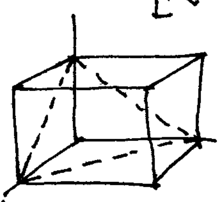
if $\mathcal{A}$ is bpf & separates pts. then $\bar{\mathcal{A}}^{\text{unif}} \equiv C(K)$.

Cor. Using ~~$\mathcal{A} = \{f = \prod f_i(x_i)\}$~~ has this prop. $\Rightarrow$ Another pf of Fubini.

Diff forms

Def: ~~k-surface~~ : $\Phi : D \longrightarrow E$: C^1 . i.e. $\Phi \in C^1(W)$
will do $D = I^k$ or Q^k $\underset{\mathbb{R}^k}{\cap}$ cpt $\underset{\mathbb{R}^n}{\cap}$ $D \subset W_{\text{open}}$

I^k



$\left\{ \begin{array}{l} x_i \geq 0 \\ \Sigma x_i \leq 1 \end{array} \right\} = Q^k$

Def: $\omega \in$ ~~$\Omega^k(E)$: k form~~ in E
denoted by $\omega = \sum_{|I|=k} a_I(x)\, dx_I$

is a function: for any $x = \Phi(u)$ a k-surface,

$$\omega(\Phi) \equiv \int_\Phi \omega := \int_D \sum_I a_{i_1 \cdots i_k}(\Phi(u)) \frac{\partial(x_{i_1}, \cdots x_{i_k})}{\partial(u_1, \cdots, u_k)}\, du$$

Properties: ① $dx_i \wedge dx_j = -dx_j \wedge dx_i$. i.e. all "~~k-area comp~~" in $\mathbb{R}^n$.

② for $\omega = \sum_I b_I(x)\, dx_I$ standard presentation. i.e. I ordered st. $i_1 < i_2 < \cdots < i_k$
then $\omega = 0 \iff b_I = 0 \ \forall I$.

pf: $b_J(x) > 0$ in a nbd of x_0, let $\Phi(u) = x_0 + \sum_{r=1}^k u_r \vec{e}_{j_r}$ ⁂

Def : wedge product : $\omega \wedge \eta$. Ex. why $(\omega \wedge \eta) \wedge \sigma = \omega \wedge (\eta \wedge \sigma)$?

Def : Cartan's d operator : $d(f\, dx_I) := df \wedge dx_I$.

Theorem : (a) $\omega \in \Omega^k \Rightarrow d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta$.
(b) for C^2 diff forms, $d^2\omega := d(d\omega) \equiv 0$.

Change of variables : (pull back)

$\underset{\mathbb{R}^n}{E} \xrightarrow{T} \underset{\mathbb{R}^m}{V}$ $\quad \omega \in \Omega^k(V) \xrightarrow{T^*} \Omega^k(E)$; $\omega = \sum b_I(y)\, dy_I$

$y = T(x) = (t_1(x), \cdots, t_m(x))$ $\quad \omega_T \equiv \varphi^* \omega = \sum b_I(T(x))\, dt_{i_1} \wedge \cdots \wedge dt_{i_k}$

Notice that, in our def'n, $\int_\Phi \omega := \int_D \Phi^* \omega \equiv \int_\Delta \omega_\Phi$
$\Delta(u) := u$ identity map.

Theorem : T^* is a ring hom. (easy)
If $T \in C^2$ then $d(T^*\omega) = T^*(d\omega)$. Functoriality

pf: Let $\omega = f\, dy_I$. If true for functions (chain rule)

$d\omega = df \wedge dy_I \Rightarrow T^* d\omega = T^* df \wedge T^* dy_I$

$d^2 = 0 \Rightarrow$ $T^*\omega = f \circ T \wedge d(y_I \circ T) \Rightarrow dT^*\omega = d(f \circ T) \wedge d(y_I \circ T)$ ⁂

Theorem (Chain Rule) : $E \xrightarrow{T} V \xrightarrow{S} W \Rightarrow (ST)^* = T^* S^*$.

pf: let $\omega = dz_i$ then $\omega_S = S^* dz_i = \sum \frac{\partial z_i}{\partial y_j} dy_j \xrightarrow{T^*} \sum \frac{\partial z_i}{\partial y_j} \frac{\partial y_j}{\partial x_k} dx_k$

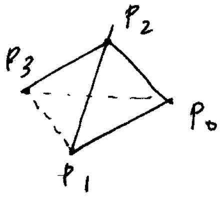
$= \sum \frac{\partial z_i}{\partial x_k} dx_k = (ST)^* \omega$. ⁂

Cor. $D \xrightarrow{\Phi} E \xrightarrow{T} V$ $\Rightarrow \int_{T\Phi} \omega = \int_D (T\Phi)^* \omega = \int_D \Phi^* T^* \omega = \int_\Phi T^* \omega$ ⁂
k-surface,

~~affine map~~ $f(x) = Ax + b$ A linear $L(\mathbb{R}^k, \mathbb{R}^n)$

oriented affine k-simplex in $\mathbb{R}^n$:

$\sigma = [p_0, p_1, \cdots, p_k]$ is a k-surface $Q^k \subset \mathbb{R}^k \xrightarrow{\sigma}$



$\sigma(\sum \alpha_i \vec{e}_i) := p_0 + \sum_{i=1}^{k} \alpha_i (p_i - p_0)$ affine map

for $\pi \in S_{k+1}$ on $\{0, 1, \cdots, k\}$, equiv. rel. on orientation

$\bar{\sigma} := \sigma_\pi = [p_{\pi(0)}, p_{\pi(1)}, \cdots, p_{\pi(k)}] \equiv s(\pi)\, \sigma$

" ± 1 (sign)

Thm : σ k-simplex in E, $\omega \in \Omega^k(E)$

then $\int_{\sigma_\pi} \omega = s(\pi) \int_\sigma \omega$. (for $k=0$ define $\int_{\pm p} f = \pm f(p)$)

~~pf~~ : if $\pi = (i, j)$, $0 < i < j \le k$, $\sigma_\pi(u) = p_0 + Cu = p_0 + A E_{ij} u$

C has column vect $[p_1 - p_0, \cdots, p_k - p_0]$ $= \sigma(E_{ij} u)$

$\Rightarrow \sigma_\pi^* \omega = E_{ij}^* \underbrace{\sigma^* u}_{k \text{ form in } k\text{-dim space}} = -\sigma^* u$ i, j exchange

if $\pi = (0, j)$, $\sigma_\pi(u) = p_j + Bu$ $\begin{cases} B e_i = p_i - p_j & i \ne j \\ B e_j = p_0 - p_j \end{cases}$

$= p_j + ASu$

why? $= (p_j - p_0) + \sigma(Su)$ Let $S = (-1) \times j$-th col. and sum to all col's.

$\Rightarrow \sigma_\pi^* \omega \overset{\downarrow}{=} S^* \sigma^* u = -\sigma^* u$. ※

affine k-chain : $\Gamma = \sigma_1 + \cdots + \sigma_r$ ~~(formal)~~ not sum in $\mathbb{R}^n$

$S = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ -1 & -1 & \cdots & -1 & -1 \\ & & & & \ddots & 1 \end{bmatrix}_{k \times k}$

$\int_\Gamma \omega := \sum_{i=1}^r \int_{\sigma_i} \omega$.

~~Eg~~ if $\Gamma = \sigma_1 + \sigma_2$, $\sigma_2 \equiv -\sigma_1$ then $\int_\Gamma \omega \equiv 0$ but $\sigma_1(u) + \sigma_2(u) \ne 0$.

Boundary : Let $k \ge 1$, $\sigma = [p_0, p_1, \cdots, p_k]$

$\partial \sigma := \sum_{j=0}^k (-1)^j [p_0, \cdots p_{j-1}, p_{j+1} \cdots p_k] \equiv \sum (-1)^j \sigma_j \equiv \partial_j \sigma$

~~Q~~ what is the affine map associated to σ_j ?

C^2 simplexes / chains : $Q^k \xrightarrow{\sigma} E \xrightarrow{T} V$, $E \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ $\Phi = T\sigma$ a C^2 k-surface in V

Φ is called (oriented) C^2 k-simplex.

$\Phi = \sum \Phi_i$ a C^2 k-chain, $\Phi_i = T_i \sigma_i$. If T_i all the same, $\Phi = T \cdot \Gamma$.

eg. $\partial \Phi := T(\partial \sigma)$; $\partial \Phi = \sum \partial \Phi_i$, still C^2.

Fact (Ex) : ~~$\partial^2 \Phi = 0$~~

Def$^{\text{n}}$: Positively oriented boundaries $\Omega = \cup E_i \subset \mathbb{R}^n$, $E_i = T_i(Q^n)$

$J_{T_i} > 0$ $\ni$ middle ∂ cancels. eg. $\partial I^2 =$  ※.

