Chapter 7

Characteristic Classes and Cobordism

A lesson from algebraic topology tells us that it is useful to attach algebraic structures on the spaces and exploit the algebraic machinery to study the question of topology. We have seen that de Rham cohomology and singular cohomology are powerful tools to study topology of manifolds. In the this chapter, we present two more constructions on this direction, which play a dominant position in modern geometry, topology, and physics.

The first five sections are devoted to characteristic classes, which is a natural way to associate a vector bundle certain cohomology classes on the base manifolds. These cohomology classes measure the extent of vector bundles deviating from trivial. We first introduce Chern classes on complex vector bundle via connection which is due to S.S Chern and A. Weil. We then move to properties, computation technique of the Chern classes. and relation with others Characteristic Classes. Finally, we discuss the topological aspect and some application for the characteristic classes.

The next theme is the cobordism due to Pontryagin and Thom. Roughly speaking, two closed n-manifold are cobordant if they can deformed through a $(n+1)$-manifold. This construction allows us to partition all closed manifolds into equivalence classes and endow a ring structure Ω on the set of equivalence classes, called the cobordism ring. We will prove a remarkable theorem of Thom that reveals the structure of $\Omega \otimes \mathbb{Q}$ and deduce Hirzebruch signature theorem from it. One of the key in the proof is to use Thom's tranversality theorem to represent the cobordism ring as homotopy group of a certain space (Thom space).

7.1. Chern Classes via Chern–Weil Theory

7.1.1. Connections and Curvature on Complex Vector Bundle. Let E^r be a real or complex vector bundle of rank r over a C^∞ manifold M, ∇ be a $\mathbb{R}$ or $\mathbb{C}$-linear connection[1]. Let $\{s_\alpha\}_{\alpha=1}^r \subset C^\infty(U,E)$ be a local frame of E over an open set $U \subset M$, then we write

$$\nabla s_\alpha = \omega_\alpha^\beta \otimes s_\beta \in A^1(E), \tag{7.1}$$

where ω_α^β is a 1-form on U. Alternatively, we can write $s = (s_1, \dots, s_r)^t$ as column vector formed by the local frame and $\omega = \left(\omega_\alpha^\beta\right)$ as a matrix of 1-form, we can write (7.1) as $\nabla s = \omega s$. The matrix of 1-form ω is called a *connection 1-form*. Notice that the connection 1-form in general depends on the choice of local frame s.

Exercise 7.1. Let $s' = (s'_1, \dots, s'_r)$ be another local frame of E on U, and $\tilde{s} = gs$, for some $g = (g_i^j) \in C^\infty(U, GL(r,\mathbb{C}))$. Suppose $\nabla s' = \omega' s'$. Show that we have the following transformation formula:

$$\omega' = dgg^{-1} + g\omega g^{-1}.$$

Given a connection ∇ on E, we can induces connection[2] on $\bigotimes^r E \otimes \otimes^s E^*$ by (3.1) and (3.2). Also, one can extend the connection $\nabla : C^\infty(E) \to A^1(E)$ to $A^k(E) \to A^{k+1}(E)$ by Leibniz rule:

$$\nabla(\eta \wedge \theta) = d\eta \wedge \theta + (-1)^p \eta \wedge \nabla\theta, \quad \eta \in A^p(M), A^q(E).$$

Thus, we obtain a sequence similar to de Rham complex:

$$C^\infty(E) \xrightarrow{\nabla} A^1(E) \xrightarrow{\nabla} A^2(E) \xrightarrow{\nabla} \cdots .$$

[1] Recall that in section 3.1, for a real vector bundle $E \to M$, we define a connection $\nabla : C^\infty(E) \to A^1(E) = C^\infty(E \otimes T^*M)$ which is $\mathbb{R}$-linear and satisfies Leibniz rule: $\nabla(fs) = (df)s + f\nabla s$. It is easy to see that we can extend the definition to a complex vector bundle $E \to M$ and a $\mathbb{C}$-linear connection.

[2] Particularly, if $\tilde{s} = (\tilde{s}^1, \dots, \tilde{s}^r)$ is the dual frame of E^* with respect to a frame $s = (s_1, \dots, s_r)^t$, and assume that $\nabla \tilde{s}^j = \tilde{\omega}_i^j \tilde{s}^i$, then

$$0 = d(\delta_i^j) = d(\tilde{s}^j(s_i)) = \tilde{s}^j(\nabla s_i) + \nabla \tilde{s}^j(s_i) = \omega_i^j + \tilde{\omega}_i^j.$$

Hence, $\widetilde{\omega} = -\omega$ and $\nabla \tilde{s} = -\tilde{s}\omega$.

However, for $s \in C^\infty(U,E)$, $\nabla^2 s \neq 0$ in general. Instead, we have

$$\begin{aligned}\nabla^2 s &= \nabla(\nabla s) = \nabla(\omega s)\\ &= (d\omega - \omega\wedge\omega)s.\end{aligned}$$

We define the *curvature 2-form* of ∇ by $\Omega := d\omega - \omega\wedge\omega$. In terms of component, $\Omega = \left(\Omega_\alpha^\beta\right)$ is given by

$$\Omega_\alpha^\beta = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta. \tag{7.2}$$

Though ω is not a globally defined section, Ω is a globally defined endomorphism valued 2-form on M.

Lemma 7.1. $\Omega \in A^2(\mathrm{End}E)$.

PROOF. Observe that if $f \in C^\infty(U)$, then

$$\nabla^2(fs) = \nabla(fds + f\nabla s) = -df\nabla s + df\nabla s + f\nabla^2 s = f\nabla^2 s.$$

This shows that $\nabla^2 s = \Omega s$ is function–linear. Hence, $\Omega \in A^2(\mathrm{End}E)$. $\square$

Exercise 7.2. [Properties of Curvature 2-Form]

(1) Suppose $s' = gs$ is another frame of E, then $\Omega' = g\Omega g^{-1}$.
(2) For $X, Y \in C^\infty(TM)$, show that Ω is indeed the curvature:

$$\Omega(X,Y)s = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})s.$$

By exterior differentiating again on $\Omega = d\omega - \omega\wedge\omega$, we get following form of the *second Bianchi identity*:

$$d\Omega = \omega\wedge d\omega - d\omega\wedge\omega = \omega\wedge\Omega - \Omega\wedge\omega = [\omega,\Omega]. \tag{7.3}$$

Exercise 7.3. Show that for any $p \in M$, there exists a "normal frame" $s = (s_1,\dots,s_r)^t$ on a chart U of p such that $\omega(p) = 0$, $d\Omega(p) = 0$.

Notice that for $\Phi \in A^k(\mathrm{End}E)$, it is easy to see that

$$\nabla\Phi = d\Phi + (-1)^k\Phi\wedge\omega - \omega\wedge\Phi = d\Phi + (-1)^k[\Phi,\omega]. \tag{7.4}$$

In particular, $\nabla\Omega = d\Omega + [\Omega,\omega]$. Therefore, (7.3) can be written as

$$\nabla\Omega = 0. \tag{7.5}$$

Exercise 7.4.

(1) Let $\Phi \in A^k(\mathrm{End}E)$. Show that (7.4) and $\nabla^2\Phi = [\Phi, \Omega]$.
(2) Explain that the second Bianchi identity above is equivalent to the version given in Exercise 3.7 (2).

7.1.2. Chern Forms and Chern Classes. We now in place to define *(total) Chern form* associated to a complex vector bundle E. From exercise 7.2 (1), we see that for different choice of local frames $s' = gs$ of E, the curvature transforms as $\Omega' = g\Omega g^{-1}$. Thus, $\mathrm{tr}(\Omega^k) \in A^k(M)$ and $\det(\Omega^k) \in A^{\mathrm{even}}(M)$ are well-defined.

Definition 7.2. The *total Chern form* associated to a complex vector bundle $E \to M$ and a connection ∇ on E is defined by

$$c(E,\nabla) := \det(I + \frac{i}{2\pi}\Omega) = c_0(E,\nabla) + c_1(E,\nabla) + \cdots + c_r(E,\nabla),$$

where $c_k(E,\nabla) = \left(\frac{i}{2\pi}\right)^k \sigma_k(E,\nabla) \in A^{2k}(M)$ is called the *k-th Chern form* and $\sigma_k(E)$ is the k-th elementary symmetric polynomial of eigenvalues of Ω.

For $c_k(E,\nabla)$ to define a class in de Rham cohomology, we first need to check that $c_k(E,\nabla)$ is closed.

Proposition 7.3. *$dc_k(E,\nabla) = 0$, for all $k = 1,\ldots,r$.*

PROOF. Observe that both $c_k(E,\nabla)$ and $\mathrm{tr}(\Omega^k)$ form bases for symmetric function for eigenvalues of Ω. It is equivalent to show $\mathrm{tr}(\Omega^k) = 0$. It is easy to see that for $\alpha \in A^k(\mathrm{End}E)$,

$$d\mathrm{tr}(\alpha) = \mathrm{tr}(\nabla\alpha). \tag{7.6}$$

Thus, we have $d\mathrm{tr}(\Omega^k) = \mathrm{tr}(\nabla(\Omega^k)) = k\mathrm{tr}(\nabla\Omega \wedge \Omega^{k-1}) = 0$, by second Bianchi identity (7.5). □

Exercise 7.5. Verify (7.6).

A priori, Chern classes $c_k(E,\nabla)$ depends on the choice of the connection ∇. However, the following proposition shows its cohomology class is in fact independent of the choice of ∇.

Proposition 7.4. *For any two connections ∇_0, ∇_1 on $E \to M$, there exists $Q_k(\nabla_0, \nabla_1) \in H^{2k-1}_{\mathrm{dR}}(M, \mathbb{C})$ such that*

$$c_k(E, \nabla_1) - c_k(E, \nabla_2) = dQ_k(\nabla_1, \nabla_2).$$

PROOF. Given two connections ∇_0, ∇_1 on E, by exercise 3.2, we know that $\eta = \nabla_0 - \nabla_1 \in A^1(\mathrm{End}E)$. For each $t \in [0,1]$, let

$$\nabla_t := (1-t)\nabla_0 + t\nabla_1 = \nabla_0 + t(\nabla_1 - \nabla_0) = \nabla_0 + t\eta.$$

If we denote ω_t by the connection 1-form for the connection ∇_t and thus $\omega_t = \omega_0 + t\eta$. In terms of ω_t, Ω_t is given by

$$\begin{aligned}
\Omega_t &= d\omega_t - \omega_t \wedge \omega_t \\
&= d\omega_0 + td\eta - \omega_t \wedge \omega_t \\
&= d\omega_0 - \omega_0 \wedge \omega_0 - td\eta - t(\omega_0 \wedge \eta + \eta \wedge \omega_0) - t^2 \eta \wedge \eta \\
&= \Omega_0 + t\nabla_0 \eta - t^2 \eta \wedge \eta \quad \text{(by (7.4))}
\end{aligned}$$

Particularly, the time derivative of Ω_t is given by

$$\Omega_t' := \frac{d\Omega_t}{dt} = \nabla_0 \eta - 2t\eta \wedge \eta. \tag{7.7}$$

Again, we need only to prove the statment for $[\mathrm{tr}(\Omega^k)] \in H^{2k}_{\mathrm{dR}}(M; \mathbb{C})$. First, notice that $\Omega_t' = \nabla_t \eta$. Thus,

$$\begin{aligned}
\frac{d}{dt}(\Omega_t^k) &= k\Omega_t' \wedge \Omega_t^{k-1} = k(\nabla_t \eta) \wedge \Omega_t^{k-1} \\
&= k\nabla_t(\eta \wedge \Omega_t^{k-1}) - k\eta \wedge \nabla_t \Omega_t^{k-1}.
\end{aligned}$$

From the second Bianchi identity, $\nabla_t \Omega_t^{k-1} = 0$. As a result,

$$\begin{aligned}
\frac{d}{dt}\left(\mathrm{tr}\Omega_t^k\right) &= \mathrm{tr}\left(\frac{d}{dt}(\Omega_t^k)\right) \\
=& k\mathrm{tr}(\nabla_t(\eta \wedge \Omega_t^{k-1})) = kd\mathrm{tr}(\eta \wedge \Omega_t^{k-1}).
\end{aligned}$$

Therefore, we write $\mathrm{tr}(\Omega_1^k) - \mathrm{tr}(\Omega_2^k)$ as

$$\int_0^1 \frac{d}{dt}\left(\mathrm{tr}\Omega_t^k\right) dt = kd\left[\int_0^1 \mathrm{tr}\left(\eta \wedge \Omega_t^{k-1}\right) dt\right] = dQ_k(\nabla_0, \nabla_1),$$

where $Q_k := k\int_0^1 \mathrm{tr}\left(\eta \wedge \Omega_t^{k-1}\right) dt$. $\square$

Consequently, the class $[c(E,\nabla)]$ and $[c_k(E,\nabla)] \in H^{2k}(M,\mathbb{C})$ are called the *total Chern class*[3] and *k-th Chern class* of the complex vector bundle $E \to M$ respectively, and are denoted by $c(E)$ and $c_k(E)$ for short. Conventionally, we set $c_0(E) = 1$ and $c_k(E) = 0$, for $k > r$.

In section 6.6, for a complex manifold M with a Riemannian metric which is compatible with its complex structure, we can define a hermitian metric on its holomorphic tangent bundle. In general, for a smooth complex vector bundle $E \to M$ over a C^∞ manifold M, a *hermitian metric* h on E is a section $h \in C^\infty(E^* \otimes E^*)$ such that

$$h_p(u,\bar{v}) = \overline{h_p(v,\bar{u})}; \quad h_p(w,\bar{w}) > 0,$$

for any $p \in M$, $u,v \in E_p$, $w \in E_p \setminus \{0\}$. Similar to the case of Levi-Civita connection, a connection ∇ on E is *hermitian* or *metrical* with respect to h if

$$d(h(s_1,s_2)) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2), \quad \forall s_1, s_2 \in C^\infty(E).$$

Exercise 7.6. Let $E \to M$ be a complex vector bundle.

(1) There exists a hermitian metric on E.
(2) For any hermitian metric h on E, there exists a hermitian connection ∇ on E with respect to h.

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[3]We can generalize the previous constructions of Chern classes as following. A polynomial $P : M_r(\mathbb{C}) \to \mathbb{C}$ is called *invariant* if $P(AXA^{-1}) = P(X)$, for any $A \in GL(r,\mathbb{C})$. Obviously, both det and tr are invariant polynomial. By the same token above, $P(\Omega^k) \in A^{\text{even}}(M)$ is well–defined. Similarly, for any invariant polynomial P, one can show that $P(\Omega)$ is a closes form. Also, for any two connections ∇_0 and ∇_1, there exists a differential form $Q(\nabla_0,\nabla_1)$, called the *transgression* of P, such that $P(\nabla_0) - P(\nabla_1) = dQ(\nabla_0,\nabla_1)$ (cf. Problem 7.20). Thus, the cohomology class $[P(\Omega)]$ is well–defined and is called a *characteristic class*. Given any invariant polynomial P, we may decompose $P = \sum_{i=1}^{d} P_i$ into homogeneous part. Since each P_i is symmetric polynomial of eigenvalues of Ω and $\{c_k(E,\nabla)\}$, $\{\mathrm{tr}(\Omega^k)\}$ are bases, there exists unique $Q_1, Q_2 \in \mathbb{C}[X_1,\ldots,X_r]$ such that

$$P_i = Q_1(c_1,\ldots,c_r) = Q_2(\mathrm{tr}(\Omega),\ldots,\mathrm{tr}(\Omega^r)).$$

Thus, every characteristic class of E is some polynomial over Chern classes or trace powers of E inside cohomology ring.

