Chapter 8

Atiyah–Singer Index Theorem

8.1. Index of an Elliptic Operator and Heat Kernel

Let $E, F \to M$ be real or complex vector bundle over a compact manifold M, $P : C^\infty(M, E) \to C^\infty(M, F)$ be an elliptic differential operator (cf. section 4.3). We denote $P^* : C^\infty(F) \to C^\infty(E)$ by its formal adjoint. Then P^* is also an elliptic operator (cf. Exercise 4.7). We define the *index* of P by

$$\mathrm{ind}P := \dim\ker P - \dim\mathrm{coker}P \in \mathbb{Z},$$

where $\mathrm{coker}P := C^\infty(M, F)/\mathrm{Im}P$. We first see that the index of an elliptic operator is well–defined.

Exercise 8.1. Adapt the proof in theorem 4.13 to show that $\mathrm{Im}P = (\ker P^*)^\perp$.

Hence, $\mathrm{coker}P = C^\infty(F)/\ker(P^*)^\perp \cong \ker(P^*)$. Since P and P^* are both elliptic, by compactness theorem, we know that both $\dim\ker P, \dim\ker P^*$ are finite. Therefore, $\mathrm{ind}P$ is well–defined. Moreover, if $s \in \ker P^*P$, then $0 = (P^*Ps, s) = (Ps, Ps) = \|Ps\|^2$ implies that $s \in \ker P$. Thus,

$$\ker P = \ker P^*P; \quad \ker P^* = \ker P^*P.$$

Notice that $P^*P : C^\infty(E) \to C^\infty(E)$ and $PP^* : C^\infty(F) \to C^\infty(F)$ are self–adjoint and elliptic, we see that

(8.1)

$$\mathrm{ind}P := \dim\ker P - \dim\mathrm{coker}P = \dim\ker P^*P - \dim\ker PP^*.$$

Thus, to calculate the index of an elliptic operator P of order d, it suffices to calculate the dimension of kernels of self–adjoint, elliptic operator PP^* and P^*P of order $2d$.

To see the connection between index and hear equation, let us consider a self–adjoint elliptic operator $L : C^\infty(E) \to C^\infty(E)$ and its "heat equation":

$$\frac{\partial}{\partial t} f + Lf = 0. \tag{*}$$

First, we observe that the problem 4.7 can be generalized to any self–adjoint elliptic operator L of order d, and we get the following conclusion:

Proposition 8.1 (Spectrum for Elliptic Operator)**.** *Let $L : C^\infty(E) \to C^\infty(E)$ be an elliptic self-adjoint operator of order $d > 0$.*

(1) *We can find a complete orthonormal basis $\{\phi_n\}_{n=1}^\infty$ of $L^2(E)$ of eigenvectors of L.*
(2) *The eigenvectors ϕ_n are smooth.*
(3) *The eigenvalues λ_i of L are discrete and $\lim_{n\to\infty} |\lambda_n| = \infty$.*

We then define the *heat operator* of $e^{-tL} : C^\infty(E) \to C^\infty(E)$ by

$$e^{-tL}(g)(x) = \int_M H(x, y, t)(g(y))dv_M(y)$$

where dv_M is the volume form on M and $H(x, y, t) \in E_x \otimes E_y^*$ is called the *heat kernel* of L:

$$H(x, y, t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \phi_k(x) \otimes \phi_k^*(y).$$

Exercise 8.2. Let $L : C^\infty(E) \to C^\infty(E)$ be an elliptic self-adjoint operator of order $d > 0$.

(1) Show proposition 8.1.
(2) Show that $H(x, y, t) \in C^\infty(M \times M, E \boxtimes E)$ by showing that there exists constants $C, \delta > 0$ such that $\lambda_n \geq Cn^\delta$, for $n \ll 0$, where $E \boxtimes E = p_1^* E \otimes p_2^* E$ and $p_i : M \times M \to M$ is the projection, for $i = 1, 2$.

Hence, we can exchange the summation and integration legally such that

$$e^{-tL}(g)(x) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \phi_k(x) \int_M \phi_k^*(y)(g(y))dv_M(y).$$

Then observe that

$$\begin{aligned}\frac{\partial}{\partial t}\left(e^{-tL}(g)\right) &= \int_M \frac{\partial}{\partial t}H(x,y,t)(g(y))d\nu_M(y)\\ &= -\int_M L_x H(x,y,t)(g(y))d\nu_M(y)\\ &= -L\left(\int_M H(x,y,t)(g(y))d\nu_M(y)\right) = -L(e^{-tL}(g)),\end{aligned}$$

and $\lim_{t\to 0} e^{-tL}(g)(x) = \sum_{k=1}^{\infty}(g,\phi_k)\phi_k(x) = g(x)$ since $\{\phi_k(x)\}_{k=1}^{\infty}$ is a complete basis on $L^2(M,E)$. This shows that $e^{-tL}g$ is the solution of (*) with integrable initial condition g.

Now, we define the *trace of heat kernel* by

$$\mathrm{tr} e^{-tL} := \sum_{k=1}^{\infty} e^{-\lambda_k t} = \int_M \mathrm{tr}_{E_x} H(t,x,x)d\nu_M(x),$$

and notice that $\sum_{k=1}^{\infty} e^{-\lambda_k t} = \dim\ker L + \sum_{\lambda_k\neq 0} e^{-\lambda_k t}$.

Now, back to the case when $L = P^*P$. Observe that for $\lambda_i \neq 0$, $P^*P\phi_i = \lambda_i\phi_i$, then

$$(PP^*)(P\phi_i) = P(P^*P\phi_i) = P(\lambda_i\phi_i) = \lambda_i P\phi_i \text{ (note that } P\phi_i \neq 0).$$

This establishes a bijection between λ_i-egienspace of PP^* and P^*P and thus

$$\mathrm{tr} e^{-tP^*P} - \mathrm{tr} e^{-tPP^*} = \dim\ker P^*P - \dim\ker PP^*.$$

Combining with (8.1), we then get the *McKean–Singer formula*:

Corollary 8.2 (McKean–Singer Formula)**.**

$$\mathrm{ind} P = \int_M \left(\mathrm{tr}_{E_x} H_{P^*P}(x,x,t) - \mathrm{tr}_{F_x} H_{P^*P}(x,x,t)\right) d\nu_M(x).$$

Alternatively, we can define an operator $D : C^\infty(E \bigoplus F) \to C^\infty(E \bigoplus F)$ by $D = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix}$. Then clearly, D is self–adjoint and

$$D^2 = \begin{pmatrix} P^*P & 0 \\ 0 & PP^* \end{pmatrix}.$$

If we regard $V := E \bigoplus F$ as a $\mathbb{Z}_2$-graded vector bundle with $V^+ = E$, $V^- = F$, then the heat kernel $k_t(x,x) := H_{D^2}(x,x,t)dv_M(x) \in (\mathrm{End}E_x \bigoplus \mathrm{End}F_x) \otimes \bigwedge^n T_x^*M$ is of the form

$$\begin{pmatrix} H_{P^*P}(x,x,t) & 0 \\ 0 & H_{P^*P}(x,x,t) \end{pmatrix} dv_M(x).$$

We then define the *supertrace*[1] of $k_t(x,x)$ by

$$\mathrm{str}k_t(x,x) := \left(\mathrm{tr}_{E_x} H_{P^*P}(x,x,t) - \mathrm{tr}_{F_x} H_{P^*P}(x,x,t)\right) dv_M(x)$$

and the supertrace of e^{-tD^2} by

$$\mathrm{str}\left(e^{-D^2}\right) := \mathrm{tr}e^{-tP^*P} - \mathrm{tr}e^{-tPP^*}.$$

Then the McKean–Singer formula can be written as

$$\mathrm{ind}P = \mathrm{str}\left(e^{-tD^2}\right) = \int_M \mathrm{str}k_t(x,x) \tag{8.2}$$

Notice that the left–hand side is independent of $t \in \mathbb{R}^+$ while $k_t(x,x)$ is dependent on t. Therefore, the computing index of an elliptic operator via heat kernel lies in the same circle of idea as Witten deformation discussed in problem 4.13, namely the ”supersymmetry”[2]. When we let $t \to 0+$, the index of P is robust under the deformation of t, while we can asymptotically exapnd $k_t(x)$ into explicitly computable differential forms. This is the central idea of local index theorem, which we will carry out in section 8.4.

8.2. Heat Kernel for Harmonic Operator in Euclidean Spaces

In this section, we construct explicitly the heat kernel for generalized Harmonic oscillator on Euclidean space $V = \mathbb{R}^n$, which will

[1] In general, for a $\mathbb{Z}_2$-graded vector space $V = V^+ \bigoplus V^-$, $T \in \mathrm{End}V$ can be decomposed into $T_{\mathrm{even}} : V^\pm \to V^\pm$ and $T_{\mathrm{odd}} : V^\pm \to V^\mp$. We then define the supertrace of T by $\mathrm{str}T = \begin{cases} \mathrm{tr}T\big|_{V^+} - \mathrm{tr}T\big|_{V^-} & T \text{ is even} \\ 0 & T \text{ is odd} \end{cases}$

[2] In fact, Alvarez-Gaumé gave a physicists’ ”proof” of Atiyah–Singer theorem by supersymmetry in 1983 and the proof we will present in section 8.4 due to E.Getzler [Get83] is also inspired by this physical proof.

be our ”local model” for differential operators on compact manifolds considered in the later sections.

Let R be $n \times n$ skew–symmetric matrix and F be $N \times N$ matrix with coefficients in a commutative algebra[3] $\mathcal{A}$. The *generalized harmonic operator* H acting on $C^\infty(V, \mathcal{A} \otimes \mathbb{C}^N)$ is given by

$$H := -\sum_i (\partial_i + \frac{1}{4} R_{ij} x_j)^2 + F.$$

To begin, let us start with the one–dimensional case ”without potential”, i.e. $V = \mathbb{R}$ and $H = -\partial^2/\partial x^2$. The starting point is the Gaussian integral in elementary calculus:

$$A = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} = \int_{-\infty}^{\infty} e^{-\frac{x^2}{4t}} d\left(\frac{x}{\sqrt{4t}}\right),$$

for $A^2 = 2\pi \int_0^\infty e^{-r^2} r dr = 2\pi \left(-\frac{1}{2} e^{-r^2}\right)\Big|_0^\infty = \pi$. Thus, the scaled Gaussian integral is given by

$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{4t}} dx = 1.$$

Also, notice that $\lim_{t \to 0+} \frac{1}{\sqrt{4\pi t}} e^{-x^2} 4t = \delta_0(x)$. Therefore, let

$$p(x, y, t) := \frac{1}{\sqrt{4\pi t}} e^{\frac{-(x-y)^2}{4t}}.$$

Lemma 8.3. *$p(x, y, t)$ is the heat kernel for $H = -\frac{\partial}{\partial x^2}$.*

PROOF. First, for $t > 0$, by direct computation, we see that

$$p_t = \frac{-1}{2} (4\pi t)^{-\frac{3}{2}} \cdot 4\pi e^{\frac{-(x-y)^2}{4t}} + p \cdot \frac{1}{4} \frac{(x-y)^2}{t^2} = -\frac{p}{2t} + p \frac{1}{4} \frac{(x-y)^2}{t^2},$$

$p_x = -p \frac{x-y}{2t}$, and thus $p_{xx} = p \frac{(x-y)^2}{4t^2} - p \frac{1}{2t}$.

Hence, we see that $p_t - p_{xx} = 0$, for $t > 0$. On the other hand, we define the heat operator e^{-tH} by

$$e^{-tH}(g)(x) := \int_{\mathbb{R}} p(x, y, t) g(y) dy.$$

[3] In the application, $\mathcal{A}$ will be the even part of exterior algebra.

We leave readers to verify[4], that $\lim_{t\to 0+} e^{-tH}(g)(x) = g(x)$, for bounded $g \in C^0(\mathbb{R})$. $\square$

Exercise 8.3. Show that $\lim_{t\to 0+} e^{-tH}(g)(x) = g(x)$, for $g \in C^0(\mathbb{R})$. (Hint: let $K(x,t) = K_t(x) := p(x,0,t)$ and note that $K_t(x)$ is a mollifier in t.)

Remark 8.4. The computation in the above lemma works for n-dimension and shows that $p(\mathbf{x},\mathbf{y},t) = (4\pi t)^{-n/2}e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4}}$ is the heat kernel for $H = -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$.

Next, we consider the heat kernel of harmonic oscillator $H = -\frac{\partial}{\partial x^2} + x^2$, i.e. $p_t(x,y)$ such that $\left(\frac{\partial}{\partial t} + H\right) p = 0$ with the initial condition

$$\lim_{t\to 0+} \int_{-\infty}^{\infty} p_t(x,y)g(y)dy = g(x).$$

First, observe that H is self–adjoint, one must have $p_t(x,y) = p_t(y,x)$. We try to solve $p_t(x,y)$ by the following ansatz

$$p_t(x,y) = e^{A\frac{x^2}{2}+Bxy+A\frac{y^2}{2}+C},$$

where A, B, C are functions of t only. Then one directly computes

$$\left(\frac{\partial}{\partial t} + H\right) p = \left[A'\frac{x^2}{2} + B'xy + A'\frac{y^2}{2} + C' - (Ax+By)^2 - A + x^2\right] p.$$

From above, we see that $\left(\frac{\partial}{\partial t} + H\right) p = 0$ is equivalent to the solving the following ODE

$$\begin{aligned} &\frac{A'}{2} - A^2 + 1 = 0, \quad B' - 2AB = 0,\\ &\frac{A'}{2} - B^2 = 0, \quad C' - A = 0. \end{aligned}$$

[4]Notice that $e^{-tH}g$ is well–defined for $g \in L^1(\mathbb{R})$, yet we need $g \in C^0(\mathbb{R})$ to prove this.

One can solve directly that $A(t) = -\coth(2t+c)$ and $B^2 = \frac{A'}{2} = \operatorname{csch}^2(2t+c)$. Thus, we have

$$
\begin{aligned}
B(t) &= \operatorname{csch}(2t+c), \\
C(t) &= \int A(t)dt + d = -\frac{1}{2}\log(\sinh(2t+c)) + d,
\end{aligned}
$$

where c and d are some constants. To fix c and d, we put $x = 0$ and as $t \to 0+$, the initial condition implies $p_t(0,y)$ should "converge" to $\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$. Therefore, we see that the asymptotics of $A(t)$ and $C(t)$ are given by

$$A(t) = -\coth(2t+c) \sim \frac{-1}{2t}; \quad e^{C(t)} = (\sinh(2t+c))^{-1/2}e^d \sim (4\pi t)^{-1/2}.$$

One has $c = 0$, $d = -\frac{1}{2}\log(2\pi)$.

Exercise 8.4. Solve the ODE $A' - 2(A^2+1) = 0$ and fill in the detail that the initial condition determines $c = 0$, $d = -\frac{1}{2}\log(2\pi)$.

In conclusion, we obtain the *Mehler's formula*

$$p(x,y,t) = (2\pi\sinh 2t)^{-1/2}e^{-\frac{x^2+y^2}{2}\coth 2t - xy\operatorname{csch} 2t}. \tag{8.3}$$

In particular, let $y = 0$ and change of variable $t \mapsto \frac{tr}{4}$, $x \mapsto \sqrt{\frac{r}{4}}x$, Mehler's formula shows the equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + \frac{r^2}{16}x^2 + f\right)p_t(x) = 0 \tag{*}$$

has a solution

$$\frac{1}{\sqrt{4\pi t}}\left[\left(\frac{tr/2}{\sinh(tr/2)}\right)^{1/2}\right]e^{-\frac{tr}{2}\coth\left(\frac{tr}{2}\right)\frac{x^2}{4t}}e^{-tf}. \tag{8.4}$$

Finally, we tackle the case of generalized Laplacian mentioned in the beginning of this section.

Theorem 8.5. *Let $H := -\sum_i(\partial_i + \frac{1}{4}R_{ij}x_j)^2 + F$ on $\mathbb{R}^n$.*

$$p(\mathbf{x},t) = \frac{1}{(4\pi t)^{n/2}}\left(\det\frac{tR/2}{\sinh(tR/2)}\right)^{1/2}e^{\frac{-1}{4t}\left\langle \mathbf{x}\middle|\frac{tR}{2}\coth\frac{tR}{2}\middle|\mathbf{x}\right\rangle}e^{-tF}$$

is the formal heat kernel[5] *for the equation* $\partial_t + H = 0$, *i.e.* $(\frac{\partial}{\partial t} + H)p(\mathbf{x}, t) = 0$ *and* $\lim_{t\to 0} p(x,t) = \delta(x)$.

PROOF. Observe $\partial_t p = -Hp$ is a purely algebraic formula and both sides are analytic functions in R_{ij}, we may assume that $R_{ij} \in \mathbb{R}$. Moreover, since R is a skew–symmetric real matrix, we can choose a basis of V such that R is of the block form:

$$\begin{pmatrix} 0 & -r & & & \\ r & 0 & & & \\ & & 0 & -s & \\ & & s & 0 & \\ & & & & \ddots \end{pmatrix}.$$

We then reduce the problem to the 2-dimensional case, which H is given by

$$H = -(\partial_1^2 + \partial_2^2) - \frac{r}{2}(x_2\partial_1 - x_1\partial_2) - \frac{r^2}{16}(x_1^2 + x_2^2) + F$$

and

$$p_t(x_1, x_2) = \frac{1}{4\pi t}\frac{tr/2}{\sin(tr/2)} e^{-\frac{tr}{2}\cot\frac{tr}{2}\frac{\|x\|^2}{4t}} e^{-tF}.$$

We then see that $\partial_t + H$ is just (*) with addition term $-\frac{r}{2}(x_2\partial_1 - x_1\partial_2)$ while $p_t(x_1, x_2)$ is just (8.4) in two variables case with r replaced by ir and an addition term $e^{\frac{\|x\|^2}{4t}}$. This additional term $\|x\|^2$ is annihilated by $x_2\partial_1 - x_1\partial_2$. Therefore, we have proved that $p_t(x_1, x_2)$ is the heat kernel of H. □

8.3. Clifford Algebra and Dirac Operators

In this section, we introduce Clifford algebra and Dirac operators which are essential in the formulation of local index theorem presented in the next section.