$\int d\omega = (-1) \int_{Q^k} \frac{\partial}{\partial x_k} \cdots$

Invariant under coord change (~~oriented~~):

$$\int_T \omega \qquad \underline{U} \xrightarrow{S} \underline{V} \xrightarrow{T} W$$

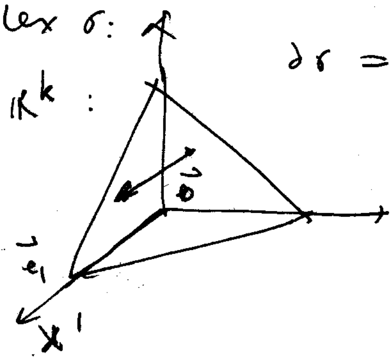
~~change of variable~~ ~~i.e. $\det J_S > 0$~~

$U = V = Q^k$

$$\int_{TS} \omega = \int_{Q^k} S^*(\underbrace{T^*\omega}_{\text{in } \mathbb{R}^k}) = \int_{Q^k} \det J_S \cdot T^*\omega = \int_{Q^k} T^*\omega = \int_T \omega \quad ※$$

Theorem: Stokes' thm for chains: $\int_\Phi d\omega = \int_{\partial \Phi} \omega$. $\omega \in \Lambda^k(\mathbb{R}^n)$.

~~pf:~~ Reduction to standard k-simplex σ:



$$\partial\sigma = \underset{\substack{\| \\ [e_1,\dots,e_k]}}{\partial_0\sigma} - \underset{\substack{\| \\ [e_0, e_2, \dots, e_k]}}{\partial_1\sigma} + \dots \quad (\text{terms } \textstyle\int = 0)$$

$$\int_{\partial\sigma} \omega = \int_{\partial_0\sigma} \omega - \int_{\partial_1\sigma} \omega$$

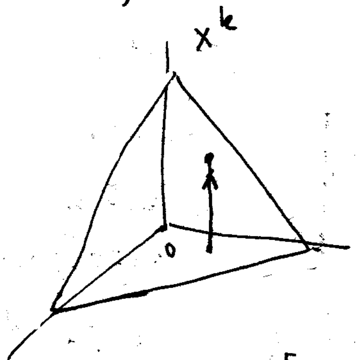
Applications:
~~de Rham homomorphism~~:

$$H_k(E, \mathbb{Z}) \times H^k_{dR}(E) \longrightarrow \mathbb{R}.$$

for $\omega = f\, dx_2 \wedge \dots \wedge dx_k$

$$\Rightarrow d\omega = \frac{\partial f}{\partial x_1} dx_1 \wedge \dots \wedge dx_k$$

~~for~~ Ease of notations:



$$\omega = f\, dx_1 \wedge \dots \wedge dx_{k-1}$$

$$d\omega = \frac{\partial f}{\partial x_k} dx_1 \wedge \dots \wedge dx_k \cdot (-1)^{k-1}$$

$$\partial\sigma = \partial_0\sigma + (-1)^k \partial_k \sigma + \dots \quad (\text{terms } \textstyle\int = 0)$$

$$\int_{\partial\sigma} \omega = (-1)^k \int_{Q^{k-1}} f\, dx_1 \dots dx_{k-1} + (-1)^{k-1} \int_{Q^{k-1}} f(x_1, \dots x_{k-1}, \underline{\underline{x_k}})\, dx_1 \dots dx_{k-1}$$

$\partial_0\sigma = [e_1 \dots e_k] \approx (-1)^{k-1} [e_k, e_1, \dots, e_{k-1}]$

$\partial_k\sigma = [e_0 \dots e_{k-1}]$

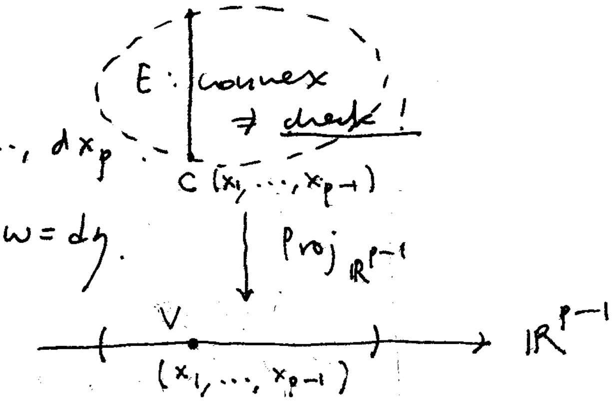
$$\int_\sigma d\omega = (-1)^{k-1} \int_{Q^k} \frac{\partial f}{\partial x_k} dx_1 \wedge \dots \wedge dx_k$$

$$= (-1)^{k-1} \int_{Q^{k-1}} \Big(f(x_1, \dots x_{k-1}, \overset{1 - \sum x_i}{x_k}) - f(x_1, \dots x_{k-1}, 0) \Big)\, dx_1 \dots dx_{k-1}$$

$$= \int_{\partial\sigma} \omega \quad ※$$

Thm. Poincaré lemma.

$\omega \in \Omega^k(E)$, $k \geq 1$. if $d\omega = 0$ then $\omega = d\eta$, $\eta \in \Omega^{k-1}(E)$.
C^1. open convex in $\mathbb{R}^n$



~~pf.~~ For $1 \leq p \leq n$, let
Ω^k_p = C^1 forms in E in $dx_1, \dots, dx_p$.

<u>Induction on p</u>: $\omega \in \Omega^k_p$, $d\omega = 0 \Rightarrow \omega = d\eta$.
(for all $k \geq 1$).

$p = 1$: then $k = 1$, $\omega = f\,dx_1$

$$0 = d\omega = \sum_{j=2}^n \partial_j f \cdot dx_j \wedge dx_1 \quad \Rightarrow \quad \partial_j f \equiv 0, \; j = 2 \dots n$$

<u>check!</u> i.e. $f(x) = f(x_1)$. Let $F(x) = \int_c^{x_1} f(t)\,dt \Rightarrow dF = \omega$.

Now let $p \geq 2$. Assume proved for Ω^k_{p-1} (all k up to $p-1$).

let $\omega \in \Omega^k_p$, $\quad 0 = d\omega = \sum_I \sum_{j=1}^n \partial_j f_I \; dx_j \wedge dx_I$
$\quad \| $
$\sum f_I \, dx_I$
for $j > p$ we get $\partial_j f_I \equiv 0$.

i.e. $f_I(x) = f_I(x_1, \dots, x_p)$. Let $F_I(x_1, \dots, x_p) = \int_c^{x_p} f_I(x_1 \dots x_{p-1}, t)\,dt$

$\Rightarrow \partial_p F_I = f_I$; $\partial_j f_I \equiv 0$. $j > p$ $\quad$ $c(x_1, \dots, x_{p-1})$ lower limit of ~~$V = \text{Proj}_{\mathbb{R}^{p-1}}(E)$ convex~~.

Write $\omega = \alpha + \sum_{I_0 \subset \{1, \dots p-1\}} f_I \, dx_{I_0} \wedge dx_p$

$\alpha \in \Omega^k_{p-1}$ $\qquad I = (I_0, p)$.

put $\beta = \sum_{I_0} F_I \, dx_{I_0} \in \Omega^{k-1}_{p-1}$ (what is $d\beta = ?$)

key step: $\gamma := \omega - (-1)^{k-1} d\beta \quad (\Rightarrow d\gamma = 0)$

$$= \omega - \sum_{I_0} \sum_{j=1}^p \partial_j F_I \; dx_{I_0} \wedge dx_j \quad (\text{look at } j = p)$$

$$= \alpha - \sum_{I_0} \sum_{j=1}^{p-1} \partial_j F_I \; dx_{I_0} \wedge dx_j \in \Omega^k_{p-1}$$

$\Rightarrow \gamma = d\mu \Rightarrow \omega = d\mu + (-1)^{k-1} d\beta$. (set $\eta = \mu + (-1)^{k-1}\beta$)

<u>Cor.</u> Holds for $E \simeq U$ by a C^2 map T. eg. star.

Examples: ~~closed, but not exact forms.~~

$E = \mathbb{R}^2 \setminus 0$: $\quad \omega = \dfrac{x\,dy - y\,dx}{x^2 + y^2} = d \tan^{-1} y/x \equiv d\theta \quad (\mathbb{R}^3 \setminus \text{line})$.

$E = \mathbb{R}^3 \setminus 0$: $\quad \omega = \dfrac{x\,dy \wedge dz + y\,dz\,dx + z\,dx\,dy}{(x^2 + y^2 + z^2)^{3/2}} \equiv d(\text{surf angle})$.

Maxwell's eq'n in $\mathbb{R}^{3,1}$ using diff forms p. 11

Minkowski $\mathbb{R}^{1,3}$: $ds^2 = c^2 dt^2 - dx^2 - dy^2 -$
now set $c = 1$.