Now, if we choose $s_1, \dots, s_r$ be a unitary frame of E with respect to h (i.e., $h(s_i, s_j) = \delta_{ij}$), then

$$0 = d(h(s_i, s_j)) = h(\omega_i^k s_k, s_j) + h(s_i, \omega_j^l s_l) = \omega_i^j + \overline{\omega_j^i}. \tag{7.8}$$

In other words, the connection 1-form ω with respect to the unitary frame $s = (s_1, \dots, s_r)^t$ is skew–hermitian. Hence, $\Omega = d\omega - \omega \wedge \omega$ is also skew–hermitian. Since eigenvalues of skew–hermitian matrices are purely imaginary, $c_k(E, \nabla) \in A^{2k}(M, \mathbb{R})$ are real differential forms. By proposition 7.4, we obtain

Corollary 7.5. *For any complex vector bundle E, Chern classes*[4] *$c(E)$, $c_k(E) \in H^*(M, \mathbb{R})$, for $k = 1, \dots, r$.*

7.2. Properties of Chern Classes

7.2.1. Functoriality of Chern Classes.

Let $\pi : E \to M$ be a complex vector bundle over a C^∞ manifold M, $f : N \to M$ be a C^∞ map, In problem 2.18, we have defined the pull–back bundle of E via f

$$f^*E := \{(x, v) \in N \times E : \pi(v) = f(x)\},$$

which is a vector bundle over N. Given any connection ∇ on E, one can induce a canonical connection $f^*\nabla$ on f^*E as follows: for any open set $U \subset M$, a section $s \in C^\infty(U, E)$, there is an induced section $f^*s := s \circ f \in C^\infty(f^{-1}(U), f^*E)$. Moreover, one can check that for any connection $s' \in C^\infty(N, f^*E)$, s' can be locally expressed as linear combination of induced sections of the form f^*s above (cf. exercise below for precise statement). We can then define the *pull–back connection* $f^*\nabla$ on f^*E by first defining

$$f^*\nabla_X(f^*s) = f^*(\nabla_{f_*X} s) \tag{7.9}$$

and then extending linearly.

Exercise 7.7. Let $E \to M$ be a complex vector bundle, $f : N \to M$ be a C^∞ map, $f^*E \to N$ be the pull–back bundle over N.

[4] As we will see in the next section, $c(E)$ and $c_k(E)$ are in fact lie in $H^*(M, \mathbb{Z})$ via topological theory of characteristic classes.

(1) For any open set $V \subset N$, a section $s' \in C^\infty(V, f^*E)$, there exists an open set $U \subset M$ with $f^{-1}(U) \subset V$, a frame $s = (s_1, \ldots, s_r)$ of E over U such that $s' = \sum_{j=1}^r a_j f^* s_j$, where $a_j \in C^\infty(f^{-1}(U))$.

(2) Verify that the pull-back connection is indeed a connection on f^*E and is uniquely characterized by (7.9).

Henceforth, locally if $\omega = (\omega_i^j)$ is a connection 1-form of (E, ∇), then $f^*\omega := (f^*\omega_j^i)$ is a connection 1-form on $(f^*E, f^*\nabla)$. Also, $f^*\Omega := (f^*\Omega_j^i)$ is the curvature 2-form of $(f^*E, f^*\nabla)$. From this, the following result is immediately.

Corollary 7.6 (Functoriality of Chern Classes)**.** *Let $E \to M$ be a complex vector bundle, $f : N \to M$ be a C^∞ map. Then for $k \geq 0$, $c_k(f^*E) = f^*(c_k(E)) \in H^{2k}_{\mathrm{dR}}(N, \mathbb{R})$.*

Particularly, if $E \cong F$ as vector bundles, then $c(E) = c(F)$.

Example 7.7. If E is trivial bundle, i.e., $E = M \times \mathbb{C}^r$. Notice that on $M \times \mathbb{C}^r$, we can endow a trivial connection $\nabla = d$, i.e. $\omega \equiv 0$. Hence, $\Omega = 0$ for trivial connection. Thus, combining with above corollary, if $F \cong E$, we have $c_k(F) = 0$, for $k > 0$. This justifies the assertion in the beginning that characteristic classes measure the deviation of a bundle being trivial.

Next, let E_1, E_2 be complex vector bundles over M. In problem 2.21, we define the direct sum $E_1 \bigoplus E_2$, the tensor product $E_1 \bigotimes E_2$ of E_1 and E_2, and the dual bundle E_1^* of E_1. We have seen that for any given connection ∇_i on E_i, $i = 1, 2$, we can define canonical connections on $E_1 \otimes E_2$ and on E_1^* via (3.2) and (3.1). For $E_1 \bigoplus E_2$, we can also induce a connection by

$$\nabla(s_1, s_2) = (\nabla_1 s_1, \nabla_2 s_2), \quad (s_1, s_2) \in C^\infty(E_1 \bigoplus E_2).$$

It is easy from definition to prove the following:

Exercise 7.8. Let ω_i, Ω_i be the connection 1-forms and curvature 2-forms of (E_i, ∇_i), $i = 1, 2$. Show that

(1) The connection and curvature of canonical connection on $E_1 \bigotimes E_2$ are given by $\omega_1 \otimes 1 + 1 \otimes \omega_2$, $\Omega_1 \otimes 1 + 1 \otimes \Omega_2$.

(2) The connection and curvature of canonical connection on E_1^* are given by $-\omega_1^t, -\Omega_1^t$.

(3) The connection and curvature of canonical connection on $E_1 \bigoplus E_2$ are given by $\begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}, \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix}$.

From the exercise, we immediately obtain

Proposition 7.8. *If E', E'' be complex vector bundles over M, then*

(1) $c(E' \bigoplus E'') = c(E')c(E'')$ *or*

$$c_k(E' \bigoplus E'') = \sum_{i=0}^{k} c_i(E')c_{k-i}(E''), \quad \forall k \geq 0.$$

(2) $c_k(E'^*) = (-1)^k c_k(E'), \forall k \geq 0.$

The formula (1) is called the *Whitney product formula*. More generally, let $0 \to E' \to E \to E'' \to 0$ be a short exact sequence[5]. In fact, a short exact sequence of C^∞ vector bundle always splits, i.e. $E \cong E' \bigoplus E''$. Thus, applying Whtiney product formula, we still have $c(E) = c(E')c(E'')$.

Exercise 7.9. Show that a short exact sequence of smooth vector bundle splits smoothly (Hint: consider a bundle metric on E).

For tensor product of line bundles, we have the following.

Proposition 7.9. *Let $L_i \to M$ be complex line bundle, for $i = 1, 2$. Then $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$.*

Proof. By exercise 7.8, the curvature for canonical connection on $L := L_1 \otimes L_2$ is given by $\Omega_1 \otimes 1 + 1 \otimes \Omega_2 = \Omega_1 + \Omega_2$, where we identify $\mathrm{End} L = L^* \otimes L \cong \mathbb{C}$. Hence,

$$c_1(L_1 \otimes L_2) = \frac{i}{2\pi}\mathrm{tr}(\Omega_1 + \Omega_2) = \frac{i}{2\pi}(\mathrm{tr}\Omega_1 + \mathrm{tr}\Omega_2) = c_1(L_1) + c_1(L_2).$$

□

[5] $0 \to E' \to E \to E'' \to 0$ is a short exact seuqence if for any $p \in M$, $0 \to E'_p \to E_p \to E''_p \to 0$ is exact as a vector spaces.

7.2.2. Torsion-Free and Chern Connection. We have considered hermitian connections of a complex vector bundle over C^∞ manifold M. In analogy with the Levi–Civita connection on Riemannian manifolds, it is natural to ask whether there exists a "torsion-free condition" which uniquely characterize hermitian connections on complex vector bundle. A natural analogue for this is holomorphic connections on a holomorphic vector bundle.

Definition 7.10 (Holomorphic Vector Bundle). Let M be a complex manifold. A complex vector bundle $E \to M$ is a *holomorphic vector bundle* if one can find an trivialization $\{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ of E such that the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C})$ are holomorphic maps, $\forall \alpha, \beta \in A$.

Obviously, a holomorphic vector bundle $E \to M$ is a complex manifold. A section $s : U \to E$ is called *holomorphic* if it is a holomorphic map. We denote the space of holomorphic sections of E over U by $\mathcal{O}(U, E)$. If $s = (s_1, \ldots, s_r)$ is a holomorphic frame of E over U, then any $s \in \mathcal{O}(U, E)$, there exists (uniquely) $f_1, \ldots, f_r \in \mathcal{O}(U)$ such that $s = \sum_{i=1}^r f_i s_i$.

Exercise 7.10. Let M be a complex manifold. Show that $T^{1,0}M$ and $\bigwedge^{p,0} T^*M$ ($\forall p \geq 1$) are holomorphic vector bundles.

For any holomorphic vector bundle $E \to M$, we can define *Dolbeault operator* $\bar{\partial}_E : A^{p,q}(E) := C^\infty(\bigwedge^{*p,q} T^*M \otimes E) \to A^{p,q+1}(E)$ as follows. Let $s = (s_1, \ldots, s_r)$ be a holomorphic frame of E over U, then any $\alpha \in A^{p,q}(E)$, $\alpha\big|_U = \sum_{i=1}^r \alpha_i s_i$, where $\alpha_i \in A^{p,q}(M)$. We define

$$\bar{\partial}_E(\alpha\big|_U) = \sum_{i=1}^r (\bar{\partial}\alpha_i) s_i.$$

Exercise 7.11. Show that we can glue $\bar{\partial}_E(\alpha\big|_U)$ into a global section $\bar{\partial}_E \alpha \in A^{0,q+1}(E)$

Now, given a $\mathbb{C}$-linear connection ∇ on a holomorphic vector bundle $E \to M$. Using (6.18), we can decompose $\nabla : C^\infty(E) \to A^1(E)$ as $\nabla = \nabla^{1,0} + \nabla^{0,1}$, where

$$\nabla^{1,0} := \pi_{1,0} \circ \nabla; \quad \nabla^{0,1} := \pi_{0,1} \circ \nabla.$$

A connection ∇ on a holomorphic vector bundle $E \to M$ is *holomorphic* if $\nabla^{0,1} = \bar{\partial}_E$. The following theorem shows that this is indeed the analogue for torsion–free condition of Levi-Civita connection.

Theorem 7.11 (Fundamental Theorem of Hermitian Geometry). *Let $E \to M$ be a holomorphic vector bundle, h be a hermitian metric on E. Then there exists a unique connection ∇ on E such that it is holomorphic and metrical with respect to h.*

PROOF. We prove only the uniqueness. Let ∇ be such a connection, $s = (s_1, \dots, s_r)$ be a holomorphic frame of E over an open set $U \subset M$. We denote $h_{ij} := h(s_i, s_j)$, $H := (h_{ij})$, and $H^{-1} := (h^{ij})$. Since ∇ is holomorphic,

$$\nabla s_i = \nabla^{1,0} s_i + \nabla^{0,1} s_i = \nabla^{1,0} s_i + \bar{\partial}_E s_i = \nabla^{1,0} s_i = \omega_i^j s_j.$$

Hence, the connection 1-form $\omega = (\omega_i^j)$ for ∇ with respect to s has only $(1,0)$-part. Next, by metrical condition,

$$dh_{ij} = h(\nabla s_i, s_j) + h(s_i, \nabla s_j) = \omega_i^k h_{jk} + \overline{\omega_j^k} h_{ik}.$$

From (6.18), we see that $\partial h_{ij} = \omega_i^k h_{jk}$. This implies that $\omega_i^k = h_{ij} h^{jk}$ or $\omega = (\partial H) H^{-1}$ in matrix notation. $\square$

The unique connection is called the *Chern connection* on (E, h).

Exercise 7.12. Let E be a holomorphic vector bundle over a complex manifold M, h be a hermitian metric on E.

(1) Show that a differential operator locally defined by $\nabla := d + (\partial H) H^{-1}$ defines a connection on E.
(2) If M is a Kähler manifold, and $E = T^{1,0}M$, then under the bundle isomorphism $T^{1,0}M \cong TM$, the Chern connection ∇ on $T^{1,0}M$ corresponds to the Levi-Civita connection on TM.

Let Ω be the curvature 2-form corresponding to the Chern connection ∇. Since $\omega = (\partial H) H^{-1}$ and $\partial(H^{-1}) = -H^{-1}(\partial H) H^{-1}$,

$$\begin{aligned}
\Omega &= d\omega - \omega \wedge \omega = (\partial + \bar{\partial})(\partial H H^{-1}) - (\partial H) H^{-1} \wedge (\partial H) H^{-1} \\
&= -\partial H \wedge \partial(H^{-1}) + (\bar{\partial}\partial H) H^{-1} - \partial H \wedge \bar{\partial} H^{-1} - (\partial H) H^{-1} \wedge (\partial H) H^{-1} \\
&= (\bar{\partial}\partial H) H^{-1} + (\bar{\partial} H) H^{-1} \wedge (\partial H) H^{-1} = \bar{\partial}_E \omega \in A^{1,1}(\mathrm{End} E).
\end{aligned}$$

Particularly, let $L \to M$ be a holomorphic line bundle, h be a hermitian metric on it. If s is non–vanishing holomorphic section of L over U, then $H = h(s,s) \in C^\infty(U, \mathbb{R})$ and $H > 0$. In this case, the curvature form is given by

$$\Omega = -\partial\bar{\partial}\log H, \tag{7.10}$$

and thus $c_1(L) = \frac{i}{2\pi}\bar{\partial}\partial\log H$.

7.3. Examples and Calculation of Chern Classes

7.3.1. Chern Classes on Projective Spaces.

Let us apply above calculations to $\mathbb{CP}^n$, the complex projective space of dimension n. Recall that it is the set of all lines in $\mathbb{CP}^{n+1}$ through 0. On $\mathbb{CP}^n$, we have a god-given line bundle, called *tautological bundle* $\mathcal{O}(-1)$, as following:

$$\mathcal{O}(-1) := \{(p,v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : v \in p\}. \tag{7.11}$$

Exercise 7.13. Show that $\mathcal{O}(-1)$ is a holomorphic line bundle (Hint: Use standard chart given in example 6.39)

On the product bundle $\underline{\mathbb{C}}^{n+1} := \mathbb{CP}^n \times \mathbb{C}^{n+1}$, we have a natural hermitian metric given by standard hermitian metric $\sum_{i=0}^n |Z_i|^2$ on $\mathbb{C}^{n+1}$. Restricting it to the subbundle $L := \mathcal{O}(-1)$ gives a bundle metric h on it. Thus, by (7.10), the first Chern form for the Chern connection with respect to h is given by

$$c_1(L) = \frac{i}{2\pi}\partial\bar{\partial}\log(\sum_{i=0}^n |Z_i|^2).$$

One finds that $c_1(L) = -\omega_{FS}$, the Fubini–Study metric given in example 6.39.

Hence, $\omega_{FS} = c_1(L^*)$ (cf. exercise 7.8). The dual bundle L^* is denoted by $\mathcal{O}(1)$, called the *hyperplane bundle*. Let $h := c_1(L^*)$. Suppose H is a hyperplane in $\mathbb{CP}^n$, i.e. $H \cong \mathbb{CP}^{n-1}$. The Poincaré dual $[\eta_H] \in H^2(\mathbb{CP}^n, \mathbb{R})$ is called the *hyperplane class*. Recall that the cohomology group of $\mathbb{CP}^n$ is given by[6]:

[6] One way to see this is through cellular decomposition of $\mathbb{CP}^n$ introduced in section 7.5

Fact 7.12. $H^k(\mathbb{CP}^n,\mathbb{R}) = \begin{cases} \mathbb{R} & ,k = 0,2,4,\ldots,2n; \\ 0 & ,k = 1,3,5,\ldots,2n-1. \end{cases}$

Hence, we can regard $[\eta_H]$ as the generator of $H^2(\mathbb{CP}^n,\mathbb{R})$ and $h = \lambda[\eta_H]$. If L is a complex line (i.e. $L \cong \mathbb{CP}^1$), then

$$L.H := \eta_H([L]) = \int_L \eta_H = 1.$$

However, by exercise 6.20, we see that $\int_L \omega_{FS} = 1$ and thus $h = \eta_H$ is the hyperplane class. As a result, we have proved:

Proposition 7.13 (Normalization Axiom)**.** *Let $L := \mathcal{O}_{\mathbb{P}^n}(-1)$ be the tauotological line bundle on $\mathbb{CP}^n$. Then $c(L) = 1 - h$, where h is the hyperplane class.*

Moreover, by the same proof as in corollary 6.43, $[h^k] \neq 0$ in $H^{2k}(\mathbb{CP}^n,\mathbb{R})$. As a result, we see that as a ring[7] $H^*(\mathbb{CP}^n,\mathbb{R}) = \mathbb{R}[h]/(h^{n+1})$.