[5]The hear kernel here is the formal sense since $\left(\det \frac{tR/2}{\sinh(tR/2)}\right)^{1/2}$ and $\coth(tR/2)$ are defined in terms of power series in R_{ij} which converge for tR, for some $|t| \ll 1$.

Definition 8.6. Let V be a real vector space[6] with a quadratic form Q. The *Clifford algebra* $C(V,Q)$ of (V,Q) is defined by

$$C(V,Q) = T(V)/I,$$

where I is the 2-sided ideal generated by $v \otimes v + Q(v)$.

For $v, w \in V$, we denote $v \cdot w$ or just vw by the image of $v \otimes w$ in $C(V,Q)$. Then $v^2 = Q(v)$ by construction. Also, for any $v, w \in V$, we have

$$vw + wv = -2\langle v, w\rangle,$$

where $\langle v, w\rangle = \frac{1}{2}[Q(v+w) - Q(v) - Q(w)]$ is the symmetric bilinear form associated to Q given by the polarization.

Exercise 8.5. Show that $C(V,Q)$ has the universal property of the following: let A be an $\mathbb{R}$-algebra and $c : V \to A$ be a linear map satisfying $v \cdot w + w \cdot v = -2Q(v,w)$. Then there exists a unique algebra homomorphism $C(V,Q) \to A$ extending c.

The Clifford algebra $C(V,Q)$ has an induced $\mathbb{Z}_2$–grading induced from $T(V)$ and we write $C(V,Q) = C^+(V,Q) \bigoplus C^-(V,Q)$, where $V \subset C^-(V,Q)$.

Example 8.7. If $Q = 0$, then $C(V,Q)$ is just exterior algebra $\bigwedge^* V$.

In other words, $C(V,Q)$ is the deformation of exterior algebra $\bigwedge^* V$. When Q is positive definite, we denote $C(V)$ for simplicity.

Definition 8.8. A $\mathbb{Z}_2$-graded vector space $E = E^+ \bigoplus E^-$ is called a *Clifford module* if E has a $C(V)$-module structure which is compatible with $\mathbb{Z}_2$-grading, i.e.

$$C^+(V) \cdot E^\pm = E^\pm; \quad C^-(V) \cdot E^\pm = E^\mp.$$

For $v \in V$, we denote $c(v)$ by the Clifford multiplication of v on a Clifford module.

[6]The definition and the general results for Clifford algebra work for any vector space over a field F with char$F \neq 2$

Example 8.9. Let $E = \bigwedge^*(V)$. By the universal property, it suffices to define $c : V \to E$. For $\alpha \in \bigwedge^* V$, we set $c(v)\alpha := v \wedge \alpha - \iota_v \alpha$, where ι_v means the contraction with $\langle v, \cdot \rangle \in V^*$. Then from

$$v \wedge \iota_w + \iota_w(v \wedge \cdot) = v \wedge \iota_w + Q(w, v) - v \wedge \iota_w = Q(v, w),$$

we see that this defines a $C(V)$-module structure on $\bigwedge^* V$ since

$$\begin{aligned}(c(v)c(w) + c(w)c(v))\alpha &= -v \wedge \iota_w \alpha - w \wedge \iota_v \alpha - \iota_v(w \wedge \alpha) - \iota_w(v \wedge \alpha) \\ &= -2Q(v, w)\alpha.\end{aligned}$$

Also, E has a natural $\mathbb{Z}_2$-grading $E^+ := \bigwedge^{\mathrm{ev}} V$, $E^- := \bigwedge^{\mathrm{odd}} V$, it is clear that $c(v) : E^{\pm} \to E^{\mp}$.

We now define the *symbol map* $\sigma : C(V) \to \bigwedge^* V$ by

$$\sigma(a) = c(a)1 \in \bigwedge^* V,$$

where $1 \in \bigwedge^* V$ is the identity in the exterior algebra $\bigwedge^* V$. Then σ has an obvious inverse $\mathbf{c} : \bigwedge^* V \to C(V, Q)$ given by

$$\mathbf{c}(e_{i_1} \wedge \cdots \wedge e_{i_k}) = c_{i_1} \cdots c_{i_k},$$

where $\{e_i\}$ is an orthogonal basis for $\langle\, , \rangle$ and c_i is the element of $C(V, Q)$ corresponding to e_i. The map $\mathbf{c}$ is called the *quantization map*.

Exercise 8.6. Write $\sigma(v_1 v_2)$, $\sigma(v_1 v_2 v_3)$ and $\mathbf{c}(v_1 \wedge v_2)$, $\mathbf{c}(v_1 \wedge v_2 \wedge v_3)$ explicitly. Also, for $v_1, \ldots, v_k \in V$, show that

$$\mathbf{c}(v_1 \wedge \cdots \wedge v_k) = \frac{1}{k!} \sum_{\tau \in S_k} (-1)^{\operatorname{sign} \tau} v_{\tau(1)} \cdots v_{\tau(k)}$$

Hence, $\bigwedge^* V \cong C(V, Q)$ as a vector space (but with different algebra sturcutre). Particularly, this shows $\dim C(V, Q) = \dim \bigwedge^* V = 2^{\dim V}$.

On the other hand, observe that both $\bigwedge^* V$ and $C(V, Q)$ have natural filtrations:

$$C_0(V, Q) \subset C_1(V, Q) \subset \cdots \subset C_i(V, Q) \subset \cdots$$
$$\bigwedge^0 V \subset \bigwedge^1 V \subset \cdots \subset \bigwedge^i V \subset \cdots,$$

where each $C_i(V,Q)$ is spanned by elements of the form $v_1 \dots v_k$, where $v_j \in V$ and $k \le i$. Obviously, the filtration is compatible with Clifford multiplication: $C_i(V,Q) \cdot C_j(V,Q) \subset C_{i+j}(V,Q)$. We can then define an associated graded algebra

$$\mathrm{gr}C(V,Q) := \bigoplus_{i \ge 0} \mathrm{gr}_i C(V,Q)$$

with $\mathrm{gr}_i C(V,Q) := C_i(V,Q)/C_{i-1}(V,Q)$. Obviously, $\mathrm{gr}_k C(V,Q) \cong \bigwedge^k(V)$. We then define $\sigma_k : C_k(V) \to \bigwedge^k(V)$ by the composition of the isomorphism with the quotient $C_k(V) \to \mathrm{gr}_k C(V,Q)$.

Exercise 8.7. Show that $\mathrm{gr}C(V,Q) \cong \bigwedge^* V$ as algebra and σ extends σ_k in the sense that if $a \in C_k(V)$, $\sigma(a)_{[k]} = \sigma_k(a)$.

Now, we are in place to study the complex representation of $C(V)$.

Theorem 8.10 (Spinor Module)**.** *Let V be an even dimensional, oriented vector space, Q be a positive definite quadratic form induced from an inner product $\langle\, , \rangle$ on V.*

(1) *There exists a unique Clifford module $S = S^+ \oplus S^-$ such that $C(V) \otimes \mathbb{C} \cong \mathrm{End}(S)$.*
(2) *For any finite-dimensional Clifford module E over $\mathbb{C}$, there exists a $\mathbb{Z}_2$-graded vector space W with trivial $C(V)$-action such that $E = W \otimes S$.*

The module S is called the *spinor module* or *(half)-spinor representation* and (2) implies that it is the building block for any complex representation of $C(V)$. The space W in (2) is called the *twisting space* for the Clifford module E.

SKETCH OF PROOF. For (1), we endow V with an almost complex structure J such that J is $\langle\, , \rangle$-invariant. As in section 6.5, we know that

$$V \otimes_{\mathbb{R}} \mathbb{C} \cong V^{1,0} \bigoplus V^{0,1}.$$

We define $S = \bigwedge^*(V^{1,0})$ and notice that $S = S^+ \oplus S^-$ where $S^+ = \bigwedge^{\mathrm{even}} V$ and $S^- = \bigwedge^{\mathrm{odd}} V$. We next define a $C(V)$-action on S by: for

$v \in V \otimes_{\mathbb{R}} \otimes \mathbb{C}$, $v = w + \bar{w}$ with $w \in V^{1,0}$, $\bar{w} \in \overline{V^{1,0}} \cong V^{0,1}$,

$$c(w) \cdot s = \sqrt{2} w \wedge s$$
$$c(\bar{w}) \cdot s = -\sqrt{2} \iota_{\bar{w}} s.$$

Observe that $C(V, Q) \otimes_{\mathbb{R}} \mathbb{C} = C(V \otimes_{\mathbb{R}} \mathbb{C}, Q \otimes_{\mathbb{R}} \mathbb{C})$. Combining with the calculation in example 8.9 and the fact that $\langle \, , \, \rangle$ is hermitian (cf. lemma 6.37), one can see easily that this defines a Clifford action on S. In summary, we have $c : C(V) \otimes \mathbb{C} \hookrightarrow \mathrm{End}(S)$. This is an isomorphism by dimension counting:

$$\dim_{\mathbb{C}} C(V) \otimes_{\mathbb{R}} \mathbb{C} = 2^{\dim V} = (\dim_{\mathbb{C}} S)^2.$$

For (2), one simply take $W := \mathrm{Hom}_{C(V)}(S, E)$, the space of linear maps from S to E commuting with the Clifford action. Thus, W carries a trivial Clifford action and has a $\mathbb{Z}_2$-graded given by $W^+ = \mathrm{Hom}_{C(V)}(S^\pm, E^\pm)$, $W^- = \mathrm{Hom}_{C(V)}(S^\pm, E^\mp)$. In fact, any any finite dimensional End$S$-module is of the form $W \otimes S$. □

Remark 8.11. In problem 7.21, we define the spin group $\mathrm{Spin}(n)$ as double covering of $SO(n)$. As one will see in problem 8.7, $\mathrm{Spin}(n)$ can be defined as a subset of $C(V)$ given by

$$\mathrm{Spin}(V) = \{v_1 \ldots v_{2k} : v_i \in V, k \geq 0, |v_i| = 1\}.$$

Hence, any representation of $C(V)$ gives rise to a linear representation of $\mathrm{Spin}(V)$. This is the reason why S is called the spinor representation and we denote $\Delta : \mathrm{Spin}(V) \to GL(S)$ by the corresponding representation.

Now, we generalize everything above on a Riemannian manifold. Let (M^{2m}, g) be an oriented Riemannian manifold.

Definition 8.12. The *Clifford bundle* is defined by $C(M) := \bigcup_{x \in M} C(T_x^* M)$, the Clifford algebras of the cotangent space $T_x^* M$.

By choosing a local orthonormal frame $e_1, \ldots, e_{2m} \in C^\infty(U, T^* M)$, the Clifford bundle $\pi : C(M) \to M$ is trivialized by $\pi^{-1}(U) \cong U \times C(\mathbb{R}^n)$. On $C(M)$, one can define the symbol map $\sigma : C(M) \to$

$\bigwedge T^*M$ by pointwise symbol map $\sigma_x : C(T_x^*M) \to \bigwedge T_x^*M$. It is easy to see that this is a bundle isomorphism between $C(M)$ and $\bigwedge T^*M$.

Exercise 8.8. Show that $C(M) = \bigoplus_{r\geq 0}(T^*M)^{\otimes r}/\mathcal{I}$, where $\mathcal{I}$ is the subbundle whose fiber $\mathcal{I}_p$ is the 2-sided ideal generated by $v \otimes v + g(v,v)$, where $v \in T_p^*M$

Definition 8.13. A *Clifford module* E on an even diemensional Riemannian manifold M is a $\mathbb{Z}_2$-graded vector bundle $E = E^+ \oplus E^-$ on M with the smooth action of Clifford bundle $C(M)$, which we denote

$$C^\infty(M, C(M)) \times C^\infty(M,E) \to C^\infty(M,E); \quad (a,s) \mapsto c(a)s.$$

If W is a vector bundle, the *twisted Clifford bundle* of E is just the bundle $W \otimes E$ with Clifford action $1 \otimes c(a)$.

It is natural to generalize spinor module S in theorem 8.10 to a vector bundle $S \to M$, called *spinor bundle*. If so, any Clifford module E is just a twisted Clifford module of S by taking $W = \mathrm{Hom}_{C(M)}(S,E)$. Locally, on a coordinate chart U, $\pi^{-1}(U) \cong U \times C(\mathbb{R}^n)$, one can always define $S(U)$ locally. However, there are topological obstructions to patch these $S(U)$ globally. In general, we say M has a *spin structure*[7] or M is a *spin manifold* if an associated spinor bundle $S \to M$ is defined.

Remark 8.14. For the application to local index theorem, we do not need M to be spin since our proof will be local. Thus, we denote S by locally defined and unique spinor bundle and decompose Clifford module E as $\mathrm{Hom}_{C(M)}(S,E) \otimes S$.

[7]To define the spin structure on M properly, we need the language of principal bundle (cf. problem 7.9)). We say M has a spin structure if the frame bundle $P_{SO(TM)}$ of TM can be lifted to a principal $\mathrm{Spin}(2m)$-bundle $P_{\mathrm{Spin}(M)}$. With the spin structure, one can then construct spinor bundle by $P_{\mathrm{Spin}(M)} \times_\Delta S$, where $\Delta : \mathrm{Spin}(2m) \to S$ is the spinor representation, cf. remark 8.11. Using the theory of Čech cohomology (cf. [LM89] Chapter II, §1), one can show that the topological obstruction for this lifting is exactly $w_2(M) := w_2(TM)$, the second Stiefel–Whitney class of M (cf. fact 7.30). In other words, M is spin if and only if $w_2(M) = 0$.

Let ∇^{LC} be the Levi–Civita connection of (M, g). Then since ∇^{LV} is metrical, it is easy to see that ∇^{LC} descends to a connection on the quotient bundle $C(M)$ (cf. exercise 8.8) satisfying

$$\nabla_X^{LC}(ab) = (\nabla_X^{LC} a)b + a(\nabla_X^{LC} b), \forall a, b \in C^\infty(M, C(M)), X \in C^\infty(M, TM).$$

Similarly, on a Clifford module $E \to M$, we define

Definition 8.15. A connection ∇^E on E is called a *Clifford connection* if $\forall s \in C^\infty(M, E), X \in C^\infty(M, E), a \in C^\infty(M, C(M))$,

$$\nabla_X^E(c(a)s) = c(\nabla_X^{LC} a)s + c(a)\nabla_X^E s.$$

Lemma 8.16. *For any Clifford module E, there always exists a Clifford connection ∇^E on E.*

PROOF. If M is spin, then the Levi–Civita connection ∇^{LC} on TM induces[8] to a canonical connection still denoted by ∇^{LC} on the spinor bundle $S \to M$ and it is a Clifford connection. Since $E = S \otimes W$, pick any connection ∇^W on W, this induces a Clifford connection ∇^E by

$$\nabla^{LC} \otimes id + id \otimes \nabla^W. \tag{8.5}$$

In general, if M is not spin, one can still define spinor bundle S locally and define ∇^{LC} on S locally. Thus, on any Clifford module $E \to M$, we can locally split E into $S \otimes W$ and define local connection as (8.5). We then use partition of unity to glue these local operators to get a Clifford connection ∇^E on E. □

We now in place to define *Dirac operator* on a Clifford module E with a Clifford connection ∇^E. We define the Dirac operator $D : C^\infty(M, E) \to C^\infty(M, E)$ by the composition

$$C^\infty(M, E) \xrightarrow{\nabla^E} C^\infty(M, T * M \otimes E) \xrightarrow{c} C^\infty(M, E).$$

In local coordinate, the operator is given by $D = \sum_i c(dx^i)\nabla^E_{\partial_i}$.

Proposition 8.17. *D is an elliptic operator.*

[8]Since ∇^{LC} corresponds to a connection on the orthonormal frame bundle $P_{SO(TM)}$. Since M is spin, the connection 1-form on $P_{SO(TM)}$ lifts to $P_{\mathrm{Spin}(M)}$. Hence, it gives rise to a connection on the associated vector bundle $P_{\mathrm{Spin}(M)} \times_\Delta S$.

PROOF. Let $\xi = \xi_i dx^i \in T_p^*M \setminus \{0\}$, the symbol of the differential operator $p_D(x, \xi) = \sum_i c(dx_i) \cdot \xi_i = c(\xi) \neq 0$. □

Exercise 8.9. Show that if $c(a)$ is skew–adjoint on E, then D is self–adjoint.

It is clear from definition that $D : C^\infty(M, E^\pm) \to C^\infty(M, E^\mp)$. That is, we can write $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$.

Now, back to McKean–Singer formula in section 8.1, let E be a Clifford bundle, D be a Dirac operator on E. We denote $k_t(x, y)$ by the heat kernel of D^2. Since E locally splits into $S \otimes W$, the diagonal of heat kernel $k_t(x, x)$ has values in $\mathrm{End}(E_x) = \mathrm{End}(S_x) \otimes \mathrm{End}(W_x) = C(M)_x \otimes \mathrm{End}(W_x)$. By McKean–Singer formula, we know that

$$\mathrm{ind} D = \int_M \mathrm{str}_{S\otimes W} k_t(x, x) dv_M(x),$$

where the $\mathrm{str}_{S\otimes W}$ means that the supertrace is taken on $C(M)_x \otimes \mathrm{End}(W_x)$. Obviously, we have

$$\mathrm{str}_{S\otimes W} k_t(x, x) = \mathrm{str}_S k_t(x, x) \cdot \mathrm{str}_W k_t(x, x).$$

We end this section by a relation between supertrace of $C(V)$ acting on S and symbol maps $\sigma : C(V, Q) \to \bigwedge^* V$. For $a \in C(V)$, we define

$$\mathrm{str}(a) = \begin{cases} \mathrm{tr}_{S^+}(a) - \mathrm{tr}_{S^-}(a) & a \in C^+(V) \\ 0 & a \in C^-(V). \end{cases}$$

Lemma 8.18. *Let $V = \mathbb{R}^n$ with even n. $\mathrm{str}(a) = (-2i)^{n/2} T(\sigma(a))$, where $T : \bigwedge^* V \to \mathbb{R}$ is defined by the coefficients of α in the monomial $e_1 \wedge \cdots \wedge e_n$, where $\{e_1, \ldots, e_n\}$ is an oriented orthonormal basis of V.*

SKETCH OF PROOF. On $C(V)$, we have $\mathbb{Z}_2$-commutator $[u, v] := uv - (-1)^{|u||v|} vu$, where $|u|$, $|v|$ are parities of u, v, respectively. One can check by case by case that $\mathrm{str}([u, v]) = 0$, for any $u, v \in C(V)$.

