Introduction to Geometric Analysis

- NCU 2005
- NTU 2021

notice that

our convention $\qquad$ P. Li's convention

$B(e_i, e_j) = \nabla^N_{e_i} e_j \qquad B(e_i,e_j) = -\nabla^N_{e_i} e_j$

$\vec{H} = \operatorname{tr} B \qquad \vec{H} = \operatorname{tr} B$

for hypersurface, N = outer normal

$\vec{H} = H(-\vec{N}) \qquad \vec{H} = H\vec{N}$

inner normal

Thus we still have the same H.

* There are 3 preliminary lectures in Spring term before P. Li's Lecture notes:

1. Review of tensor calculus, Bochner formula
2. Lichnerowicz thm on λ_1, intro to eigenvalues
3. Thm of Choi-Wang, intro to heat kernel.

2004-2005 Diff Geom at NCU

Spring Term, part I: geometric analysis

- Peter Li (Lecture notes). (Lect 1. 3/4)

R. Schoen. S-T Yau. $\qquad N^n \hookrightarrow M^m$,

Recall variation of area: let V = var field

$$A(t) = \int_{N_t} dA = \int_N dA_t = \int_N J(x,t)\, dA_0$$

$$\frac{d}{dt} dA_t \Big|_{t=0} = \left[\operatorname{div} \vec{V}^T - \langle \vec{V}^N, \vec{H} \rangle \right] dA_0$$

For normal variation of hyp surface, let $\vec{H} = -H\vec{N}$, $\vec{V} = \varphi \vec{N}$

$\Rightarrow \quad J'(0) = \varphi H J(0)$

$$\frac{d^2}{dt^2} dA_t \Big|_{t=0} = \left[|\nabla\varphi|^2 - \varphi^2\left(|B|^2 + \operatorname{Ric}(N,N)\right)\right] dA_0$$

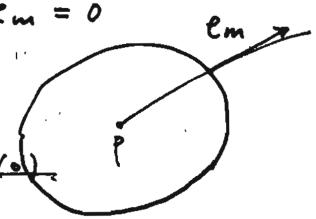
$-\langle (\nabla_{\vec{V}}\vec{V})^N, \vec{H}\rangle \leftarrow +\langle \vec{V}, \vec{H}\rangle^2$ this disappears only for min sub.

Corollary: for N_0 = geod sphere

$V = N = e_m \ (\varphi \equiv 1)$, $\nabla_{e_m} e_m = 0$

get: $J'(0) = H J(0)$, "$J(0) = 1$"

$J''(0) = \left(H^2 - |B|^2 - \operatorname{Ric}(\vec{N}, \vec{N})\right) J(0)$



Def'n: Fix $p \in M$, $Q = \gamma(t)$ is a cut point of p if $\gamma(s)$ is min length $\forall s < t$, but not $s > t$.

Fact: conj pt occurs after cut pt.

Example: M cpt, $C(p)$ exists, $M \setminus C(p) \underset{\text{homeo}}{\sim} \mathbb{R}^n$

but may $\nexists$ conj pt, eg T^m.

indeed, for any space of non-pos curv. (Cartan-Hadamard Thm).

polar coor. $(\theta = (\theta_1, \dots, \theta_{m-1}), \underset{\substack{\shortparallel \\ t}}{r})$, $e_m = \frac{\partial}{\partial r}$

write $dA = J(\theta, r)\, dr \wedge d\theta$

$R_{mm} = Ric(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = R_{rr}$.

$J'' = (H^2 - |B|^2 - R_{rr}) J$ Now "$J(0) \neq 1$"

$\le H^2 (1 - \frac{1}{m-1}) J - R_{rr} J$ since $\dfrac{H^2 \le (m-1)|B|^2}{\text{(last time)}}$

$= \frac{m-2}{m-1} (J')^2 J^{-1} - R_{rr} J$

If $\bar{M}_K$ is a space form, i.e. const sect curv $= K$

then $B = \frac{H}{m-1} I_{m-1}$ and equality holds, with $R_{rr} = (m-1)K$

(Exercise! Hint: Use Gauss' Eq'n)

$\exists$ at least 3 ways to compute $\bar{J}$ (hence $\bar{H}$) explicitly for $\bar{M}_K$. ① is the Exercise.

② In classical geometry (2-dim'l case)

$ds^2 = dr^2 + G\, d\theta^2$; $J = \sqrt{G}$; $K = -\frac{\sqrt{G}_{rr}}{\sqrt{G}}$.

$$\Rightarrow J(r) = \begin{cases} \frac{1}{\sqrt{K}} \sin \sqrt{K} r & K > 0 \\ 1 & K = 0 \\ \frac{1}{\sqrt{-K}} \sinh \sqrt{-K} r & K < 0 \end{cases}$$

using initial condition $J \sim r^{m-1}$, $J' \sim (m-1) r^{m-2}$

since normal coor approx Euclid to 2nd order.

for higher dim $m > 2$, simply take product.