We denote Q by the quotient bundle $(\underline{\mathbb{C}}^{n+1})/\mathcal{O}(-1)$, which fits into an short exact sequence of vector bundle

$$0 \to \mathcal{O}(-1) \to \underline{\mathbb{C}}^{n+1} \to Q \to 0$$

or upon tensoring $\mathcal{O}(1)$,

$$0 \to \underline{\mathbb{C}} \to \mathcal{O}(1)^{\oplus(n+1)} \to \mathcal{O}(1) \otimes Q \to 0.$$

Exercise 7.14. Show that $\mathcal{O}(1) \otimes Q = \mathrm{Hom}(\mathcal{O}(-1), Q) = T\mathbb{P}^n$.

We then obtain the celebrated *Euler sequence*:

$$0 \to \underline{\mathbb{C}} \to \mathcal{O}(1)^{\oplus(n+1)} \to T\mathbb{P}^n \to 0. \tag{7.12}$$

From Whitney product formula, we have

$$c(\mathbb{P}^n) := c(T\mathbb{P}^n) = (1+h)^{n+1}. \tag{7.13}$$

[7]In fact, h is integral cohomology class. Hence, the result holds for the integral coefficients.

7.3.2. Projective Bundle and Splitting Principle.

More generally, given a complex vector bundle $\rho : E^r \to M$ with transition functions $g_{ij} : U_i \cap U_j \to GL(r, \mathbb{C})$, we can define $\mathbb{P}(E)$, the *projective bundle* of E, by making each fiber $E_p \cong \mathbb{C}^r$ into projective space $\mathbb{P}(E_p) \cong \mathbb{CP}^{r-1}$ and glueing via $\tilde{g}_{ij} : U_i \cap U_j \to PGL(r, \mathbb{C})$. A point of $\mathbb{P}(E)$ is a line $l_p \subset E_p$ through origin. It is not hard to show that $\mathbb{P}(E)$ is a smooth manifold with a submersion $\pi : \mathbb{P}(E) \to M$.

Exercise 7.15. Let $\mathbb{P}(E)$ be the projective bundle of $\rho : E \to M$.

(1) Show that $\mathbb{P}(E)$ is a smooth manifold and π induces a submersion $\pi : \mathbb{P}(E) \to M$.
(2) Show that there exists an open covering $\{U_i\}_{i \in I}$ of M such that $\pi^{-1}(U_i) \cong U_i \times \mathbb{P}^{r-1}$.

On $\mathbb{P}(E)$, we have pull–back bundle $\pi^* E$ whose fiber at l_p is E_p, which $\pi^* E\big|_{\pi^{-1}(p)} = \pi^{-1}(p) \times E_p$. We generalize the *tautological line bundle* $S := \mathcal{O}_{\mathbb{P}(E)}(-1)$ to $\mathbb{P}(E)$ by

$$\mathcal{O}_{\mathbb{P}(E)}(-1) := \{(l_p, v) \in \pi^* E : v \in l_p\}.$$

The *universal quotient bundle* Q is then defined by $\pi^* E / S$ and fits into the *Euler sequence on* $\mathbb{P}(E)$:

$$0 \to S \to \pi^* E \to Q \to 0. \tag{7.14}$$

Similarly, we denote $\mathcal{O}_{\mathbb{P}(E)}(1) = S^*$ by the *hyperplane bundle* on $\mathbb{P}(E)$, $\xi = c_1(S^*) \in H^2(\mathbb{P}(E))$. By construction, $S\big|_{\mathbb{P}(E_p)} = \mathcal{O}_{\mathbb{P}(E_p)}(-1)$. Using functoriality of Chern classes, $\xi\big|_{\mathbb{P}(E_p)}$ is the hyperplane class on $\mathbb{P}(E_p)$. Therefore, $1, \xi, \xi^2, \ldots, \xi^{r-1}$ are cohomology class on $\mathbb{P}(E)$ whose restriction on each fiber freely generate the cohomology of the fiber $\mathbb{P}(E_p)$ We make use the following fact, which is the generalization of Künneth formula (cf. Exercise 2.23).

Theorem 7.14 (Leray–Hirsch)**.** *Let E be a fiber bundle (cf. problem 5.14) over M with fiber F. Assume that there exists $e_1, \ldots, e_s \in H^*(E)$ such that $\iota^* e_1, \ldots, \iota^* e_s$ forms a basis of $H^*(F)$, where $\iota : F \hookrightarrow E$ is the inclusion map. Then $H^*(E)$ is a free module over $H^*(M)$ with basis $\{e_1, \ldots, e_s\}$, i.e.*

$$H^*(E) \cong H^*(M) \otimes \mathbb{R}[e_1, \ldots, e_s] \cong H^*(M) \otimes H^*(F).$$

Exercise 7.16. Assuming M has finite good cover, show Leray–Hirsch theorem by Mayer–Vietoris argument.

By Leray–Hirsch theorem, $H^*(\mathbb{P}(E))$ is a free $H^*(M)$-module with basis $\{1, \xi, \xi^2, \ldots, \xi^{r-1}\}$. Hence, ξ^r can be expressed into

$$\xi^r + \pi^* a_1 \xi^{r-1} + \cdots + \pi^* a_r = 0,$$

where $a_1, \ldots, a_r \in H^*(M)$, or

$$H^*(\mathbb{P}(E)) = H^*(M)[\xi]/(\xi^r + \pi^* a_1 \xi^{r-1} + \cdots + \pi^* a_r). \tag{7.15}$$

Miraculously, we have

Theorem 7.15 (Projective Bundle Formula). *$a_i = c_i(E)$, for $i = 1, \ldots, r$. That is,*

$$H^*(\mathbb{P}(E)) = H^*(M)[\xi]/(\xi^r + \pi^* c_1(E) \xi^{r-1} + \cdots + \pi^* c_r(E)).$$

Before proving this, let us first discuss an important technique in computing Chern classes, known as *splitting principle*.

Theorem 7.16 (Splitting Principle). *Let $\rho : E \to M$ be a C^∞ complex vector bundle of rank r over a C^∞ manifold M. There exists a C^∞ manifold $F(E)$, called splits manifold for E, and a C^∞ map $\sigma : F(E) \to M$ such that*

(1) $\sigma^* E = L_1 \bigoplus L_2 \bigoplus \cdots \bigoplus L_r$, where $L_1, \ldots, L_r$ are complex line bundles on $F(E)$.
(2) $\sigma^* : H^*(M) \to H^*(F(E))$ is injective.

Proof. We prove by induction on the rank r of E. Let $\pi : \mathbb{P}(E) \to M$ be the projective bundle of E, $S_E := \mathcal{O}_{\mathbb{P}(E)}(-1)$ be the tautological bundle of $\mathbb{P}(E)$. Then we have the short exact sequence

$$0 \to S_E \to \pi^* E \to Q_E \to 0,$$

where Q_E is the universal quotient bundle on $\mathbb{P}(E)$. By exercise 7.9, we know that $\pi^* E = S_E \bigoplus Q_E$. Also, $\pi^* : H^*(M) \to H^*(\mathbb{P}(E))$ is injective by Leray–Hirsch theorem. Now, since Q_E is a a vector bundle of rank $r - 1$, there exists a split manifold $\sigma_0 : F(Q_E) \to \mathbb{P}(E)$ such that $\sigma_0^* Q_E = L_2 \bigoplus \cdots \bigoplus L_r$ and $\sigma_0^* : H^*(M) \to H^*(F(Q_E))$ is injective. Consequently, we consider $\sigma = \sigma_0 \circ \pi : F(Q_E) \to M$, then $\sigma^* E = \sigma_0^* S_E \bigoplus L_2 \bigoplus \cdots \bigoplus L_r$ and $\sigma^* = \pi^* \circ \sigma_0^*$ is injective. $\square$

More generally, by repeating the construction above, we see that given any number of complex vector bundle $E_1, \ldots, E_m$ over M, there exists a split manifold N and $\sigma : N \to M$ such that $\sigma^* E_i$ are all splits into line bundles and $\sigma^* : H^*(M) \hookrightarrow H^*(N)$. By splitting principle and functoriality of Chern classes, we obtain:

Corollary 7.17. *To prove a polynomial identity in the Chern classes of complex vector bundles, it suffices to prove it under the assumption that all vector bundles are direct sum of line bundles.*

Therefore, given a complex vector bundle $E \to M$ of rank r, we may write formally

$$c(E) = \prod_{i=1}^{r}(1 + x_i),$$

where x_i is called the *Chern root* of $c(E)$. These $x_i's$ can be regarded as the first Chern classes of the line bundles of $\sigma^* E = \bigoplus_{i=1}^{r} L_i$ on the split manifold $F(E)$. Since $c_1(E), \ldots, c_r(E)$ are elementary symmetric polynomial in $x_1, \ldots, x_r$, any symmetric polynomial in $x_1, \ldots, x_r$ is a polynomial in $c_1(E), \ldots, c_r(E)$. It is more convenient to express Chern classes in terms of Chern roots.

Example 7.18 (Chern Classes of Tensor Product). If if $E = \bigoplus_{i=1}^{r} L_i$ with $c(E) = \prod_{i=1}^{r}(1 + x_i)$, $F = \bigoplus_{j=1}^{s} M_j$ with $\prod_{j=1}^{s}(1 + y_j)$, where L_i, M_j are line bundles. Since $E \otimes F = \bigoplus_{i,j} L_i \otimes M_j$, by splitting principle, we see that

$$c(E \otimes F) = \prod_{i=1}^{r}\prod_{j=1}^{s}(1 + x_i + y_j),$$

for any rank r bundle E and rank s bundle F. Similarly, $\mathrm{Sym}^p(E) = \bigoplus_{1 \le i_1 \le \cdots \le i_p \le r} L_{i_1} \otimes \cdots L_{i_p}$. By splitting principle, we see that

$$c(\mathrm{Sym}^p E) = \prod_{1 \le i_1 \le \cdots \le i_p \le r}(1 + x_{i_1} + \cdots + x_{i_p}).$$

Particularly, if L is a complex line bundle with $c_1(L) = y$, then

$$c(E \otimes L) = \prod_{i=1}^{r}(1 + y + x_i) = \sum_{i=0}^{r} c_i(E)(1 + y)^{n-i}, \tag{7.16}$$

Exercise 7.17. Let E be a complex vector bundle of rank r.

(1) Express $c(\bigwedge^p E)$ in terms of Chern roots x_i's of E.
(2) For $r = 3$, compute $c(\bigwedge^2 E)$ in terms of $c_i(E)$'s.

Example 7.19 (Chern Character). Let $E \to M$ be a complex vector bundle with a connection ∇. We define the Chern character $\mathrm{ch}(E, \nabla)$ by

$$\mathrm{ch}(E, \nabla) = e^{\mathrm{tr}(i\Omega/2\pi)} = \sum_{j=0}^{\infty} \mathrm{ch}_j(E, \nabla),$$

where $\mathrm{ch}_j(E, \nabla) := \mathrm{tr}\left(\frac{i\Omega}{2\pi}^j\right)/j!$. Similar to Chern classes, $\mathrm{ch}(E) := [\mathrm{ch}(E, \nabla)]$ is well-defined.. If $x_1, \ldots, x_r$ are the Chern roots of E, then

$$\mathrm{ch}(E) = \sum_{j=1}^{r} e^{x_j} = \sum_{m=0}^{\infty} \frac{x_1^m + \cdots + x_r^m}{m!}.$$

Exercise 7.18. Let E, F be a complex vector bundle of rank r.

(1) Show that $\mathrm{ch}(E \bigoplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$.
(2) Show that $\mathrm{ch}(E \bigotimes F) = \mathrm{ch}(E)\mathrm{ch}(F)$
(3) Expand $\mathrm{ch}(E)$ in terms of $c_k(E)$ up to degree 6.

We conclude this section by completing the proof of projective bundle formula.

PROOF OF THEOREM 7.15. From the Euler sequence (7.14), we have

$$0 \to \underline{\mathbb{C}}_{\mathbb{P}(E)} \to S^* \otimes \pi^* E \to S^* \otimes Q \to 0,$$

where $\underline{\mathbb{C}}_{\mathbb{P}(E)} = \mathbb{C} \times \mathbb{P}(E)$. By Whitney product formula,

$$c_r(S^* \otimes \pi^* E) = \sum_{k=0}^{r} c_k(\underline{\mathbb{C}}) c_{r-k}(S^* \otimes Q) = c_r(S^* \otimes Q) = 0,$$

since $S^* \otimes Q$ has rank $r - 1$. From (7.16) and $c_1(S^*) = \xi$, we obtain

$$c_r(S^* \otimes \pi^* E) = \xi^r + \pi * c_1(E)\xi^{r-1} + \cdots + \pi^* c_r(E) = 0.$$

□

7.4. Characteristic Classes on Real Vector Bundles

7.4.1. Pontryagin Classes on Real Vector Bundles.

We so far only consider complex vector bundle. For a real vector bundle $E \to M$ with a given $\mathbb{R}$-linear connection ∇, we define analogously the *total Pontryagin class* of E by

(7.17)
$$p(E, \nabla) := \det\left(I + \frac{1}{2\pi}\Omega\right) = \widetilde{p}_0(E) + \widetilde{p}_1(E) + \widetilde{p}_2(E) + \cdots + \widetilde{p}_{[r/2]}(E),$$

where $\widetilde{p}_k(E) = \frac{1}{(2\pi)^k}\sigma_k(E)$ and $\sigma_k(E)$ is the k-th elementary symmetric polynomial of eigenvalues of Ω.

Clearly, the proof of proposition 7.3 and 7.4 show that $\widetilde{p}_k(E)$ is closed and $[\widetilde{p}_k(E, \nabla)]$ is independent of the choice of connection. Similarly, one can always endow the real vector bundle E a Riemannian metric h and a connection ∇ which is metrical with respect to h. The similar calculation as in (7.8) shows that the connection 1-form ω and curvature 2-form Ω with respect to an orthonormal frame of E are skew–symmetric.

However, in contrast with the case of Chern classes, notice that for k is odd, $\mathrm{tr}(\Omega^k)$ is also skew–symmetric. Hence, $\mathrm{tr}(\Omega^k) = 0$. We then conclude that for any odd degree k invariant polynomal P, the $2k$-form $P(\Omega)$ is exact.. Particularly, for odd k, we have

$$[\sigma_k(E)] = [\widetilde{p}_k(E)] = 0 \in H^{2k}_{\mathrm{dR}}(M, \mathbb{R}).$$

We then define the *k-th Pontryagin class* $p_k(E) \in H^{4k}_{\mathrm{dR}}(M, \mathbb{R})$ by the cohomology class of the form

$$p_k(E) := \frac{1}{(2\pi)^{2k}}\sigma_{2k}. \tag{7.18}$$

In fact, Pontryagin classes can also be obtained from Chern classes. Let E^r be a real vector bundle and $E \otimes \mathbb{C}$ be its complexification.

Proposition 7.20. *For $k = 1, \ldots, r$, $p_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C})$.*

PROOF. Let ∇ be a $\mathbb{R}$-linear connection on E. Clearly, ∇ induces a $\mathbb{C}$-linear connection $\nabla \otimes \mathbb{C}$ on $E \otimes \mathbb{C}$ and the connection 1-form

ω and curvature 2-form Ω for $\nabla \otimes \mathbb{C}$ are the same as ∇. We then deduce that

$$p_k(E) = \frac{1}{(2\pi)^{2k}}\sigma_{2k}(\Omega) = (-1)^k \left(\frac{i}{2\pi}\right)^{2k} \sigma_{2k}(\Omega) = (-1)^k c_{2k}(E \otimes \mathbb{C}).$$

Also, we see that $c_{2k+1}(E \otimes \mathbb{C}) = 0$. □

Similarly, Pontryagin classes satisfy Whitney product formula and functoriality. Moreover, given a complex vector bundle $E \to M$, we denote $E_{\mathbb{R}}$ by the underlying real vector bundle. Similar to (6.16), the complex structure on E gives a decomposition $E_{\mathbb{R}} \otimes \mathbb{C} = E \bigoplus \overline{E}$, where $\overline{E}$ is the conjugate bundle of E. By introducing a hermitian metric h on E, one can identify $\overline{E}$ with dual bundle E^*. Therefore, we have

$$c(E_{\mathbb{R}} \otimes \mathbb{C}) = c(E \bigoplus \overline{E}) = c(E)c(E^*).$$

By (7.20), we have

$$\begin{aligned} &1 - p_1(E) + p_2(E) + \cdots + (-1)^r p_r(E) \\ =&(1 + c_1(E) + c_2(E) + \cdots + c_r(E))(1 - c_1(E) + c_2(E) - \cdots + (-1)^n c_r(E)). \end{aligned}$$

Exercise 7.19. Let E be a real vector bundle. Let $x_1, \ldots, x_r$ be the Chern roots of $E \otimes \mathbb{C}$. Show that $p(E) = \prod_{i=1}^r (1 + x_i^2)$.