MAXWELL'S EQUATIONS

We use relativistic units and let $(x^\mu) = (t, x, y, z)$ be a Lorentz frame. We have an electromagnetic field given by $(F_{\mu\nu})$ and a 4-current (J^μ). We have the free space values ε_0, μ_0 where $\varepsilon_0\mu_0 = c^{-2}$,

$$\varepsilon_0 = 8.85 \times 10^{-12} \quad \frac{\text{coul}^2\text{-sec}^2}{\text{kg-m}^3}$$

in MKS units or

$$7.96 \times 10^5 \quad \frac{\text{coul}^2}{\text{kg-m}}$$

in relativistic units. *Maxwell's equations* now read, using $c = 1$,

(a) $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$, Maxwell-Faraday
(b) $\nabla \cdot \mathbf{B} = 0$, Gauss
(c) $\nabla \times \mathbf{B} = (\partial\mathbf{E}/\partial t) + \mu_0\mathbf{J}$, Ampère with correction $\frac{\partial E}{\partial t}$ by Maxwell's (12.19)
(d) $\nabla \cdot \mathbf{E} = (1/\varepsilon_0)\rho = \mu_0\rho$. Gauss

Now we have

$$F = E_x\,dt\wedge dx + E_y\,dt\wedge dy + E_z\,dt\wedge dz - B_x\,dy\wedge dz - B_y\,dz\wedge dx - B_z\,dx\wedge dy.$$

We have then

$$\begin{aligned} dF = &\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} - \frac{\partial B_z}{\partial t}\right) dt\wedge dx\wedge dy \\ &+ \left(\frac{\partial E_x}{\partial t} - \frac{\partial E_z}{\partial x} + \frac{\partial B_y}{\partial t}\right) dz\wedge dt\wedge dx \\ &+ \left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} - \frac{\partial B_x}{\partial t}\right) dy\wedge dz\wedge dt \\ &+ \left(-\frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y} - \frac{\partial B_z}{\partial z}\right) dx\wedge dy\wedge dz. \end{aligned}$$

Thus the Maxwell equations (a), (b) combine in the single equation

$$dF = 0. \tag{12.20}$$

Now consider Maxwell equations (c), (d). If ρ, $\mathbf{J}$ are 0, these are

$$\nabla \times \mathbf{B} = \partial\mathbf{E}/\partial t, \qquad \nabla\cdot\mathbf{E} = 0.$$

These are just (a), (b) if $\mathbf{B}$ is replaced by $\mathbf{E}$ and $\mathbf{E}$ is replaced by $-\mathbf{B}$. Let

$*$: α basic $\ni$ $\alpha \wedge *\alpha = \pm\, dt\wedge dx\wedge dy\wedge dz$

 Hodge $*$ dual : (-1) if 247 at $\in \alpha$.

$(M_{\mu\nu})$ be obtained from $(F_{\mu\nu})$ by replacing $\mathbf{B}$ by $\mathbf{E}$, $\mathbf{E}$ by $-\mathbf{B}$ so that

$$(M_{\mu\nu}) = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{bmatrix}.$$

If $(M_{\mu\nu})$ can be shown to be the matrix of components of a tensor M, then

$$\begin{aligned} dM = &\left(-\frac{\partial B_x}{\partial y} + \frac{\partial B_y}{\partial x} - \frac{\partial E_z}{\partial t}\right) dt\wedge dx\wedge dy \\ &+ \left(-\frac{\partial B_x}{\partial z} + \frac{\partial B_z}{\partial x} + \frac{\partial E_y}{\partial t}\right) dz\wedge dt\wedge dx \\ &+ \left(-\frac{\partial B_y}{\partial z} + \frac{\partial B_z}{\partial y} - \frac{\partial E_x}{\partial t}\right) dy\wedge dz\wedge dt \\ &+ \left(-\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z}\right) dx\wedge dy\wedge dz. \end{aligned}$$

The Maxwell equations (c), (d) require comparing the four quantities in parentheses with the components $(\mu_0 J^\mu)$. To do this we need a 3-form, so we lower the index on J^μ and form the dual.

$$\begin{aligned} \mu_0^* J &= \mu_0^*(\rho\,dt - J_x\,dx - J_y\,dy - J_z\,dz) \\ &= \mu_0(\rho\,dx\wedge dy\wedge dz + J_x\,dy\wedge dt\wedge dz + J_y\,dt\wedge dx\wedge dz \\ &\quad + J_z\,dx\wedge dt\wedge dy) \\ &= -\mu_0 J_z\,dt\wedge dx\wedge dy + \mu_0 J_y\,dz\wedge dt\wedge dx - \mu_0 J_x\,dy\wedge dz\wedge dt \\ &\quad + \mu_0\rho\,dx\wedge dy\wedge dz. \end{aligned}$$

The Maxwell equations (c), (d) become

$$dM = -\mu_0^* J, \tag{12.21}$$

so $*dM = -\mu_0 {**}J = -\mu_0 J$.

Note that $\underline{M = *F}$, so M is indeed a tensor. Then we have $*d*F = -\mu_0 J$. We can write this as

$$\delta F = -\mu_0 J.$$

Thus *Maxwell's equations* are

$$dF = 0, \qquad \delta F = -\mu_0 J. \tag{12.22}$$

In the source free case, i.e., in case $J = 0$, we see Maxwell's equations are

$$dF = 0, \qquad \delta F = 0. \tag{12.23}$$

$\Box := d\delta + \delta d = (d+\delta)^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \quad \ni \Box F = 0$.

Def'n: $R \subset 2^X$ is a ring if $A \cup B,\ A \setminus B \in R$ ($\Rightarrow A \cap B \in R$)

a σ-ring if $\bigcup_{n=1}^{\infty} A_n \in R$ ($\Rightarrow \bigcap_{n=1}^{\infty}$)

Set function $\phi : R \to \mathbb{R} \cup \{\infty\}$ (or $\mathbb{R} \cup \{-\infty\}$) st. not all $+\infty$.

- Additive: $A \cap B = 0$ ($\equiv \phi$) $\Rightarrow \phi(A \cup B) = \phi(A) + \phi(B)$
- countably additive $\Rightarrow$ Fact: $A_n \nearrow \Rightarrow \phi(A_n) \xrightarrow[n\to\infty]{} \phi(A)$, $A = \bigcup A_n$.

  $\phi(\sqcup A_n) = \sum \phi(A_n)$ hence abs. conv. if conv.

Def'n: $X = \mathbb{R}^p$, $\mathcal{E} \ni A$: elementary $\equiv A = \bigcup_{i=1}^{n} I_i$ is a ring (not σ)

$A = \sqcup I_j$ and $m(A) = \sum m(I_j)$ well-defined, m additive on $\mathcal{E}$.

Def'n: $\phi : \mathcal{E} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ additive, is regular if

$\forall A \in \mathcal{E}$, $\varepsilon > 0$, $\exists$ closed $F \in \mathcal{E}$, open $G \in \mathcal{E}$ st. $F \subset A \subset G$ and

$\phi(G) - \varepsilon \leq \phi(A) \leq \phi(F) + \varepsilon$. Eg. m is regular.

Def'n: Outer measure $\mu^*(E) := \inf \sum_{n=1}^{\infty} \mu(A_n)$ for μ add. ≥ 0, regular and finite on $\mathcal{E}$.

any set in $\mathbb{R}^p$ — among $E \subset \bigcup A_n$ open elementary sets.

Prop: $A \in \mathcal{E} \Rightarrow \mu^*(A) = \mu(A)$; $E = \bigcup_1^{\infty} E_n \Rightarrow \mu^*(E) \leq \sum_1^{\infty} \mu^*(E_n)$. Sub-addit.

Def'n: Symmetric difference $S(A,B) := (A \setminus B) \cup (B \setminus A)$, $d(A,B) := \mu^*(S(A,B))$.

Write $A_n \to A$ if $d(A, A_n) \to 0$. finitely measurable (later)

- In case $A_n \in \mathcal{E}$, call $A \in \mathfrak{M}_F(\mu)$, $A \in \mathfrak{M}(\mu)$ if it is a countable such union

Metric space structure compatible with ring str:

- $d(A,A) = 0$, $d(A,B) = d(B,A)$, $d(A,B) \leq d(A,C) + d(C,B)$
- $d(A_1 \cup A_2, B_1 \cup B_2)$, $d(A_1 \cap A_2, B_1 \cap B_2)$, $d(A_1 \setminus A_2, B_1 \setminus B_2) \leq d(A_1,B_1) + d(A_2,B_2)$

  eg. using $S(A,B) = S(A^c, B^c)$.

  if define $A \sim B \iff d(A,B) = 0$ then get metric sp. & $\mathfrak{M}_F(\mu) = \overline{\mathcal{E}}$
- If $\mu^*(B) < \infty$ then $|\mu^*(A) - \mu^*(B)| \leq d(A,B)$. using $d(C,0) = \mu^*(C)$.

Theorem. $\mathfrak{M}(\mu)$ is a σ-ring, μ^* is countably additive on $\mathfrak{M}(\mu)$.