③ Solving ODE ★ (when $m = 2 \mapsto J'' + KJ = 0$),

$J'' = \frac{m-2}{m-1} (J')^2 J^{-1} - (m-1) K J$

Method (Euler or Bernoulli ?):

Set $\underline{f = J^{1/m-1}}$.

$$\Rightarrow \begin{cases} f' = \frac{H}{m-1} f \\ f'' \le -K f \end{cases}.$$

Theorem (Bishop Comparison Theorem):
M complete, $p \in M$ st.

$Ric(x) \ge (m-1) \underline{K(r)}$; $r = r(p, x)$

let $\bar{J}$ be solution of equation ★, $\bar{H} := \bar{J}'/\bar{J}$.

(i.e. we define a space $\bar{M}_K$, $K = K(r)$)

Then, within the cut locus of p:

(i) $J/\bar{J}$ non-increasing in r

(ii) $H \le \bar{H}$.

pf: Similarly: $\bar{f}'' = -K \bar{f}$; $\bar{f}(0) = 0$, $\bar{f}'(0) = 1$.

Let $F = f/\bar{f}$

$F' = \dfrac{f'\bar{f} - f\bar{f}'}{\bar{f}^2}$ (Wronskian)

$(\bar{f}^2 F')' = f''\bar{f} + \cancel{f'\bar{f}'} - \cancel{f'\bar{f}'} - f\bar{f}'' \le -K f \bar{f} + K f \bar{f} = 0$

$0 < \varepsilon \le r \Rightarrow \bar{f}^2(r) F'(r) \le \bar{f}^2(\varepsilon) F'(\varepsilon)$

$= f'(\varepsilon)\bar{f}(\varepsilon) - f(\varepsilon)\bar{f}'(\varepsilon) \to 0$ as $\varepsilon \to 0$

$\Rightarrow F'(r) \le 0$, get (i), use F^{m-1}.

but (ii) is equiv to (i):

$$\left(\frac{J}{\bar{J}}\right)' = \frac{J'\bar{J} - J\bar{J}'}{\bar{J}^2} = \frac{J}{\bar{J}}\left(\frac{J'}{J} - \frac{\bar{J}'}{\bar{J}}\right) = \frac{J}{\bar{J}}(H - \bar{H})$$ □

Cor (Bonnet-Myer): $Ric \ge (m-1)K \Rightarrow d \le \pi/\sqrt{K}$.

pf: $J(r)/\bar{J}(r) \le (J/\bar{J})|_{r \to 0} = 1$,

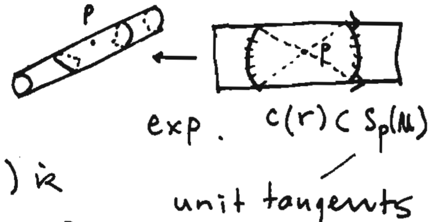
but $\bar{J}(r) = \left(\frac{1}{\sqrt{K}}\right)^{m-1} \sin^{m-1}(\sqrt{K} r)$. □

(Lect 2. 3/8).

geodesic ball $B_p(r) = \{ q \in M \mid d(p,q) \le r \}$

not diff to a ball in general:

$A_p(r) := \text{area}(\partial B_p(r))$



$\partial B_p(r) = \exp_p(r\, c(r))$

$c(r) := \{ \theta \in S_p(M) \mid \exp_p(s\theta)$ is minimizing up to $s = r \}$.

Corollary: $0 \le r_1 \le r_2 < \infty \Rightarrow$

(i) $\dfrac{A_p(r_2)}{\bar A(r_2)} \le \dfrac{A_p(r_1)}{\bar A(r_1)}$

Also for $V_p(r) = \text{vol}(B_p(r))$; $0 \le r_1 \le r_2, r_3 \le r_4$,

(ii) $\dfrac{V_p(r_4) - V_p(r_3)}{\bar V(r_4) - \bar V(r_3)} \le \dfrac{V_p(r_2) - V_p(r_1)}{\bar V(r_2) - \bar V(r_1)}$.

pf: from $\dfrac{J(\theta, r_2)}{\bar J(r_2)} \le \dfrac{J(\theta, r_1)}{\bar J(r_1)}$, to get (i)

integrate over $c(r_2)$ (which $\subset c(r_1)$) and use $\bar A(r) = A(S^{m-1})\bar J(r)$,

For (ii), if $r_1 \le r_2 \le r_3 \le r_4$, then use MVT.

if $r_1 \le r_3 \le r_2 \le r_4$, then simply use

$(V_4 - V_3)(\bar V_2 - \bar V_1) = (V_4 - V_3)[(\bar V_2 - \bar V_3) + (\bar V_3 - \bar V_1)]$

分二段 $\le (V_4 - V_3)[(\bar V_2 - \bar V_3) + (V_3 - V_1)]$

$\le$ similarly for 1st term $= (\bar V_4 - \bar V_3)(V_2 - V_1)$. done.