Next, observe that $C_{n-1}(V) = [C(V), C(V)]$. Let $e_1, \ldots, e_n$ be an oriented orthonormal basis of V, then for any $I \subset \{1, \ldots, n\}$ with

$|I| < n$, say $j \notin I$, then one can easily show that

$$e_I = -\frac{1}{2}[e_j, e_j e_I],$$

where $e_I = \prod_{i\in I} e_i$. Obviously, this implies that $C_{n-1}(V) = [C(V), C(V)]$.

Thus, we see that str must proportional to $T \circ \sigma$, say $\mathrm{str}(a) = cT\sigma(a)$, for some constant c. To determine this constant, let us consider the *chirality element* $\epsilon := i^p e_1 \dots e_n \in C(V) \otimes \mathbb{C}$, where

$$p = \begin{cases} n/2 & n \text{ even} \\ (n+1)/2 & n \text{ odd} \end{cases}.$$

Since n is even, then $\epsilon^2 = 1$ obviously. It is easy to see that $S^\pm = \{v : \epsilon v = \pm v\}$. Therefore, $\mathrm{str}\epsilon = \dim S^+ + \dim S^- = 2^{n/2}$ while $T(\sigma(\epsilon)) = T(i^p e_1 \wedge e_n) = i^p = i^{n/2}$. Hence, the constant $c = (-2i)^{n/2}$. □

Exercise 8.10. Complete the proof of lemma 8.18 by showing the following

(1) Show that $\mathrm{str}([u, v]) = 0$, for any $u, v \in C(V)$.
(2) Let $\{e_1, \dots, e_n\}$ be an oriented orthonormal basis of V and $I \subset \{1, \dots, n\}$. If $\exists j \notin i$, prove that

$$\prod_{i\in I} e_i = -\frac{1}{2}[e_j, e_j e_I].$$

(3) Show that $S^\pm = \{v \in S : \epsilon v = \pm v\}$.

8.4. Local Index Theorem

Let M be an oriented[9] Riemannian manifold of even dimension n, D be a Dirac operator on a Clifford module $E \to M$ associated to a Clifford connection ∇^E. We now let $k_t(x, y)$ be the the heat kernel of D^2 coupled with the volume form $dv_M(x)$ and denote $\langle x|e^{-tD^2}|x\rangle := k_t(x, x)$ by the restriction of it to the diagonal. Note that $k_t(x, x) \in$

[9]In fact, M need not be oriented. We only need to define integration on M. When M is not orientable, we just replace $\bigwedge^n TM$ by $|\bigwedge^n T^*M|$ to get the volume form.

$C^\infty(M, \mathrm{End}(E) \otimes \bigwedge^* TM)$. Since $E = S \otimes W$ locally, $\mathrm{End}(E)_x = C(M)_x \otimes \mathrm{End}(W)_x$, for all $x \in M$. We denote $C_i(M)$ by the subbundle of $C(M)$ with degree$\leq i$.

Now, we are ready to state and prove the local index theorem.

Theorem 8.19 (Local Index Theorem)**.** *$k_t(x,x)$ has an asymptotic expansion*

$$k_t(x,x) \sim (4\pi t)^{-n/2} \sum_{i=0}^{\infty} t^i k_i(x).$$

(1) *The coefficient $k_i(x) \in C^\infty(M, C_{2i}(M) \otimes \mathrm{End}(W) \otimes \bigwedge^n TM)$.*
(2) *Let $\sigma(k) := \sum_{i=0}^{n/2} \sigma_{2i}(k_i) \in A^{\cdot}(\mathrm{End}(W))$ be the negative degree pieces in t, then*

$$\sigma(k) = \det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right) e^{-F^W},$$

where F^W is the curvature of W, R is the Riemannian curvature on M.

Before proving theorem 8.19, let us first show that how it deduce the usual form of the Index theorem for Dirac operator. By McKean–Singer formula (8.2), if M is furthermore compact, then for $t > 0$,

$$\mathrm{ind} D = \int_M \mathrm{str}(k_t(x,x))dx.$$

Notice that by lemma 8.18 and theorem 8.19, supertraces and $(-2i)^{n/2} T \circ \sigma$ coincides and vanishes on $C_i(M)$ for $i < n$. Hence, we have

$$\mathrm{str} k_t(x,x) \sim \sum_{i \geq n/2} t^i \mathrm{str}(k_i(x)),$$

and thus there are no poles in the asymptotic expansion for $\mathrm{str}(k_t(x,x))$. Consequently, the integrand has a limit as $t \to 0+$ and $\mathrm{ind} D$ is independent of t, and we conclude

$$\mathrm{ind} D = (2\pi i)^{-n/2} \int_M \mathrm{tr}(\sigma(k))_{[n]} = \int_M \left[\det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right) \mathrm{tr}_W(e^{-F^W})\right]_n$$

or in the more familiar form in the literatures:

Theorem 8.20 (Atiyah, Singer). *Let M^n be compact, oriented even-dimensional manifold, the index of a Dirac operator D on a Clifford module E (locally splits as $S \otimes W$) is given by*

$$\operatorname{ind} D = (2\pi i)^{-n/2} \int_M \det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right) \operatorname{tr}_W(e^{-F^W}),$$

Remark 8.21. If we put back the factor $(2\pi i)^{-n/2}$ into the integrand, then it will become $\det^{1/2}\left(\frac{R/2\pi}{\sinh(R/4\pi i}\right) \operatorname{tr}_W(e^{F^W})$. In view of problem 7.8 and example 7.19, we recognize that the Atiyah–SInger index theorem is then given by

$$\operatorname{ind} D = \int_M \hat{A}(M)\operatorname{ch}(F).$$

The rest of of this section is dedicated to prove theorem 8.19. We first need a Bochner type formula for Dirac operator (cf. section 4.4). Let $E \to M$ be a Clifford module, ∇^E be a Clifford connection on E, $\operatorname{tr}(\nabla^E)^2$ be the connection Laplacian of ∇^E defined in definition 4.18, and s_M be the scalar curvature of M. Assume $E = W \otimes S$ locally. The Lichnerowicz formula states that

Theorem 8.22 (Lichnerowicz Formula).

$$D^2 = -\operatorname{tr}(\nabla^E)^2 + c(F^W) + \frac{s_M}{4},$$

where $F^W \in A^2(M, \operatorname{End}(W))$ is the curvature of W. Here, $c(F) := \sum_{i<j} F(e_i, e_j)c(e^i)c(e^j)$, where $\{e_i\}$ is an orthonormal frame of TM and $\{e^i\}$ is its dual frame.

Sketch of Proof. It suffices to prove the formula on a coordinate chart U at each point $p \in M$. Pick a normal coordinate $\mathbf{x}$ on U which coincides with an orthonormal frame e_i at p. Let $c^i := c(e^i)$. Then

$$\begin{aligned} D^2 &= c^i \nabla_{e_i}(c^j \nabla_{e_j}) \\ &= c^i c^j \nabla_{e_i}\nabla_{e_j} + c^i c(\nabla_{e_i} e^j)\nabla e_j \quad (\nabla_{e_i} e^j = 0 \text{ at } p) \\ &= -\nabla_{e_i}^2 + \sum_{i<j} c^i c^j (\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i}) \quad (\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} = F^E(e_i, e_j)). \end{aligned}$$

Since $\mathrm{End}(E) \cong C(M) \otimes \mathrm{End}(W)$, one can show that

$$F^E(e_i, e_j) = -\frac{1}{8}\sum_{k,l} R_{klij} c^i c^j c^k c^l + F^W(e_i, e_j) c^i c^j. \tag{8.6}$$

Moreover, using definition of Clifford algebra and symmetry of Riemann curvature tensor (cf. exercise 3.7), one can show that the first term in (8.6) equals to $s_M/4$. $\square$

Exercise 8.11. Show the formula (8.6) and $-\sum_{i,j,k,l} frac18 R_{klij} c^i c^j c^k c^l = s_M/4$.

Recall that $E \to M$ is a Clifford module on an even–dimensional (oriented) Riemannian manifold M with a Dirac operator D associated to a Clifford connection ∇^E on E. Fix $x_0 \in M$, let $\mathbf{x} = (x^1, \dots, x^n)$ be a normal coordinate on a Gauss ball U centered at x_0. We pick any basis of E_{x_0}, then for any $x \in U$, $\tau(x_0, x) : E_x \cong E_{x_0}$ via parallel transpor the basis with respect to ∇^E along the geodesic $\exp_{x_0} \mathbf{x}$ (recall that $\mathbf{x} \in T_{x_0}M$). In this way, we trivialize the bundle $E\big|_U$ as $E_{x_0} \times U$ along radial tangent vectors. Thus, $C^\infty(U, E) \cong C^\infty(U, E_{x_0})$. Since E splits locally as $S \otimes W$, we again identify $\mathrm{End}(E_{x_0}) = C(M)_{x_0} \otimes \mathrm{End}(W_{x_0})$. Let $c^i = c(dx^i) \in \mathrm{End}(E_{x_0})$, e_i be local orthonormal frame obtained by parallel transport $\partial_i\big|_{x_0}$ along radial geodesics. We first observe that:

Lemma 8.23. *The function $c(e^i) \in C^\infty(U, \mathrm{End}(E_{x_0}))$ is a constant map with value c^i.*

PROOF. Let $\mathcal{R}$ be a radial vector field on $T_{x_0}M$.

$$\mathcal{R} \cdot c(e^i) = \nabla^E_{\mathcal{R}} c(e^i) = [\nabla^E_{\mathcal{R}}, c(e^i)] = c(\nabla_{\mathcal{R}} e^i) = 0.$$

$\square$

Let $p_t(x, x_0)$ b the heat Kernel of D^2, and $k(t, \mathbf{x}) := \tau(x_0, x) p_t(x, x_0)$, where $x = \exp_{x_0}(\mathbf{x})$. Thus, we can regard $k(t, \mathbf{x})$ as smooth functions with values in $\mathrm{End}(E_{x_0}) = C(T^*_{x_0}M) \otimes \mathrm{End} W$ and $C(T^*_{x_0}M)$ acts on $\bigwedge T^*_{x_0}M$ by $c(\alpha)\beta = \alpha \wedge \beta - \iota_\alpha \beta$ as in example 8.9. We denote $\alpha \wedge \cdot$ by $\epsilon(\alpha)$ and ι_α by $\iota(\alpha)$.

Proposition 8.24. *Let L be the differential operator on U with coefficients in $C(T^*_{x_0}M)\otimes \mathrm{End}W$, defined by the formula*

$$L=-\sum_i\left((\nabla^E_{e_i})^2-\nabla^E_{\nabla_{e_i}e_i}\right)+\frac{s_M}{4}+c(F^W).$$

Then $(\partial_t+L)k(t,\mathbf{x})=0$.

PROOF. By definition, $p_t(x,x_0)\in E_x\otimes E^*_{x_0}$ satisfies the heat equation

$$\left(\frac{\partial}{\partial t}+D_x^2\right)p_t(x,x_0)=0.$$

Fom Lichnerowicz formula and the above lemma $\tau(x_0,x)p_t(x,x_0)\in C^\infty(U,\mathrm{End}E_{x_0})$ satisfies the equation $(\partial_t+L)\tau(x_0,x)p_t(x,x_0)=0$. $\square$

Now, here comes the heart of proof for local index theorem. Namely, the rescaling procedure due to Getzler in [Get83] and [Get86] with respect to the degree in exterior algebra $\bigwedge T^*_{x_0}M$. For $\alpha\in\mathcal{A}:=C^\infty(\mathbb{R}^+\times U,\bigwedge T^*_{x_0}M\otimes\mathrm{End}(W_{x_0}))$, we define the rescaling of α by

$$(\delta_u\alpha)(t,\mathbf{x})=\sum_{i=0}^n u^{-i/2}\alpha(ut,u^{1/2}\mathbf{x})_{[i]},$$

where $\alpha_{[i]}$ is the degree i component of α. Then one can check easily that the rescaling δ_u acts on operators on $\mathcal{A}$ by

$$\delta_u\phi(\mathbf{x})\delta_u^{-1}=\phi(\sqrt{u}\mathbf{x}),\quad \text{for }\phi\in C^\infty(U),\tag{8.7}$$

$$\delta_u\frac{\partial}{\partial t}\delta_u^{-1}=u^{-1}\frac{\partial}{\partial t},\tag{8.8}$$

$$\delta_u\frac{\partial}{\partial x^i}\delta_u^{-1}=u^{-1/2}\frac{\partial}{\partial x^i},\tag{8.9}$$

$$\delta_u\epsilon(\alpha)\delta_u^{-1}=u^{-1/2}\epsilon(\alpha),\quad\text{for }\alpha\in T^*M_{x_0},\tag{8.10}$$

$$\delta_u\iota(\alpha)\delta_u^{-1}=\sqrt{u}\iota(\alpha).\tag{8.11}$$

Exercise 8.12. Verify (8.7) to (8.11).

Definition 8.25. The *rescaled heat kernel* $r(u,t,\mathbf{x})$ is defined by

$$r(u,t,\mathbf{x})=u^{n/2}(\delta_u k)(t,\mathbf{x}).$$

The factor $u^{n/2}$ is included because $k_t(t,\mathbf{x})$ has volume form[10] $\bigwedge^n T^*M$ part . Let us first indicate how we will approach the proof of theorem 8.19. Assume that $k(t,\mathbf{x})$ has an asymptotic expansion of the form at $x_0 \in U$ as indicated in theorem 8.19 (1):

$$k(t,\mathbf{x}) \sim (4\pi t)^{-n/2} q_t(\mathbf{x})(k_0 + k_1 t + \cdots + k_{\frac{n}{2}} t^{n/2} + \cdots),$$

where $k_i(x) \in C^\infty(U, \bigwedge^{\leq 2i}(T^*_{x_0}M) \otimes \mathrm{End}(W_{x_0}) \otimes \bigwedge^n T^*_{x_0}M)$ under the identification $\sigma : C(T^*_{x_0}M) \cong \bigwedge(T^*_{x_0}M)$ and $q_t(\mathbf{x}) = e^{\frac{-|\mathbf{x}|^2}{4t}}$ is the heat kernel for standard Laplacian on $\mathbb{R}^n$. Then notice that $(4\pi t)^{-n/2} q_t(x)$ is invariant under rescaling and thus

$$r(u,t,\mathbf{x}) \sim (4\pi t)^{-n/2} q_t(\mathbf{x}) u^{n/2}(k_0 + \delta_u(k_1)ut + \cdots + \delta_u(k_{n/2})(ut)^{n/2} + \cdots)$$

Since $\delta_u(k_i)(ut)^i = t^i \sum_{j=0}^{i} u^{i-j/2} k_i(ut, \sqrt{u}\mathbf{x})_{[(j)]}$ and higher order terms have positive u-degree, we then see that the total negative part is singled out by

$$\lim_{u\to 0} r(u, t=1, \mathbf{x}=0) = (4\pi)^{-\frac{n}{2}} \left((k_0)_{[0]} + (k_1)_{[2]} + \cdots + (k_{\frac{n}{2}})_{[n]} \right).$$

Thus, the content of (2) in theorem 8.19 is that the image of of right–hand side under σ is exactly $\det^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right) e^{-F^W}$.

To start, let us notice that the rescaled heat kernel $r(u,t,\mathbf{x}) := \sqrt{u}^n (\delta_u k)(t,\mathbf{x})$ is the heat kernel of a recaled heat equation.

Lemma 8.26.

$$\left(\frac{\partial}{\partial t} + L(u)\right) r(u,t,\mathbf{x}) = 0,$$

where $L(u) := u\delta_u L \delta_u^{-1}$.

PROOF. This is a direct consequence of (8.8). Since $\partial_t = u\delta_u \partial_t \delta_u^{-1}$, we have

$$\begin{aligned}(\partial_t + u\delta_u L\delta_u^{-1})r &= u\delta_u(\partial_t + L)\delta_u^{-1} r = u\delta_u(\partial_t + L)\delta_u^{-1} u^{n/2}(\delta_u k)(t,\mathbf{x}) \\ &= u\delta_u u^{n/2}(\partial_t + L)k(t,\mathbf{x}) = 0.\end{aligned}$$

□

[10] or density $|\bigwedge^n T^*M|$ if M is not orientable

Recall that $L = -\sum_i \left((\nabla^E_{e_i})^2 - \nabla^E_{\nabla_{e_i} e_i} \right) + \frac{s_M}{4} + c(F^W)$ on U, and $L(u) = u\delta_u L \delta_u^{-1}$. Using (8.7) to (8.10) and lemma 8.23 , we now decompose $L(u)$ into $L_1(u) + L_2(u)$, where

$$
\begin{aligned}
L_1(u) &= -\sum_i \left(\sqrt{u}\delta_u \nabla^E_{e_i} \delta_u^{-1} \right)^2 \\
&\quad + \sum_{i<j} F^W_{ij}(\sqrt{u}\mathbf{x})\sqrt{u}\left(\sqrt{u}^{-1}\epsilon^i - \sqrt{u}\iota^i \right) \cdot \sqrt{u}\left(\sqrt{u}^{-1}\epsilon^j - \sqrt{u}\iota^j \right), \\
L_2(u) &= \frac{u}{4}s(\sqrt{u}\mathbf{x}) + \sqrt{u}\left(\sqrt{u}\delta_u \nabla^E_{\nabla_{e_i} e_i} \delta_u^{-1} \right).
\end{aligned}
$$

Now, we have the following explicit formula for covariant derivative $\nabla^E_{\partial_i}$ acting on $C^\infty(U, E)$.

Lemma 8.27. *In the normal coordinate $(U, \mathbf{x})$ centered at x_0 with the trivialization $TM\big|_U \cong T_{x_0}M \times U$, $E\big|_U \cong E_{x_0} \times U$ (via parallel transport along radial line), the Clifford connection ∇^E of the Clifford bundle $E = S \otimes W$ is locally given by*

$$
\nabla^E_{\partial_i} = \partial_i + \frac{1}{4}\sum_{j;k<l} R_{klij}\mathbf{x}^j c^k c^l + \sum_{k<l} f_{ikl}(\mathbf{x})c^k c^l + g_i(\mathbf{x}),
$$

where $R_{klij} = \langle R(\partial_i, \partial_j)\partial_l, \partial_k \rangle(x_0)$ is the Riemannian curvature at x_0, $c^i := c(dx^i) \in \mathrm{End}(E)$, $f_{ikl}(\mathbf{x}) = O(|\mathbf{x}|^2) \in C^\infty(U)$ and $g_i(\mathbf{x}) = O(|\mathbf{x}|) \in C^\infty(U, \mathrm{End}(E))$.