Pf of Prop: μ regular $\Rightarrow$ $A \subset G \in \mathcal{E}$ st. $\mu(G) \leq \mu(A) + \varepsilon$ $\forall \varepsilon > 0$, G open $\Rightarrow \mu^*(A) \leq \mu(A)$.

$A \supset F$, F closed

by def'n, $\mu(A) \leq \mu(F) + \varepsilon \leq \mu(\bigcup_1^N A_i) + \varepsilon \leq (\mu^*(A) + \varepsilon) + \varepsilon \Rightarrow$ "$=$".

$\sum_1^{\infty} \mu(A_i) \leq \mu^*(A) + \varepsilon$. cpt: $F \subset A_1 \cup \cdots \cup A_N$

Now let $E = \bigcup E_n$; $\mu^*(E_n) < \infty$ $\forall n$ (otherwise trivial)

Given $\varepsilon > 0$, $\exists$ open $A_{nk} \in \mathcal{E}$ covers E_n st. $\sum_{k=1}^{\infty} \mu(A_{nk}) \leq \mu^*(E_n) + \frac{\varepsilon}{2^n}$

$\Rightarrow \mu^*(E) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(A_{nk}) \leq \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon$ ※

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~~pf of Thm~~ Let $\mathcal{E} \ni A_n \to A \in \mathfrak{M}_F(\mu)$, $B_n \to B$

then $A_n \cup B_n \to A \cup B$, $A_n \cap B_n \to A \cap B$, $A_n \setminus B_n \to A \setminus B$

(*) $\mu^*(A_n) \to \mu^*(A) < \infty$ since $|\mu^*(A_n) - \mu^*(A)| \le d(A_n, A) < \infty$ $(\to 0)$
$\wedge$
∞

$\Rightarrow \mathfrak{M}_F(\mu)$ is a ring.

Now on $\mathcal{E}$, $\mu(A_n) + \mu(B_n) = \mu(A_n \cup B_n) + \mu(A_n \cap B_n)$

$n \to \infty$ get $\mu^*(A) + \mu^*(B) = \mu^*(A \cup B) + \mu(A \cap B)$ $\Rightarrow$ μ^* additive.

apply (*) to all 4 terms

Next, ~~$A \in \mathfrak{M}(\mu)$~~ $\Rightarrow A = \coprod_{n=1}^{\infty} A_n$, ~~$A_n \in \mathfrak{M}_F(\mu)$~~.

eg. $A = \bigcup A_n'$, $A_n' \in \mathfrak{M}_F(\mu)$. Then $A_1 := A_1'$, $A_n := A_n' - (A_1' \cup \cdots \cup A_{n-1}')$.

$\Rightarrow \sum_{n=1}^{\infty} \mu^*(A_n) \geq \mu^*(A) \geq \mu^*(A_1 \cup \cdots \cup A_m) = \mu^*(A_1) + \cdots + \mu^*(A_m)$

$m \to \infty$ get $\mu^*(A) = \sum_{n=1}^{\infty} \mu^*(A_n)$. (**)

claim: if $\mu^*(A) < \infty$ then $A \in \mathfrak{M}_F(\mu)$.

apply (**)

pf: Let $B_n = A_1 \cup \cdots \cup A_n \Rightarrow d(A, B_n) = \mu^*\left(\bigcup_{i=n+1}^{\infty} A_i\right) = \sum_{i=n+1}^{\infty} \mu^*(A_i) \to 0$

i.e. $\mathfrak{M}_F(\mu) \ni B_n \to A$. But $\mathfrak{M}_F(\mu) = \overline{\mathcal{E}}$ is closed $\Rightarrow A \in \mathfrak{M}_F(\mu)$.

※

~~Cor:~~ μ^* is countably additive on $\mathfrak{M}(\mu)$.

say, $A = \coprod A_n$. If $\mu^*(A_n) < \infty$ $\forall n$, then $A_n \in \mathfrak{M}_F(\mu)$

and (**) holds. otherwise get $\infty = \infty$ ※

It remains to prove $\mathfrak{M}(\mu)$ is a σ-ring.

$A_n \in \mathfrak{M}(\mu) \Rightarrow \bigcup A_n \in \mathfrak{M}(\mu)$ by easy diagonal process.

Let $A = \bigcup A_n$, $B = \bigcup B_n$; $A_n, B_n \in \mathfrak{M}_F(\mu)$

$\Rightarrow A_n \cap B = \bigcup_{i=1}^{\infty} (A_n \cap B_i) \in \mathfrak{M}(\mu)$ in fact $\in \mathfrak{M}_F(\mu)$

since $\mu^*(A_n \cap B) \le \mu^*(A_n) < \infty$.

$\Rightarrow A - B = \bigcup_{n=1}^{\infty} (A_n - B) \in \mathfrak{M}(\mu)$.

" $A_n \setminus (A_n \cap B)$ in $\mathfrak{M}_F(\mu)$ ※

From now on, we write $\mu(A) := \mu^*(A)$ when $A \in \mathfrak{M}(\mu)$.

the case $\mu = m$ is the Lebesgue measure on $\mathbb{R}^p$.

Ex. Borel sets: A open $\Rightarrow A \in \mathfrak{M}(\mu)$ $\rightsquigarrow$ $\mathcal{B} \subset \mathfrak{M}(\mu)$

the smallest σ-ring containing open sets.

$A \in \mathfrak{M}(\mu)$, $\varepsilon > 0$, $\exists$ $\underset{\text{closed}}{F} \subset A \subset \underset{\text{open}}{G}$

st. $\mu(G - A) < \varepsilon$ (by def'n of μ^*); $\mu(A - F) < \varepsilon$ (take complement)

if $F, G \in \mathcal{B}$ can even make $\mu(G - A) = \mu(A - F) = 0$

hence $A = F \cup (A - F)$ and $\mu(A - F) = 0$. Q: $\exists A \in \mathfrak{M}(\mu)$ but not in $\mathcal{B}$?

4/10. Measure space $(X, \mathcal{M}, \mu)$ $\quad \mathcal{M}$ σ-ring, $\mu \geq 0$, σ-additive

X measurable space if $X \in \mathcal{M}$. eg. $X = \mathbb{Z}$. counting m.

$f \in M(X)$ if $\{x \mid f(x) > a\} \equiv f^{-1}(a, \infty] \in \mathcal{M}$ $\quad (f : X \to \mathbb{R} \cup \{\pm\infty\})$

So, if X has a top. conti $\Rightarrow$ measurable ~~hence all Borel sets~~

~~Fact: TFAE~~: $\forall a \in \mathbb{R}$: $f^{-1}(a,\infty]$, $f^{-1}[a,\infty]$, $f^{-1}[-\infty, a)$, $f^{-1}[-\infty, a] \in \mathcal{M}$

$f \in M(X) \Rightarrow |f| \in M(X)$: $\quad |f(x)| < a \Leftrightarrow f(x) < a$ & $f(x) > -a$.

Thm: $f_n \in M(X) \Rightarrow$ ~~$g(x) := \sup_n f_n(x)$, $h(x) = \lim\sup_{n\to\infty} f_n(x)$~~ $\in M(X)$.

in particular, $\max(f, g)$, $\min(f, g)$, f^+, f^- all $\in M(X)$

~~pf:~~ $g^{-1}(a, \infty] = \bigcup_{n=1}^\infty f_n^{-1}(a, \infty]$, $h(x) = \inf_m \sup_{n \geq m} f_n(x)$ $\quad$ *

Thm: $F \in C(\mathbb{R}^2)$, $f, g \in M(X) \Rightarrow h(x) := F(f(x), g(x)) \in M(X)$,

in particular, $f+g$, $fg \in M(X)$.

In Apostol's approach, this is trivial!

~~Caution~~: But not for $f \circ g$ (with $f \in M$, $g \in C$) since in our def'n $f^{-1}(\mathcal{B}) \notin \mathcal{B}$!

~~pf:~~ $\{f(x) + g(x) > a\} = \bigcup_{r \in \mathbb{Q}} \left(\{f(x) > r\} \cap \{g(x) > a - r\}\right)$

eg. $\Big($ $f^2 > a \Leftrightarrow f > \sqrt{a}$ or $f < -\sqrt{a}$

$fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$. ~~The general case~~ use similar idea:

let $G_a = F^{-1}(a, \infty)$ open in $\mathbb{R}^2$, $G_a = \bigcup_{n=1}^\infty I_n$ - open intervals $(a_n, b_n) \times (c_n, d_n)$

$x : (f(x), g(x)) \in I_n \Leftrightarrow f(x) \in (a_n, b_n)$ & $g(x) \in (c_n, d_n)$

measurable, hence $\{x \mid h(x) > a\} = \bigcup_{n=1}^\infty \{x \mid (f(x), g(x)) \in I_n\} \in \mathcal{M}$ $\quad$ *

~~Rmk:~~ $M(X)$ depends only on $\mathcal{M}$ (σ-ring), not on μ.