Corollary: Equality in (ii) holds $\Leftrightarrow$

$C(r_1) = C(r_4)$ and $\dfrac{J(\theta, r)}{\bar J(r)} = \text{const}$ $\forall r \in [r_1, r_4]$, $\theta \in C(r_1)$;

If $r_1 = 0$, then $J(\theta, r) = \bar J(r)$ $\forall r \le r_4$ & $\forall \theta \in S_p(M)$ $B_p(r_4) \underset{\text{isometric!}}{\simeq} \bar B(r_4)$.

(i) Application to ~~S-Y. Cheng's thm~~:

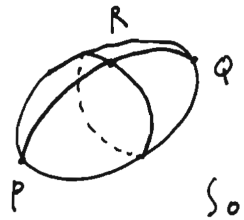
M complete, $Ric \ge (m-1)K$ $(K > 0)$

if $d := \text{diam}\, M = \pi/\sqrt{K}$, then $M \underset{\text{isometric}}{\simeq} S^m\left(\frac{1}{\sqrt K}\right)$.

pf: let $d(P,Q) = d$, then

$\dfrac{V_P(d)}{V_P(\frac d2)} \le \dfrac{\bar V(d)}{\bar V(\frac d2)} = 2$ ← since $\bar M_K = S^m\left(\frac{1}{\sqrt K}\right)$. apply to $r_1 = r_3 = 0$

So $V_P(d) \le 2 V_P(\frac d2)$, $V_Q(d) \le 2 V_Q(\frac d2)$ too.



But $B_P(\frac d2) \cap B_Q(\frac d2) = \phi$

otherwise $d = d(P,Q)$

$\le d(P,R) + d(Q,R) < d$ ※.

So $2\,\text{Vol}(M) = V_P(d) + V_Q(d)$

$\le 2(V_P(\frac d2) + V_Q(\frac d2)) \le 2\,\text{Vol}(M)$

since $B_P(\frac d2) \amalg B_Q(\frac d2) \subset M$

So "=" holds, i.e. $\overline{B_P(d)}$ isometric to $\overline{B}(d) = S^m(\frac{1}{\sqrt K})$ ※

(ii) Application to Lichnerowicz-Obata's thm:

M complete, $Ric \ge (m-1)K$ $(K > 0)$

then $\lambda_1 \ge mK$, $\lambda_1 = mK \Leftrightarrow M$ isometric to $S^m(\frac{1}{\sqrt K})$

pf: (L) Let $\lambda = \lambda_1$, $\Delta f = -\lambda f$, then

$\frac12 \Delta |\nabla f|^2 = |H(f)|^2 + \langle \nabla f, \nabla(\Delta f)\rangle + Ric(\nabla f, \nabla f)$

$\ge \frac{\lambda^2}{m} |f|^2 - (\lambda - (m-1)K)|\nabla f|^2$

since $m|H(f)|^2 \ge |\Delta f|^2$

Key idea: Absorbing the term $|f|^2$ into $\Delta(\dots)$.

Since $\frac{1}{2}\Delta|f|^2 = \Delta f \cdot f + |\nabla f|^2$,

$$\Rightarrow \frac{1}{2}\Delta\left(|\nabla f|^2 + \frac{\lambda}{m}f^2\right) \geq \cancel{\frac{\lambda^2}{m}f^2} - (\lambda - (m-1)K)|\nabla f|^2 - \cancel{\frac{\lambda^2}{m}f^2} + \frac{\lambda}{m}|\nabla f|^2$$

$$= -\frac{m-1}{m}(\lambda - mK)|\nabla f|^2$$

At the maximal point of $|\nabla f|^2 + \frac{\lambda}{m}f^2$, $\Rightarrow$ $\lambda \geq mK$.

(0) If $\lambda = mK$, then $F := |\nabla f|^2 + \frac{\lambda}{m}f^2$ has $\Delta F \geq 0$.

Lemma: On cpt M, $F\Delta F \geq 0 \Rightarrow F$ = constant.

Pf: $F\cdot\Delta F \geq 0 \Rightarrow \operatorname{div}(F\nabla F) = F\Delta F + |\nabla F|^2 \geq 0$, $\int * = 0$.

Hence $|\nabla f|^2 + \frac{\lambda}{m}f^2 = Kf_{max}^2 = Kf_{min}^2$,

Thus may assume $f_{max} = 1$, $f_{min} = -1$ by scaling.

$$\Rightarrow |\nabla f|^2 = K(1-f^2) \text{ i.e. } \frac{|\nabla f|}{\sqrt{1-f^2}} = \sqrt{K}$$

let $f(P) = \min = -1$

$f(Q) = \max = 1$

γ joins P, Q with shortest length, then

$$d\sqrt{K} \geq d(P,Q)\sqrt{K} = \int_\gamma \frac{|\nabla f|}{\sqrt{1-f^2}} \geq \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}}$$

$$= \sin^{-1} u\Big|_{-1}^{1} = \pi$$

together with $d \leq \frac{\pi}{\sqrt{K}}$ (Bonnet-Myer) $\Rightarrow$ $d = \pi/\sqrt{K}$.