7.4.2. Euler Classes on Oriented Vector Bundle.

If E is an oriented real vector bundle [8], then we can define another characteristic class $e(E)$, known as *Euler class*. If we endow E any Riemannian metric h and choose an oriented orthonormal frame $s = (s_1, \ldots, s_r)$ of E over an open set $U \subset M$. Then the corresponding curvature $\Omega = (\Omega^i_j)$ of any metrical connection with respect to s is skew–symmetric. The top Pontryagain class is given by

$$p_r(E) = \left[\det\left(\frac{1}{2\pi}\Omega\right)\right] = \frac{1}{(2\pi)^{2n}}[\det \Omega] \in H^{4r}_{\mathrm{dR}}(M, \mathbb{R}).$$

Recall that in linear algebra, for a skew–symmetric matrix $X = (X_i^j) \in M_{2r}(\mathbb{R})$, one can associate a polynomial $\mathrm{Pf}(A)$ in X^i_j, called

[8] cf. Exercise 2.22 for orientation of a vector bundle.

the *Pfaffian* of X satisfying $\mathrm{Pf}(A)^2 = \det(X)$. The Pfaffian is defined by

$$\mathrm{Pf}(X) := \frac{1}{2^r r!} \sum_{\sigma \in S_{2r}} (-1)^{\mathrm{sign}\sigma} X^{\sigma(1)}_{\sigma(2)} \cdots X^{\sigma(2r-1}_{\sigma(2r)}. \tag{7.19}$$

Exercise 7.20. Show that $\mathrm{Pf}(X)^2 = \det(X)$. Also, for $A \in O(2r, \mathbb{R})$, show that $\mathrm{Pf}(A^{-1}XA) = \det(A)\mathrm{Pf}(A)$.

Particularly, if we substitute A by the curvature 2-form $\Omega = (\Omega^j_i)$ with respect the oriented orthonormal frame s, then from exercise 7.20, $\mathrm{Pf}(\Omega)$ is invariant under the change of oriented orthonormal frames $s' = gs$ with $g : U \to SO(2r, \mathbb{R})$. Thus, the *Euler form*

$$e(\Omega) = \frac{1}{(2\pi)^r}\mathrm{Pf}(\Omega) \in A^{2r}(M)$$

is well–defined and $p_r(E) = [e(\Omega)^2]$. Again, one can show that $e(\Omega)$ defines a cohomology class $e(E) := [e(\Omega)] \in H^{2r}_{\mathrm{dR}}(M, \mathbb{R})$ and is independent of the choice of metrics and metrical connection. Moreover, for complex vector bundle E^r, which always has a natural orientation, the Euler class of its underlying real vector bundle $E_{\mathbb{R}}$ is identical with the top Chern class $c_r(E)$.

Exercise 7.21.
1. Use similar argument as in proposition 7.3 and 7.4 to show that $e(\Omega)$ is closed and $e(E)$ is independent of the choice of metric and metrical connection.
2. Let $E^r \to M$ be a complex vector bundle and $E_{\mathbb{R}}$ be its underlying real vector bundle. Show that $e(E_{\mathbb{R}}) = c_r(E)$.
3. Show that if E, F are oriented vector bundle, then so is $E \bigoplus F$, and $e(E \bigoplus F) = e(E)e(F)$.

Remark 7.21. If we impose extra structures (metrics, orientation) on a vector bundle E, then we can pick a trivialization of E such that the transition functions have values in the linear subgroup G preserving the extra structures. The group G is called the *structure group* of E. As we have seen in the section, the connection 1-form and curvature 2-form of the connections compatible with the extra structure have values in $\mathfrak{g} = Lie(G)$ (cf. problem 7.1).

7.5. Topological Aspect of Characteristic Classes

In this section, we introduce topological theory of characteristic classes as cohomology class on classifying spaces of vector bundles. We refer the details to [MS74] or Hatcher's note[9].

7.5.1. Classifying Spaces of Vector Bundles.
We first discuss the classifying space of vector bundles. Recall that in section 5.5, $G_{n,k}(\mathbb{C})$ is the space of k-dimensional subspaces of $\mathbb{C}^n$. We can generalize tautological line bundle on projective spaces (cf. 7.3) to a tautological bundle $\gamma_n^k(\mathbb{C})$ of rank k on $G_{n,k}(\mathbb{C})$ as following. Let $\underline{\mathbb{C}}^k$ be the trivial bundle $G_{n,k}(\mathbb{C}) \times \mathbb{C}^k$. We define

$$\gamma_n^k(\mathbb{C}) := \{([V], v = (v_1, \ldots, v_k)) \in \underline{\mathbb{C}}^k : \mathrm{span} v = V\}. \tag{7.20}$$

We denote $Q = (\underline{\mathbb{C}}^k)/\gamma_n^k$ by the *universal quotient bundle* and call the short exact sequence

$$0 \to \gamma_n^k \to \underline{\mathbb{C}}^k \to Q \to 0$$

the *tautological sequence* on $G_{n,k}(\mathbb{C})$. Similarly, one can define real tautological bundle $\gamma_n^k(\mathbb{R}) \to G_{n,k}(\mathbb{R})$ and oriented real tautological bundle $\tilde{\gamma}_n^k \to \tilde{G}_{n,k}$.

Exercise 7.22. Show that $\gamma_n^k(\mathbb{C})$ is a rank k complex vector bundle[10].

γ_n^k is also called *universal bundle* in the sense of following:

Proposition 7.22. *Let $\pi : E \to M$ be a rank k complex vector bundle over a compact manifold M. Then there exists a smooth map $f : M \to G_{n,k}(\mathbb{C})$ for some $n \in \mathbb{N}$ such that $E = f^*\gamma_n^k$, the tauotological bundle on $G_{n,k}(\mathbb{C})$.*

PROOF. Let $\{U_i\}_{i=1}^m$ be a finite trivializing covering for E. For each $i = 1, \ldots, m$, $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^k$. We choose a partition of unity $\{\rho_i\}_{i=1}^m$ subordinate to $\{U_i\}_{i=1}^m$ with $\mathrm{supp}\rho_i \subset U_i$.

Let $g_i : \pi^{-1}(U_i) \to \mathbb{C}^k$ be the composition of ϕ_i with the projection $U_i \times \mathbb{C}^k \to \mathbb{C}^k$. Then the map $(\rho_i \circ \pi)g_i$ extends to a C^∞ map $E \to \mathbb{C}^k$ by zeros outside $\pi^{-1}(U_i)$. We then define $g : E \to \mathbb{C}^n$ by

[9] http://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf

[10] In fact, $G_{n,k}(\mathbb{C})$ is a complex manifold and $\gamma_n^k(\mathbb{C})$ is a holomorphic bundle of rank k.

$g = ((\rho_i \circ \pi) g_i)_{i=1}^m$, where $n = mk$. This is a linear injection on each fiber by construction. We then define a map $f : M \to G_{n,k}(\mathbb{C})$ by $f(x) = g(E_x)$. Clearly, $f^* \gamma_n^k(\mathbb{C}) = E$. $\square$

We call the map $f : M \to G_{n,k}(\mathbb{C})$ the *classifying map* and the map $g : E \to \mathbb{C}^n$ in the proof *generalized Gauss map*[11] for the bundle E. Let $\mathrm{Vect}_k(M, \mathbb{C})$ be the isomorphism classes of complex vector bundle of rank k over M, $[M, G_{n,k}(\mathbb{C})]$ be the homotopy class of maps from M to $G_{n,k}(\mathbb{C})$. From exercise 2.19, we establish a surjective map

$$[M, G_{n,k}(\mathbb{C})] \to \mathrm{Vect}_k(M, \mathbb{C}) \text{ by } [f] \mapsto f^* \gamma_n^k.$$

In fact, we have

Fact 7.23. if $n \geq k + \dim M/2$ and $f, g : M \to G_{n,k}(\mathbb{C})$ satisfying $f^*Q \cong g^*Q$, then f and g are homotopic.

The proof of this fact can be found in [Hus94], proposition 6.1. For general manifold, we need *infinite Grassmannian*. Let $\mathbb{C}^\infty$ be the linear space of infinite sequence of complex number $(z_0, z_1, z_2, \dots)$ which $z_i \neq 0$, for all but finite $i \in \mathbb{N}$. Clearly, $\mathbb{C}^\infty = \bigcup_{k=1}^\infty \mathbb{C}^k$. We then define infinite Grassmannian $G_{\infty,k}(\mathbb{C})$ by the set of all k-dimensional subspaces of $\mathbb{C}^\infty$ and thus we have a sequence of inclusion

$$G_{k,k}(\mathbb{C}) \subset G_{k+1,k}(\mathbb{C}) \subset \cdots G_{\infty,k}(\mathbb{C}).$$

We then topologize $G_{\infty,k}(\mathbb{C})$ by direct limit topology: $U \subset G_{\infty,k}(\mathbb{C})$ is open if and only if $U \cap G_{k+n,k}(\mathbb{C})$ is open, for any n. We denote $G_{\infty,1}(\mathbb{C})$ by $\mathbb{CP}^\infty$.

On $G_{\infty,k}(\mathbb{C})$, one can still define a tautological bundle $\gamma^k(\mathbb{C})$ of rank k by

$$\gamma^k(\mathbb{C}) = \{(V, (v_1, \dots, v_k) : V \in G_{\infty,k}(\mathbb{C}^n), v_i \in \mathbb{C}^\infty,\ V = \mathrm{span}(v_1, \dots, v_k)\}.$$

---

[11] By Whitney embedding theorem, a k-dimensional manifold M^k can be embedded in $\mathbb{R}^N$, for some $N \geq 2k$. For any $x \in M$, we identify $T_xM \subset \mathbb{R}^N$ as a k-plane in $\mathbb{R}^N$. Thus, for each point $x \in M$, T_xM defines a point in $G_{N,k}(\mathbb{R})$. Hence, we can generalize Gauss map by $\bar{g} : M \to G_{N,k}(\mathbb{R})$ by $x \mapsto T_xM$. We then see that this is covered by a bundle map $\tilde{g} : TM \to \gamma_N^k(\mathbb{R})$ by $(x, v) \mapsto (T_xM, v)$.

Then one can show $\gamma^k(\mathbb{C}) \to G_{\infty,k}(\mathbb{C})$ is a complex vector bundle of rank k. Furthermore, using similar line of proof above, one can show

$$[M, G_{\infty,k}(\mathbb{C})] \to \mathrm{Vect}_k(M, \mathbb{C}), \quad [f] \mapsto f^*\gamma^k(\mathbb{C})$$

is a bijection, for any (topological) manifold M.

Similarly, let $\mathrm{Vect}_k(M, \mathbb{R})$ and $\mathrm{Vect}_k^+(M, \mathbb{R})$ be isomorphism classes for real vector bundle and oriented vector bundle of rank k over M, we have $[M, G_{\infty,k}(\mathbb{R})] \cong \mathrm{Vect}_k(M, \mathbb{R})$, $[M, \tilde{G}_{\infty,k}(\mathbb{R})] \cong \mathrm{Vect}_k^+(\mathbb{R})$.

7.5.2. Cellular Decomposition of Grassmannian.
Next, we discuss a certain stratification of Grassmannian which allows us to compute its cohomology readily. We first recall

Definition 7.24. A *CW-complex* consists of a Hausdorff space K with a partition $\{e_\alpha\}_{\alpha \in A}$ such that

(1) each e_α is topologically an open disk of dimension $n(\alpha) \geq 0$, and there exists a continuous map $f_\alpha : \mathbb{D}^{n(\alpha)} \to K$ such that $f(\mathring{\mathbb{D}}^{n(\alpha)}) = e_\alpha$.
(2) for any $x \notin \overline{e_\alpha} \setminus e_\alpha$, $x \in e_\beta$, for some $\beta \in A$ with $n(\beta) < n(\alpha)$.

If $|A| < \infty$, then (1) and (2) will suffice. In general, we need two more conditions. A subset of K is called a (finite) subcomplex if it is closed and is a union of (finitely many) e_α's.

(3) each $x \in K$, x belongs to a finite subcomplex.
(4) a subset of K is closed if and only if it is closed in each finite subcomplex.

We call each e_α an *open cell* and f_α the *characteristic map* for e_α.

Given a CW complex structure, one can compute its homology readily. Particularly, we have

Fact 7.25. Let X be a CW complex.

(1) If X has no n-cell, then $H_n(X, \mathbb{Z}) = 0$.
(2) If X has no cells in $n-1$ and $n+1$, then $H_n(X, \mathbb{Z})$ is a free abelian groups with basis given by n-cells of X.

For the proof of the fact, one can consult any standard textbook on algebraic topology, eg. [Hat02], p.140.

Example 7.26. The sphere S^n can be considered as a CW-complex as following: $e_0 = \{N\}$ be the north pole and $S^n \setminus e_0 \cong \mathbb{R}^n \cong \mathring{\mathbb{D}}^n$. That is, $S^n = e_0 \cup e_n$. Hence, we have $H_k(S^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & \text{,otherwise.} \end{cases}$

Example 7.27. A CW complex structure on complex projective space $\mathbb{CP}^n$ is inductively by: $\mathbb{CP}^n \setminus \mathbb{CP}^{n-1} = \mathbb{C}^n \cong \mathring{\mathbb{D}}^{2n}$. Thus, $\mathbb{CP}^n = e_0 \cup e_2 \cup e_4 \cup \cdots \cup e_{2n}$. By above fact, we have

$$H_k(\mathbb{CP}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & ,k = 0, 2, 4, \ldots, 2n; \\ 0 & ,k = 1, 3, 5, \ldots, 2n-1. \end{cases}$$

Particularly, for $\mathbb{CP}^\infty$, we see that $H^*(\mathbb{CP}^\infty, \mathbb{Z}) = \mathbb{Z}[\alpha]$, where $\alpha \in H^2(\mathbb{CP}^\infty, \mathbb{Z})$.

We next construct a cell decomposition for $G_{m,n}(\mathbb{C})$ as follows. Let

$$\mathbb{C}^0 \subset \mathbb{C}^1 \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^m$$

be a fixed filtration. For any $X \in G_{m,n}(\mathbb{C})$, the sequence of m numbers

$$0 \leq \dim_{\mathbb{C}}(X \cap \mathbb{C}^1) \leq \cdots \leq \dim_{\mathbb{C}}(X \cap \mathbb{C}^m) = n$$

has n jumps, and denote the sequence of jumps by $j(X)$. We call a sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ with $1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_n \leq m$ a *Schubert symbol*. For each Schubert symbol σ, we associate a subset $e(\sigma)$ in $G_{m,n}(\mathbb{C})$ as the collection of all X with $j(X) = \sigma$. Then $e(\sigma)$ is topologically a cell of complex dimension

$$(\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_n - n)$$

This dimension formula is easy to see by deform X a little bit, but it needs some work to show it is really an open cell (cf. [MS74] p.76). There are totally $\binom{m}{n}$ such cells, and the whole collection of these cells form a CW complex structure of $G_{m,n}(\mathbb{C})$.

In particular, there is no cell of (real) odd dimension, and for cells of even dimension $2r$, they corresponds to those σ such that (let $\tau_i =$

$\sigma_i - 1$):

$$0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq m - n$$
$$\tau_1 + \tau_2 + \cdots + \tau_n = r.$$

When m is large, say $m - n \geq r$, and $n \geq r$, the number of all such σ is exactly the partition number $p(r)$ of r. Since no cells are of odd dimension, the cohomology groups are then clear:

$$H^{2r+1}(G_{m,n}(\mathbb{C}); \mathbb{Z}) = 0$$
$$H^{2r}(G_{m,n}(\mathbb{C}); \mathbb{Z}) \simeq \mathbb{Z}^{p(r)}.$$

In problem 7.22 to problem 7.24 , we will use Morse theory to show that every smooth compact manifold has a homotopy type of CW complex.