~~Eg~~ $s = \sum_{i=1}^n c_i \chi_{E_i}$ (simple function) $\in M(X) \Leftrightarrow E_i \in \mathcal{M}$.
$\qquad c_i \neq c_j$

Thm. Any $f : X \to \mathbb{R}$, $\exists s_n$ simple st. $s_n(x) \to f(x)$ $\forall x \in X$

if $f \in M(X)$ then can choose $s_n \in M(X)$, $f \geq 0$ may get $s_n \nearrow$.

~~pf:~~ let $f \geq 0$, $F_n = f^{-1}[n, \infty)$, $E_{ni} := f^{-1}[\frac{i-1}{2^n}, \frac{i}{2^n})$ $\quad i = 1, \cdots, n \cdot 2^n$

Put $s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{ni}} + n \chi_{F_n}$. In general, $f = f^+ - f^-$.

~~Rmk:~~ the ~~conv. is unif. if f is bounded~~ *.

Def'n: $f \in M(X) \geq 0$, $E \in \mathcal{M}$, $\int_E f \, d\mu := \sup_{0 \leq s \leq f} I_E(s)$

where $I_E(s) = \sum_{i=1}^n c_i \mu(E \cap E_i)$. $\qquad$ simple measurable

if $f \in M(X)$, $\int_E f \, d\mu := \int_E f^+ d\mu - \int_E f^- d\mu$ if ~~at least one is finite~~.

if both finite. get $f \in \mathcal{L}(\mu)$ or L if $\mu = m$. (otherwise $= +\infty$ or $-\infty$)

Thm : let $f \in M(X) \geq 0$ (or $\mathcal{L}(\mu)$) then

$\phi(A) := \int_A f \, d\mu$ is σ-additive on $A \in \mathcal{M}$.

~~pf:~~ $f = \chi_E$, simple ($\Leftarrow$ μ : σ-additive) $\underset{\coprod A_n}{\phi(A)} = \sum_{n=1}^{\infty} \phi(A_n)$

general $f \in M(X) \geq 0$, $\forall\, 0 \leq s \leq f$

$$\int_A s \, d\mu = \sum_{n=1}^{\infty} \int_{A_n} s \, d\mu \leq \sum_{n=1}^{\infty} \phi(A_n) \Rightarrow \phi(A) \leq \sum \phi(A_n)$$

Suppose $\phi(A_n) < \infty$ $\forall n$. (otherwise trivial.)

Given $\varepsilon > 0$, $\exists\, 0 \leq s \leq f$ st. $\int_{A_1} s \, d\mu \geq \int_{A_1} f - \varepsilon$, $\int_{A_2} s > \int_{A_2} f - \varepsilon$

$\Rightarrow \phi(A_1 \cup A_2) \geq \int_{A_1 \cup A_2} s = \int_{A_1} s + \int_{A_2} s \geq \phi(A_1) + \phi(A_2) - 2\varepsilon$

let $\varepsilon \to 0$ get $\underset{\phi(A) \geq}{\phi(A_1 \cup \cdots \cup A_n)} \geq \phi(A_1) + \cdots + \phi(A_n)$ $\forall n$ ✻

~~Cor~~ If $B \subset A$ in $\mathcal{M}$, $\mu(A-B) = 0$, $\int_A f = \int_B f + \underset{0}{\int_{A-B} f}$.

~~Cor~~ $f \in \mathcal{L}(\mu) \Rightarrow |f| \in \mathcal{L}(\mu)$ & $|\int_E f| \leq \int_E |f|$.

pf: $\int_E |f| = \int_A f^+ + \int_B f^- < +\infty$ where $f \geq 0$ on A, $f < 0$ on B ✻

~~Cor:~~ $f \in M(E)$, $|f| \leq g \in \mathcal{L}(\mu, E) \Rightarrow f \in \mathcal{L}(\mu, E)$. (Since $f^{\pm} \leq g$).

Lebesgue's monotone conv. thm (cf. Levi)

$E \in \mathcal{M}$, $0 \leq f_n \nearrow$ on E. Let $f_n(x) \to f(x) \in [0, \infty]$, $\Rightarrow \int_E f_n \to \int_E f$.

~~pf:~~ $\int_E f_n \to \alpha \in [0, \infty]$. $\int f_n \leq \int f \Rightarrow \alpha \leq \int f$

consider $0 \leq s \leq f$. $0 < c < 1$. $E_n = \{x \mid f_n(x) \geq c\, s(x)\} \nearrow E$

$$\int_E f_n \geq \int_{E_n} f_n \geq c \int_{E_n} s \Rightarrow \alpha \geq c \int_E s \quad \text{(why?)}$$

let $c \to 1$ get $\alpha \geq \int_E s$. take sup $\Rightarrow \alpha \geq \int f$ ✻.

Thm (linearity of $\int d\mu$) $f_1, f_2 \in \mathcal{L}(\mu, E) \Rightarrow f = f_1 + f_2 \in \mathcal{L}(\mu, E)$

and $\int f = \int f_1 + \int f_2$.

~~pf:~~ let $f_1, f_2 \geq 0$, if f_i simple then ok by def^n.

otherwise $s_n' \nearrow f_1$, $s_n'' \nearrow f_2$ then $s_n = s_n' + s_n'' \nearrow f$.

- let $f_1 \geq 0$, $f_2 \leq 0$. Let $A = \{x \mid f(x) \geq 0\}$, $B = \{x \mid f(x) < 0\}$

then $\int_A f_1 = \int_A f + \int_A (-f_2) = \int_A f - \int_A f_2$ (why "$-$"?

$\int_B (-f_2) = \int_B f_1 + \int_B (-f)$. subtract ✻ by def^n!)

in general, decomp E to $E_1 \amalg E_2 \amalg E_3 \amalg E_4$, f_i has sign, sum up.

~~Cor / Thm~~ Levi-Lebesgue in series form.

As before $\Rightarrow$ Fatou's lemma $\Rightarrow$ Lebesgue DCT.

Thm ~~(Comp with Riem int)~~

$f \in \mathcal{R}[a,b] \Rightarrow f \in \mathcal{L}[a,b]$ & $\int_{[a,b]} f\, d\mu = \int_a^b f$

this is the case iff f is conti a.e.

~~Pf:~~ Let $P_k \in \mathcal{P}[a,b]$, $P_k \subset P_{k+1}$, $|P_k| \to 0$; f bdd on $[a,b]$.

$m_i = L_k(x) \le f(x) \le U_k(x) = M_i$ on $(x_i, x_{i+1}]$

$\nearrow \qquad \searrow$ $k \to \infty$ get $L(x) \le f(x) \le U(x)$

$\int L\, d\mu = \underline{\int} f$, $\int U d\mu = \overline{\int} f$ $\quad$ L, U bdd measurable $f \in M$.

by monotone conv. thm.

so $f \in \mathcal{R} \Leftrightarrow \int L d\mu = \int U d\mu$ ie. $\int (U-L)\, d\mu = 0$, $U \overset{a.e.}{=} L$

Now $x \notin \bigcup P_k$ (countable) & $U(x) = L(x) \Leftrightarrow f$ conti at x (why? Ex 1.) ∗

Thm ~~(FTC)~~ (1) $f \in \mathcal{L}[a,b]$, $F(x) := \int_a^x f\, d\mu \Rightarrow F' = f$ a.e.

(2) F' exists $\forall x \in (a,b]$ & $F' \in \mathcal{L}$, then $\int_a^x F' d\mu = F\big|_a^x$.

~~Int on complex functions~~: $f = u + iv \in M(x) \underset{def}{\Longleftrightarrow} u, v \in M(x)$.

$\Rightarrow |f| = (u^2+v^2)^{1/2} \in M(x)$ (alg. op)

Def'n: $f \in \mathcal{L}(\mu) \triangleq |f| \in \mathcal{L}(\mu) \Longleftrightarrow u, v \in \mathcal{L}(\mu)$. $\int f := \int u + i \int v$.

eg. Cauchy ineq'n: $|\int f| \le \int |f|$: let $|c| = 1$, $c\int f \ge 0$. $g = cf = u + iv$

$\mathcal{L}^2(\mu)$ space, $\langle f, g\rangle := \int f\bar{g}$, <u>Hilbert space</u> ($\Leftarrow$ <u>Riecz-Fischer</u>)

Def'n: ~~complete basis~~: $f \in \mathcal{L}^2$, $(f, \phi_n) = 0\ \forall n \Rightarrow \|f\| = 0$, ie in the funct case, $f = 0$ a.e.

Thm (Parseval)

On $[-\pi, \pi]$, e^{inx}, $n \in \mathbb{Z}$ is complete. Namely, $f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}$

$f \in \mathcal{L}^2$, partial sum S_n. Then $\|f - S_n\|^2 \to 0$

$+ \|S_n\|^2 = \|f\|^2$

$\| \qquad \Downarrow$

~~Pf:~~ Given $\varepsilon > 0$, "$\overline{C} = \mathcal{L}^2$" $\quad \sum_{k=-n}^{n} |c_n|^2$ $\quad$ $n \to \infty$ becomes "=".