Cheng's thm $\Rightarrow$ M is isometric to $S^m\left(\frac{1}{\sqrt{K}}\right)$. □

Rmk: it's possible to prove Obata's part using exp map conj. pt etc. without via Cheng's thm.
But it's much longer (cf. Book II by W.-H. Huang)

Lecture 3 (March 11). Gradient Estimates I.

Thm (Li-Yau). M cpt, $\partial M = \phi$, $Ric \geq 0$, then $\lambda_1 \geq \frac{\pi^2}{(1+a)d^2}$, where $a = \frac{M+m}{M-m} < 1$.

Here $\Delta\phi = -\lambda_1\phi$, $M = \sup\phi$, $m = \inf\phi$.

Pf: ~~Key idea~~: inspired by the pf of Cheng's max diam thm, we need

- gradient estimate $|\nabla u|^2 \leq \cdots$ on u st.
- $\sup u = 1$, $\inf u = -1$.

This can be achieved by require on $\phi \cdot l$ st

$$\begin{aligned} a-1 &= ml \\ a+1 &= Ml \end{aligned} \Rightarrow \frac{a-1}{a+1} = \frac{m}{M} \Rightarrow a = \frac{M+m}{M-m} \Rightarrow \exists l.$$

For this new ϕ, then $u := \phi - a$ is desired.

Try to estimate $P := |\nabla u|^2 + cu^2$, some $c > 0$.

Let $\lambda \equiv \lambda_1$, x_0 be max pt of P.

~~CLAIM: $P(x) \leq c$ $\forall x \in M$~~ (for $c = \lambda(1+a)$).

pf: If $\nabla u(x_0) = 0$ then it's OK since $|u| \leq 1$.

If $\nabla u(x_0) \neq 0$, by choosing the ONF, may assume that $\nabla u(x_0) = u_1(x_0)e_1$. Then

cf. Remark B at the end first!

$\frac{1}{2}P_i = u_m u_{mi} + cuu_i$.

at x_0 get $0 = u_1(u_{11} + cu)$, i.e. $u_{11}(x_0) = -cu(x_0)$

At x_0: $0 \geq \frac{1}{2}\Delta P = u_{mi}u_{mi} + u_m u_{mii}$

$+ c|\nabla u|^2 + cu\Delta u$

$\geq c^2u^2 + u_m(\Delta u)_m + Ric(\nabla u, \nabla u) + c|\nabla u|^2 - c\lambda u(u+a)$

$\geq (c-\lambda)|\nabla u|^2 + c(c-\lambda)u^2 - ca\lambda u$

$= (c-\lambda)P(x_0) - ca\lambda u(x_0)$

if at the beginning we choose $\underline{c = \lambda + \lambda a = \lambda(1+a)}$, in fact this is the best possible choice.

then get $P(x) \leq P(x_0) \leq C u(x_0) \leq C$.

As before, $|\nabla u|^2 + cu^2 \leq C \Rightarrow \frac{|\nabla u|^2}{1-u^2} \leq C$

let $u(p) = \min = -1$

$u(Q) = \max = 1$, γ shortest geod joins p, q

$\exists$ $\pi = \int_{-1}^{1} \frac{dv}{\sqrt{1-v^2}} \leq \int_\gamma \frac{|\nabla u|}{\sqrt{1-u^2}}\, ds \leq \sqrt{C}\, d(p,Q) \leq \sqrt{C}\, d$

ie. $\lambda(1+a) = C \geq \left(\frac{\pi}{d}\right)^2$, done □

Since we do not know a, this only gives $\underline{\lambda_1 > \frac{\pi^2}{2d^2}}$.

Remark: Cheng shows $\underline{\lambda_1 \leq 2n(n+4)/d^2}$ using estimate of heat kernels and on space forms.

* Now we shift to <u>non-compact</u> manifolds. The standard maximal principle needs to be modified (using cut-off functions). It turns out that <u>gradient estimate</u> is the basic form in harmonic theory:

<u>Theorem (Yau)</u>: M^m complete, $B_p(2\rho) \cap \partial M = \phi$. if $Ric \geq -(m-1)R$ on $B_p(2\rho)$ for some $R > 0$ and $\Delta u = 0$, $u > 0$ on $B_p(2\rho)$, then $\exists\, c = C(m) > 0$:

$$\sup_{B_p(\rho)} |\nabla \log u|^2 \leq C\left(R + \frac{1}{\rho^2}\right).$$

Cor (Yau's Liouville thm): M complete, $Ric \geq 0$. Then any positive harmonic fcn is constant.

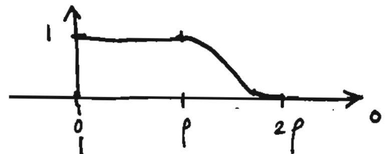
Pf of thm:

let $v = \log u$, then $\nabla v = \frac{\nabla u}{u}$

$\Delta v = \cancel{\frac{\Delta u}{u}} - \frac{|\nabla u|^2}{u^2} = -|\nabla v|^2$.

consider cut-off function $\phi(x) := \bar{\phi}(r(p,x))$

where $\bar{\phi}(r) :=$



satisfies (Exercise!)