7.5.3. Characteristic Classes on Universal Bundles.

We now ready to define characteristic classes topologically.

Theorem 7.28 (Uniqueness of Chern Classes). *For each complex vector bundle $E \to B$, there exists a unique sequence of cohomology class $c_i(E) \in H^{2i}(B, \mathbb{Z})$ depending only on the isomorphism classes of E such that*

- (a) *Functoriality: $f^*(c_i(E)) = c_i(f^*(E))$.*
- (b) *Whitney product formula: $c(E \cup F) = c(E) \cup c(F)$, where $c(E) = 1 + c_1(E) + \cdots \in H^*(B, \mathbb{Z})$.*
- (c) *$c_i(E) = 0$, for $i >$ rank$E$.*
- (d) *Normalization: For tautological bundle $\mathcal{O}(-1) \to \mathbb{CP}^n$, $c(\mathcal{O}(-1)) = 1 - H$, where H is the hyperplane class on $\mathbb{CP}^n$.*

PROOF. We first prove by induction on the rank. Let $L \to B$ be a line bundle, then there exists sufficiently large $n \in \mathbb{N}$ and $f : B \to \mathbb{CP}^n$ such that $L = f^*\mathcal{O}(-1)$. By (a) and (d), we see that $c(L) = f^*(1 + H)$. In particular, $c_i(L) = 0$, for $i > 1$. Let $E \to B$ be a complex vector bundle of rank k. We can then use theorem 7.16 to construct a split space $\sigma : S(E) \to B$ such that $\sigma^* E = \bigoplus_{i=1}^k L_i$, where $L_i \to S(E)$ is line bundle. Then (a) and (b) show that

$$\sigma^* c(E) = c(\sigma^*(E)) = \prod_{i=1}^{k} (1 + \gamma_i),$$

where $c(L_i) = 1 + \gamma_i$. Since $\sigma^* : H^*(B) \to H^*(S(E))$ is injective, we have determined $c(E)$ uniquely. This also shows $c_i(E) = 0$, for $i > \mathrm{rank}(E)$. $\square$

The proof shows that the Chern classes defined in terms of connection given in section 7.1 coincide with this topological definition. Now, on universal bundle $\gamma^k \to G_{\infty,k}(\mathbb{C}^n)$, one has $c_1 := c_1(\gamma^k), \dots, c_k := c_k(\gamma^k)$. With a little effort, one can actually prove

Fact 7.29. The cohomology ring $H^*(G_{\infty,k}(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_k]$ and there are no algebraic relations between them.

Combining with the fact $\gamma^k \to G_{\infty,k}(\mathbb{C}^n)$, we see that $c_1, \dots, c_k$ are the universal Chern class in the sense for any complex vector bundle of rank k $E \to B$, there exists $f : B \to G_{\infty,k}(\mathbb{C}^n)$ such that $E = f^*\gamma^k$. By functoriality, we have $c_i(E) = f^*c_i$.

The advantage for topological theory is that one can deal with torsion. There is a complete analogue for real vector bundles in $\mathbb{Z}_2$-coefficient.

Fact 7.30. For each real vector bundle $E \to B$, there exists a unique sequence of cohomology class $w_i(E) \in H^i(B, \mathbb{Z}_2)$ depending only on the isomorphism classes of E such that

(a) Functoriality: $f^*(w_i(E)) = w_i(f^*(E))$.
(b) Whitney product formula: $w(E \cup F) = w(E) \cup w(F)$, where $w(E) = 1 + w_1(E) + \cdots \in H^*(B, \mathbb{Z}_2)$.
(c) $w_i(E) = 0$, for $i > \mathrm{rank} E$.
(d) Normalization: For $\mathcal{O}(-1) \to \mathbb{RP}^n$, $w(\mathcal{O}(-1)) = 1 - H$, where H is the hyperplane class on $\mathbb{RP}^n$.

$w_i(E)$ are called the i-th *Stiefel–Whitney class* of $E \to B$. Moreover, we also have $H^*(G_{\infty,k}(\mathbb{R}), \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \dots, w_k]$. For details on Stiefel–Whitney class, we refer to [MS74].

The topological definition for Pontryagin classes p_i and Euler classes e on oriented vector bundles are more involved. We only remark that they are also universal cohomology class in the sense of following:

Fact 7.31. Let R be an integral domain with 2 to be invertible, and denote p_i, e by Pontryagin classes and Euler class of the universal bundle $\tilde{\gamma}^k \to \tilde{G}_{\infty,k}(\mathbb{R})$,

$$H^*(\tilde{G}_{\infty,2k+1}(\mathbb{R}), R) \cong R[p_1, \ldots, p_k]$$
$$H^*(\tilde{G}_{\infty,2k}(\mathbb{R}), R) \cong R[p_1, \ldots, p_{k-1}, e].$$

7.6. Cobordism Ring

7.6.1. Oriented Cobordism. Let M be an oriented C^∞ manifold, then we denote $-M$ by the same manifold with opposite orientation. If M, N are C^∞ manifolds, we denote $M + N$ by the disjoint union $M \sqcup N$, which has natural C^∞ manifold structure. In this section, we assume all manifolds are oriented.

Definition 7.32. Let M_1, M_2 be closed n-manifolds. We say M_1 and M_2 are (oriented) *cobordant* if there exists $(n+1)$-dimensional compact manifold W with boundary such that

$$M_1 - M_2 = \partial W,$$

where the equality means up to an orientation preserving diffeomorphism. We call W an (oriented) *cobordism* between M_1 and M_2.

Exercise 7.23. Show that this defines an equivalence relation on the set $\mathbf{\Omega}_n$ of all closed n-manifolds (Hint: for transitivity, one might need collar neighborhood, cf. exercise 2.14) and forms an abelian group with respect to disjoint union $+$.

Particularly, we say a closed oriented n-manifold M is *null-cobordant* if M lies in the equivalence class of 0, the additive identity of $\mathbf{\Omega}_n$. Obviously, M is null-cobordant if and only if $M = \partial W$, for some oriented $(n+1)$-manifold W. On $\mathbf{\Omega} := \bigoplus_{n\geq 0} \mathbf{\Omega}_n$, we define a bilinear map $\mathbf{\Omega}_n \times \mathbf{\Omega}_m \to \mathbf{\Omega}_{m+n}$ by Cartesian product $(M, N) \mapsto M \times N$. Notice that the product is well–defined since if $M_1 - M_1' = \partial W_1$, $M_2 - M_2' = \partial W_2$, then we have

$$M_1 \times M_2 - M_1' \times M_2' = \partial(W_1 \times M_2 - M_1' \times W_2).$$

Exercise 7.24. Show that the Cartesian product $\times$ makes Ω into a graded ring with identity $1 = [pt]$.

7.6.2. Characteristic Numbers and Cobordism.

Let M^{4n} be a closed $4n$-dimensional C^∞ manifold, we write $p_k(M) := p_k(TM)$ be the Pontryagin classes on TM. For a partition $I = \{i_1 \geq \cdots \geq i_r\}$ of n, we define the *I-th Pontryagin number* $p_I[M]$ by

$$p_I[M] := \left(p_{i_1}(M) \cdots p_{i_r}(M)\right)[M].$$

Similarly, if (M^{2n}, J) is any almost complex manifold, then we write $c_k(M) := c_k(TM, J)$ be the Chern classes on the complex vector bundle (TM, J). For any partition I of n, the *I-th Chern number* $c_I[M]$ by

$$c_I[M] := \left(c_{i_1}(M) \cdots c_{i_r}(M)\right)[M].$$

We have the following important observation.

Proposition 7.33 (Pontryagin). *Pontryagin numbers are cobordant invariant. Particularly, if any Pontryagin number of M^{4n} is non–vanishing, then it cannot bound any $(4n+1)$-manifold.*

PROOF. Observe that for $-M$, $p_k(-M) = p_k(M)$ but $[-M] = -[M]$. Hence, $p_I[M] = -p_I[M]$, for any partition I of n. It suffices to prove if M is null-cobordant, i.e. $M = \partial W$, then $p_I[M] = 0$, for all I. Let $\iota : M \hookrightarrow M$ be the inclusion map. By definition of orientation on boundary $M = \partial W$, we have

$$0 \to TM \to \iota^* TW \to N_{M/W} = \underline{\mathbb{R}} \to 0,$$

where the normal bundle $N_{M/W}$ is trivial since it has a non–vanishing out–ward pointing vector field. By functoriality and Whitney product formula, $p(i^*W) = \iota^* p(W) = p(M)$. By using Stokes' theorem,

$$\begin{aligned} p_I[M] &= \int_M p_{i_1}(M) \cdots p_{i_r}(M) = \int_{\partial W} \iota^* p_{i_1}(M) \cdots \iota^* p_{i_r}(M) \\ &= \int_W d(p_{i_1}(W) \cdots p_{i_r}(W)) = 0. \end{aligned}$$

$\square$

For $n \in \mathbb{N}$, let $I = \{i_1, \ldots, i_r\}$ be any partition of n, we define

$$M_{\mathbb{C}}^I = \mathbb{CP}^{i_1} \times \cdots \times \mathbb{CP}^{i_r}$$
$$M_{\mathbb{R}}^I = \mathbb{CP}^{2i_1} \times \cdots \times \mathbb{CP}^{2i_r},$$

where $\dim_{\mathbb{R}} M_{\mathbb{C}}^I = 2n$, $\dim_{\mathbb{R}} M_{\mathbb{R}}^I = 4n$.

Lemma 7.34. *The $p(n) \times p(n)$ matrix $[c_I(M_{\mathbf{C}}^J)]_{IJ}$ of characteristic numbers is non-singular, where $p(n)$ is the number of the partition of n. And the same statement holds for $[p^I(M_{\mathbf{R}}^J)]_{IJ}$.*

PROOF. Let $\mathrm{ch}_I := \mathrm{ch}_{i_1}(M) \cdots \mathrm{ch}_{i_r}(M)$. We only have to prove the matrix formed by $a(I,J) = \mathrm{ch}_I(M_{\mathbf{C}}^J)$ is nonsingular. Then the first part of the lemma follows.

Since $\mathrm{ch}_i(M) = 0$ if $i > \dim_{\mathbb{C}}(M)$, by the property (*), we get $a(I,J) = 0$ if $|I| < |J|$. (At least one i_s is larger than all j_t.) Furthermore, even in the case $|I| = |J|$, $a(I,J) = 0$ unless $I = J$. Introduce a total order on all partitions which extends the partial order $|I|$, then the matrix $[a(I,J)]$ is in a triangular form. So to prove it to be non-singular, only have to prove the diagonal elements (ie. $I = J$) are nonzero. Let $I = J = i_1, \ldots, i_r$, then $a(I,I)$ is

$$\begin{aligned}
&\prod_{\ell=1}^{r} \mathrm{ch}_{i_\ell}(\mathbb{CP}^{i_1} \times \cdots \times \mathbb{CP}^{i_r})[M_{\mathbf{C}}^I] \\
=&\prod_{\ell=1}^{r} (\mathrm{ch}_{i_\ell}(\mathbb{CP}^{i_1}) + \cdots + \mathrm{ch}_{i_\ell}(\mathbb{CP}^{i_\ell}))[M_{\mathbf{C}}^I],
\end{aligned}$$

where the obvious zero terms are omitted. By evaluating on the $[M_{\mathbf{C}}^I]$ from $\ell = 1, \cdots, r$ gradually, again use $\mathrm{ch}_i(M) = 0$ if $i > \dim_{\mathbf{C}}(M)$, we find

$$a(I,I) = \prod_{\ell=1}^{r} \mathrm{ch}_{i_\ell}(\mathbb{CP}^{i_\ell})[\mathbb{CP}^{i_\ell}].$$

Hence, we reduce the problem to compute $\mathrm{ch}_n(\mathbb{CP}^n)$. From Euler sequence,

$$\mathrm{ch}_n(\mathbb{CP}^n) = (n+1)\mathrm{ch}_n(\mathcal{O}(1)) = (n+1)c_1(\mathcal{O}(1))^n/n! = \frac{(n+1)h^n}{n!},$$

and thus $\mathrm{ch}_n(\mathbb{CP}^n)[\mathbb{CP}^n] = (n+1)/n!$.

For the Pontryagin case, for a real bundle E, we define $\mathrm{ph}_i(E) := \mathrm{ch}_{2i}(E \otimes \mathbb{C})$. For a complex manifold M,

$$\begin{aligned} &\mathrm{ph}_i(TM) = \mathrm{ch}_{2i}((TM)_{\mathbb{R}} \otimes \mathbb{C}) \\ =&\mathrm{ch}_{2i}(TM \oplus \overline{TM}) = \mathrm{ch}_{2i}(TM) + \mathrm{ch}_{2i}(\overline{TM}) = 2\mathrm{ch}_{2i}(TM). \end{aligned}$$

Then by the same manner we also get the matrix with entries $\mathrm{ph}_I(M^J_{\mathbb{R}})$ is nonsingular. Since these classes ph_i are an equivalent basis of p_i over $\mathbb{Q}$, the result follows. $\square$

From the proposition 7.33 and above lemma, we see that $M^I_{\mathbb{R}}$ are distinct cobordism classes in Ω_{4n}, for any partition I of n.

7.7. Transversality and Thom Spaces

7.7.1. Transversality.

Let X, Y be smooth manifolds, $f : X \to Y$ be a smooth map, $A \subset X$ be any subset, $Z \subset Y$ be a submanifold.

Definition 7.35. We say f is *transversal* to Z on A, denoted by $f \pitchfork_A Z$ if $\forall x \in f^{-1}(Z) \cap A$,

$$df(T_x X) + T_{f(x)} Z = T_{f(x)} Y.$$

We denote $f \pitchfork Z$ if $A = X$.

Note that when $Z = \{pt\}$ and $A = X$, this is just the usual definition of a regular value. Observe that if $f \pitchfork Z$ and Z is of codimension k in Y, then by the implicit function theorem we have $f^{-1}(Z)$ a smooth submanifold of X of codimension k (empty set is allowed). Also, the normal bundle of $f^{-1}(Z)$ in X is isomorphic to the pull back bundle f^*N of the normal bundle N of Z in Y.

Now we prove the Thom's transversality theorem which says that any smooth map can be approximated by transversal ones.

Theorem 7.36 (Thom)**.** *Let $f : X \to Y$ be smooth, and $f \pitchfork_A Z$, where A is a closed set of X and Z is a submanifold of Y. Let d be any metric compatible with the underlying topology of Y and $\epsilon > 0$, then there is a smooth map $g : X \to Y$ such that*

$$g \pitchfork Z, \quad d(f(x), g(x)) < \epsilon,$$

and $f\big|_A = g\big|_A$.

PROOF. We divide the proof in four steps:

Step1: Since transversality is an open condition, there is an open set $U \subset A$ such that $f \pitchfork_U Z$. Now, we will choose some appropriate coordinate coverings to reduce the problem to the case of Euclidean sapces. First of all, let $Y_0 = Y \setminus Z$ and $\{Y_i\}$ be charts cover Z with $Z \cap Y_i$ coordinate planes. Secondly, we choose charts $\{V_i\}$ charts such that $\{V_i\}$ is a refinement of both $\{X \setminus A, U\}$ and $\{f^{-1}(Y_i)\}$. Since we cannot modify f on A, we need an even more refine covering of those V_i that do not interesect A.

By paracompactness of X, we choose $\{W_i\}$ a family of locally finite relatively compact charts with $\{\overline{W}_i\}$ finer than $\{V_j\}$. Finally we disgard those W_i which are contained in U. We still denote the final family by $\{W_i\}$, which is clearly a covering of $X \setminus U$.

Step2: We will construct f_i inductively such that

a) $d(f_i(x), f_{i-1}(x)) \leq \epsilon/2^i \quad \forall x \in X$.

b) $f_i \equiv f_{i-1}$ outside a compact neighborhood of $\bar{W}_i$ in V_i.

c) $f_i \pitchfork Z$ on $f_i^{-1}(Z) \cap (\bar{W}_1 \cup \cdots \cup \bar{W}_i)$.