$\exists g \in C$ st. $\|f - g\| < \frac{\varepsilon}{2}$

adjust $g(\pi) = g(-\pi)$

$\Rightarrow \exists$ triangular poly T of degree N

st. $\|g - T\| < \frac{\varepsilon}{2}$ (Fejér)

Now $n \ge N \Rightarrow \|S_n - f\| \le \|T - f\| < \varepsilon$ ∗

Lemma: $\underline{\overline{C} = \mathcal{L}^p, p \ge 1}$ <u>(May assume $f \ge 0$)</u>

~~Pf:~~ $A \subset [a,b]$ closed, $g_n(x) = \dfrac{1}{1 + n\, d(x, A)}$ $\begin{cases} = 1 & x \in A \\ \to 0 & x \notin A \end{cases}$ $\Rightarrow \|g_n - \chi_A\|^p$

hence for χ_E, $E \in M$, hence simple $S_n \nearrow f$, use DCT ∗ $= \int_{[a,b]\setminus A} g_n^p \to 0$

4/19. FTC. Lebesgue version

Lemma. $f \in L[a,b]$, then $F(x) = \int_a^x f(t)\,dt$ is absolut conti.

~~pf:~~ $|F(x) - F(y)| = |\int_y^x f| \le |x-y| \cdot M$ if f is bounded.

in general, let $f_n = \min(|f|, n) \longrightarrow |f|$

$\forall \varepsilon > 0$, $\exists N$ st. $\int_a^b f_n + \frac{\varepsilon}{2} > \int_a^b |f|$, i.e. $\int_a^b (|f| - f_n) < \frac{\varepsilon}{2}$ $\forall n \ge N$

Now take $\delta < \frac{\varepsilon}{2N}$ and $|x-y| < \delta$ (or any $m(A) < \delta$)

then $\int_A |f| \le \int_A (|f| - f_N) + \int_A f_N < \frac{\varepsilon}{2} + \frac{\varepsilon}{2N} \cdot N = \varepsilon$.

in fact we have proved F is ~~absolutely conti.~~ ($\Rightarrow BV$) ※

Lemma. $f \in L[a,b]$, $F(x) \equiv 0 \Rightarrow f = 0$ a.e. on $[a,b]$.

~~pf:~~ If $f > 0$ on E, $m(E) > 0$, $\exists$ closed $F \subset E$, $m(F) > 0$

then $U := [a,b] - F = \coprod_n (a_n, b_n)$

$\int_U f = -\int_F f < 0$, σ-additive $\Rightarrow \int_{(a_n,b_n)} f < 0$ for some n.

but then $F(b_n) \ne F(a_n)$, can't be both 0 ※

Lemma. If $f \in M[a,b]$ & bdd ($\Rightarrow \in L$), then $F' = f$ a.e.

~~pf:~~ $F \in BV[a,b]$ so $F = u - v$, $u, v \nearrow$, u', v' exists a.e.

$\Rightarrow F' = u' - v'$ exists a.e. on $[a,b]$. ※ the most difficult!

let $|f| \le K$. Define $f_n(x) := \dfrac{F(x+h) - F(x)}{h}$, $h := \frac{1}{n}$

then $|f_n| \le K$. $= \frac{1}{h}\int_x^{x+h} f(t)\,dt$ still abs. cont.

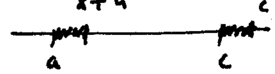
Now $f_n \longrightarrow F'$ a.e. in particular, $F' \in L[a,b]$.

and (Lebesgue's bounded conv thm)

$$\int_a^c F' = \lim_{n\to\infty} \int_a^c f_n(x)\,dx$$

$$= \lim_{h\to 0} \frac{1}{h}\int_a^c \big(F(x+h) - F(x)\big)\,dx$$

$$= \lim_{h\to 0}\left(\frac{1}{h}\int_c^{c+h} F - \frac{1}{h}\int_a^{a+h} F\right)$$



since $F \in C$ $\searrow$ $= F(c) - F(a) = \int_a^c f$ $\quad \forall c \in [a,b]$

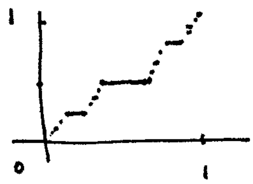
$\Rightarrow F' = f$ a.e. ※

* Thm: let $f \nearrow$ on $[a,b]$, then f' exists a.e.

f' is measurable, and $\int_a^b f'(x)\,dx \le f(b) - f(a)$.

eg. ~~Cantor-Lebesgue function~~:

$f' = 0$ a.e. since $m(C) = 0$



$C = \bigcap_{i=1}^\infty C_i$ closed

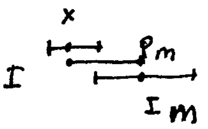
$m(C_i) = \left(\frac{2}{3}\right)^i \to 0$

Def^{n}: $\mathcal{G}$ Vitali cover of E, if $\forall \varepsilon>0$, $x\in E$, $\exists\ x\in I\in\mathcal{G}$, $|I|<\varepsilon$.
collection of intervals of any type ($\neq$ pt)

Lemma (Vitali). $\mathcal{G}$ Vitali cover of E, $m^*(E)<\infty$,
then $\forall\varepsilon>0$, $\exists\{I_1,\cdots,I_N\}\subset\mathcal{G}$ st. $\underline{m^*(E\setminus \amalg_{i=1}^N I_i)<\varepsilon}$
disjoint intervals

~~pf:~~ May assume all $I\in\mathcal{G}$ are closed.
let $E\subset U$ open, $m(U)<\infty$, may assume all $I\subset U$.
let I_1 be any. Assume $I_1\amalg\dots\amalg I_n$ is chosen
$k_n := \sup |I|$ for $I\in\mathcal{G}$, $I\cap I_i=\phi$ $\forall i=1\dots n$.
If $E\subset\amalg_{i=1}^n I_i$ then done, otherwise such I exists
and we may choose I_{n+1} st $|I_{n+1}|>\frac12 k_n$.
Get $I_1, I_2,\cdots$, $\sum|I_i|\le m(U)<\infty$, in parti, $|I_n|\to 0$.
say $\underline{\sum_{n=N+1}^\infty |I_n|<\frac{\varepsilon}{5}}$, let $R:=E-\bigcup_{n=1}^N I_n$.

claim: $m^*(R)<\varepsilon$.
$x\in R$, $\exists I\in\mathcal{G}$, $I\cap\amalg_1^N I_i=\phi$. If $I\cap\amalg_1^n I_i=\phi$, or $|I|\le k_n<2|I_{n+1}|$
then $I\cap I_n\neq\phi$ for some n, let $\underset{N}{\underset{\vee}{m}}$ be the 1st such integer.
then $|I|<k_{m-1}\le 2|I_m|$
$|x-p_m|\le |I|+\frac12|I_m|\le\frac52|I_m|$



i.e. $R\subset\bigcup_{m=N+1}^\infty B_{\frac52|I_m|}(p_m)$ $\Rightarrow m^*(R)\le 5\cdot\sum_{m=N+1}^\infty|I_m|<\varepsilon$ ※

pf of Thm *: There are $D^+f(x)$, $D_+f(x)$, $D^-f(x)$, $D_-f(x)$,
$f'(x)$ exists $\Leftrightarrow$ all four equal. $\underline{\text{Will show eg. } D^+f(x)>D_-f(x) \text{ has } m^*=0}$.
Enough to show $E_{u,v}$ of $\{x \mid D^+f(x)>u>v>D_-f(x)\}$ has $m^*=0$. $\forall u>v\in\mathbb{Q}$
Let $s=m^*(E_{u,v})$, $\epsilon>0$, $\exists U\supset E_{u,v}$, $m(U)<s+\epsilon$.
$\forall x\in E_{u,v}$ $\underline{\exists [x-h, x]\subset U}$ st. $f(x)-f(x-h)<vh$ ($h>0$) could be small $\to 0$
Lemma $\Rightarrow\exists$ such $I_1,\cdots,I_N$ st. $m^*(\amalg_1^N I_i\cap E)>s-\epsilon$ and
$=: A$
$\sum_{n=1}^N\left(f(x_n)-f(x_n-h_n)\right)<v\sum h_n<v(s+\epsilon)$.
could $\to 0$
$\forall y\in A$, $\underline{\exists (y,y+k)\subset I_n}$ for some n & $f(y+k)-f(y)>uk$ ($k>0$)
$\ast x_n$
Lemma $\Rightarrow\exists J_1,\cdots,J_M$ st. $\amalg_1^M J_i\cap A$ has $m^*>(s-\epsilon)-\epsilon=s-2\epsilon$.
$\sum_{i=1}^N\left(f(y_i+k_i)-f(y_i)\right)>u\sum k_i>u(s-2\epsilon)$.
set $f(x)=f(b)$ for $x\ge b$.
$\wedge$ $\leftarrow$ $\underline{\text{since } f\nearrow}$ $\underset{\epsilon\to0}{\Rightarrow}$ $vs\ge us$ so $u>v\Rightarrow s=0$.
$v(s+\epsilon)$

So, $g_n(x):=\left(f(x+1/n)-f(x)\right)/\frac1n\ \ge 0\ \xrightarrow[\text{a.e.}]{} g(x):=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}\Rightarrow g=f'\in M[a,b]$
Fatou $\Rightarrow\int g\le\underline{\lim}\int g_n=\underline{\lim}\left(n\int_b^{b+1/n}f-n\int_a^{a+1/n}f\right)\le f(b)-f(a)\Rightarrow f'\underset{\text{a.e.}}{<}\infty$ ※

Lebesgue's version of FTC (conti). 4/26

Thm. $f \in L[a,b]$, $F(x) := \int^x f \Rightarrow F' = f$ a.e.