$\underline{\frac{\bar{\phi}'^2}{\bar{\phi}} \leq \frac{C}{\rho^2}\,;\ |\bar{\phi}''| \leq \frac{C}{\rho^2}}$; where C is indep of ρ.

We plan to estimate $Q := \phi|\nabla \log u|^2 \equiv \phi|\nabla v|^2$.

and to evaluate at the max pt of Q st

$0 \geq \Delta Q(x_0) \geq F(\underbrace{\nabla Q}_{=0 \text{ at } x_0}, Q, \cdots)$ curvature condi.

since $x_0 \in B_p(2\rho)$.

Remark A. In the cpt case we estimate $\frac{|\nabla u|^2}{1-u^2}$. In the non-compact case we use $|\nabla \log u| = \frac{|\nabla u|}{|u|}$ is indeed very similar, as u is usually unbounded.

Remark B. In estimating $P = |\nabla u|^2 + cu^2$, if use the same method in Obata's thm, get

$\frac{1}{2}\Delta P \geq |Hu|^2 + \langle \nabla u, \nabla(\Delta u)\rangle + cu\Delta u + c|\nabla u|^2$

$\geq (c-\lambda)|\nabla u|^2 + \left(\frac{\lambda^2}{m} - c\lambda\right)u^2 + \text{lower terms}$

$= \underline{(c-\lambda)}|\nabla u|^2 + \frac{\lambda}{m}\underline{(\lambda - mc)}u^2 + \cdots$

opposite sign. only get worse estimate $\frac{\pi^2}{8d^2}$.

Lecture 4 (March 15). Gradient Estimate II.

Pf continued: $Q = \phi|\nabla v|^2 = -\phi\Delta v$

$$\Delta Q = \Delta\phi|\nabla v|^2 + 2\langle\nabla\phi, \underbrace{\nabla(|\nabla v|^2)}_{1}\rangle + \phi\Delta|\nabla v|^2$$

$$\geq \frac{\Delta\phi}{\phi}Q + \frac{2}{\phi}\langle\nabla\phi, \nabla Q - \frac{\nabla\phi}{\phi}Q\rangle + \underbrace{2\phi|Hv|^2}_{2} + \underbrace{2\phi\langle\nabla v, \nabla(\Delta v)\rangle}_{3} - 2(m-1)R\phi|\nabla v|^2$$

$$\geq Q\left(\frac{\Delta\phi}{\phi} - \frac{2}{\phi^2}|\nabla\phi|^2 - 2(m-1)R\right) + \frac{2}{m\phi}Q^2$$

$$-2\langle\nabla v, \nabla Q\rangle + \frac{2}{\phi}\langle\nabla v, \nabla\phi\rangle Q$$

Here we use

1. $\nabla|\nabla v|^2 = \nabla\left(\frac{Q}{\phi}\right) = \frac{\nabla Q}{\phi} - \frac{\nabla\phi}{\phi^2}Q$,
2. $|Hv|^2 \geq \frac{|\Delta v|^2}{m} = \frac{|\nabla v|^4}{m} = \frac{Q^2}{m\phi^2}$,
3. $\langle\nabla v, \nabla\Delta v\rangle = -\langle\nabla v, \nabla\left(\frac{Q}{\phi}\right)\rangle = \frac{-1}{\phi}\langle\nabla v, \nabla Q\rangle + \frac{Q}{\phi^2}\langle\nabla v, \nabla\phi\rangle$.

Suppose that the max pt x_0 of Q is not a cut point, ie. Q is C^∞ at x_0, then at x_0:

$$0 \geq Q + m\left(\frac{1}{2}\Delta\phi - \frac{|\nabla\phi|^2}{\phi} - (m-1)R\phi\right) + m\underline{\langle\nabla v, \nabla\phi\rangle}.$$

From $|\vec{a}\cdot\vec{b}| \leq |\vec{a}||\vec{b}| \leq \frac{1}{2}(|\vec{a}|^2 + |\vec{b}|)$, get

$$|m\langle\nabla v, \nabla\phi\rangle| = |\phi\langle\nabla v, m\frac{\nabla\phi}{\phi}\rangle| \leq \phi\,\frac{1}{2}\left(|\nabla v|^2 + m^2\frac{|\nabla\phi|^2}{\phi^2}\right)$$

$$= \underline{\frac{1}{2}Q} + \frac{m^2}{2\phi}|\nabla\phi|^2,$$

$$\Rightarrow \frac{Q}{2}(x_0) \leq \frac{-m}{2}\Delta\phi + \frac{m}{2\phi}(m+2)|\nabla\phi|^2 + m(m-1)R\phi$$

now $0 \leq \phi \leq 1$, $\frac{|\nabla\phi|^2}{\phi} = \frac{\bar{\Phi}'^2}{\phi} \leq \frac{C}{\rho^2}$, and $|\bar{\Phi}''| < \frac{C}{\rho^2}$.