Once this is done, we get $\lim_i f_i = g : X \to Y$ the desired smooth map. Notice there is no limit process since the covering is chosen to be locally finite. Actually when X is compact, $\{W_i\}$ is a finite family. We also remark that we never change the value of f near A in the whole process.

Step3: On each $i \geq 1$, $f_{i-1}(V_i) \subset Y_{j(i)}$ for some $j(i)$. Since V_i, Y_j are all coordinate charts, by the induction steps, we only have to treat everything in the Euclidean spases. Namely, $K \equiv \overline{W} \subset V \subset \mathbb{R}^n$, $Z = Y \cap \mathbb{R}^q \subset Y \subset \mathbb{R}^p$, and $f : V \to Y$ smooth with $f \pitchfork Z$ on a relatively closed set S (S is to be thought as $\overline{W}_1 \cup \cdots \cup \overline{W}^{i-1}$).

Consider the projection $p : \mathbb{R}^p \to \mathbb{R}^{p-q}$, then $Z = p\big|_Y^{-1}(0)$, so $p \circ f : V \to \mathbb{R}^{p-q}$ has 0 as a regular value if and only if f is transversal to Z, thus it suffices to consider the case $Z = 0$, that is, a point.

Step4: Since 0 is now a regular value of f on S, what we have to do is to modify it to be regular on all $S \cup K$. Using a smooth partition of unity, construct a smooth map $\lambda : V \to [0, 1]$ which equals 1 on a neighborhood of K and equals 0 outside a compact neighborhood K' of K in V. By Sard's theorem, there are always points arbitrarily near 0 and are still regular values, pick y with $|y| < \epsilon$ and consider

$$g(x) = f(x) - \lambda(x)y.$$

then by the definition of λ, clearly have

(1) g has 0 as a regular value (ie. $g \pitchfork 0$) on K.
(2) $g \equiv f$ outside K'.
(3) $|g(x) - f(x)| < \epsilon$.

Since y can be chosen arbitrarily near 0 and $|\frac{\partial}{\partial x_i}\lambda|$ is globally bounded, we can make g, dg near f, df uniformly. By (2). $g \pitchfork 0$ on $S \setminus K'$, so we only have to care about the set $(S \cap K') \cap g^{-1}(0)$. Notice $S \cap K'$ is a compact set, so when $|y|$ small, df_x is surjective implies dg_x is surjective. So $g \pitchfork 0$ on $S \cup K$ as required. $\square$

7.7.2. Thom Space and Thom Isomorphism.
Let $E \to B$ be an oriented vector bundle of rank k. We endow E with a bundle metric $\langle \, , \rangle$. Let $D(E) = \{(b, v) \in E : |v| \leq 1$ be the disk bundle over B, $S(E) := \{(b, v) \in E : |v| = 1\}$ be the sphere bundle over B. We define the *Thom space* $T(E)$ of $E \to B$ by the quotient

$$T(E) := D(E)/S(E) = E/A,$$

where $A := \{(b, v) \in E : |v| \geq 1\}$. We denote the base point corresponds to A by $t_0 \in T(E)$. Also, B embeds C^∞ into E and thus B embeds into $T(E)$.

Exercise 7.25. Show that $T(E)$ is independent of the choice of metric. Also, when B is compact, show that $T(E)$ is the one-point compatification of the total space E.

Now, we assume B is compact. Given a C^∞ map $f : S^{n+k} \to T(E)$ with $f(\infty) = t_0$, we have $f^{-1}(B) \subset S^{k+n} - f^{-1}(t_0) \cong \mathbb{R}^{n+k}$. Thus, $f^{-1}(B) \subset U$, for some precompact open set $U \subset \mathbb{R}^{n+k}$. By transversality theorem, we may modify f within its homotopy class and keep it unchanged outside U, so we may assume $f \pitchfork B$. Then $f^{-1}(B)$ is an n dimensional closed oriented submanifold of U. We then define *Thom homomorphism* by

$$\tau_{k,n} : \pi_{n+k}(T(E)) \to \Omega_n, \quad [f : S^{n+k} \to T(E)] \mapsto f^{-1}(B). \tag{7.21}$$

Proposition 7.37. *$\tau_{k,n}$ is a well-defined group homomorphism.*

SKETCH OF PROOF. To prove that $\tau_{k,n}$ is well-defined, suppose $F : S^{k+n} \times [0,1] \to T(E)$ be a homotopy such that $F(x,0) = f_0(x)$, $F(x,1) = f_1(x)$. We may choose F such that $F(\cdot,[0,1/3]) = f_0$, $F(\cdot,[2/3,1]) = f_1$. Applying the transversality theorem to $X = F^{-1}(E \cap (S^{k+n} \times (0,1))$, we get a map $\widetilde{F}$ coincides with f_0 near 0, and coincides with f_1 near 1, and $\widetilde{F}^{-1}(B)$ is a manifold with boundary $f_1^{-1}(B) \setminus f_0^{-1}(B)$. So τ is well defined as a map. We leave readers to prove that $\tau_{k,n}$ is a group homomorphism. $\square$

Exercise 7.26. Complete the proof of proposition 7.37.

When $B = \tilde{G}_{k,k+p}(\mathbb{R})$, the oriented Grassmannian, and $E = \tilde{\gamma}^k_{k+p}$, the universal oriented k-plane bundle, we obtain

Theorem 7.38 (Thom). *$\tau_{n,k}$ is an isomorphism if $k \geq n+2$ and $p \geq n$.*

PROOF. For surjectivity, we only need $k \geq n$, $p \geq n$. Let $[M] \in \mathbf{\Omega}_n$. By (strong) Whitney embedding theorem, M can be embeded in $\mathbb{R}^{k+n}$. Let ν be the normal bundle of M in $\mathbb{R}^{k+n}$. By the existence of tubular neighborhood of M (cf. problem 3.22), denoted by U, we can construct the generalized Gauss map

$$g : U \cong \nu \longrightarrow \tilde{\gamma}^k_n \hookrightarrow \tilde{\gamma}^k_{k+p}.$$

We extend g to be a map $g : S^{n+k} \to T(\tilde{\gamma}^k_{k+p})$ by sending all points outside U to t_0. Then by the same procedure above, we can assume g to be transversal to B. It is then clear that $\tau_{k,n}([g]) = [M]$.

For injectivity, let $g : S^{k+n} \to T(\tilde{\gamma}^k_{k+p})$ be transversal to B and $g^{-1}(B) = M = \partial N$. We have to prove g is homotopic to a constant map. We may assume M is a closed n-manifold in $\mathbb{R}^{k+n} \subset S^{k+n}$.

Claim 7.39. We can embed N in $\mathbb{R}^{k+n} \times [0,1/2]$ such that $N \cap \mathbb{R}^{k+n} \times [0,1/4]) = M \times [0,1/4]$.

With claim 7.39, let V be a tubular neighborhood of N in $\mathbb{R}^{k+n} \times [0,1]$ with $d(V,N) < \epsilon$. Then $U = V \cap (\mathbb{R}^{k+n} \times \{0\})$ is a tubular neighborhood of M in $\mathbb{R}^{k+n}$. To proceed, we need two claims.

Claim 7.40. We can deform g such that $g|_U : U \to \gamma^k_{k+p}$ is a bundle map.

Once this is done, one can extend g to a bundle map $\tilde{g} : V \to \gamma^k_{k+p}$ by a more delicate argument similar to proposition 7.22 (cf. [Hir94], p,100). We then extend $\tilde{g}$ to whole $\mathbb{R}^{k+n} \times [0,1]$ by sending the complement of V to t_0. This establish the deseired homotopy from g to the constant map g_0.

For claim 7.39, let $h : M \times [0,1)$ diffeomorphic onto a collar neighborhood of ∂N. Define $\beta : \mathbb{R} \to [0,1]$ to be a smooth increasing cutoff function such that $\beta(x) = 0$ for $x < 1/2 + \epsilon$ and $\beta(x) = 1$ for $x > 1 - \epsilon$. Then define $h_1 : N \to \mathbb{R}^{k+n} \times [0, \frac{1}{2}]$ by

$$h_1(y) = \begin{cases} (x, \frac{1}{2}s) & \text{for } y \in h(M \times [0, \frac{1}{2}]) \\ p & \text{for } y \notin h(M \times [0,1)) \\ (1-\beta(s))(x, \frac{1}{4}) + \beta(s)p & \text{for } 1/2 \le s < 1, \end{cases}$$

where $p = (x_0, \frac{1}{2})$ with $x_0 \in M$ an arbitrary point. Although h_1 is a smooth map, the resulting image $h_1(N)$ is never a manifold. Still, it contains the collar $M \times [0, \frac{1}{4}]$. Since $k \ge n+2$, $k+n+1 \ge 2(n+1)+1 = 2\dim N + 1$. Notice that a close reexamine on the proof of Whitney embedding theorem 1.22 shows that in such a dimension, embeddings are dense in the space of smooth maps (cf. [Hir94], p.35). Thus, we can find a map

$$h_2 : N \hookrightarrow \mathbb{R}^{k+n+1}$$

with $h_2 \equiv h_1$ on $h(M \times [0, 1/4])$. The first claim is proved.

For the second claim, we first deform the map g such that it sends all points outside U to t_0 and keep g unchanged on a smaller neighborhood of $M = g^{-1}(B)$. We denote the universal bundle by $\pi : \xi \to B$ and denote the bundle projection of $U \to M$ by p. Let $x \in U$, by using the linear structures on $U_{p(x)}$, which is induced by the normal bundle ν (it is just a rescaling of each fiber), and the linear structure on $\xi_{\pi(g(x))}$, can define for $s \in [0,1]$,

$$H(x,s) = \frac{g(sx)}{s},$$

with $H(x,0) = dg_{\pi(x)}(x)$, which is surely a linear map on the normal space over $p(x)$ (since Dg is linear on the whole tangent space), and this linear map is an isomorphism by the transversality of g Thus $g_0 \equiv H(\cdot,0)$ defines a bundle map $U \to \xi$ and $H(\cdot,s), s \in [0,1]$ defines the homotopy from $g \equiv g_1$ to g_0. The construction behavior well on points near ∂U, so the second claim is proved. □

7.8. Thom's Cobordism Theorem and its Application

7.8.1. Thom Cobordism Theorem. Although we have established the isomorphism of Thom homomorphism, in general it is still difficult to compute the higher homotopy groups. Let us recall a few facts on homotopy groups for later use. The first is a standard exact sequence for homotopy groups of a fiber bundle[12]

Fact 7.41 (Homotopy Exact Sequence of Fibration). : let $E \to B$ be a locally trivial fibration with fiber F, then there exists a long exact sequence in homotopy groups

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots \to \pi_0(E) \to 0.$$

The second one is to reduce the problem of homotopy groups to homology. In fact, there exists a natural map $\pi_i(X) \to H_i(X,\mathbb{Z})$, called the Hurewicz homomorphism. The celebrated Hurewicz theorem asserts that if X is $(n-1)$-connected[13], then the Hurewicz homomorphism $\pi_n(X) \to H_n(X,\mathbb{Z})$ is an isomorphism. However, the assumption here is too strong to be satisfied in our situation. Anyway, since we only concern with the non-torsion part $\mathbf{\Omega} \otimes \mathbb{Q}$, so we

[12]In fact, we only need *Serre's fibration* which is weaker than a locally trivial fiber bundle. We refer the concept and the proof to the following fact to [DFN85], section 22.

[13]That is, $\pi_r(X)$ is trivial, for all $r \leq n-1$.

only have to compute the so-called *rational homotopy groups.* In this case, we have (cf. [DFN90], p.129)

Fact 7.42. Let X be a finite CW complex which is r-connected with $r \geq 1$, then the rational Hurewicz homormophism

$$\pi_i(X) \otimes \mathbb{Q} \to H_i(X, \mathbb{Q})$$

is an isomorphism for $i \leq 2r$.

In order to apply this to our case, we must show $T(\tilde{\gamma}^k_{k+p})$ have some higher connectivity, but since $B = \tilde{G}_{k+p,k}(\mathbb{R})$ whose cell decomposition is well known, say e_l is an open j cell of it, then the inverse image $\pi^{-1}(e_l)$ is clearly an open $j+k$ cell of $T(\tilde{\gamma}^k_{k+p})$, Together with the zero cell t_0, we obtain a CW complex structure of $T(\tilde{\gamma}^k_{k+p})$ without cells of dimension between 0 and k, that is, $T(\tilde{\gamma}^k_{k+p})$ is $(k-1)$-connected. So we get, for $n \leq k-2$:

$$\pi_{k+n}(T(\tilde{\gamma}^k_{k+p})) \otimes \mathbb{Q} \cong H_{k+n}(T(\tilde{\gamma}^k_{k+p}), \mathbb{Q}).$$

Using the usual "Thom isomorphism" (cf. [MS74], section 10) and simple algebraic topology, it is not hard to prove that for an oriented k-plane bundle $E \to B$,

Fact 7.43 (cf. [MS74], p,206). $H_{k+i}(T(E), t_0; \mathbb{Z}) \cong H_i(B, \mathbb{Z})$.

So by connecting these isomorphisms, we finally obtain for $k \geq n+2$,

$$\Omega_n \otimes \mathbb{Q} \cong H_n(B, \mathbb{Q}).$$

As noted in section 7.5, by letting k large enough, $H^n(B, \mathbb{Q})$ is freely generated by Pontryagin classes of the universal bundle $\tilde{\gamma}^k_{k+p}$, so it is zero when $4 \nmid n$ and is of dimension $p(m)$ if $n = 4m$. Together with the calculation in proposition 7.33 and lemma 7.34, which shows that $M^I_{\mathbb{R}}$ are all in distinct corbordism classes. Since there are also $p(m)$-many $M^I_{\mathbb{R}}$'s, we finally conclude

Theorem 7.44. $\Omega \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^2, \cdots]$.

Remark 7.45. In the proof we do not need each step to be isomorphism. We only need

$$\begin{aligned}p(r) &= \dim H^n(\tilde{G}_{k+p,k}(\mathbb{R});\mathbb{Q}) = \dim H^{k+n}(T\tilde{\gamma}^k_{k+p}, t_0;\mathbb{Q}) \\ &\geq \dim(\pi_{k+n}(T\tilde{\gamma}^k_{k+p})\otimes\mathbb{Q}) \geq \dim(\mathbf{\Omega}_n\otimes\mathbb{Q}) \geq p(r)\end{aligned}$$

So we need only the surjective part of the Thom homomorphism τ (which is the easier part), and the injective part of the rational Hurwicz homomorphism.

Remark 7.46. There exists a complex analogue for cobordism group, called *complex cobordism group*. For an oriented smooth manifold

7.8.2. Genus and Multiplicative Sequence. Let R be a commutative ring over $\mathbb{Q}$. An R-*genus* is defined to be a ring homomorohism $\phi : \mathbf{\Omega}\otimes\mathbb{Q} \to R$.

A systematic way to construct genus is due to Hirzebruch in his monograph *Neue topologische Methoden in der algebraischen Geometrie* in 1956 (cf. [Hir95] for recent reprint). Let $Q(x) = \sum_{j=0}^{\infty} q_j x^{2j} \in R[[x]]$ be an even formal power series with $q_0 = 1$. We form

$$\begin{aligned}\textstyle\prod_{i=1}^n Q(x_i) &= 1 + q_2\textstyle\sum_{i=1}^n x_i^2 + \cdots \\ &= 1 + K_1(p_1) + K_2(p_1,p_2) + \cdots \\ &\quad + K_n(p_1,\ldots,p_n) + K_{n+1}(p_1,\ldots,p_n,0) + \cdots .\end{aligned}$$

Here, we regard the variables x_i's as have weight 2, and the weight $4r$ parts of the first product is a symmetric polynomial of x_i^2. Let p_i be the i-th elementary symmetric polynomial of x_i^2, then we define K_r to be the unique polynomial in p_i's corresponding to the weight $4r$ part in $\prod_{i=1}^n Q(x_i)$. In application, x_i will be Chern roots of a vector bundle and p_i is the i-th Pontryagin classes of of it. We call $p = 1 + p_1 + p_2 + \cdots$ the formal total Pontryagin classes.