Sketch of Proof. Let $\mathcal{R} := \sum_i \mathbf{x}^i \partial_i$ be a radial vector, $\nabla^E = d + \omega$, where ω is the connection 1-form of ∇^E. We write $\omega = (\sum_i \Gamma^k_{ij} d\mathbf{x}^j)$ into matrix-valued 1-form and let $\omega_j = (\Gamma^k_{ij})$ be the matrix. Notice that in such choice of frame, $\iota_{\mathcal{R}}\omega = 0$ since we trivialize the bundle along $\mathcal{R}$. Thus, by Cartan's formula

$$
\mathcal{L}_{\mathcal{R}}\omega = (\iota_{\mathcal{R}} d + d\iota_{\mathcal{R}})\omega = \iota_{\mathcal{R}} d\omega = \iota_{\mathcal{R}}(d\omega + \omega \wedge \omega) = \iota_{\mathcal{R}}(F^E).
$$

Next, we take Taylor's expansion on both sides.

$$
\begin{aligned}
\mathcal{L}_{\mathcal{R}}\omega &= \mathcal{L}_{\mathcal{R}}\left(\sum_{j,\alpha} \partial^\alpha \omega_j(x_0) \frac{x^\alpha}{\alpha!} dx^j \right) \\
&= \sum_{j,\alpha} \frac{\partial^\alpha \omega_j(x_0)}{\alpha!} \left(\mathcal{R}(x^\alpha) dx^j + x^\alpha \mathcal{L}_{\mathcal{R}}(dx^j) \right).
\end{aligned}
$$

Since $\mathcal{R} = \sum_i x^i \partial_i$ is the Euler vector field, $\mathcal{R}(x^\alpha) = |\alpha| x^\alpha$ and $\mathcal{L}_\mathcal{R} = d\iota_\mathcal{R} dx^j = dx^j$. Therefore, $\mathcal{L}_\mathcal{R}\omega = \sum_{\alpha,j}(|\alpha|+1)\partial^\alpha \omega_j(x_0)\frac{\mathbf{x}^\alpha}{\alpha!}dx^j$. For the right hand side, we have $\sum_{\alpha,j,k}\partial^\alpha F^E(\partial_k,\partial_j)(x_0)\frac{\mathbf{x}^k\mathbf{x}^\alpha}{\alpha!}dx^j$. Comparing the coefficients.

$$\sum_\alpha (|\alpha|+1)\partial^\alpha \omega_j(x_0)\frac{\mathbf{x}^\alpha}{\alpha!} = \sum_{\alpha,k}\partial^\alpha F^E(\partial_k,\partial_j)(x_0)\frac{\mathbf{x}^k\mathbf{x}^\alpha}{\alpha!},$$

Now, we pick $\alpha = l$, we get $2\partial_l \omega_j(x_0) = F(\partial_j l, \partial_j)(x_0)$ by comparing the coefficients of x^α. Finally, by (8.6), we have

$$\begin{aligned} F^E(e_i,e_j)(x_0) = \frac{1}{2}\sum_{i<j;k<l}&\langle R(\partial_i,\partial_j)e_k,e_l\rangle(x_0)c^k c^l d\mathbf{x}^i \wedge d\mathbf{x}^j \\ &+ F^W(e_i,e_j)c^i c^j(x_0), \end{aligned}$$

where $R(\partial_i,\partial_j)e_k,e_l\rangle(x_0) = -R_{klij}$ since $e_i = \partial_i$ at x_0. $\square$

Hence by above lemma and (8.7) to (8.11), $\nabla^{E,u}\partial_i := \sqrt{u}\delta_u\nabla^E_{\partial_i}\delta_u^{-1}$ equals to

$$\begin{aligned} &\partial_i + \sqrt{u}\frac{1}{4}\sum_{j;k<l}R_{klij}\sqrt{u}x^j(\sqrt{u}^{-1}\epsilon^k - \sqrt{u}\iota^k)(\sqrt{u}^{-1}\epsilon^l - \sqrt{u}\iota^l) \\ &\quad + \sqrt{u}\sum_{k<l}f_{ikl}(\sqrt{u}\mathbf{x})(\sqrt{u}^{-1}\epsilon^k - \iota^k)(\sqrt{u}^{-1}\epsilon^l - \sqrt{u}\iota^l) + \sqrt{u}g_i(\sqrt{u}\mathbf{x}) \\ &= \partial_i + \frac{1}{4}\sum_{j;k<l}R_{klij}x^j(\epsilon^k - u\iota^k)(\epsilon^l - u\iota^l) \\ &\quad + \sqrt{u}^{-1/2}\sum_{k<l}f_{ikl}(\sqrt{u}\mathbf{x})(\epsilon^k - u\iota^k)(\epsilon^l - u\iota^l) + \sqrt{u}g_i(\sqrt{u}\mathbf{x}). \end{aligned}$$

Since $f_{ikl}(\sqrt{u}\mathbf{x}) = O(|\sqrt{u}\mathbf{x}|^2) = uO(|\mathbf{x}|^2)$, we see that as $u \to 0$, $\nabla^{E,u}_{\partial_i}$ has a limit

$$\nabla^{E,0}_{\partial_i} = \partial_i + \frac{1}{4}R_{klij}\mathbf{x}^j\epsilon^k\epsilon^l = \partial_i + \frac{1}{4}R_{ij}\mathbf{x}^j,$$

where $R_{ij} = \sum_{k<l}R_{klij}\epsilon^k\epsilon^l$ is the curvature 2-form.

Clearly, $L_2(u) = \frac{u}{4}s(\sqrt{u}\mathbf{x}) + \sqrt{u}(\sqrt{u}\nabla^E_{\nabla_{e_i}e_i}\delta_u^{-1}) \to 0$ as $u \to 0$. For $L_1(u)$, as $u \to 0$, $e_i \to e_i(0) = \partial_i$, we see that

$$-\sum_i(\sqrt{u}\delta_u\nabla^E_{e_i}\delta_u^{-1})^2 \to -\sum_i(\nabla^{E,0}_{\partial_i})^2 = -(\partial_i + \frac{1}{4}R_{ij}\mathbf{x}^j)^2$$
$$\sum_{i<j}F^W_{ij}(\sqrt{u}\mathbf{x})(\epsilon^i - u\iota^i)(\epsilon^j - u\iota^j) \to \sum_{i<j}F^W_{ij}(0)\epsilon^i\epsilon^j = F^W(x_0).$$

In conclusion, we have shown that

$$L(u) = K + O(\sqrt{u}),$$

where $K := \sum_i \left(\partial_i - \frac{1}{4}\sum_j R_{ij}\mathbf{x}^j\right) + F^W(x_0)$. Recall that K is exactly a generalized harmonic oscillator considered in section 8.2. Therefore, by theorem 8.5, we already construct an explicit solution:

$$p(\mathbf{x}, t) = \frac{1}{(4\pi t)^{n/2}}\left(\det\frac{tR/2}{\sinh(tR/2)}\right)^{1/2} e^{\frac{-1}{4t}\langle \mathbf{x}\left|\frac{tR}{2}\coth\frac{tR}{2}\right|\mathbf{x}\rangle} e^{-tF}.$$

Thus, we get

$$\lim_{u\to 0} r(u, t, \mathbf{x}) = \frac{1}{(4\pi t)^{n/2}}\left(\det\frac{tR/2}{\sinh(tR/2)}\right)^{1/2} e^{\frac{-1}{4t}\langle \mathbf{x}\left|\frac{tR}{2}\coth\frac{tR}{2}\right|\mathbf{x}\rangle} e^{-tF}.$$

Particularly, put $\mathbf{x} = 0$, $t = 1$, we get

$$\sigma(k) = \frac{1}{(4\pi)^{n/2}}\det{}^{1/2}\left(\frac{R/2}{\sinh R/2}\right)e^{-F}.$$

This completes the proof of local index theorem (theorem 8.19).

8.5. Applications of Atiyah–Singer Index Theorems

In this section, we see that Atiyah–Singer index theorem (cf. theorem 8.20) is the generalization for many important formulae in the history of geometry. More precisely, we will see that both Gauss–Bonnet–Chern theorem and Hirzebruch signature theorem are consequences of it. For Hirzebruch–Riemann–Roch theorem, which is the higher dimension generalization for classical Riemann–Roch formula on compact Riemann surface, one can consult problem 8.10.

First of all, by example 8.9, for any compact oriented even-dimensional Riemannian manifold M^n, $\bigwedge T^*M$ is a Clifford module with Clifford action given by

$$c(\alpha)\beta = \epsilon(\alpha) - \iota(\alpha)\beta, \alpha \in A^1(M), \beta \in A(M).$$

Clearly, the Levi–Civita connection on $\bigwedge T^*M$ is compatible with this Clifford connection.

Lemma 8.28. *The Dirac operator D associated to the Cliiford module $\bigwedge T^*M$ and its Levi–Civita connection ∇ is the operator $d + d^*$, where d is Cartan's exterior derivative and d^* is the formal adjoint of d.*

PROOF. This is a direct consequence of exercise 4.6 since

$$(d + d^*)\beta = \sum_{i=1}^{n} (\epsilon(e^i) - \iota(e^i))\nabla_{e_i}\beta.$$

$\square$

Hence, $D^2 = (d + d^*)^2 = dd^* + d^*d = \triangle$ is exactly the Hodge Laplacian considered in chapter 4. By Hodge decomposition, we know that $\ker D = \ker D^2 = \ker \triangle = H^*_{\mathrm{dR}}(M, \mathbb{R})$. Hence, the index of Dirac operator $D = d + d^*$ on $\bigwedge T^*M = \bigwedge^{\mathrm{ev}} T^*M \bigoplus \bigwedge^{\mathrm{odd}} T^*M$ is given by

$$\begin{aligned}
\operatorname{ind}(d + d^*) &= \dim \ker(D|_{\bigwedge^{\mathrm{ev}} T^*M}) - \dim \ker(D|_{\bigwedge^{\mathrm{odd}} T^*M}) \\
&= \sum_{i=0}^{n} (-1)^i \dim H^i_{\mathrm{dR}}(M, \mathbb{R}) = \chi(M),
\end{aligned}$$

where $\chi(M) = \sum_{i=0}^{n}(-1)^i b_i$ is the Euler characteristic of M.

Now, we compute the twisting curvature F^W for $E = \bigwedge T^*M$. Recall that by theorem 8.10, we have the following identification:

$$S^* \otimes S = \operatorname{End}(S) = C(V)_{\mathbb{C}} \xrightarrow{\sigma} \bigwedge\nolimits_{\mathbb{C}} V$$

where $S = S^+ \bigoplus S^- = \bigwedge V^{1,0}$. Thus, we can identify S^* as the twisting space W of $E = \bigwedge V \otimes_{\mathbb{R}} \mathbb{C}$. Here, the Z_2-grading on $(S^*)^{\mathrm{ev}} \oplus (S^*)^{\mathrm{odd}}$ is the Z_2-grading induced from $\bigwedge V$. Let $\{e^i\}$ be an oriented basis for $V = T_x^*M$. Let $c^i := c(e^i) = \epsilon(e^i) - \iota(e^i)$, $b^i := b(e^i) = \epsilon(e^i) + \iota(e^i)$. Obviously, we have

$$[c^i, c^j] = 2\delta^{ij}, \quad [b^i, b^j] = -2\delta^{ij}, \quad [c^i, b^j] = 0.$$

The curvature 2-form for E is given by

$$\begin{aligned} F^E &= \sum_{i,j;k<l} R_{ijkl}\epsilon^i\iota^j e^k \wedge^l = -\frac{1}{4}\sum_{ij;k<l} R_{ijkl}(c^i + b^i)(c^j - b^j)e^k \wedge e^l \\ &= \frac{-1}{4}\sum_{ij;k<l} R_{ijkl}(c^ic^j - b^ib^j)e^k \wedge e^l = F^S + F^W. \end{aligned}$$

where we use the skew-symmetry R_{ijkl} in i and j to eliminate the cross terms. Hence, $F^W = \frac{-1}{4}\sum_{ij;k<l} R_{ijkl}e^i \wedge e^j b^i b^j$.

Now, observe that the supertrance $\mathrm{str}_{\bigwedge V}$ with respect to above grading is closely related to operator b.

Exercise 8.13. Let n be even, ϵ be the chirality element.

(1) $(-1)^p = c(\epsilon)b(\epsilon)$ on $\bigwedge^p V$.

(2) $\mathrm{str}_{\bigwedge V}(b(e_I)) = \begin{cases} 0 & , I \neq= \{1, 2, \dots, n\} \\ (-2i)^{n/2} & I = \{1, 2, 3, \dots, n\}. \end{cases}$.

Let $a = \sum_{i<j} a_{ij}e^ie^j \in \bigwedge^2 V \cong C^2(V)$ via σ with $a = (a_{ij}) \in so(n)$, we define $b(a) := \sum_{i<j} a_{ij}b^ib^j \in \mathrm{End}(\bigwedge V)$. We have the following key result.

Lemma 8.29. *The supertrace* str *of $e^{b(a)}$ with respect to the grading above is given by*

$$\mathrm{str}(e^{b(a)}) = (-2i)^{n/2} \det{}^{1/2}\left(\frac{\sinh(a/2)}{a/2}\right) \mathrm{Pf}(a).$$

PROOF. By splitting principle, we may assume that a is block diagonal of the form $\begin{pmatrix} 0 & -y_i \\ y_i & 0 \end{pmatrix}$, for $i = 1, \dots, n/2$. Then

$$\begin{aligned} \mathrm{str}(e^{b(a)}) &= \mathrm{str}(e^{\sum_{i=1}^{n/2} y_i b^{2i-1}b^{2i}}) = \mathrm{str}\left(\prod_{i=1}^{n/2} e^{y_i b^{2i-1}b^{2i}}\right) \\ &= (-2i)^{n/2}\prod_{i=1}^{n} \sin(y_i). \end{aligned}$$

On the other hand, one can immediately verified that

$$\det{}^{1/2}\left(\left(\frac{\sinh(a/2)}{a/2}\right)\right) = \prod_{i=1}^{n/2}\frac{\sin y_i}{y_i},$$

and $\mathrm{Pf}(a) = \prod_{i=1}^{n/2} y_i$. This completes the proof. $\square$

Hence by plugging F^W by a , we see that $\hat{A}(M)\mathrm{str}(e^{-F^W}) = i^{n/2}\mathrm{Pf}(TM)$. We then obtain

Theorem 8.30 (Chern–Gauss–Bonnet)**.** *The Euler characteristic $\chi(M)$ of a compact even–dimensional oriented manifold is given by*

$$\chi(M) = \int_M e(M),$$

where $e(M) = \frac{1}{(2\pi)^{n/2}}\mathrm{Pf}(TM)$ is the Euler class of M.

Next, we deduce Hirzebruch signature theorem from Aityah–Singer theorem by introducing a different $\mathbb{Z}_2$-grading on $\bigwedge T^*M$. Recall that in section 4.1, we define the Hodge star operator $* : \bigwedge^k T^*M \to \bigwedge^{n-k} T^*M$ with $*^2 = (-1)^{p(n-p)}$ (cf. exercise 4.1). We now define

$$\tilde{*} : \bigwedge^k T^*M \otimes_{\mathbb{R}} \mathbb{C} \to \bigwedge^k T^*M \otimes_{\mathbb{R}} \mathbb{C}$$

by $\tilde{*} := i^{\frac{n}{2}+p(n-p)}*$. Obviously, $\tilde{*}^2 = id$ on $\bigwedge T^*M \otimes \mathbb{C}$. In fact, we have

Exercise 8.14. $\tilde{*} = \epsilon$, where ϵ is chirality element on $C(T^*M) \otimes \mathbb{C}$. Also, under $\bigwedge_{\mathbb{C}} T^*M \cong S \otimes W$, $W = W^+$ is of pure even grading.

Since $\tilde{*}^2 = id$, we can splits the bundle $\bigwedge_{\mathbb{C}} T^*M = \bigwedge_{\mathbb{C}}^+ T^*M \bigoplus \bigwedge_{\mathbb{C}}^- T^*M$ as ±1-eigenspaces of $\tilde{*}$. Notice that if $\dim M = n = 4k$, then ϵ is an element in real Clifford algebra and $\tilde{*}$. We have

$$\bigwedge T^*M = \bigwedge^+ T^*M \oplus \bigwedge^- T^*M,$$

where $\alpha \in A^\pm(M) := C^\infty(M, \bigwedge^\pm T^*M)$ if and only if $\tilde{*}\alpha = \pm\alpha$. We call forms in $A^+(M)$ by *self-dual form* and $A^-(M)$ by *anti self-dual form*. Observe that $d^* = -\tilde{*}d\tilde{*}$ and hence

$$\tilde{*}(d+d^*) + (d+d^*)\tilde{=}0, \quad \tilde{*}\triangle = \triangle\tilde{*}. \tag{8.12}$$

This implies that $D = (d + d^*) : A^+(M) \to A^-(M)$ and

$$\mathbb{H}(M) = \mathbb{H}^+(M) \oplus \mathbb{H}^-(M),$$

where we denote $\mathbb{H}^\pm(M)$ by the space of self-dual/anti self-dual harmonic forms. The index of $D = (d + d^*) : A^+(M) \to A^-(M)$ is given by

Exercise 8.15. If $n = 4k = \dim M$, show that

$$\mathrm{ind}D = \sigma(M),$$

where $\sigma(M)$ is the signature of M. Also, show that if $4 \nmid n$, then $\mathrm{ind}D = 0$.