$$\Delta\phi = \Delta\bar{\Phi}(r(x)) = (\bar{\Phi}'(r(x))r_i)_i$$

$$= \bar{\Phi}''|\nabla r|^2 + \bar{\Phi}'\Delta r = \bar{\Phi}'' + \bar{\Phi}'\Delta r$$

Lemma: $\Delta r = H \equiv J'/J$. (Exercise)

Bishop comparison thm $\Rightarrow$

$$\Delta r = H \leq \bar{H} = \bar{J}'/\bar{J} = (\log\bar{J})' = (m-1)\sqrt{R}\,\underline{\coth\sqrt{R}r}$$

$$\leq (m-1)\sqrt{R}\left(\frac{1}{\sqrt{R}r} + 1\right) = (m-1)\left(\frac{1}{r} + \sqrt{R}\right).$$

$$\Rightarrow \frac{-m}{2}\Delta\phi = \frac{-m}{2}\bar{\Phi}'' + \frac{m}{2}\underbrace{(-\bar{\Phi}')}_{\geq 0,\ <\frac{1}{\rho}}\Delta r \leq \frac{C_1}{\rho^2} + \frac{C_2}{\rho}\Big(\underbrace{\frac{1}{\rho}}_{r\geq\rho \text{ otherwise } \bar{\Phi}'=0} + \sqrt{R}\Big),$$

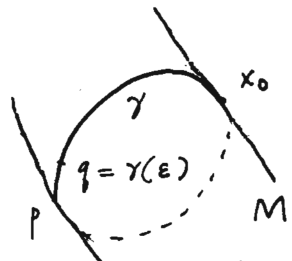
$$\Rightarrow \sup_{B_p(\rho)}|\nabla\log u|^2 \leq \sup_{B_p(2\rho)}\phi|\nabla v|^2 \leq Q(x_0) \leq C\left(\frac{1}{\rho^2} + R\right).$$

If x_0 is a cut pt of p, we use the "METHOD OF SUPPORT FUNCTIONS":

Let γ = min geod. joins p, x_0. Let $q = \gamma(\varepsilon)$, $\varepsilon > 0$.

$\psi(x) := \bar{\Phi}(r_q(x) + \varepsilon) \geq 0$; $\bar{\Phi}\searrow$

$r_q(x) + \varepsilon = r_q(x) + r_q(p) \geq r_p(x)$



$\Rightarrow$ $\psi(x) \leq \phi(x)$, also

$\psi(x_0) = \phi(x_0)$

and ψ is C^∞ at x_0 (since r_q is C^∞ at x_0, ie. x_0 is not a cut pt of r_q).

So x_0 is also max for $\psi|\nabla v|^2$, which is C^∞ at x_0.

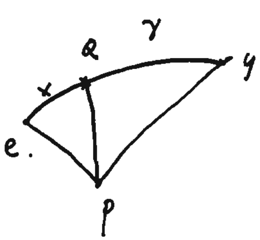
By the same computation and let $\varepsilon \to 0$. □

Rmk: Weaker estimate: $\sup_{B_p(\rho)}|\nabla u| \leq c\left(\frac{1}{\rho} + \sqrt{R}\right)\sup_{B_p(\rho)}|u|$

And this holds without assuming $u > 0$.

Cor (Harnack inequality), for $\Delta u = 0$, $u > 0$ on $B_p(2\rho)$, get $\sup_{B_p(\rho/2)} u \le e^{C(1+\rho\sqrt{K})} \inf_{B_p(\rho/2)} u$.

pf: let $u(x) = \inf$, $u(y) = \sup$, γ the min geod. joins x, y, then $|\gamma| \le \rho/2 + \rho/2 = \rho$.



$\forall Q \in \gamma$, either $d(x,Q) \le \rho/2$ or else.
so $d(p,Q) \le d(p,x) + d(x,Q) \le \rho$.
ie. $\gamma \subset B_p(\rho)$ $\Rightarrow$

$$\log u \Big|_x^y = \int_\gamma (\log u)' ds \le \int_\gamma |\nabla \log u| \le \rho \cdot C\left(\frac{1}{\rho} + \sqrt{K}\right). \quad \square$$

Cor (Cheng): M complete, $Ric \ge 0$, then any sub-linear growth harmonic fcn is constant.

pf: This follows from the weaker estimate since $\lim_{\rho\to\infty} \sup u/\rho \to 0$ $\square$.

Remark: Polynomial growth har. fcn's are finite dim (Colding, P. Li, Song ...)