By the theory of symmetric polynomials, we know that K_r is independent of the number of variables n if $n \geq r$. In the following, we will always assume n is large enough. In fact, we can take $n = \infty$. the resulting series is denoted by $K(p) = 1 + K_1(p_1) + K_2(p_1, p_2) + \cdots$, called the *multiplicative sequence* associated to Q. In the special case $p = 1 + p_1 = 1 + x^2$, we have $K(1 + x^2) = Q(x)$.

Let $K(p)$ be a multiplicative sequence associated to Q, we define a map $\phi_Q : \mathbf{\Omega} \to R$ by $\phi_Q([M]) = K(p_M)[M]$, where $K(p_M)$ means that we substitute p by total Pontryagin class p_M of TM.

Lemma 7.47. *ϕ_Q is a R-genus.*

PROOF. By the same proof of lemma 7.33, if $M^{4n} = \partial W^{4n+1}$ then $\phi_Q([M]) = 0$. It remains to show the additivity and multiplicativitiy of ϕ_Q. The additivity is obvious, yet the multiplicativity is more subtle. We have to prove: Let $p' = 1 + p'_1 + p'_2 + \cdots$, $p'' = 1 + p''_1 + p''_2 + \cdots$ with p'_k, p''_k of weight $4k$. If $p = p'p'' = 1 + p_1 + p_2 + \cdots$, by collecting corresponding terms, then

$$K(p) := K(p'p'') = K(p')K(p'').$$

Let $Q(x) = \sum_{i=0}^{\infty} q_i x^{2i}$ as before. For any partition $I = \{i_1, \ldots, i_r\}$ of n, let $q_I = q_{i_1} \cdots q_{i_r}$, and let

$$s_I(p_1, \ldots, p_n) = \sum x_1^{2j_1} \cdots x_r^{2j_r}$$

where the sum is over all distinct permutations $(j_1, \ldots, j_r)$ of I. Since it is symmetric in x_i^2's, it is uniquely represented by a polynomial in $p_i's$, this is the definition of s_I. Then we have

$$s_I(p'p'') = \sum_{\{H,J\}=I} s_H(p')s_J(p''),$$

which is just a partition of standard monomials into two parts of fewer variables. Again by comparing corresponding terms, we have

$$K_r(p_1, \ldots, p_n) = \sum_I q_I s_I(p_1, \ldots, p_n).$$

Take summation over all n, we get

$$\begin{aligned} K(p'p'') &= \textstyle\sum_I q_I s_I(p'p'') \\ &= \textstyle\sum_I \sum_{H,J=I} q_{H,J} s_H(p') s_J(p'') \\ &= \textstyle\sum_{H,J} q_H q_J s_H(p') s_J(p'') \\ &= \textstyle\sum_H q_H s_H(p') \sum_J q_J s_J(p'') \\ &= K(p')K(p''). \end{aligned}$$

□

Remark 7.48. In most literatures, one defines a multiplicative sequence by a sequence of polynomial $\{K_1(p_1), K_2(p_1, p_2), \cdots\}$ with coefficients in R such that p_i has weight $4i$ satisfying

(1) $K_r(p_1, \ldots, p_r)$ is homogeneous of degree $4r$
(2) Let $K(p) := 1 + K_1(p_1) + K_2(p_1, p_2) + \cdots$. Then K is multiplicative in formal total Pontryagin class $p = 1 + p_1 + p_2 + \cdots$ in the sense

$$p = p'p'' \Rightarrow K(p) = K(p')K(p'').$$

What we have proved is that given any even power series $Q(x)$ begins with 1, there exists an unique multiplicative sequence K such that $K(1 + x^2) = Q(x)$. We have proved the existence. For the uniqueness, use a formal decomposition $p = \prod_i (1 + x_i^2)$, then $K(p) = \prod_i Q(x_i)$. This is exactly our construction.

If we start with any power series $Q(x) = \sum_{k=0}^{\infty} q_k x^k$ with $q_0 = 1$, then one can consider similarly

$$\prod_{i=1}^{n} Q(x_i) = 1 + \tilde{K}_1(c_1) + \tilde{K}_2(c_1, c_2) + \cdots,$$

where c_i is the i-th elementary symmetric function of x_j's. Let $c = 1 + c_1 + c_2 + \cdots$ be the formal total Chern class. Then one can derive similarly the formula:

$$\tilde{K}(1 + x) = Q(x), \quad \tilde{K}(c) = 1 + \tilde{K}_1(c_1) + \tilde{K}_2(c_1, c_2) + \cdots.$$

Let us end this section with an important example of multiplicative sequence which arises later.

Example 7.49 (L genus). Consider the Taylor series of $x/\tanh(x)$:

$$Q(x) = \frac{x}{\tanh(x)} = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k}}{(2k)!} B_k x^{2k},$$

which B_k is the *Bernoulli's number* (cf. [MS74], Appendix B). The first few terms of B_k are $1/6, 1/30, 1/42, 1/30, 5/66, 691/2730 \ldots$. Then the associated multiplicative sequence $L(p) = 1 + L_1(p_1) + L_2(p_1, p_2) + \cdots$ is called the *L genus or L polynomial*. One can directly calculate the first few terms of L genus are given by:

$$\begin{aligned} L_1(p_1) =& \frac{1}{3} p_1 \\ L_2(p_1, p_2) =& \frac{1}{45}(7p_2 - p_1^2) \\ L_3(p_1, p_2, p_3) =& \frac{1}{945}(62p_3 - 13p_1p_2 + 2p_1^3). \end{aligned}$$

The L genus will appear in the Hirzebruch signature theorem which we will discuss later.

For more examples of genus associated to a multiplicative sequence, see problem 7.7 and 7.8.

Exercise 7.27. Verify the first three terms of L genus.

7.8.3. Hirzebruch Signature Theorem. Now, we are ready to give a topological proof of Hirzebruch signature theorem[14] as an important application of Thom's cobordism theorem.

First, we recall the definition of signature of a manifold. Let M be a $2n$ dimensional oriented closed manifold. The intersection pairing q_M is then the cup product and evaluated on the fundamental class $[M]$. Alternatively, using de Rham cohomology, the intersection pairing then can be viewed as the integration of wedge of closed differential forms over M. Anyway, it is a non–degenerate pairing[15],

[14] We will give an analytic proof of Hirzebruch's theorem as a special case of Atiyah–Singer theorem in section 8.6. However, historically the Index The- orem was first proved by a "twisted version" of Hirzebruch Signature Theorem.

[15] In fact, it is unimodular. Since we have some classification theory of integral quadratic forms and alternating forms (cf.[Ser73], Chapter V), the study of the

which is alternating when n is odd, symmetric when n is even, in the later case. Now we define the signature $\sigma = \begin{cases} \mathrm{sgn}(q_M) & 4 \mid \dim M \\ 0 & 4 \nmid \dim M, \end{cases}$ where the $\mathrm{sgn}(q_M)$ is the signature of q_M.

We now show that the signature of manifolds is a genus.

Lemma 7.50. *The signature is a $\mathbb{Z}$-genus. That is,*

(1) $\sigma(V + W) = \sigma(V) + \sigma(W), \sigma(-V) = -\sigma(V)$.
(2) $\sigma(V \times W) = \sigma(V) \times \sigma(W)$.
(3) $\sigma(M) = 0$ *if M bounds some compact oriented manifolds.*

PROOF. (1) is clear since the intersection form of disjoint union of manifolds splits as the direct sum of the individual ones. Also, let $\alpha_1, \ldots, \alpha_k$ be a basis of $H^n(M, \mathbb{Z})$, then $(q_M)_{ij} = (\alpha_i \alpha_j)[M] = -(\alpha_i \alpha_j)[-M] = -(q_{-M})_{ij}$. Hence, $\sigma(-V) = -\sigma(V)$.

For (2), we use the cohomology with coefficient in $\mathbb{R}$. Let $M^{4k} = V^n \times W^m$, by the Künneth formula (cf. Problem 2.23),

$$H^{2k}(M) = \bigoplus_{s+t=2k} H^s(V) \otimes H^t(W).$$

Let $\{v_i^s\}, \{w_j^t\}$ be the basis of $H^s(V), H^t(W)$ such that $\int_V v_i^s \wedge v_j^{n-s} = \delta_{ij}$, $\int_W w_i^t \wedge w_j^{m-t} = \delta_{ij}$ for $s \neq n/2, t \neq m/2$.

Now, we define $A = H^{n/2}(V) \otimes H^{m/2}(W)$ if n, m are even and $A = 0$ otherwise, and let $B := A^{\perp}$ in $H^{2k}(M)$ with respect to q_M. Observe that B is spanned by elements not in A since $q_M\big|_A = q_V \otimes q_W$m which is non-degenerate. Hence, the set

$$\{v_i^s \otimes w_j^t : s + t = 2k, s \neq \frac{n}{2}, t \neq \frac{m}{2}\}$$

is an orthogonal basis for B. Observe that

$$\int_M (v_i^s \otimes w_j^{2k-s}) \wedge (v_{i'}^{s'} \otimes w_{j'}^{2k-s'}) = \begin{cases} \pm 1 & s + s' = n, i = i', j = j' \\ 0 & \text{otherwise.} \end{cases}$$

intersection form in fact reveals some important information about the topology of M as we will see later in the case of 4-manifold topology

It follows that $q_M\big|_B$ is represented by a matrix with diagonal blocks $\pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Therefore, $\operatorname{sgn}(q_M\big|_B) = 0$. We claim that $\sigma(A) := \operatorname{sgn}(q_M\big|_A) = 0$ if $n \nmid 4$. If so, then when $n \mid 4$ and $m \mid 4$, $\sigma(M) = \sigma(A) + \sigma(B) = \sigma(A) = \sigma(V)\sigma(W)$.

Observe that if $n \nmid 4$, then so does $m \nmid 4$. To show $\sigma(A) = 0$, it suffices to show that the symmetric bilinear form obtained from the tensor product of two alternating forms must has zero signature. However, it is clear from the structure theorem of non-degenerate alternating form over $\mathbb{R}$, one can represent it by a matrix with block diagonal $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Hence, the tensor product of two such matrices is still with block diagonal $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and thus has signature 0.

Finally, for (3), Let $i : M^{4k} \to W^{4k+1}$ be the boundary inclusion. We have the following commutative diagram:

$$\cdots \longrightarrow H^{2k}(W) \xrightarrow{i^*} H^{2k}(M) \xrightarrow{\partial^*} H^{2k+1}(W,M) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_{2k+1}(W,M) \xrightarrow{\partial_*} H_{2k}(M) \xrightarrow{i_*} H_{2k}(W) \longrightarrow \cdots$$

The verticle maps are all isomorphisms by Poincaré-Lefschetz duality ([Hat02], Theorem 3.43). Let $A = \operatorname{Im} i^*, B = \operatorname{Ker} i_*$. Since i^*, i_* are dual vector space maps, A is dual to $H_{2k}(M)/B$ under the duality between $H^{2k}(M)$ and $H_{2k}(M)$. So $\dim(A) = \dim H_{2k}(M) - \dim(B)$. By the exactness of the above diagram, $\dim(A) = \dim(B)$, so we get $\dim(A) = \dim(B) = \frac{1}{2}\dim H_{2k}(M)$. Now by Stokes' theorem, for any $\omega \in A$,

$$\int_M (i^*\omega)^2 = \int_W d(i^*\omega \wedge i^*\omega) = 0$$

so the zero cone of the intersection form contains A. Since A has half the dimension, this can happen only when $\sigma^+ = \sigma^-$, that is, $\sigma(M) = 0$. $\square$

Now, we are in place to state the Hirzebruch signature theorem. Let p_M be the total Pontrayagin class of M, $L(p_M)$ be the L genus with p substituting by p_M.

Theorem 7.51 (Hirzebruch Signature Theorem)**.** *Let M^{4k} be a smooth compact oriented $4k$-manifold. Then $\sigma(M) = L(p_M)[M]$.*

By Thom's cobordism theorem, we know that $\Omega^* \otimes \mathbb{Q}$ is generated by $\mathbb{CP}^{2k}$. To check two $\mathbb{Q}$-genus coincides, it suffices to check on the generator $\mathbb{CP}^{2k}$. In section 7.3.1, we have seen that the cohomology ring $H^*(\mathbb{CP}^{2k}, \mathbb{Z}) \cong \mathbb{Z}[h]/(h^{2k+1})$. Clearly, $\sigma(\mathbb{CP}^{2k+1}) = 1$. To make connection with L genus, notice the following lemma.

Lemma 7.52. *Let $Q(x) \in \mathbb{R}[[x]]$ be an even formal power series, $f(x) := x/Q(x) = x + \cdots \in \mathbb{R}[[x]]$ is clearly invertible. Let $g(y) = f^{-1}(y)$. Then*

$$g'(y) = \sum_{n=0}^{\infty} \phi_Q([\mathbb{CP}^n])y^n$$

PROOF. Since $p(\mathbb{CP}^n) = (1 + p_1)^{n+1}$, where $p_1 = h^2$. We have $K(p) = K(1 + p_1)^{n+1} = Q(p_1)^{n+1}$. Thus,

$$\begin{aligned}
\phi_Q[\mathbb{CP}^n] &= Q(p_1)^{n+1}[\mathbb{CP}^n] \\
&= \text{ coefficient of } x^n \text{ in } Q(x)^{n+1} = \left(\frac{x}{f(x)}\right)^{n+1} \\
&= res_{x=0}\frac{1}{f(x)^{n+1}}dx \\
&= \frac{1}{2\pi i}\int_C \frac{1}{f(x)^{n+1}}dx = \frac{1}{2\pi i}\int_{f(C)} \frac{g'(y)}{y^{n+1}}dy \\
&= \text{ coefficient of } y^n \text{ in } g'(y).
\end{aligned}$$

Since $f(x) = x + \cdots$, when the circle C around 0 is small enough, $f(C)$ is also a curve around 0 with winding number 1. In fact, we even do not need f to be convergent since the above argument is essentially a sequence of substitution of formal power series. □

Consequently, if we start with a genus ϕ and consider the generating power series $P_\phi = \sum_{n=0}^{\infty} \sigma(p\mathbb{CP}^{2n})y^{2n}$, then we set $g = \int P_\phi$

with constant term 0, and set $f := g^{-1}$. By Thom's cobordism theorem, the power series $Q = x/f(x)$ whose associated multiplicative sequence defines the genus $\phi_Q = \phi$

PROOF OF SIGNATURE THEOREM. We now consider the generating series $P_\sigma(y) = \sum_{n=0}^{\infty} \sigma(p\mathbb{CP}^{2n}) y^{2n} = 1 + y^2 + y^4 + \cdots = \frac{1}{1-y^2}$. Then

$$\tanh^{-1}(y) = \int \frac{dy}{1-y^2}, \quad f(x) = \tanh(x).$$

Thus, $Q(x) = x/\tanh(x)$ is the power series associated to L genus!. This proves the theorem. $\square$

7.9. Problems

7.1. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, given any Lie group $G \subset GL(r, \mathbb{K})$ as a structure of group of a vector bundle E, show that we can choose a connection such that its curvature has values in $\mathfrak{g} = Lie(G)$

7.2 (Flat Bundle)**.** A vector bundle is *flat* or *integrable* if there exists a connection ∇ with curvature $\Omega_\nabla = 0$. Such connection is called a *flat connection*.

(1) Show that a rank r vector bundle $E \to M$ with structure group G is flat if and only if there are k linearly independent parallel sections near each point $p \in M$.
(2) Show that a flat G-bundle corresponds to a representation $\rho \in \mathrm{Hom}(\pi_1(M), G)$.

7.3 (Chern Classes on Flat Bundle)**.** Let $E \to M$ be a flat vector bundle.

(1) Show that the Chern classes $c(E)$ is a torsion class in $H^*(M, \mathbb{Z})$.
(2) Conversely, show that if $L \to M$ is a line bundle with torsion Chern class $c_1(L) \in H^2(M, \mathbb{Z})$, then it is flat.
(3) Investigate the validity of (2) for rank 2 bundle (Hint: consider bundles over S^5).

7.4. In this problem, we show a result due to Ehresmann and Hopf that S^4 admits no almost complex structure. Let M^4 be a closed, oriented 4-manifold. Assume J is an almost complex structure on M, then (TM, J) is a complex vector bundle over M.