~~pf. Idea: reduce to bounded case.~~ May assume $f \geq 0$.

$f_n := \min(f, n) \nearrow f$

$g_n(x) := \int_a^x (f - f_n) \nearrow \Rightarrow g_n'$ exists a.e. ≥ 0

$\Rightarrow F'(x) = \underset{\overset{\vee\!\vee}{0}}{g'(x)} + \frac{d}{dx}\int_a^x f_n \underset{a.e.}{=} g'(x) + f_n(x) \underset{a.e.}{\geq} f_n(x), \ \forall n$

$\Rightarrow F' \geq f$ a.e. $\Rightarrow \int_a^b F' \geq \int_a^b f = F(b) - \underset{\overset{\vee\!\vee}{0}}{F(a)}$

but $F \nearrow$, hence $\int_a^b F' \leq F(b) - F(a)$, get equality

i.e. $\int_a^b \underset{\overset{\vee\!\vee}{0}}{(F' - f)} = 0 \Leftrightarrow F'(x) = f(x)$ a.e. ※

Thm/Cor. $F(x)$ is an indefinite integral $\Leftrightarrow$ F is abs. conti.

The pf requires one more crucial fact:

Lemma: f abs. conti on $[a,b]$ & $f' = 0$ a.e. $\Rightarrow f = \text{const}$.

~~pf:~~ let $c \in [a,b]$, $E \subset (a,b)$ be the set $f'(x) = 0$.

let $\eta, \varepsilon > 0$ arbitrary.

$x \in E \Rightarrow \exists [x, x+h] \subset [a,c]$ st. $|f(x+h) - f(x)| < \eta h$

$\forall \delta > 0$, Vitali $\Rightarrow \exists$ finite disj. $\{[x_k, y_k]\}_{k=1}^n$, $m^*(E - \coprod_1^n [x_k, y_k]) < \delta$.

For later use, here δ is chosen to be the module of ε-abs. conti of f.

we label $y_0 = a \leq x_1 < y_1 \leq x_2 < y_2 \cdots \leq y_n \leq c = x_{n+1}$

then $\sum_{k=0}^n |x_{k+1} - y_k| < \delta$, $\sum_{k=0}^n |f(x_{k+1}) - f(y_k)| < \varepsilon$

But $\sum_{k=1}^n |f(y_k) - f(x_k)| \leq \eta \sum_{k=1}^n (y_k - x_k) < \eta(c-a)$

$\Rightarrow |f(c) - f(a)| \leq \varepsilon + \eta(c-a)$. let $\varepsilon, \eta \to 0$ get $f(c) = f(a)$ ※

~~pf of Thm/Cor:~~ $\Rightarrow$ is done.

$\Leftarrow$: F abs. conti $\Rightarrow$ BV $\Rightarrow F = F_1 - F_2$, $F_i \nearrow$. F_i' exists a.e. and $\in L[a,b]$. (Thm *)

$\Rightarrow F' = F_1' - F_2' \in L[a,b]$.

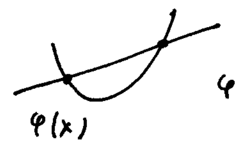
let $G(x) = \int_a^x F'(t)\,dt$ abs. conti $\Rightarrow H := F - G$ is abs. conti.

$H' = F' - G' = F' - F' = 0$ a.e. $\Rightarrow H = \text{const}$. ※

Applications to convex functions 5/1

Def'n: $\varphi(\lambda x+\mu y) \le \lambda\varphi(x)+\mu\varphi(y)$

$\forall \lambda,\mu \ge 0,\ \lambda+\mu=1$



$\varphi(x)$ $\varphi(y)$ Fig 1.

fact: the slope ↗ in both x and y.

e.g. $x \quad y \quad y_1$ $\quad y=\lambda x+\mu y_1 \Rightarrow \frac{\varphi(y_1)-\varphi(x)}{y_1-x} - \frac{\varphi(y)-\varphi(x)}{y-x}$

$y-x=\mu(y_1-x)$ $\quad \ge \frac{1}{y_1-x}\left(\mu\varphi(y_1)-\mu\varphi(x)-\lambda\varphi(x)-\mu\varphi(y)+\varphi(x)\right)$

$= 0$.

Prop: φ convex on (a,b) ⇒ abs. conti on all $[c,d] \subset (a,b)$

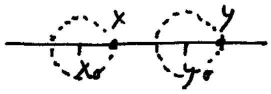
$D^+\varphi = D_+\varphi$ ↗, $D^-\varphi = D_-\varphi$ ↗. Also φ' exists except a countable set.

pf: $\varphi(y)-\varphi(x) \le (y-x)\cdot\frac{\varphi(b)-\varphi(d)}{b-d}$ ⇒ Lipschitz ⇒ abs. conti.

As an ↗ funct. $g(y)=\frac{\varphi(y)-\varphi(x_0)}{y-x_0}$ has limits $g(x_0+)$, $g(x_0-)$.

i.e. $D^+\varphi=D_+\varphi$, $D^-\varphi=D_-\varphi$ are well-defined.

Now $\frac{\varphi(x)-\varphi(x_0)}{x-x_0} \le \frac{\varphi(y)-\varphi(y_0)}{y-y_0}$ for



⇒ $D^\pm\varphi(x_0)\le D^\pm\varphi(y_0)$ ⇒ $D^\pm\varphi$ are conti outside a countable set.

At a conti pt x_0 of say $D^-\varphi$, then for $0<\delta_1<\delta$:

$D^-\varphi(x_0-\delta)\le D^+\varphi(x_0-\delta_1)\le D^+\varphi(x_0)\le D^+\varphi(x_0+\delta_1)\le D^-\varphi(x_0+\delta)$

$\delta\to 0$ ⇒ $D^+\varphi$ also conti at x_0 and $\varphi'(x_0)$ exists. ※

Prop (Converse): $\varphi\in C(a,b)$ and $D^+\varphi$ exists ↗, then φ is convex.

pf: Idea: using Rollé argument. (cf. Fig 1.)

for any $a<x<y<b$, let $\psi(t)=\varphi((1-t)x+ty)-(1-t)\varphi(x)-t\varphi(y)$

$\psi(0)=\psi(1)=0$. Claim $\psi\le 0$ on $t\in[0,1]$.

let $\psi(\gamma)=\max$. $D^+\psi=(y-x)D^+\varphi+\varphi(x)-\varphi(y)$ ↗

If $\gamma=1$, done. otherwise $D^+\psi(\gamma)\le 0$ ⇒ $D^+\psi\le 0$ on $[0,\gamma]$

i.e. (Prop 2. Ex.6) ⇒ ψ ↘ ⇒ $\psi\equiv 0$ on $[0,\gamma]$ and $\psi\le 0$ on $[0,1]$

in fact if we take γ the largest max pt. then $\gamma=1$ ※

Cor: If φ'' exists on (a,b) then φ convex ⟺ $\varphi''\ge 0$.

Thm (Jensen inequality)

let φ convex on $(-\infty,\infty)$, $f\in L[0,1]$, then $\int\varphi(f)\ge\varphi\int f$.

pf: $\varphi(f)\in M[0,1]$ since $\varphi\in C$. Let $\alpha=\int f$

$y=m(x-\alpha)+\varphi(\alpha)$ a supporting line of φ at $(\alpha,\varphi(\alpha))$

i.e. $D^-\varphi(\alpha)\le m\le D^+\varphi(\alpha)$.

⇒ $\varphi(f(t))\ge m(f(t)-\alpha)+\varphi(\alpha)$

φ  m

$\int$ ⇒ Thm. ※

Classical ~~Banach spaces~~

$L^p[0,1]$, $0<p<\infty$ $f\in M[0,1]$, $\int |f|^p<\infty$

VS since $|f+g|^p \le |2f|^p + |2g|^p$

$\|f\|_p := \left(\int |f|^p\right)^{1/p}$ so $\|c\cdot f\| = |c|\cdot\|f\|$

if $0<p<1$: $\Rightarrow$ No 2^p, metric space

But for $p=\infty$ $\|f\|_\infty := \text{ess.sup}\,|f|$. L^∞ = "bad" $M[0,1]$

Ex. $\|f\|_p \xrightarrow[p\to\infty]{} \|f\|_\infty$.