Since $W = W^+$, $\mathrm{str}_W(e^{-F^W}) = \mathrm{tr}_W(e^{-F^W}) = 2^{-n/2}\mathrm{tr}_{\bigwedge T^*M}(e^{-F^W})$. Similar to the proof of lemma 8.29, we have

$$\mathrm{tr}_W(e^{b(a)}) = 2^{n/2}\det{}^{1/2}(\cosh(a/2)). \tag{8.13}$$

Put $a = F^W$ in (8.13), we have

$$\hat{A}(M)\mathrm{tr}_W(e^{-F^W}) = 2^{n/2}L(M),$$

where $L(M) = \det^{1/2}\left(\frac{R/2}{\tanh(R/2)}\right)$ is the L genus which we have seen in example 7.49. We then obtain the "analytic proof" of Hirzebruch signature theorem:

$$\sigma(M) = (\pi)^{-n/2}\int_M L(M).$$

Exercise 8.16.

(1) Prove (8.13).
(2) Let E be a complex vector bundle, $D : C^\infty(M, \bigwedge^\pm T^*M \otimes E) \to C^\infty(M, \bigwedge^\mp T^*M \otimes E)$ be the twisted signature operator. Show the twisted signature theorem:

$$\mathrm{ind}D = \int_M L(M)\mathrm{ch}(E).$$

8.6. Milnor's Exotic 7-Spheres

In 1956, Milnor constructed a non-standard smooth structures on S^7 in [Mil56]. It is the first known example that a manifold can admit more than one smooth structures. This section serves to be an introduction to this construction.

The rough idea is the following: starting from the sphere bundle $S(E)$ of an oriented four plane bundle E over S^4, one can show the total space M of $S(E)$ is a topological S^7 if the Euler number $e(E) = 1$. In this case, if M is diffeomorphic to the standard S^7, we can attach an 8-disk to the disk bundle $D(E)$ along the boundary M via this diffeomorphism to get a smooth closed 8-manifold W. By applying the Hirzebruch signature theorem to W, we will obtain some divisibility condition on its Pontryagin numbers. By a detailed computation of the characteristic classes, this can not be true for some E, and we get the "exotic spheres".

As a start, let us consider quaternion projective spaces $\mathbb{HP}^m := (\mathbb{H}^{m+1} \setminus \{0\})/\mathbb{H}^\times$ by the right action of $\mathbb{H}^\times$. Similar to real or complex projective space, it has a tautological line bundle $\gamma := \gamma^1_{\mathbb{H}^m} \to \mathbb{PH}^m$, called *Hopf bundle*. It has the sphere bundle $S(\gamma) \cong S^{4m+3} \subset \mathbb{H}^{m+1}$. Notice that the action $\mathbb{H}^\times$ restricts to $S(\gamma)$ is just unitary quaternion $Sp(1) = S^3$. Hence, $S(\gamma)$ is a S^3-principal bundle. This construction is known as the *quanternion Hopf fibration*. Especially, when $m = 1$, then observe that $\mathbb{PH}^1 = S^4$, then this gives the classical Hopf fibration

$$\begin{array}{ccc} S^3 & \hookrightarrow & S^7 \\ & & \downarrow \\ & & S^4 \end{array}$$

Exercise 8.17. Show that $H^*(\mathbb{HP}^m, \mathbb{Z}) = \mathbb{Z}[e]/e^{m+1}$, where $e \in H^4(\mathbb{HP}^m, \mathbb{Z}) = \mathbb{Z}$ is the generator (Hint: You may try to find a CW complex of it). Show that $c(\gamma_\mathbb{C}) = 1 + e$ and $p(\gamma_\mathbb{R}) = 1 - 2e + e^2$, where $\gamma_\mathbb{C}$ and $\gamma_\mathbb{R}$ means the underlying complex vector bundle and real vector bundle of quaternion line bundle γ.

In general, from problem 7.26 and 7.27, we know that oriented real vector bundle of rank 4 (or $SO(4)$-bundle in the language of principal bundle, cf problem refprincipal bundle) over S^4 are classified by $\pi_3(SO(4))$. Moreover, from problem 5.8, we know that $SO(3) \cong \mathbb{RP}^3$ by considering the map $\rho : S^3 \to SO(3)$ defined by using quaternion multiplication

$$\rho(u)v = uvu^{-1}, \tag{8.14}$$

where $v \in S^2$ is considered the unit sphere spanned by purely imaginary quaternions i, j, k. Hence, $\pi_1(SO(3)) = \mathbb{Z}_2$ and $\pi_i(SO(3)) = \pi_I(S^3)$ for $i \geq 2$.

Now consider the principal bundle structure:

$$\begin{tikzcd} SO(3) \arrow[r, hook, "j"] & SO(4) \arrow[d, "p"] \\ & S^3 \end{tikzcd}$$

where p is defined by $p(g) = g \cdot 1$, and define a map $\sigma \colon S^3 \to \mathbf{SO}(4)$ by still using quaternion multiplication:

$$\sigma(u)v = uv. \tag{8.15}$$

Since $p(\sigma(u)) = \sigma(u) \cdot 1 = u \cdot 1 = u$, σ is a section. By problem 7.9 (3), we conclude $SO(4) \cong S^3 \times SO(3)$ is a trivial bundle. So we have

$$\pi_3(SO(4)) \cong \pi_3(S^3) \times \pi_3(SO(3)) = \mathbb{Z} \oplus \mathbb{Z}.$$

Since $SO(3)$ is a Lie group, we have the following general result on multiplications of its homotopy groups.

Lemma 8.31. *Let G be a topological group, then for $k \geq 2$ the (pointwise) multiplication of two homotopy classes in $\pi_k(G)$ corresponds to the composition law of homotopy classes.*

PROOF. Let $\phi_1, \phi_2 : (I^k, \partial I^k) \to (G, e)$ and let ϕ_0 be the constant map e, then clearly have homotopies

$$\phi_1 + \phi_0 \sim \phi_1, \quad \phi_0 + \phi_2 \sim \phi_2$$

Multiply these two homotopies, we get

$$(\phi_1 + \phi_0) \cdot (\phi_0 + \phi_1) \sim \phi_1 \cdot \phi_2$$

By the definition of composition law, the left hand side is exactly $\phi_1 + \phi_2$. This proves the lemma. $\square$

In view of (8.14), (8.15), and above lemma, we see that all clutching maps $\rho : S^3 \to SO(4)$ are of the form

$$f_{hj}(u)v = u^h v u^j,$$

where $h, j \in \mathbb{Z}$. In fact, it is very clear that σ corresponds to the Hopf bundle discussed before because we use right action there, and as a $S^3 = Sp(1)$ bundle its coordinate transform will be the left transformation. So we have an even simpler set of generators, namely the right Hopf bundle γ and the left Hopf bundle $\bar{\gamma}$ defined by left action. It correspons to the homotopy class $\bar{\sigma} := \sigma\rho^{-1}$, that is

$$\bar{\sigma}(u)v = vu. \tag{8.16}$$

Since γ, $\bar{\gamma}$ are isomorphic as real bundles, they have the same Euler class $e = u$, but since the "quaternion orientation" is changed, we will have $p_1 = 2u$.

Moreover, recall that oriented rank n bundle over a manifold X are calssified by $[X, \tilde{G}_{\infty,n}(\mathbb{R})]$. When $X = S^m$, this coincides with our previous discussion.

Exercise 8.18. $[S^m, \tilde{G}_{\infty,n}(\mathbb{R})] \cong \pi_m(\tilde{G}_{n,\infty}(\mathbb{R}) \cong \pi_{m-1}(SO(n))$ (Hint: consider the principal $SO(n)$ fibration given by $SO(n+N)/SO(N) \to \tilde{G}_{n,N+n}(\mathbb{R})$ and use fact 7.41).

Now, we will show both $e, p_1 : \pi_4(\tilde{G}_{4,\infty}(\mathbb{R})) \to H^4(S^4)$ can be regarded as group homomorphism. For example, let $[f] \in \pi_4(\tilde{\mathbf{G}}_4)$, p_1 is the map

$$[f] \mapsto p_1(f^*\tilde{\gamma}^4)([S^4])$$

so $p_1(f^*\tilde{\gamma}_4)([S^4]) = f^*(p_1\tilde{\gamma}^4)([S^4]) = p_1\tilde{\gamma}^4(f_*[S^4])$ by the definition of f^* and f_*, and the last map $[f] \mapsto f_*([S^4])$ is exactly the Hurwicz homomorphism

$$\pi_4(\tilde{G}_{4,\infty}(\mathbb{R})) \to H_4(\tilde{G}_{4,\infty}(\mathbb{R})).$$

Now, here comes a crucial arithmetic fact on characteristic classes of these $SO(4)$-bundle over S^4.

Proposition 8.32. *The $SO(4)$ bundle E_{hj} defined by f_{hj} has $e(E_{hj}) = (h+j)u$ and $p_1(E_{hj}) = 2(h-j)u$. In another word, for $k \equiv l \pmod 2$, there is an unique $SO(4)$ bundle E such that $p_1(E) = 2ku$ and $e(E) = lu$.*

SKETCH OF PROOF. The proposition is ovbious by looking at the characteristic classes of the right and left hopf bundles γ and $\bar{\gamma}$. Another way to see this is to see the tangent bundle TS^4, which has $e(TS^4) = 2u$ (the Euler number) and $p_1(TS^4) = 0$ since $TS^4 \oplus \epsilon = TS^4 \oplus NS^4 = \epsilon^{\bigoplus 5}$ and by the Whitney product formula (Notice that TS^4 corresponds to f_{11}, which is the "sum" of γ and $\bar{\gamma}$). □

Let ξ_{hj} be the sphere bundle of E_{hj}, that is, $\partial D(E_{hj})$, we will show when $h+j=1$, (so $h-j=k$ is odd), the total spase of ξ_{hj}, denoted by M_k^7. In proposition 8.32, we have already proved these two numbers $h+j, 2(h-j)$ correspond to e, p_1. Since $h+j=1$, k determines the pair (h,j) uniquely, so in the following we write the lower indices as k instead of h, j.

Now, we show that M_k^7 is a topological 7-sphere. The idea is to construct a Morse function f on M_k^7 with exactly two critical points. Then M_k^7 is homeomorphic to S^7 by Reeb's theorem (cf. problem 7.23). First, we need the following realization of M_k^7.

Lemma 8.33. *M_k^7 is an identification of two $\mathbf{R}^4 \times S^3$ along $(\mathbf{R}^4 - 0) \times S^3$ via the diffemorphism g of $(\mathbf{R}^4 - 0) \times S^3$:*

$$g : (u,v) \mapsto (u', v') = \left(\frac{u}{|u|^2}, \frac{u^h v u^j}{|u|} \right)$$

PROOF. First, we need to check that $v' \in S^3$ so that the expression makes sense. Note that

$$\left| \frac{u^h v u^j}{|u|} \right| = \frac{|u|^h |v| |u|^j}{|u|} = \frac{|u|^{h+j}}{|h|} = 1,$$

since $h+j=1$. The formula $u' = u/|u|^2$ is nothing but the coordinate change of the two stereographic projections: $S^4 \to \mathbf{R}^4$, one from the south and one from the north.

To check the glueing really gives M_k^7, we consider the equator S^3, that is, $|u| = |u'| = 1$, in fact $u = u'$. The map g restrict on this equator then defines a map $\tilde{g} : S^3 \to SO(4)$ by $\tilde{g}(u)v = u^h v u^j$. which is exactly the map f_{hj}, since any bundle over S^4 is classified by this map as mentioned before, this space is exactly M_k^7. $\square$

Now, we construct the desired Morse function f on M_k^7.

Lemma 8.34. *Consider the following two coordinate charts, (u,v) and (u'',v') of M_k^7, where $u'' = u'(v')^{-1}$. We define $f\colon M_k^7 \to \mathbb{R}$ by*

$$f(x) = \frac{\mathrm{Re(v)}}{\sqrt{1+|u|^2}} = \frac{\mathrm{Re(u'')}}{\sqrt{1+|u''|^2}},$$

which has exactly two non–degenerate critical points.

PROOF. First of all, we check that the two expression of f is indeed identical. Notice that the right hand side equals to

$$\frac{\mathrm{Re(u'(v')^{-1})}}{\sqrt{1+|u'|^2}} = \frac{\mathrm{Re(|u|u'(v')^{-1})}}{\sqrt{1+|u|^2}},$$

and $|u|u'(v')^{-1}) = u(|u|v')^{-1} = u(u^h v u^j)^{-1} = u \cdot u^{1-j} v^{-1} u^{-h} = u^h v^{-1} u^{-h}$. Observe that when we represent $\mathbb{H}$ as 4×4 matrices over $\mathbb{R}$, $\mathrm{Re}(x) = \frac{1}{4}\mathrm{tr}(x)$. Therefore,

$$\mathrm{Re}(u^h v^{-1} u^{-h}) = \frac{1}{4}\mathrm{trace}(u^h v^{-1} (u^h)^{-1}) = \frac{1}{4}\mathrm{trace}(v^{-1}) = \mathrm{Re}(v^{-1}).$$

Since $|v| = 1$, $v^{-1} = \bar{v}$, we have $\mathrm{Re}(v^{-1}) = \mathrm{Re}(\bar{v}) = \mathrm{Re}(v)$. Thus, this is equals to the left hand side.

Now we consider the critical points. From the right expression of f we easily see that no critical points exists in the chart (u'',v'): the function $x_1/\sqrt{1+|x|^2} \nearrow$ in the direction x_1, so $\partial_1 f(x) > 0$. Hence all critical points lie in the (u,v) chart, and in fact lie in the set $(0,v)$. However, in this set, $f(x)$ reduces to be $\mathrm{Re}(v)$ (the height function) on S^3, the unit sphere of $\mathbb{H}$. Thus, the critical points are clearly the two points $v = \pm 1$, that is, $(0, \pm 1)$. We leave the readers to verify that they are indeed non–degenerate. □

Exercise 8.19.

(1) Find the identification of $\mathbb{H}$ as subsets of $M_4(\mathbb{R})$ and prove that $\mathrm{Re}(x) = \frac{1}{4}\mathrm{tr}(x)$.
(2) Verify that the two critical points of f on M_k^7 are non–degenerate.

In summary, we have shown that M_k^7 are all homeomorphic to S^7 by problem 7.23. Now, we show that there exists some k such that M_k^7 is not diffeomorphic to the standard S^7. Recall that in the proof of problem 7.23, suppose M_k^7 is diffeomorphic to S^7, then we can attach an standard 8-disk D^8 onto the boundary of the total space of the disk bundle $D(E^k)$ along $M_k^7 \cong S^7$ via the assumed diffeomorphism. We denote the resulting closed 8-manifold by W_k^8. We notice W_k^8 is nothing but the Thom space $\mathbf{T}(E_k)$, by the usual Thom isomorphism theorem (again, see [MS74], sec.10), we get (by excision and $\cup e(E_k)$):

$$H^i(S^4) \cong H^{4+i}(D(E_k), S(E_k)) \cong H^{4+i}(T(E_k), t_0).$$

The integral cohomology groups of W_k^8 therefore equal $\mathbb{Z}$ in dimension 0, 4, and 8, and zero in other dimensions. Actually, it is $\mathbb{Z} \oplus \mathbb{Z}e(E_k) \oplus \mathbb{Z}e(E_k)^2$. This implies $\sigma(W_k^8) = \pm 1$. Choosing an orientation, may assume $\sigma(W_k^8) = 1$. Now, apply the Hirzebruch signature theorem, we have

$$1 = \sigma = \frac{7p_2 - p_1^2}{45}.$$

Recall that proposition 8.32, which states that

$$e(E_{hj}) = (h+j)u, \quad p_1(E_{hj}) = 2(h-j)u.$$

In the present case, $e(E_k) = u$ and $p_1(E_k) = 2ku$. To pass this result to W_k^8, denote by $\pi : E_k \to S^4$ the bundle projection, we always have $TE_k \cong \pi^*(TS^4) \oplus \pi^*(E_k)$, so apply the Whitney product formula and naturality as in section 7.2.1 and $p(TS^4) = 1$, we get $p(TE_k) = \pi^* p(E_k)$ and $p_1(TE_k) = \pi^* p_1(E_k) = \pi^*(2ku) = 2ku = 2ke(E_K)$. Thus, $p_1^2(TW_k^8) = p_1^2(TE_k) = 4k^2$. This is true because of naturality, the pontryagin classes of E_k are the restriction of Pontryagin classes of W_k^8 which is a smooth closed manifold and only have one more point than E_k, and so have the same value when we evlauate them on the fundamental class.

Now, put everything into the signature formula, we get $4k^2 + 45 = 7p_2 \equiv 0 \pmod 7$ (since Pontryagin numbers are integers!). This implies $4(k^2-1) \equiv 0 \pmod 7$ and so $k \equiv \pm 1 \pmod 7$. But k can be any odd integers! We get a contradiction for those E_k with $k \not\equiv \pm 1 \pmod 7$. That is, M_k^7 is not diffeomorphic to S^7!

Remark 8.35. Although there are a lot of exotic seven spheres, they may be diffeomorphic! For example, are M_3^7 and M_5^7 diffeomorphic? In Milnor's original approach [Mil56], he put everything in the category of manifolds with boundary, and from this he constructed a diffeomorphism invariant λ which is exactly $k^2 - 1 \pmod 7$. In this way, he can distinguish some of these exotic spheres.

There is still another even more sophisticated question: How many smooth structures can a topological sphere have? The following paragraph is a summary of Kervaire and Milnor's result to this question. We will omit the proof and give the reference for the proof.

In the rest of this section, we assume all manifolds are smooth and oriented with dimension $n \geq 5$ and all vector bundles are smooth and oriented.

Definition 8.36. Two closed n-manifolds M_1, M_2 are *h-cobordant* if $M_1 \setminus M_2 = \partial W$ and M_1, M_2 are both deformation retracts of some $(n+1)$-manifold W.

As in the case of cobordism, this defines an equivalence relation on manifolds.

Exercise 8.20. Check that h-cobordism is indeed an equivalence relation.

Recall that in exercise 7.25, we define the connected sum $M_1 \# M_2$ of two connected n manifolds M_1, M_2. The h-cobordism is compatible with the connected sum (cf. [KM63], Lemma 2.2). In summary, all closed n-manifolds form a commutative monoid under # with identity the standard S^n and descends to the h-cobordism classes. Now, we call a closed manifold M a *homotopy n-sphere* if M is closed and has the homotopy type of S^n. Two main results of [KM63] are the following:

Fact 8.37 ([KM63], Theorem 1.1). The h-cobordism classes of homotopy n-spheres forms an abelian group Θ_n under #.