Yau: In general, no L^α subharmonic functions f with $\alpha > 1$, other than constants.

pf: $\operatorname{div}(\phi^2 f^{\alpha-1}\nabla f) = 2\phi f^{\alpha-1}\langle\nabla\phi, \nabla f\rangle + (\alpha-1)\phi^2 f^{\alpha-2}|\nabla f|^2 + \phi^2 f^{\alpha-1}\Delta f$

$$\Rightarrow (\alpha-1)\int \phi^2 f^{\alpha-2}|\nabla f|^2 \le -2\int \phi f^{\alpha-1}\langle\nabla\phi,\nabla f\rangle \le \frac{\alpha-1}{2}\int \phi^2 f^{\alpha-2}|\nabla f|^2 + \frac{2}{\alpha-1}\int f^\alpha |\nabla\phi|^2$$

$$\Rightarrow \underline{\int \phi^2 f^{\alpha-2}|\nabla f|^2 \le \left(\frac{2}{\alpha-1}\right)^2 \int f^\alpha |\nabla\phi|^2 \le \frac{C(\alpha)}{\rho^2}\int_{B_p(2\rho)} f^\alpha}$$

$\rho \to \infty$ get the result. $\square$

Lecture 5 (March 18)

Final lect of "Intro to geom. analysis"

Isoperimetric inequalities vs λ_1.

Back to Euclidean space $\mathbb{R}^{m+1}$

$X = (x_1, \dots, x_{m+1})^T$ position vector (function)

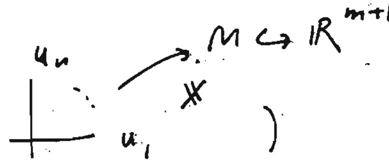
Key1: $V \in T\mathbb{R}^{m+1} \Rightarrow VX = V$.

pf: let $V = v^i \frac{\partial}{\partial x_i}$, then $V(x^i) = v^i$ $*$

Key2: Hessian $H_{\mathbb{R}^{m+1}}(X) = 0$.

bec $H_{\mathbb{R}^{m+1}}(f)$ is a 2tensor, for standard coor. set $H(x^i)_{jk} = \partial_j\partial_k x^i = 0$ $*$ (degree $= 1$)

Lemma 1: $M \overset{X}{\hookrightarrow} \mathbb{R}^{m+1}$ (classical picture



)

then $H_M(X) = \vec{II} \equiv \vec{B}$;
$\Delta_M X = \vec{H}$. ($\Rightarrow$ no cpt min sub mfd)

pf: for any frame e_i on M,

$$\begin{aligned} H_M(X)_{ij} &= (e_i e_j - \nabla_{e_i} e_j) X \\ &= (e_i e_j - \nabla^{\mathbb{R}^{m+1}}_{e_i} e_j) X + \nabla^{\mathbb{R}^{m+1}}_{e_i} e_j{}^{N} X \\ &= \cancel{H_{\mathbb{R}^{m+1}}(X)_{ij}} + B(e_i, e_j) X \equiv B(e_i, e_j) = X_{ij}^N \quad * \end{aligned}$$

Lemma 2: $N^n \hookrightarrow S^m \hookrightarrow \mathbb{R}^{m+1}$

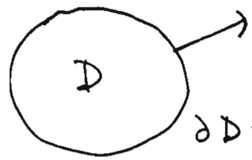
then N minimal in $S^m \iff \Delta_N X = -nX$.

pf: Similarly $H_N(X) = H_{S^m}(X)|_N + \vec{II}_{N/S^m}$,
for S^m, $B(e_i, e_j) = X_{ji}^N = (X_{ji} \cdot N) N = -(X_j \cdot N_i) N$
$= -(X_j \cdot X_i) N = -g_{ij} N$. Take tr over N. $\square$

• Thus, for min submfd $N \hookrightarrow S^m$, $x_1, \cdots, x_{m+1}$ are all eigen fcns with value $= n$. So $\lambda_1 \le n$.
Choi-Wang: $\lambda_1 \ge n/2$ for $n = m-1$, Conj is $\lambda_1 = n$.

• For spheres, do have $\lambda_1 = m$ (Lichnerowicz).

~~Isoperimetric problem~~: let D any mfd w. $\partial D \neq \emptyset$,



$\nu = fN$

$A(t) = \text{Area } M_t$

$\partial D = M$

$$A'(0) = -\int_M \langle \nu, \vec{H} \rangle = \int_M fH$$

expect $\mathrm{vol}'(0) = 0$ ∀ vol preserving variation, ie. ∀f st $\int_M f = 0$. This holds ⇔ H = constant.
(why? eg use Hodge theory. $0 = \int fH = \int \Delta g\, H = -\int g \Delta H$ true ∀ C^∞ g ⇒ $\Delta H = 0$ ⇒ H = const on M.)

~~Thm (Alexandrov 1958)~~ $M^m \hookrightarrow \mathbb{R}^{m+1}$ with
$H = \text{const} \Rightarrow M = \text{standard sphere}$.

In 1977, Reilly gave a pf using:

~~Thm (Reilly's Bochner formula for mfd with ∂)~~

Let $\partial D^{m+1} = M^m$ with Dirichlet problem:

$$\begin{cases} \Delta f = g & \text{on } D \\ f|_M = u \end{cases}$$

consider II = B and H wrt. outer normal ν.

$$\Rightarrow \frac{m}{m+1}\int_D g^2 \ge \int_M \left(H \cdot f_\nu^2 + 2 f_\nu \Delta_M u + B(\nabla u, \nabla u)\right) + \int_D Ric(\nabla f, \nabla f)$$

"=" holds ⇔ $H(f) = \frac{g}{m+1}$ "g" ← the metric tensor.