Thm (Minkowski inequality) norm cond.

$1\le p\le\infty \Rightarrow \|f+g\|_p \le \|f\|_p + \|g\|_p$

~~pf:~~ let $\alpha = \|f\| \ne 0$, $\beta = \|g\|\ne 0$

$$\int |f+g|^p \le \int \left|\alpha\frac{|f|}{\alpha} + \beta\frac{|g|}{\beta}\right|^p \le (\alpha+\beta)^p \int \left|\frac{\alpha}{\alpha+\beta}\frac{|f|}{\alpha} + \frac{\beta}{\alpha+\beta}\frac{|g|}{\beta}\right|^p$$

$$\le (\alpha+\beta)^p \int \frac{\alpha}{\alpha+\beta}\frac{|f|^p}{\alpha^p} + \frac{\beta}{\alpha+\beta}\frac{|g|^p}{\beta^p} = (\alpha+\beta)^p.$$

= holds $\Leftrightarrow$ $f = cg$, $c>0$ (clearly).

Thm (Hölder inequality) $p,q\in\mathbb{R}^+\cup\{\infty\}$, $\frac{1}{p}+\frac{1}{q}=1$ dual number

$f\in L^p$, $g\in L^q \Rightarrow fg\in L^1$ & $\int |fg| \le \|f\|_p\|g\|_q$.

~~pf:~~ concavity of log $\Rightarrow u^{1/p}v^{1/q} \le \frac{u}{p}+\frac{v}{q}$ (Ex. 2 #8) ※

Prop. A metric V.S. ~~$(V,\|\cdot\|)$ complete $\Leftrightarrow$ abs. conv. series converges~~

pf: $\Rightarrow$ Routine: f_n abs. conv. series $\Rightarrow$ S_n is Cauchy.

$\Leftarrow$: f_n Cauchy, $\|f_n - f_m\| < \frac{1}{2^k}$ $\forall n,m \ge n_k$, $n_k\nearrow$.

Set $g_1 = f_{n_1}, \dots, g_k = f_{n_k} - f_{n_{k-1}}$ $\Rightarrow$ $\sum\|g_k\| \le \|g_1\| + 1$

$\Rightarrow$ $S_k(g) := g_1+\dots+g_k = f_{n_k} \to f$

So $\|f_n - f\| \le \|f_n - f_{n_k}\| + \|f_{n_k} - f\| \to 0$ ※

Thm (Riesz–Fischer): L^p is complete $\forall\, 0<p\le\infty$. ($p=\infty$ in Ex.)

~~pf:~~ let $f_n\in L^p$, $\sum_{n=1}^\infty \|f_n\| = M<\infty$.

$g_n(x) := \sum_{k=1}^n |f_k(x)| \in L^p$ $\Rightarrow$ $\|g_n\| \le \sum_{k=1}^n \|f_k\| \le M$

ie. $\int g_n^p \le M^{\max(p,1)}$. let $g_n(x)\nearrow g(x)$

Fatou (or Levi) $\Rightarrow$ $g^p\in L^1$. so $g(x)<\infty$ a.e.

$g(x)<\infty \Rightarrow \sum_{k=1}^\infty f_k(x) \xrightarrow[\text{abs.}]{} s(x)$. Set $s(x)=0$ if $g(x)=\infty$.

so $S_n(x) := \sum_{k=1}^n f_k(x) \xrightarrow[\text{a.e.}]{} s(x) \in M[0,1]$.

$|S_n(x)| \le g_n(x) \le g \Rightarrow |s(x)| \le g(x) \Rightarrow s\in L^p$.

so $|S_n(x) - s(x)|^p \le \underline{2\cdot 2^p\cdot g(x)^p} \in L^1$, (if $0<p<1$, 2^p not needed)

Lebesgue DCT $\Rightarrow$ $\int |S_n - s|^p \xrightarrow[n\to\infty]{} 0 \Rightarrow \|S_n - s\|\to 0$ in L^p ※

(5/10) ~~conti linear functional~~ on L^p, $1 \le p \le \infty$ (Easy for Hilbert)

On a normed space $(X, \|\cdot\|)$, $F: X \to \mathbb{R}$ bdd $\Leftrightarrow$ conti.

Q: why consider X^*?

$\|F\| := \sup_{f \ne 0} \frac{|F(f)|}{\|f\|}$. Always let $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \in [1, \infty]$

Prop: $g \in L^q \Rightarrow F: L^p \to \mathbb{R}$ by $f \mapsto \int fg$ is conti, $\|F\| = \|g\|_q$.

~~pf:~~ Hölder $\Rightarrow \|F\| \le \|g\|_q$. Consider $f = |g|^{q/p} \cdot \mathrm{sgn}(g)$

(Ex. $p=1,\infty$) $\Rightarrow F(f) = \int |g|^{\frac{q}{p}+1} = q = \|g\|_q^q = \|g\|_q^1 \cdot \|g\|_q^{q/p} = \|g\|_q \cdot \|f\|_p$.

Key lemma: If ~~$g \in L[0,1]$~~, st. $\forall$ ~~bdd $f \in M[0,1]$~~, $\exists M$, $1 \le p < \infty$,
$|\int fg| \le M \|f\|_p$ then $g \in L^q$ and $\|g\|_q \le M$.

~~pf: If $1 < p < \infty$~~, $g_n(x) := \begin{cases} g(x) & |g(x)| \le n \\ 0 & \text{otherwise} \end{cases}$

$f_n := |g_n|^{q/p} \cdot \mathrm{sgn}(g_n)$

$q \Rightarrow \int f_n g \le M \|f_n\|_p = M \|g_n\|_q^{q/p} \Rightarrow \|g_n\|_q \le M$.

$|g_n|^{1+\frac{q}{p}} = f_n \cdot g_n$, $\|g_n\|_q^q$ $\Rightarrow \int |g_n|^q \le M^q$

$|g_n|^q \nearrow_{a.e.} |g|^q$ Fatou $\Rightarrow \int |g|^q \le \liminf \int |g_n|^q \le M^q$.

~~The case $p=1$~~: let $E = \{x \mid |g(x)| \ge M + \varepsilon\}$, $f = (\mathrm{sgn}\, g) \cdot \chi_E$.

$m(E) \cdot (M+\varepsilon) \le |\int fg| \le M \|f\|_1 = M \cdot m(E) \Rightarrow m(E) = 0$, $\|g\|_\infty \le M$ ※

Thm (Riesz Repr) $F: L^p \to \mathbb{R}$ bdd, $1 \le p < \infty \Rightarrow F(f) = \int fg$
for some $g \in L^q$ (and hence $\|F\| = \|g\|_q$).

~~pf:~~ let $\Phi(x) := F(\chi_{[0,x]})$. ~~claim: $\Phi(s)$ is abs. conti~~.

if $\sum_i (y_i - x_i) < \delta$. let $f = \sum_i (\chi_{[0,y_i]} - \chi_{[0,x_i]}) \cdot \mathrm{sgn}(\Phi(y_i) - \Phi(x_i))$

then $\sum_i |\Phi(y_i) - \Phi(x_i)| = F(f) \le \|F\| \cdot \|f\|_p < \|F\| \cdot \delta^{1/p}$ small ※

FTC $\Rightarrow \Phi(x) = \int_0^x g$ $(g = \Phi' \in L) \Rightarrow F(\chi_{[0,x]}) = \int_0^1 g \cdot \chi_{[0,x]}$.

$\Rightarrow F(\psi) = \int_0^1 g\psi$ for $\psi \in S[0,1]$ ~~(step fun)~~.

Now let ~~$f \in M[0,1]$, bdd~~, then $\exists \psi_n \in S[0,1] \xrightarrow{a.e.} f$

$|F(f) - F(\psi_n)| = |F(f - \psi_n)| \le \|F\| \cdot \|f - \psi_n\|_p \longrightarrow 0$ ~~Lebesgue BCT~~.

$\int_0^1 g\psi_n \longrightarrow \int_0^1 gf$ ~~Lebesgue DCT~~. $|g\psi_n| \le |g| \cdot A$

So $F(f) = \int gf$. Lemma $\Rightarrow g \in L^q$, $\|g\|_q \le \|F\|$. approx

Now for ~~$f \in L^p$~~, $\varepsilon > 0$, $\exists \psi \in S[0,1]$ st. $\|f - \psi\|_p < \varepsilon$ (Ex. 14)

$\Rightarrow |F(f) - \int fg| \le |F(f-\psi)| + |\int (\psi - f) g| < (\|F\| + \|g\|_q) \cdot \varepsilon \to 0$

※.