Fact 8.38 ([KM63], Theorem1.2). Θ_n is finite[11]

A landmark in h-cobordism theory is the following *h-cobordism theorem* due to Smale.

Fact 8.39 (Smale's h-cobordism theorem, [Sma62]). If M and N are h-cobordant via W, then W is diffeomorphic to $M \times [0, 1]$.

Smale's original proof is to decompose a manifold M into *handles*, which is the smooth analogue of cells. The procedure is known as *handlebodies decomposition*. One can consult the [Kos93] for this account. Alternatively, one can consult [Mil65] for Morse theoretic exposition for the proof of Smale's theorem.

Remark 8.40. One of the consequence of h-cobordism theorem is that that simply connected manifolds in higher dimension (at least 5) are much easier than those of dimension 3 or 4. In fact, Smale's proof fails for dimension 3 and 4. Moreover, Donaldson proved that (smooth version) h-cobordism is false in dimension 4 while topological version is true by the work of Freedman, cf. section 8.7.

An important consequence of h-cobordism theorem is the solution of Poincaré conjecture in higher dimensions:

Fact 8.41 (Generalized Poincaré Conjecture,[Sma61]). For $n \geq 5$, a homotopy n-sphere is homeomorphic to the standard sphere S^n.

Thus, we know that Θ_n is the group of smooth structures on S^n. We will not actually use these facts in the sequel. Instead, we will

[11]In the paper of Milnor and Kervaire, they cannot decide the order of Θ_3. However, if one assume the Poincaré conjecture for $n = 3$, then it can be shown that $\Theta_3 = 0$.

describe a subgroup $bP_{n+1} \subset \Theta_n$. To define it, we need the concept of parallelizable manifolds.

Definition 8.42. A smooth manifold M is *parallelizable* if TM is trivial and is *s-parallelizable* if $TM \oplus \epsilon$ is trivial, where ϵ is the trivial line bundle over M.

We need the following basic facts:

Lemma 8.43. *Let ξ be a k plane bundle over M^n, $k \geq n$. If $\xi \oplus \epsilon^r$ is trivial, then ξ is trivial.*

PROOF. It suffices to consider the case $r = 1$. The isomorphism $\xi \oplus \epsilon \cong \epsilon^{k+1}$ gives rise to a bundle map

$$\begin{array}{ccc} \xi & \longrightarrow & \gamma^k \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & S^k \end{array},$$

where γ^k is the universal oriented k plane bundle over the oriented grassmannian $\tilde{G}_{k,k+1}(\mathbb{R}) = S^k$. Since $k \geq n$, f is null homotopic, so ξ is trivial. $\square$

Corollary 8.44. *Let M^n be a submanifold of $S^{n+k}, k \geq n$, then M is s-parallelizable iff the normal bundle is trivial.*

PROOF. The bundle $T \oplus N \oplus \epsilon$ is always trivial, where ϵ is the (trivial) normal bundle of S^{n+k} in $\mathbb{R}^{n+k+1}$. If the normal bundle N is trivial, apply previous lemma to $(T \oplus \epsilon) \oplus N$, we get $T \oplus \epsilon$ is trivial. Conversely, if M is s-parallelizable, apply previous lemma to $N \oplus (T \oplus \epsilon)$, we get that N is trivial. $\square$

Corollary 8.45. *A connected manifold with nonempty boundary is s-parallelizable iff it is parallelizable.*

PROOF. We use Morse theory (cf. theorem 7.54 for the compact, without boundary case) to conclude that a smooth manifold admits a CW complex structure, and if the boundary is not empty, the dimension of this CW complex can be choosen to be $< n = \dim(M)$. In the proof of lemma 8.43, we need only $k \geq$ the CW complex dimension, so the result follows. □

Corollary 8.46. *Any oriented submanifold M of $\mathbb{R}^n$ with $\partial M \neq \emptyset$ is parallelizable.*

PROOF. Such manifold has trivial normal bundle. If we take n large, then M becomes s-parallelizable by corollary 8.44. So it is parallelizable by corollary 8.45. □

Now we define the set $bP_{n+1} \subset \Theta_n$: it consists of those homotopy n-spheres which bound a parallelizable manifold. This condition depends only on the h-cobordism class (This is clear if we use h-cobordism theorem). The main property we should know is that bP_{n+1} is a finite cyclic group and its members can be classified by simple topological invariant. For simplicity, we only consider bP_{4m} for $m \geq 2$, the collection of all parallelizable $4m$ manifolds with $\partial M = (4m-1)$-sphere. The corresponding signatures $\sigma(M)$ form a non trivial subgroup of $\mathbb{Z}$, denote it by $\sigma_m\mathbb{Z}$ where $\sigma_m \geq 0$. Then the following structure theorems are known:

Theorem 8.47 ([KM63], Theorem 7.5)**.** *Let Σ_1, Σ_2 be two $4m-1$ homotopy spheres, $\partial M_i = \Sigma_i$, with M_i parallelizable. Then Σ_1 is h-cobordant to Σ_2 if and only if $\sigma(M_1) \equiv \sigma(M_2) \pmod{\sigma_m}$. In another words, the signature $\pmod{\sigma}_m$ classifies the smooth structures on S^{4m-1}.*

Hence, bP_{4m} is a subgroup of $\mathbb{Z}/\sigma_m\mathbb{Z}$. In fact, we have

(1) $bP_{2k+1} = 0$
(2) $bP_{4m-2} = \mathbb{Z}/2\mathbb{Z}$ if $m \neq 1, 2, 4$
(3) bP_{4m} is cyclic of order $\sigma_m/8$, which equals to

$$\epsilon_m 2^{2m-2}(2^{2m-1}-1) \times \text{numerator of } \left(\frac{4B_m}{m}\right),$$

where $\epsilon_m = 1$ if m is odd, $= 2$ if m is even.

8.7. Existence of Exotic $\mathbb{R}^4$

In the section, we give a brief survey to topology for 4-manifolds as a complement for discussions on manifold topology of higher dimensions in previous section. Specifically, we will state Donaldson's and Freedman's theorem and use them to construct *exotic* $\mathbb{R}^4$–a smooth 4-manifold M which is homeomorphic to $\mathbb{R}^4$ but not diffeomorphic to $\mathbb{R}^4$ with standard differentiable structure. The details of the proof for Donaldson's theorem will be given in the next section.

First of all, in section 7.8.3, we have defined the intersection pairing q_M of a manifold M. Now, if M is a simply connected closed topological 4-manifold, one can show that $H^2(X,\mathbb{Z})$ and $H_2(M,\mathbb{Z})$ are torsion–free[12] and the intersection pairing q_M (which is symmetric) can be regarded as a quadratic form over $\mathbb{Z}$:

$$q_M : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \to \mathbb{Z}$$

via the cup product $(\alpha, \beta) \mapsto (\alpha \cup \beta)[M]$. Moreover, Poincaé duality shows that q_M is unimodular, i.e. $\det(q_M) = \pm 1$.

The intersection form q_M actually contains non–trivial information about the topology of M. For instance, here is an early result due to Whitehead and Milnor (cf. [Mil58], theorem 3):

Fact 8.48 (Whitehead, 1949). Let M be a simply connected closed topological 4-manifold. q_M determines the homotopy type of M.

Example 8.49. Now let us compute some examples.

(1) Let $M = S^4$. Since $H^2(S^4,\mathbb{Z}) = 0$, $q_M = \emptyset$.
(2) Let $M = S^2 \times S^2 \cong \mathbb{CP}^1 \times \mathbb{CP}^1$. By Künneth formula for homology which is the same form for cohomology version given in problem 2.23), we know that $H_2(M,\mathbb{Z}) \cong \mathbb{Z}^2$ generated by the cycles $a = [S^2 \times pt]$ and $b = [pt \times S^2]$. Clearly,

$$a^2 = b^2 = 0, \quad ab = 1.$$

[12]Since M is simply connected, $H_1(M,\mathbb{Z}) = 0$. The universal coefficient theorem shows that $H^2(X,\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_2(M,\mathbb{Z}),\mathbb{Z})$. The result then follows from the simple algebra fact that $\mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Z})$ kills all the torsion part of A and Poincaré duality: $H_2(M,\mathbb{Z}) \cong H^2(M,\mathbb{Z})$.

Hence, $q_M \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Notice that q_M is equivalent to the diagonal form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ over $\mathbb{Q}$ but not over $\mathbb{Z}$.

(3) Let $M = \mathbb{CP}^2$, thn $H_2(M, \mathbb{Z}) = \mathbb{Z}[H] \cong \mathbb{Z}$, where $H \cong \mathbb{CP}^1$ is the complex line. Hence, $q_M = (1)$. Let $\overline{\mathbb{CP}^2}$ be the same underlying manifold with reverse orientation. Since $[\overline{\mathbb{CP}^2}] = -[\mathbb{CP}^2]$, we have $q_{\overline{\mathbb{CP}^2}} = (-1)$.

Exercise 8.21. Let M_1, M_2 be simply connected, closed 4-manifolds.

(1) Show that $M_1 \# M_2$ is still a simply connected, closed 4-manifolds.
(2) Show that $H_2(M_1 \# M_2) \cong H_2(M_1) \oplus H_2(M_2)$ and $q_{M_1 \# M_2} = q_{M_1} \oplus q_{M_2}$.

(Hint: Use Mayer–Vietoris sequence, cf. lemma 2.30).

Hence, for $M = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, we get $q_M = (1) \oplus (-1)$. However, as we have remarked in previous example, $q_M \not\sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = q_{S^2 \times S^2}$ over $\mathbb{Z}$. Hence, $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ have different homotopy type by Whitehead's theorem. One may furthermore pose the following question: if the intersection form q_M of a 4-manifold M can be decomposed as $q_1 \oplus q_1$, can one find 4-manifold M_1, M_2 with $q_{M_i} = q_i$ for $i = 1, 2$ such that $M \cong M_1 \# M_2$?

Before answering the question, let us first discuss the classification theory of unimodular quadratic form over $\mathbb{Z}$ following the classical reference [Ser73], chapter V.

Definition 8.50. Let q be an unimodular form over $\mathbb{Z}$ on a free abelian group Λ. We define

(1) the rank $\mathrm{rank} q$ of q by the rank of Λ;
(2) the signature $\sigma(q)$ of q as the signature of $q_{\mathbb{R}}$ on $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.

As in linear algebra, we call q is *definite* if $\sigma(q) = \pm \mathrm{rank} q$. Also, we say q_M is of *even type* or is even if $q_M(\alpha, \alpha) \in 2\mathbb{Z}$ and q_M is odd if otherwise.

Exercise 8.22. Show that $q = q_1 \oplus q_2$ is even if and only if q_1 and q_2 are even.

Here is the construction of the non–trivial positive definite form. Let $k \in \mathbb{N}$, $n = 4k$, and $V = \mathbb{Q}^n$ with the standard quadratic form $\|x\|^2 = \sum_{i=1}^n x_i^2 = 0$. Let $\Lambda_0 = \mathbb{Z}^n \subset V$ be the integral points of V and q is the induced quadratic form from $\|\cdot\|$. Let $\Lambda_1 := \{x \in \Lambda_0 : q(x,x) \equiv 0 \pmod 2\}$ be the subgroup of Λ_0. In other words, $x \in \Lambda_1$ if and only if $\sum_{i=1}^n x_i \equiv 0 \pmod 2$ and thus Λ_1 is an index 2 subgroup of Λ_0.

Now, we define $\Lambda = \mathbb{Z}\langle e, \Lambda_1\rangle$, where $e = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)$. Since $n \equiv 0 \pmod 4$, one has $2e \in \Lambda_1$ but $e \notin \Lambda_1$. Hence, $[\Lambda : \Lambda_1] = 2$. For $x = (x_1, \dots, x_n) \in V$, $x \in E$ if and only if

$$2x_i \in \mathbb{Z}, \quad x_i - x_j \in \mathbb{Z}, \quad \sum_{i=1}^n x_i \in 2\mathbb{Z}. \tag{8.17}$$

Then we have $q(x,e) = \frac{1}{2}\sum_{i=1}^n x_i \in \mathbb{Z}$. Since $q(e,e) = k$, one can show that $q(x,y) \in \mathbb{Z}$, for any $x, y \in \Lambda$. Moreover, since $[\Lambda_0 : \Lambda_1] = [\Lambda : \Lambda_1] = 2$, one can show that $\det(q) = 1$ on Λ. Hence, q is a unimodular form on Λ. Moreover, when k is even (i.e., $n \equiv 0 \pmod 8$), $q(e,e)$ is even and this implies $q(x,x) \in 2\mathbb{Z}$, for any $x \in \Lambda$. Thus, we denote $\Gamma_n := (\Lambda, q)$ in this case. Then Γ_n is a positive definite unimodular form of even type with rank $n = 8m$

When $m = 1$, i.e. $n = 8$, let $\{e_1, \dots, e_n\}$ denotes the standard basis on $\mathbb{Q}^8$, then elements $x \in \Gamma_8$ with $q(x,x) = 2$ are of the form

$$\pm e_i \pm e_k (i \neq k) \quad \text{and} \ \frac{1}{2}\sum_{i=1}^8 \epsilon_i e_i, \quad \epsilon_i = \pm 1, \prod_{i=1}^8 \epsilon_i = 1.$$

One can check that

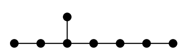
$$\frac{1}{2}(e_1 + e_8 - e_2 - \cdots - e_7), e_2 - e_1, e_3 - e_2, e_1 + e_2, e_4 - e_3, e_5 - e_4, e_6 - e_5, e_7 - e_6$$

forms an ordered basis for Γ_8 and the corresponding q with respect to the basis is given by

$$\begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & -1 & & & \\ & & -1 & 2 & 0 & & & \\ & & -1 & 0 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & 1 \\ & & & & & & -1 & 2 \end{pmatrix}$$

For $m \geq 2$, the vectors $x \in \Gamma_{8m}$ such that $q(x, x) = 2$ are simply the vectors the vectors $\pm e_i \pm e_j$, for $i \neq j$. Hence, they do not generate Γ_{8m} for $m \geq 2$. Particularly, we know that $\Gamma_8 \oplus \Gamma_8$ is not isomorphic to Γ_{16}.

Exercise 8.23. Verify the details in the above constructions.

Remark 8.51. For readers who familiar with theory of Lie groups, Γ_8 forms the *root system of type* E_8 and the matrix is the Cartan matrix associated to the Dynkin diagram .

Here are some facts on unimodular forms whose proof can be found in [Ser73], Chapter V, §2–3.

Fact 8.52. Let q be an indefinite unimodular form, then q is uniquely determined by its rank, signature, and type. Furthermore,

(i) if q is odd, then $q \sim s(1) \oplus t(-1)$, for some $s, t \in \mathbb{N}$.

(ii) if q is even and $\sigma(q) \geq 0$, then q is equivalent to $m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus n\Gamma_8$, where $n = \frac{1}{8}\sigma(q)$and $m = \frac{1}{2}(\mathrm{rank} q - \sigma(q))$. If $\sigma(q) < 0$, then the corresponding result follows from changing q to $-q$.

For q is definite, there is no easy classification. However, we have the following arithmetic restriction for any even forms.

Fact 8.53. If q is even, then $\sigma(q) \equiv 0 \pmod 8$.

Now, we compute a more involved example of intersection form. Let $M := \{[Z_0 : Z_1 : Z_2 : Z_3] : \sum_{i=0}^{3} Z_i^4 = 0\}$ be the *Kummer surface*, which is the quartic surface in $\mathbb{CP}^3$ with defining equation of Fermat type. In general, for a hypersurface $M_d \subset \mathbb{CP}^n$ of degree d, we have the normal sequence

$$0 \to TM_d \to i^*T\mathbb{CP}^n \to N_{M_d} \to 0.$$

Let us denote $\mathcal{O}_{\mathbb{P}^n}(1)$ by the hyperplane bundle of $\mathbb{CP}^n$, $h := c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ be its first Chern class which is also the Poincaré dual of a line $L \subset \mathbb{CP}^n$. We abuse the notation by still denoting h by the restriction of $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ onto M_d. The hypersurface M_d has degree d means that its normal bundle is the restriction of d-tensor power of hyperplane bundle onto M_d. In other words[13], $N_{M_d} \cong (\mathcal{O}(1)^{\otimes d})\big|_{M_d}$. Hence. from Whitney product formula, we have

$$i^*c(\mathbb{CP}^n) = c(M_d)(1 + dh).$$

From (7.13), we know that $c(\mathbb{CP}^n) = (1+h)^{n+1}$ and thus $i^*c(\mathbb{CP}^n) = (1+h)^{n+1}$ by functoriality of Chern class. From this, one can compute $c(M_d)$ by

$$\begin{aligned} c(M_d) &= (1+h)^{n+1}(1+dh)^{-1} \\ &= \Big[(1+h)^{n+1}(1 - dh + d^2h^2 - d^3h^3 + \cdots)\Big]_{n-1}, \end{aligned}$$

where $[\cdots]_{n-1}$ means taking the degree $(n-1)$-part in the expansion.

Thus, for $n = 3$, $d = 4$, the Chern class of Kummer surface $M = M_4$ is given by

$$1 + c_1(M) + c_2(M) = (1 + 4h + 6h^2)(1 - 4h + 4h^2) = 1 + 6h^2.$$

Hence, $c_1(M) = 0$ and $c_2(M) = 6h^2 = e(M)$ (cf. exercise 7.21 (2)).

By Chern–Gauss–Bonnet theorem (cf. theorem 8.30), we can compute its Euler characteristic as

$$\chi(M) = \int_M c_2(K_3) = \int_M 6h^2.$$

[13]This is known as *adjunction formula* in algebraic geometry.

Notice that another interpretation for a hypersurface $M_d \subset \mathbb{CP}^n$ having degree d is that its homology class $[M_d] \in H^{2n-2}(\mathbb{CP}^n, \mathbb{Z})$ is homologous to $d[H]$, where $[H]$ is a hyperplane in $\mathbb{CP}^n$. Also, observe that the Poincaŕe dual of $[H]$ is given by h^{n-1}, we then have

$$\int_{M_d} h^{n-1} = h^{n-1}([M_d]) = h^{n-1}(d[H]) = dh^{n-1}[H] = 1.$$

Particularly, this shows that the Euler characteristic for Kummer surface $\chi(M) = 24 = b_0 - b_1 + b_2 - b_3 + b_4$. Since M is compact and connected, $b_i = b_{4-i}$ and $b_0 = 1$. Also, by Lefschetz hyperplane theorem (cf. theorem 7.56), we know that $b_1 = 0$. Hence, $b_2 = 22$ and thus $H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{22}$.