Ex. Prove it!

* Lecture 6 (added in 5/7, 2021 at NTU)

pf: Reilly ⇒ Alexandrov: let $M = \partial D \subset \mathbb{R}^{m+1}$
rescale M st $H \equiv m$.
(since it is true for S^m)

Solve $\begin{cases} \Delta f = -1 & \text{on } D \\ f|_M = 0 \end{cases}$

$\Rightarrow \frac{|D|}{m+1} \ge \int_M f_\nu^2$. Also $|D|^2 = \left(\int_D \Delta f\right)^2 = \left(\int_M f_\nu\right)^2 \le |M| \int_M f_\nu^2$

Hence $|M| \ge (m+1)|D|$. (isoperimetric ineq')

On the other hand, Stokes' on D ⇒

$$0 = \int_D X \cdot \Delta_{\mathbb{R}^{m+1}} X = -\int_D \underbrace{|\nabla X|^2}_{= m+1} + \int_M X \cdot \underbrace{\nu X}_{= \nu} \quad \text{key 1.}$$

$$= -(m+1)|D| - \underbrace{\frac{1}{m}\int_M X \cdot \Delta_M X}_{* =}$$

Notice: $\Delta_M X = \vec{H} = -H\vec{\nu}$

(Minkowski Thm) $\int_M \frac{H}{m} X \cdot \nu = |M|$. $\frac{1}{m}\int_M |\nabla_M X|^2 = |M|$ use key 1. c.f. convention at beginning

⇒ all "≥" become "=" and $\begin{cases} f_{ij} = -\frac{\delta_{ij}}{m+1} & \text{on } D \\ f_\nu \equiv f_{m+1} = \text{const.} & \text{on } M \end{cases}$

$\Rightarrow f(x) = \frac{-1}{m+1}|X|^2 + \text{linear}$

⇒ $f^{-1}(0)$ is a round sphere ⇒ unit sphere ∗

Rmk: Hopf (1956, DG in the large) proved
$M \hookrightarrow \mathbb{R}^3$, genus = 0, H = const ⇒ sphere
by developing "Alexandrov's reflection principle" in elliptic PDE. Then conjectured $g(M) \ge 1$ cases.
If $g(M) = 1$, Wente 1984 constructed 1st count. example
See MSRI website "CMC surfaces" for VAST examples.

Lemma (Ruh, Vilms 1970 TAMS)

$M \hookrightarrow \mathbb{R}^3$ oriented, H = const ⇔ $N: M \to S^2$ is harmonic
ie. $\Delta_M N \perp T_N S^2$.

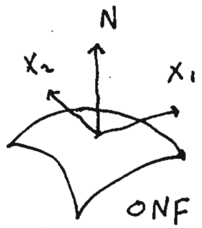
pf: Under isothermal coor near $p \in M \to \mathbb{R}^3$
induced by conformal map (Chern)

get $N_1 = -(h_{11}X_1 + h_{12}X_2)$

$N_2 = -(h_{21}X_1 + h_{22}X_2)$

$N \times N_1 = -h_{11}X_2 + h_{12}X_1 = -H X_2 - N_2$

$N \times N_2 = -h_{21}X_2 + h_{22}X_1 = H X_1 + N_1$



$N \times \Delta N = (N \times N_1)_1 + (N \times N_2)_2$

up to a conf factor

$= -H_1 X_2 + H_2 X_1 = 0 \Leftrightarrow H_1 = H_2 = 0 \Leftrightarrow H = \text{const}$ ※

Ex. Conversely, any harmonic map $D \xrightarrow{u} S^2$ is the Gauss map of 2 weakly conformal CMC immersions $X_\pm : D \to \mathbb{R}^3$, with $H \equiv 1$.

(indeed, since $0 = u \times \Delta u = (u \times u_1)_1 + (u \times u_2)_2$

$\Rightarrow$ $u \times u_2\, dx - u \times u_1\, dy$ is closed, hence exact

$= dB$ for some $B : D \to \mathbb{R}^3$

show that B defines a surface with $K = 1$ and analyze surface $B + u$ & $B - u$.)

Rmk: Far reaching generalization "completely int. sys."

eg. by Hitchin (1990), Pinkall-Sterling (1989)

CMC Tori immersed in 3D space forms via alg. geom.

The startting simplest version is

Kenmotsu's repr formula: $\varphi : V_{\subset \mathbb{C}} \to \mathbb{C} \circlearrowleft S^2$ harmonic, $H = \text{const}$

$X(z) := \mathrm{Re} \int_{z_0}^{t} X_z(z)\, dz$

$X_z(z) = \dfrac{-\bar{\varphi}_{\bar z}}{H(1+|\varphi|^2)^2} \left(1-\varphi^2,\ i(1+\varphi^2),\ 2\varphi\right)$ ※