Now, we are ready to determine the intersection form q_M of Kummer surface M. First, by Hirzebruch signature theorem, we know that $\sigma(M) = \sigma(q_M) = \frac{p_1}{3}$ while the first Pontryagin number of M is given by

$$p_1(M) = (-1)c_2(TM \otimes \mathbb{C})[M] = -2c_2(M)[M] = -48.$$

Hence, $\sigma(M) = -16$ and thus q_M is indefinite. One can show that q_M is of even type and thus by fact 8.52, we know that

$$q_M \sim (-2)\Gamma_8 \oplus 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{8.18}$$

Exercise 8.24. Show that the intersection form q_M for Kummer surface is even.

Remark 8.54. Notice that Kummer surface satisfies $c_2(M) = 0$, which implies the canonical bundle $K_M = \bigwedge^{2,0} T^*M$ is trivial. This is exactly the Calabi–Yau condition we have mentioned in section 6.7. In general, a complex compact surface[14] S with trivial K_S is called a *K3 surface*, named after Kummer, Kähler, and Kodaira. Thus, Kummer surface is an example of K3 surface.

As mentioned the remark 8.40 in previous section, we know that Smale's proof for h-cobordism theorem fails for dimension 3 and 4.

[14] A non–trivial fact due to Yum-Tong Siu is that all $K3$ surfaces are Kähler.

The classification for manifolds in dimension four remains largely mysterious until the early 1980s. In 1981, Freedman gave a remarkable result on complete classification theorem for compact, simply-connected topological 4-manifold:

Fact 8.55 (Freedman, [Fre82]). For compact, simply-connected topological 4-manifold M^4, the homeomorphism type is uniquely determined by q_M if q_M is even and up to two choices if q_M is odd.

Moreover, every unimodular form q can be realized by q_M for some M^4.

We refer the monograph [FQ90] for exposition of Freedman's work. From Freedman's result, we know that the classification for (simply–connected compact) topological 4-manifolds is essentially the same as the classification of unimodular forms. Particularly, his theorem shows the Poincaré conjecture in dimension 4.

Corollary 8.56. *Let M be a homotopy sphere of dimension* 4*. Then M is homeomorphic to S^4.*

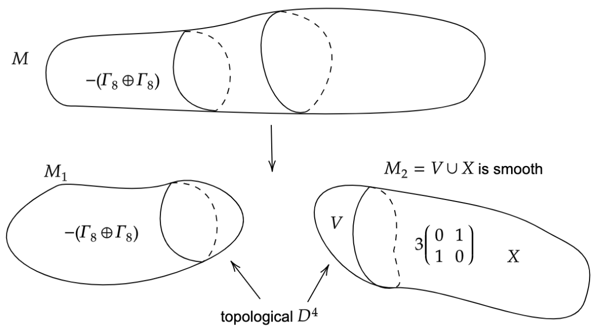
Back to our previous question. Since every unimodular form is realizable by some simply connected closed 4-manifold, if q_M decompose into $q_1 \oplus q_2$, Freedman's result shows that there exists two topological 4-manifolds M_1, M_2 with $q_i = q_{M_i}$ for $i = 1, 2$ such that M is homeomorphism to $M_1 \# M_2$.

We have not mentioned anything about differentiable structure up to this point. In fact, the topological manifolds and smooth manifolds are sharply distinct in dimension 4, as indicated by the another remarkable *diagonalizable theorem* due to Simon Donaldson [Don83] nearly at the same time.

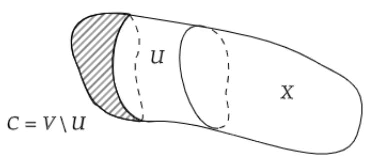
Theorem 8.57 (Donaldson, 1982)**.** *Let M be a compact, simply–connected smooth* 4*-manifold. Assume that q_M is definite, then q_M is diagonalizable over $\mathbb{Z}$.*

As a first corollary, Donaldson's theorem shows that the topological 4-manifold realizing positive definite even forms like Γ_8, $\Gamma_8 \oplus \Gamma_8$, or Γ_16 cannot admit any smooth structure.

From this, one can establish the existence of exotic $\mathbb{R}^4$ as following. Consider the Kummer surface M, from (8.18) and Freedman's result, there exists a topological surgery $M = M_1 \# M_2$, where M_1 is the topological manifold with intersection form $q_{M_1} = (-\Gamma_8) \oplus (-\Gamma_8)$ and $M_2 = 3(S^2 \times S^2)$, as shown in below.



However, Donaldson's theorem shows that such a surgery cannot be smooth. We denote V by the topological 4-disk, which is homeomorphic[15] to $\mathbb{R}^4$, in $M_2 = 3(S^2 \times S^2)$ and $X = M_2 \setminus V$. Since $M_2 = V \cup X$ is smooth, we can induce a smooth structure on V inherited from M_2.



Now, we take U to be a collar (cf. problem 2.14) of X in M_2. Donaldson's theorem then show that there exists no smooth embedding from $S^3 = \partial V \hookrightarrow U$, for otherwise the surgery will be smooth. Hence, we have shown that the compact set $C = V \setminus U$ cannot be surrounded by any smoothly embedded 3-sphere.

Of course, on $\mathbb{R}^4$ with standard differentiable structure, any compact sets can be surrounded by smooth 3-spheres of sufficiently large

[15] It is actually quite technical to show that V is homeomorphic to $\mathbb{R}^4$, which uses some topological results called *Casson handles* described in Freedman's 1981 paper cited above, see [FU84], theorem 1.6.

radius. Therefore, V is homeomorphic to $\mathbb{R}^4$ but not diffeomorphic to $\mathbb{R}^4$ with standard differentiable structure!

Remark 8.58. In fact, Taubes [Tau87] later proved that there are uncountable families of exotic $\mathbb{R}^4$s.

8.8. Introduction to Donaldson Theory

In this section, we discuss some ideas in the proof of Donaldson's diagoalizable theorem (cf. theorem 8.57). For details, we refer to the original paper [Don83], the exposition [FU84], and [**Lawsonguage**]. One can also consult the more comprehensive monograoh [DK90].

The key ingredients in Donaldson's proof is the *gauge theory*[16]. Let M be a compact, simply connected, smooth 4-manifold, $E \to M$ be a G-vector bundle, where G is a compact Lie group. In practice, we take $G = \mathrm{SU}(N)$. We denote $\mathfrak{g} = \mathfrak{su}(N) \subset M_n(\mathbb{C})$ by the Lie algebra of G with bi-invariant inner product $\langle A, B\rangle = -trAB$.

As we have seen in section 7.1, given a trivializing cover $\{U_\alpha\}$ for E, we can write a G-connection ∇ (cf. problem 7.1) locally as $\nabla = d + \theta_\alpha$ on U_α, where θ_α is a $\mathfrak{g}$-valued 1-form. Also, recall that the curvature 2-form $F_\nabla = d\theta_\alpha + \theta_\alpha \wedge \theta_\alpha$ is a globally-defined $\mathfrak{g}$-valued 2-form. In fact, both of them are differential forms valued in the vector bundle $\mathfrak{g}_E$ over M, called *adjoint bundle*[17] of the G-vector bundle E.

[16]In physics literature, this is often referred as *Yang–Mills theory* though earlier ideas have appeared in the work of Weyl. gauge theory (and its "quantization") plays the dominant role in modern physics, especially in standard model, the current model in theoretical physics to explain the fundamental interactions and particles (apart from gravitation). From the mathematical point of view, it is an open problem to construct quantization of a Yang–Mills theory mathematically, cf. [**JaffeWitten**].

[17]If one use the language of principal bundles (cf. problem 7.9), given the Lie group G, the vector bundle $E \to M$ is just the associated bundle of a principal G-bundle P with respect to a representation $\rho : G \to GL(V)$, where V is any fiber of E (cf. problem 7.11). The $\mathfrak{g}$-valued forms θ_α and F_∇ are actually differential forms values in the adjoint bundle $\mathfrak{g}_E := AdP$ (cf. problem 7.14 and **??**, whose fibers are the Lie algebra $\mathfrak{g}$ of G.

Remark 8.59. Since in the sequel we are interesting to not just one fixed connection but the spaces of connection, we would like to regard a connection as an object rather than as an operator. We differ with our notations in section 7.1 by denoting a connection by A, the associated covariant derivatives (on all associated bundle for E) by d_A, and the curvature of A by F_A. Also, to avoid the confusion with connection A, we denote the space of smooth k-forms with values in some vector bundle F by $\Omega^i(F)$ rather than $A^i(F)$.

Now, let (M, g) be a Riemannian manifold, we can define the *Yang–Mills functional* YM on the space of G-connections[18] $\mathcal{A}$ on E by

$$\mathrm{YM}(A) := \|F_A\|^2 = \int_A |F_A|^2 d\mu_g, \tag{8.19}$$

where $(\cdot,\cdot) := \int_M \langle\cdot,\cdot\rangle d\mu_g$ is the L^2-norm with respect to the inner product $\langle\cdot,\rangle$ on $\Omega^2(\mathfrak{g}_E)$. Recall that $\mathcal{A}$ is an affine space modeled on $\Omega^1(\mathfrak{g}_E)$ (cf. exerciseaffineness of connection and problem 7.14, we may investigate the critical points of YM by: let $a \in \Omega^1(\mathfrak{g}_E)$, from the proof of proposition 7.4, we know that the curvature for F_{A+ta} is then given by

$$F_{A+ta} = F_a + td_A a + t^2 a \wedge a.$$

We then calculate the derivative of $\mathrm{YM}(A + ta)$ with respect to t:

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\|F_{A+ta}\|^2 &= \frac{d}{dt}\Big|_{t=0}\int_M |F_A + td_A a + t^2 a\wedge a|^2 \d{m}u_g \\ &= 2\int_M \langle d_A a, F_A\rangle = 2(d_A a, F_A) = 2(a, d_A^* F_A).\end{aligned}$$

Hence, A is a critical point for YM if and only if $d_A^* F_A = 0$. Notice that this is a second order non–linear PDE with respect to A, called the *Yang–Mills equation*.

Exercise 8.25. Write down the Yang–Mills equation $d_A^* F_A = 0$ in local coordinates when the metric $g_{ij} = \delta_{ij}$ is flat. Also, explain why this is the *non–linear* and *non–abelian* version of Harmonic theory discussed in chapter 4.

[18]In physics literatures, a G-connection A is called a *Yang–Mills field*.

When $\dim M = 4$, as we have seen in section 8.5, (**??**) shows that the exterior bundle splits into self-dual part and anti-self dual part. Obviously, the decomposition extends to forms with values in any vector bundles. In particular, we have the orthogonal decomposition

$$\Omega^2(\mathfrak{g}_E) = \Omega^2_+(\mathfrak{g}_E) \oplus \Omega^2_-(\mathfrak{g}_E).$$

Hence, $F_A = F_A^+ \oplus F_A^-$ and $\|F_A\|^2 = \|F_A^+\|^2 + \|F_A^-\|^2$.

Now, we restrict ourselves to the case $N = 2$, i.e., $E \to M$ is a complex vector bundle of rank 2 with structure group $\mathrm{SU}(2)$, and we consider the characteristic classes on E, which are $c_1(E)$ and $c_2(E)$. Since $\mathfrak{g} = \mathfrak{su}(2)$, we have $c_1(E) = [\frac{i}{2\pi}\mathrm{tr}F_A] = 0$. Hence, we have $[\left(\frac{i}{2\pi}\right)^2 \mathrm{tr}(F_A^2)] = c_1(E)^2 - 2c_2(E) = -2c_2(E)$. The characteristic number of E is given by

$$\begin{aligned}
-2c_2(E)[M] &= \int_M \frac{-1}{4\pi^2}\mathrm{tr}F_A \wedge F_A \\
&= \frac{-1}{4\pi^2}\int_M \left(\mathrm{tr}(F_A^+ \wedge F_A^+) + \mathrm{tr}F_A^- \wedge F_A^-\right) \\
&= \frac{-1}{4\pi^2}\int_M \mathrm{tr}(F_A^+ \wedge *F_A^+) - \mathrm{tr}(F_A^- \wedge F_A^-).
\end{aligned}$$

We then get $k := c_2(E)[M] = \frac{1}{8\pi^2}\left(\|F_A^+\|^2 - \|F_A^-\|^2\right) \in \mathbb{Z}$. This is called the *topological charge* of the Yang–Mills field A. When $k > 0$, the absolute minimum of $\|F_A\|^2$ is $8\pi^2 k$, which occurs if and only if $F_A^+ \equiv 0$, or $*F_A = -F_A$. In this case, we call A is an *ASD connection*. Similarly, when $k < 0$, the minimum is $-8\pi^2 c_2(E)$ which occurs if and only if $F_A^- \equiv 0$. Thus, we then call A in this case a *self-dual* connection. We will focus on the case $k > 0$. In this case, we see that the minimizing problem for Yang–Mills functional is equivalent to the solution of $F_A^+ = 0$, a first order non-linear PDE.

Remark 8.60 (Classification of $SU(2)$-bundles and Line Bundles on 4-Manifolds). Actually, on 4-manifolds M, complex vector bundle of rank 2 over M with structure group $\mathrm{SU}(2)$ is classified by the topological charge $k \in \mathbb{Z}$. Since $\mathrm{SU}(2) \equiv \mathrm{Sp}(1)$, such complex vector bundle are in fact quaternion line bundle with classifying space $\mathbb{HP}^\infty$ (cf. section 7.5 and section 8.6). Therefore, the isomorphisms classes of SU(2)-bundles are one-to-one corresponds to $[M, \mathbb{HP}^\infty]$, the homotopy classes of maps from M to classifying space. As in exercise 8.17 indicated, the CW complex structure of $\mathbb{HP}^\infty$ is given by $\mathbb{HP}^k$ for $k = 0, 1, 2, \dots$, and $\mathbb{HP}^1 \cong S^4$. Thus, by cellular approximation theorem (cf., for instance [Hat02]), theorem 4.8)

$$[M, \mathbb{HP}^\infty] = [M, S^4] \cong \mathbb{Z},$$

where the last isomorphism comes from taking the *degree* of the map, which is compatible with taking $c_2(E)[M]$.

Similarly, the complex line bundles on M are in fact classified[19] by their first Chern classes $c_1(E)$, for any CW complex M.

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Now, for a rank 2 complex vector bundle E with $c_2(E) = k$, notice the following relation between intersection form q_M with the splitting of E.

Lemma 8.61. *E splits topologically into $L_1 \oplus L_2$, for some line bundles L_1, L_2, if and only if the equation $-k = q_M(\alpha, \alpha)$ is solvable for some $\alpha \in H^2(M, \mathbb{Z})$. Moreover, in this case, $\alpha = \pm c_1(L_1)$ and $L_2 = L_1^*$.*

PROOF. Suppose E splits, then Whitney product formula implies $0 = c_1(E) = c_1(L_1) + c_1(L_2)$ and $k = c_2(E) = c_1(L_1)c_1(L_2) = -c_1(L_1)^2$. Thus, $-k = q_M(\alpha, \alpha)$, where $\alpha = \pm c_1(L_1)$ and $L_2 = L_1^*$

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[19] Since the classifying space for complex line bundle is $\mathbb{CP}^\infty$, the isomorphism classes of line bundles are one-to-one corresponds to $[M, \mathbb{CP}^\infty]$. Similar to the construction of quaternion Hopf fibration in section 8.6, the sphere bundle $S^{2k+1} \to \mathbb{CP}^k$ for the tauotological bundle $\gamma \to \mathbb{CP}^k$ is a S^1-fibration. Hence, inductively, we have a S^1-fibration $S^\infty \to \mathbb{CP}^\infty$.Using homotopy exact sequence of fibration (cf. fact 7.41), we know that the homotopy groups of $\mathbb{CP}^\infty$ are trivial except $\pi_2(\mathbb{CP}^\infty) \cong \mathbb{Z}$. Then from Hurewicz theorem (cf. section 7.8.1) and universal coefficients theorem for cohomology, we have

$$H^2(\mathbb{CP}^\infty, \mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}\langle c_1 \rangle,$$

where $c_1 = 1_\mathbb{Z} \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ is the universal first Chern classes (cf. section 7.5.3). For any $f \in [M, \mathbb{CP}^\infty]$, we associate the cohomology class $f^* c_1 \in H^2(M, \mathbb{Z})$. Hence, we obtained an isomorphism $[M, \mathbb{CP}^\infty] = H^2(M, \mathbb{Z})$ by $f \mapsto f^* c_1$.

(since $c_1(L_2) = -c_1(L)$ and by the classification result for line bundles in previous remark).

Conversely, if $\alpha \in H^2(M, \mathbb{Z})$ satisfies $q_M(\alpha, \alpha) = -k$. By previous remark, we know that there exists a complex line bundle L with $c_1(L) = \alpha$. Above calculation then shows that $-c_2(L \oplus L^*)[M] = k$. By previous remark again, we know that $SU(2)$-bundles are classified by topological charge, and thus $E \cong L \oplus L^{-1}$ as $SU(2)$-bundles. $\square$

Now, we are ready to state how Donaldson approached the diagonalizable theorem (cf. theorem 8.57).

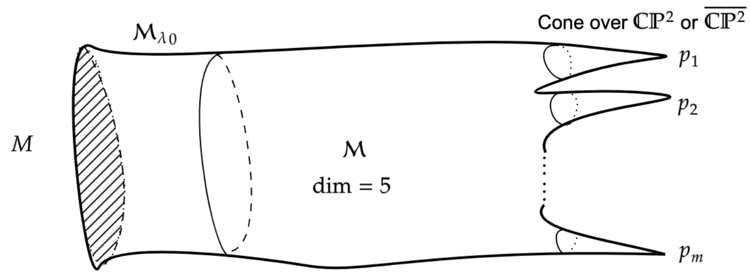
Theorem 8.62 (Donaldson, 1982)**.** *Let M be a compact, simply–connected, smooth 4-manifold with negative definite intersection form q_M, $E \to M$ be a $SU(2)$-bundle with charge $k = c_2(E) = 1$. We denote $\mathcal{M}$ by the moduli space of ASD connections on E (which depends on a choice of Riemannian metric g on M). Then for genetic Riemannian metric g on M, the following statements hold:*

(1) *There exists $p_1, \ldots, p_m \in \mathcal{M}$ such that $\mathcal{M} \setminus \{p_1, \ldots, p_m\}$ is a non–compact, oriented smooth 5-manifold.*
(2) *There exists an open set $\mathcal{M}_{\lambda_0} \subset \mathcal{M}$ such that $\mathcal{M}_{\lambda_0}$ is diffeomorphic to $M \times (0, \lambda_0)$ and $\mathcal{M} \setminus \mathcal{M}_{\lambda_0}$ is compact. In other words, $\mathcal{M}$ has a collar neighborhood $\mathcal{M}_{\lambda_0}$ and we can compacitify $\mathcal{M}$ into $\overline{\mathcal{M}}$ by adding $M \cong M \times \{0\}$ as a boundary.*
(3) *Each singularity $p_i \in \mathcal{M}$ are one-to-one corresponds to the topological splitting $E = L \oplus L^*$.*
(4) *There exists a neighborhood U_i of each p_i such that U_i homeomorphic to $\mathbb{C}^3/S^1$, a cone on $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$.*

We may illustrate the moduli space $\mathcal{M}$ satisfying (1)–(4) above by the below figure.

Granted theorem 8.62, we may prove the diagonalizable theorem as following:

Proof of theorem 8.57. Since M is oriented cobordant to disjoint union of $m(\pm\mathbb{CP}^2)$ via $\overline{M} \setminus \{p_1, \ldots, p_m\}$, where $-\mathbb{CP}^2 := \overline{\mathbb{CP}^2}$. From lemma 7.50, we know that signature is an invariant under



cobordism. Since q_M is negative definite,

$$b_2(M) = \mathrm{rank} H^2(M, \mathbb{Z}) = -\sigma(q_M) \leq m\sigma(\mathbb{CP}^2) = m.$$

However, by the process of diagonalization of integral quadratic form[20], we must have $m \leq b_2$. Hence, we conclude that $b_2 = m$ and thus q_M is diagonalizable to $\begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$. $\square$

In the remaining section, we will discuss some ideas in the proof of theorem 8.62. First, we define the moduli space $\mathcal{M}$. As in the discussion in Plateau problem (cf. section 6.4), the space of connections $\mathcal{A}$ also has many redundancy due to action of infinite dimensional symmetric group. In the present case, the symmetry group in consideration is just $\mathcal{G} := \mathrm{Aut}(E)$, the group of bundle automorphisms of E. It acts on $\mathcal{A}$ by: for each section $s \in C^\infty(E)$,

$$d_{g(A)} s := g^{-1} d_A(gs).$$

In terms of local description, by writing $d_{g(A)} = d + g(A_\alpha)$, we know that

$$g^{-1} d_A(g(s)) = g^{-1}(d + A_\alpha)(gs) = ds + (g^{-1}dg + g^{-1}A_\alpha g)s.$$

[20] if $\alpha \in H^2(M, \mathbb{Z})$ such that $q(\alpha, \alpha) = -1$, then $H^2(M, \mathbb{Z}) = \mathbb{Z}\alpha \oplus \alpha^\perp$ by $\beta \mapsto q(\alpha, \beta)\alpha + (\beta - q(\alpha, \beta)\alpha)$. Since q is definite, number of solution for $q(\gamma, \gamma) = -1$ for $\gamma \in \alpha^\perp$ equals to $m - 1$ and $\mathrm{rank}(q|_{\alpha^\perp}) = \mathrm{rank} q - 1$. The result then follows from indcution.

Hence, $g(A_\alpha) = g^{-1}dg + g^{-1}A_\alpha g$ (compare exercise 7.1) and thus $F_{g(A)} = g^{-1}F_A g = Ad_g(F_A)$. This shows that $\mathcal{G}$ acts on trivially on the "form part" of F_A and acts on the $\mathfrak{g}$-part by adjoint representation $Ad : G \to \mathrm{Aut}(\mathfrak{g})$. Hence, in view of adjoint-invariance of the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}_E$, we know that YM is in fact a well–defined functional on the orbit spaces $\mathcal{A}/\mathcal{G}$ and the condition of ASD is preserved under the action of $\mathcal{G}$. We then define the desired moduli space of ASD connection by

$$\mathcal{M} := \{A \in \mathcal{A} : *F_A = -F_A\}/\mathcal{G}.$$

ADHM Construction. Before actual proving the theorem, let us consider the case $M = S^4$, where the moduli spaces of ASD $SU(2)$ connections for $k = 1$ can be explicitly described[21]. We identify $\mathbb{R}^4$ as the quaternion $\mathbb{H}$ with the unit group $\mathrm{Sp}(1) = \mathrm{SU}(2) \cong S^3 \cong \mathrm{Spin}(3)$. Its Lie algebra $\mathfrak{sp}(1) = \mathfrak{su}(2)$ can then identified as $\mathrm{Im}\mathbb{H}$, the imaginary part of quaternions.

Then the tautological bundle $\gamma^1_{\mathbb{H}} \to \mathbb{HP}^1 \cong S^4$ is then a $\mathrm{Sp}(1) = \mathrm{SU}(2)$-bundle. Exercise 8.17 shows that the underlying rank 2 complex bundle $\gamma^1_{\mathbb{C}}$ satisfies $c_1(\gamma^1_{\mathbb{C}}) = 0$ and $c_2(\gamma^1_{\mathbb{C}})[\mathbb{HP}^1] = 1$. Now, we trivialize $\gamma^1_{\mathbb{H}}\Big|_{\mathbb{R}^4}$ with the section $\sigma(x) = \frac{(x,1)}{\sqrt{1+|x|^2}}$. We then define an ASD connection $A = d + \omega$ by the $\mathfrak{su}(2)$-valued connection 1-form

$$\omega = \frac{\mathrm{Im}(\bar{x}dx)}{1 + |x|^2}.$$

8.9. Problems

8.1. Let $E \to M = S^{2m}$ be a complex vector bundle over an $2m$-sphere. Consider the twisted signature operator $D : C^\infty(\bigwedge^{\pm}_{\mathbb{C}} T^*M \otimes E \to \bigwedge^{\mp}_{-} \mathbb{C}M \otimes E)$ in exercise 8.16.

(1) Show that $\mathrm{ind}D = \frac{1}{(m-1)!}\int_{S^{2m}} c_m(E)$.
(2) Deduce that the only spheres which could admit almost complex structures are S^2 and S^6. This generalize problem 7.4.

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[21]The construction of ASD connections on S^4 in general case is due to Atiyah, Drinfeld, Hitchin, and Manin [**ADHM**]. For more details, see [**AtiyahGeometryofYM**] or [DK90] chapter 3.

*(3) Construct an almost complex structure on S^6.

**(4) Does there exists an integrable almost complex structure on S^6? (cf. footnote 18 in Chapter 6)

Classification of Clifford Algebra. In problem 8.2 to 8.5, we consider the classification of Clifford algebra. The presentation here mainly follows the original source [ABS64] and the exposition [LM89], §4.

8.2. Let V be a real vector space of dimension n, Q be a quadratic form with signature $(r, s = n - r)$. Under the identification, $(V, Q) \cong (\mathbb{R}^n, q)$, where $q = \mathrm{diag}(\underbrace{1, \ldots, 1}_{r}, \underbrace{-1, \ldots, -1}_{s})$, we denote $C_{r,s}$ by the Clifford algebra $C(\mathbb{R}^n, q)$. Show that the following algebra isomorphisms ($\mathbb{H}$ =quaternions).

(1) $C_{1,0} \cong \mathbb{C}$, $C_{0,1} \cong \mathbb{R} \oplus \mathbb{R}$.
(2) $C_{2,0} \cong \mathbb{H}$, $C_{1,1} \cong M_2(\mathbb{R})$, $C_{0,2} \cong M_2(\mathbb{R})$.
(3) $C_{r,s} \cong C^+_{r+1,s}$, for any $r, s \geq 0$.

([ABS64], §4)

8.3. Show the following algebra isomorphisms for any $n, r, s \geq 0$:

(1) $C_{n,0} \otimes C_{0,2} \cong C_{0,n+2}$
(2) $C_{0,n} \otimes C_{2,0} \cong C_{n+2,0}$
(3) $C_{r,s} \otimes C_{1,1} \cong C_{r+1,s+1}$.

([ABS64], §4)

8.4. In this problem, we classify the $C_{n,0}$ and $C_{0,n}$ in terms of matrix algebras over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

(1) Show the following isomorphisms of $\mathbb{R}$-algebras
  (a) $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$.
  (b) $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong M_2(\mathbb{C})$.
  (c) $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(\mathbb{R})$.
(2) For any $n \geq 0$, show that there are "periodicity" isomorphisms

$$C_{n+8,0} \cong C_{n,0} \otimes C_{8,0}; \quad C_{0,n+8} \cong C_{0,n} \otimes C_{0,8}$$

(3) It remains to classify $C_{n,0}$ and $C_{0,n}$, for $n \leq 8$. Prove the following table (Here, $K(n) := M_n(K)$ with $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$).

	1	2	3	4	5	6	7	8
$C_{n,0}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
$C_{0,n}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$

With above table and $C_{1,1} \cong M_2(\mathbb{R})$, one can actually obtain the table for $C_{r,s}$, for any $0 \leq r, s \leq 8$, cf. [LM89], p.29.

([ABS64], §4)

8.5. In this problem, we consider the classification of Clifford algebra of a complex vector space.

(1) For any $r+s=n$, show that $C_{r,s} \otimes \mathbb{C} \cong \mathbb{C}l_n := C(\mathbb{C}^n, q_{\mathbb{C}})$, where $q_{\mathbb{C}}(z) = \sum_{k=1}^n z_k^2$.
(2) $\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes \mathbb{C}l_2$ and $\mathbb{C}l_2 = M_2(\mathbb{C})$.
(3) Show that $\mathbb{C}l_n = \begin{cases} M_{2^{n/2}}(\mathbb{C}) & n \text{ even} \\ M_{2^{\frac{n-1}{2}}}(\mathbb{C}) \oplus M_{2^{\frac{n-1}{2}}}(\mathbb{C}) & n \text{ odd.} \end{cases}$

([ABS64], §4)

Spin Groups. For simplicity, we consider $C_n := C_{n,0} = C(V, g)$, where $V \cong \mathbb{R}^n$, g is a positive definite symmetric bilinear form on V.

8.6. Denote $C_n^\times$ by the multiplicative group of unit of C_n.

(1) Show that $C_n^\times$ is a Lie group with Lie algebra C_n.
(2) More generally, show that if R is a finite-dimensional associative k-algebra ($k = \mathbb{R}$ or $\mathbb{C}$), then $R^\times$ is a Lie group.

8.7 (Pin and Spin Group)**.** We define groups $\mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$ as the subset of $C_n^\times$:

$$\mathrm{Pin}(n) := \{v_1 \dots v_k : v_i \in V, |v_i| = 1\}$$
$$\mathrm{Spin}(n) := \mathrm{Pin}(n) \cap C_n^+.$$

(1) We define a map $\alpha : C_n \to C_n$ by the unique map extending $V \to V$ given by $v \to -v$. Show that it is an involution on C_n and $C_n^\pm$ are ± 1-eigenspaces for α.

Following [ABS64], we define a twisted adjoint representation $\widetilde{Ad} : C_n^\times \to GL(C_n)$ by $\widetilde{Ad}_\phi(x) = \alpha(\phi) x \phi^{-1}$.

(2) Show that if $v \in V$, $v \neq 0$, then $\widetilde{Ad}_v$ is the reflection with respect to the hyerplane $v^\perp = \{w \in v^n : g(v, w) = 0\}$.
(3) Show that $\rho : \mathrm{Pin}(n) \to O(n)$ given by $\rho(\phi) = \widetilde{Ad}_\phi$ is a surjective Lie group homomorphism with $\ker \rho \cong \mathbb{Z}_2$.
(4) Deduce that $\mathrm{Spin}(n)$ is a non–trivial double cover for $SO(n)$ for $n \geq 2$ and is its universal cover if $n \geq 3$ (cf. problem 7.21).

Dirac Operator and $\bar{\partial}$-operators on Kähler Manifolds. In problem 8.8 to problem , we investigate Dirac operator on Kähler manifolds.

8.8 (Dirac Operator on Complex Manifolds)**.** Let W be a holomorphic vector bundle over a Kähler manifold M. Given any hermitian metric h on E, let ∇^W be the Chern connection on W. Since M is Kähler, the Levi-Civita connection ∇^{LC} on M is the Chern connection on $T^{1,0}M$. Hence, they induces a canonical connection $\nabla^E = \nabla^{LC} \otimes 1 + 1 \otimes \nabla^W$ on $E := \bigwedge T^{0,1}M \otimes W$.

(1) Show that E is a Clifford module with Clifford connection ∇^E, and the associated Dirac operator $D = \sqrt{2}(\bar{\partial}_W + \bar{\partial}^*_W)$.
(2) If M is only a complex manifold, define the Clifford connection and its associated Dirac operator D on $E = \bigwedge T^{*0,1}M \otimes W$. Moreover, show that the Dirac operator $D - \sqrt{2}(\partial + \partial^*) \in C^\infty(M, \mathrm{End}E)$ and such term vanishes exactly when M is Kähler.

8.9 (Dolbeault Operators and Harmonic Theory)**.** Let $W \to M$ be a holomorphic vector bundle over a complex manifold. Recall that in section 7.2.2, we have defined the Dolbeault operator $\bar{\partial} : A^{p,q}(M, E) \to A^{p,q+1}(M, E)$.

(1) Show that $\bar{\partial}^2_E = 0$.

Thus, we can form the *Dolbeault cohomology* by

$$H^{p,q}_{\bar{\partial}}(M, W) := \ker(\bar{\partial}^{(p,q)}_W)/\mathrm{im}(\bar{\partial}^{(p,q-1)}_W).$$

We define $\triangle_{\bar{\partial}_W} := \bar{\partial}^*_W\bar{\partial}_W + \bar{\partial}_W\bar{\partial}^*_W$ be the $\bar{\partial}$-Laplacian on W.

(2) Show that problem 6.13 can be generalized to forms with values in W:
  (a) $\dim \mathbb{H}^{p,q}(M, W) < \infty$, where $\mathbb{H}^{p,q}(M, W) := \ker(\triangle_{\bar{\partial}_W})$.
  (b) $A^{p,q}(M, W) = \bar{\partial}_E(A^{p,q-1}(M, W)) \oplus \mathbb{H}^{p,q}(M, W) \bigoplus \bar{\partial}^*_E(A^{p,q+1}(M, W))$.
(3) Deduce that $H^{p,q}_{\bar{\partial}}(M, W) \cong \mathbb{H}^{p,q}(M.W)$.

8.10 (Hirzebruch–Riemann–Roch Theorem)**.** Let $W \to M$ be a holomorphic vector bundle over a compact Kähler manifold M. Let D be the Dirac operator on the Clifford module $E := \bigwedge^{0,1} M \otimes W$.

(1) Show that $\mathrm{ind}D = \sum_i (-1)^i \dim H^{0,i}_{\bar{\partial}}(M, W)$.
(2) Compute the twisting curvature for W.

(3) Deduce Hirzebruch–Riemann–Rock formula rom Atiyah–Singer Theorem (cf. theorem 8.20):

$$\sum_i (-1)^i \dim H^{0,i}_{\bar\partial}(M,W) = \frac{1}{(2\pi i)^{n/2}} \int_M \mathrm{Td}(M)\mathrm{ch}(W), \tag{8.20}$$

$\mathrm{Td}(M) = \det\left(\frac{R^{1,0}}{e^{R^{1,0}}-1}\right)$ is the Todd genus (cf. problem 7.7).

(4) Argue that (8.20) holds for any compact complex manifold.

8.11 (Bochner–Kodaira–Nakano Identity)**.** Let $E \to M$ be a complex manifold. We denote by $(\nabla^{0,1})^*$ the formal adjoint of $\nabla^{0,1} = \bar\partial_E$.

(1) Show that the complex analogue for connection Laplace (cf. exercise 4.15 (2)) holds

$$(\nabla^{0,1})^*\nabla^{0,1} = -\sum_i \nabla_{Z_i}\nabla_{\bar Z_i} - \nabla_{\nabla_{Z_i}\bar Z_i},$$

where $\{Z_i\}$ is a local orthonormal frame for $T^{1,0}M$ with respect the hermitian metric on M.

Assume furthermore that M is Kähler. We define *canonical line bundle* of M by $K := \bigwedge^n(T^{*1,0}M)$.

(2) Express the curvature of K^* with respect to the Levi–Civita connection in terms of Riemannian curvature of M.

(3) Let E be a hermitian holomorphic vector bundle over M. Show that in the holomorphic coordinate $\mathbf{z}$ of M, the folllowing Bochner type formula, known as *Bocher–Kodaira–Nakano identity*, holds:

$$\bar\partial_E\bar\partial_E^* + \bar\partial_E^*\bar\partial_E = (\nabla^{0,1})^*\nabla^{0,1} + \sum_{i,j} \epsilon(d\bar z^i)\iota(dz^j)F^{W\otimes K^*}(\partial_{z^i}, \partial_{\bar z^j}), \tag{8.21}$$

where $F^{W\otimes K^*}$ is the curvature for $W \otimes K^*$.

8.12 (Kodaira Vanishing Theorem)**.** A hermitian line bundle $L \to M$ with curvature $F^L = \sum_{i,j} F_{ij} dz^i \wedge d\bar z^k$ is said to be a *positive line bundle* if F_{ij} is positive definite. Use Bochner–Kodaira–Nakano identity to deduce the *Kodaira vanishing theorem*:

(1) If $L \to M$ is a hermitian holomorphic line bundle on a compact Kähler manifold such that $L \otimes K^*$ is positive, then

$$H^{0,i}(M,L) = 0, \quad \text{for } i > 0.$$

(2) If L is a positive hermitian holomorphic line bundle and W is a hermitian holomorphic vector bundle on M, then for $m \gg 0$,

$$H^{0,i}(M, L^{\otimes m} \otimes W) = 0, \quad \text{for } i > 0.$$

8.13. Classify all intersection forms of surface M_d of degree $d > 1$ in $\mathbb{CP}^3$.

([DK90], p.13)