(1) Let $c_i(TM, J)$ be the Chern classes of (TM, J), for $i = 1, 2$. Show that $c_1^2(TM, J) - c_2(TM, J) = p_1(TM)$.
(2) Use Hirzebruch signature theorem to show that $p_1(TM)([M]) = 3\sigma(M)$, where $[M]$ is the fundmamental class of M.
(3) Deduce that S^4 cannot admit any almost complex structure.

7.5. Show that $\Omega_0 \cong \mathbb{Z}$, $\Omega_1 = 0$. How about Ω_2?

([Hir94], Exercise 7.2.3, [MS74], p.203)

7.6. If we drop all orientation requirement in the definition of cobordism, we can obtain the *unoriented cobordism group* $\mathfrak{R}_n$ of closed n-manifolds.

(1) Show that every element of $\mathfrak{R}_n$ is 2-torsion.
(2) Compute $\mathfrak{R}_0, \mathfrak{R}_1$, and $\mathfrak{R}_2$.

([Hir94], Exercise 7.2.2,3)

7.7 (Todd genus)**.** The Todd genus is the genus of the power series of $z/(1-\exp(-z))$:

$$\frac{z}{1-\exp(-z)} = 1 + \frac{1}{2}z + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{B_k}{(2k)!},$$

where B_k is still the Bernoulli's number. Then the associated multiplicative sequence $Td(c) = 1 + Td_1(c_1) + Td_2(c_1, c_2) + \cdots$ is called the *Todd genus*. Verify that the first few terms of Todd genus are given by:

$$\begin{aligned} Td_1(c_1) &= \frac{1}{2}c_1 \\ Td_2(c_1, c_2) &= \frac{1}{12}(c_1^2 + c_2) \\ Td_3(c_1, c_2, c_3) &= \frac{1}{24}(c_1 c_2). \end{aligned}$$

The Todd genus will appear in the Hirzebruch–Riemann–Roch theorem in problem 8.10.

7.8. Consider the Taylor series of $\frac{\sqrt{z}/2}{\sinh(\sqrt{z}/2)}$:

$$\frac{x/2}{\sinh(x/2)} = 1 - \frac{x^2}{24} + \frac{7x^4}{5760} + \cdots.$$

Then the associated multiplicative sequence $\hat{A}(c) = 1 + \hat{A}_1(c_1) + \hat{A}_2(c_1, c_2) + \cdots$ is called the $\hat{A}$ *genus* (called *A-roof genus*). Verify that the first few terms of $\hat{A}$ genus are given by:

$$\hat{A}_1(p_1) = \frac{-1}{24}p_1$$
$$\hat{A}_2(p_1, p_2) = \frac{1}{5760}(-4p_2 + 7p_1^2)$$
$$\hat{A}_3(p_1, p_2, p_3) = \frac{1}{967680}(-16p_3 + 44p_2p_1 - 31p_1^3).$$

The $\hat{A}$ genus will appear in Atiyah–Singer index theorem in section 8.4.

Connections and Chern–Weil theory on Principal Bundle.
From problem 7.9 to problem 7.20, we establish the theory of connections via principal bundle, which is parallel to the discussion in the text. Also, we prove the Chern–Weil theory in its general form.

7.9 (Principal Bundle)**.** Let G be a Lie group. A *principal G-bundle* $\pi : P \to M$ is a smooth fiber bundle with fiber G and structure group G (cf. problem 5.14), where G acts on itself on the left. Show the following are equivalent.

(1) $P \to M$ is a principal G-bundle.
(2) $P \to M$ is a fiber bundle with fiber G. P admits a G-action on the right such that its orbits coincides with the fiber of P and there exists a local trivialization $\{(U_i, \phi_i)\}$ such that
$$\phi_i(hg, x) = \phi_i(h, x) \cdot g, \forall g, h \in G, x \in U_i.$$
(3) Show that $P \to M$ is trivial if and only if it admits a section $s : M \to P$.

7.10 (Example of Principal Bundle)**.** Show that the following has a structure of principal bundle.

(1) Let $E \to M$ be a real vector bundle of rank r. We define the *frame bundle* $P_{GL(E)}$ by
$$P_{GL(E)} := \{(m, \mathbf{e}) : m \in M, \mathbf{e}\text{: a frame of } E_m\ .$$
and $\pi : P_{GL(E)} \to M$ by $(m, \mathbf{e}) \mapsto m$. Show that $P_{GL(E)}$ admits a structure of principal $GL(r, \mathbb{R})$-bundle.
(2) If $E \to M$ is endowed with an Euclidean metric and we take the frames in $P_{GL(E)}$ to be orthonormal frames, then show that the resulting frame bundle $P_{O(E)}$ is a principal $O(r)$-bundle. Furthermore, if we take the frame to be oriented, then the resulting frame bundle $P_{SO(E)}$ becomes a principal $SO(r)$-bundle.

(3) Show similarly that the frame bundle of a complex vector bundle $E \to M$ is a principal $GL(r, \mathbb{C})$-bundle. Also, if $E \to M$ is endowed with a hermitian metric and we take the frames in $P_{GL(E)}$ to be unitary frames, then the resulting frame$P_{U(E)}$ becomes a principal $U(r)$-bundle.
(4) Show that the homogeneous space $G \to G/H$ is a principal H-bundle.
(5) Show that the orbit space $M \to M/G$ is a principal G-bundle (cf. problem 5.13).

7.11 (Principal Bundle and Vector Bundle)**.** Let $P \to M$ be a principal G-bundle, $\rho : G \to GL(V)$ be a linear representation of a Lie group G.

(1) Show that the product space $P \times V$ admits a free G-action and $P \times_\rho V := (P \times V)/G$ admits a surjective submersion onto $M = P/G$.
(2) Show that $P \times_\rho V$ is a vector bundle over M with fiber V and structure group G.
(4) Show that when $G = GL(r, \mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $O(r)$, $SO(r)$, $U(r)$, the above construction is the inverse of frame bundle.

7.12. Let $\pi : P \to M$ be a principal G-bundle, $E := P \times_\rho V$ be the associated vector bundle to a linear representation $\rho : G \to GL(V)$.

(1) We denote $C^\infty(P, E)^G$ by the space of G-equivariant map from P to E. Show that $C^\infty(P, E)^G$ is isomorphic to $C^\infty(M, E)$ by sending $s \in C^\infty(P, E)^G$ to $s_M(x) = [p, s(p)]$, where $p \in \pi^{-1}(x)$. Make sense of this construction.
(2) A V-valued q-form $\alpha \in A^q(P, V) := A^q(P) \otimes V$ is called *basic* if α is G-invariant and $\iota_X \alpha = 0$, for any vertical vector field[16] X on P. For any basic differential form α, we define a E-valued q-form on M by

$$\alpha_M((\pi)_* X_1, \ldots, \pi_* X_q)(x) := [p, \alpha(X_1, \ldots, X_q)(p)] \in E$$

Show that this defines an isomorphism between V-valued basic forms on P and E-valued forms on M.

[16] X is vertical if $X_p \in \ker d\pi_p$, for any $p \in P$. That is, X_p is tangential to fibers.

7.13 (Connection on Principal Bundle)**.** Given a principal G-bundle $\pi : P \to M$, we always have an exact sequence $0 \to \ker d\pi \to TP \to \pi^*TM \to 0$, where $\ker d\pi$ is usually called the *vertical subbundle*. Let $\mathfrak{g} := Lie(G)$.

(1) Show that $P \times \mathfrak{g} \to \ker d\pi$ given by $(p, v) \mapsto X_v := \frac{d}{dt}\big|_{t=0}(p \cdot \exp(tv))$ is an isomorphism.

(2) Show that G-action on P lifts to each of the bundle in the exact sequence $0 \to \ker d\pi \to TP \to \pi^*TM \to 0$ and the bundle maps are G-equivariant.

We define a *connection* on P is a G-equivariant splitting of the exact sequence $A : TP \to \ker \pi$.

(3) Show that $H_A := \ker A$ is G-invariant subbundle of TP which is *horizontal* in the sense $T_pP = \ker(d\pi)_p \oplus (H_A)_p$, for any $p \in P$.

(4) Conversely, given any G-invariant subbundle H of TP such that $T_pP = H_p + \ker(d\pi)_p$, for any $p \in P$, show that H determines a G-equivariant spliiting $A : TP \to \ker \pi$ of the exact sequence and $H_A = H$.

In summary, a connection on P is equivalent to a G-invariant horizontal subbundle[17] of TP.

7.14 (Connection 1-form)**.** In this problem, we give another characterization for connections on a principal bundle.

(1) From problem 7.13 (1), we know that $P \times \mathfrak{g} \cong \ker d\pi$. Show that the isomorphism is G-equivariant with respect to the G-action on $P \times \mathfrak{g}$ given by $(p, v) \cdot g = (p \cdot g, Ad_{g^{-1}}(v))$.

(2) Show that a connection on principal G-bundle P is equivalently given by a $\mathfrak{g}$-valued 1-form A satisfies $A(X_v) = v$ and $A(d(R_g)v) = Ad_{g^{-1}}(A(v))$. Such A is called a *connection* 1-*form* on the principal G-bundle P.

(3) Show that the space of connection 1-forms on P is an affine space modeled on $A^1(M, Ad(P))$, where $Ad(P) = P \times_{Ad} \mathfrak{g}$ is called the *adjoint bundle* of P, which is a vector bundle over M with fiber $\mathfrak{g}$ (Hint: use problem 7.12 (2)).

7.15. Let G be a Lie group, $\mathfrak{g} = Lie(G)$ be its Lie algebra. For $g \in G$, $v \in T_gG$, we define $\omega(v) := (L_{g^{-1}})(v) \in \mathfrak{g}$.

[17] Obviously, the definition of connection as horizontal subbundle can be generalized to any fiber bundle. This definition is also called Ehresmann connection.

(1) Show that $\omega \in A^1(G, \mathfrak{g})$, called the *Maurer–Cartan form*.
(2) Show the *Maurer–Cartan equation* $d\omega + \frac{1}{2}[\omega, \omega] = 0$, where $[\cdot, \cdot]$ is the bracket for Lie algebra valued form[18]
(3) Show that If G is a linear Lie group, then at $g = (g_{ij})$, $\omega_g = g^{-1}dg$.
(4) Show that $q^*\omega$ is a connection 1-form on the trivial principal G-bundle $M \times G$, where $q : M \times G \to M$ is the projection.
(5) Use (4) to show that any principal G-bundle $P \to M$ admits a connection 1-form.

7.16. Let $\{U_\alpha, \phi_\alpha\}$ be a trivialization of a principal G-bundle $P \to M$, $A \in A^1(P, \mathfrak{g})$ be a connection 1-form on P. Then $\omega_\alpha := \phi_\alpha^* A \in A^1(U_\alpha, \mathfrak{g})$. Let $g_{\alpha\beta} : C^\infty(U_{\alpha\beta}) \to G$ be the transition functions for P.

(1) Show that $\omega_\beta = g_{\alpha\beta}^{-1} + Ad_{g_{\alpha\beta}^{-1}}\omega_\alpha$ (Comparing to exercise 7.1, the difference between two formulae is because we define G acting on P on the right.)
(2) Show that the collection $\{\omega_\alpha\}$ determines A.

7.17. In this problems, we establish the correspondence between connections on principal bundles and vector bundle. Let $P \to M$ be a principal G-bundle, $\rho : G \to GL(V)$ be a linear representation, $E := P \times_\rho V$ be the associated vector bundle. Let $A \in A^1(P, \mathfrak{g})$ be a connection 1-form on P. We define a connection ∇^A on $C^\infty(M, E)$ as follows: Since A determines a horizontal distribution H_A which is isomorphic to π^*TM, for each $X \in C^\infty(M, TM)$, there exists a unique horizontal lifting $\tilde{X} \in C^\infty(M, H_A)$. By problem 7.12, section $s \in C^\infty(M, E)$ can be identified as a G-equivariant map $s^P : P \to V$. We define $(\nabla^A_X)(s^P) := d(s^P)(\tilde{X})$.

(1) Show that $d(s^P)(\tilde{X})$ is a G-equivariant map from P to V.

Then $d(s^P)(\tilde{x})$ corresponds to a section $\nabla^A_X(s) \in C^\infty(M, E)$ again by problem 7.12.

(2) Show that ∇^A satisfies the Leibniz rule (Hint: first prove that $(fs)^P = \pi^*(f)s^P$, for $f \in C^\infty(M, \mathbb{R})$, $s \in C^\infty(M, E)$).
(3) Show that $\rho : G \to GL(V)$ induces a bundle map $d\rho : A^1(AdP) \to A^1(\mathrm{End}(E))$ and deduce that if A, A' are two connection 1-forms on P,

$$\nabla^A - \nabla^{A'} = d\rho(a),$$

[18] Let M be any C^∞ manifold, $\mathfrak{g}$ be a Lie algebra, $\alpha, \beta \in A(M)$, $v, w \in \mathfrak{g}$, then we define $[\alpha \otimes v, \beta \otimes w] := [v, w] \otimes \alpha \wedge \beta$.

where $A - A' = A^1(M, AdP)$.

Notice that if $P = P_{GL(E)}$, then $P \times_{ad} gl(r) = \mathrm{End}(E)$. This establishes a $1-1$ correspondence between connections on E and connections on $P_{GL(E)}$. Similarly, if E has structure group $G \subset GL(E_p)$, then the connection on P is 1-1 correspondence with connections on E with values in G.

7.18 (Curvature)**.** Let $\pi : P \to M$ be a principal G-bundle, A be a connection 1-form on P. We define the *curvature form*[19] $\Omega \in A^2(P, \mathfrak{g})$ by

$$\Omega = dA + \frac{1}{2}[A, A],$$

where $[\omega, \omega] = 2\omega \wedge \omega$ is the Lie bracket on Lie algebra-valued forms. Show that

(1) For $g \in G$, $R_g^* \Omega = Ad_{g^{-1}}(\Omega)$.
(2) For $v, w \in T_pP$, $\Omega_p(v, w) = d\omega(v^h, w^h)$, where v^h, w^h is the horizontal part with respect to the decomposition $T_pP = (H_A)_p \bigoplus d(\pi)_p$.
(3) $d\Omega = [\Omega, \omega]$ (The second Bianchi's identity)

Hence, Ω is a basic $\mathfrak{g}$-valued 2-form on P, which descends to a $Ad(P)$-valued 2-form on M, which we still denote by Ω.

(4) In terms of $\omega_\alpha := \phi_\alpha^* A \in A^1(U_\alpha, \mathfrak{g})$ with respect to a trivialization of P, show that $\Omega\big|_{U_\alpha} = d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha$.

7.19 (Invariant Polynomial)**.** Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. We denote $I^k(G)$ by the invariant subring of $\mathrm{Sym}^k \mathfrak{g}^*$ under Adjoint action of G and $I(G) := \bigoplus_{k \geq 0} I^k(G)$.

(1) Show that complete polarization (cf. exercise 2.1) extends to $I^k(G)$. That is, a polynomial $f(X)$ of degree k which is invariant under Ad-action corresponds to a $T \in I^k(G)$ such that $f(X) = T(\underbrace{X, \dots, X}_{k})$.
(2) For $T \in I^k(G)$, for any $X_1, \dots, X_k, Y \in \mathfrak{g}$,

$$\sum_{j=1}^{k} T(X_1, \dots, [Y, X_j], \dots, X_k) = 0.$$

([Che95], Appendix §2)

[19]Again, be aware of the minus sign on the curvature comparing to (7.2). This minus sign occurs since we define G acting on P on the right.

7.20 (Chern–Weil theory)**.** Let G be a Lie group with Lie algebra $\mathfrak{g}$, $P \to M$ be a principal G-bundle. Given any invariant polynomial $\phi \in I^k(G)$, any connection 1-form A on P, we define a $2k$-form $\phi(\Omega) \in A^{2k}(M)$. Since $\Omega \in A^2(M, AdP)$ and ϕ is invariant under adjoint action, the $2k$-form is well-defined. Show that

(1) $\phi(\Omega)$ is a closed form.
(2) For two connection 1-forms A_0, A_1 on P, let $\alpha = A_1 - A_2 \in A^1(M, AdP)$. Denote Ω_i by the curvature with respect to A_i. Show that

$$\phi(\Omega_1) - \phi(\Omega_0) = kd \int_0^1 \phi(\alpha, \underbrace{A_t, \ldots, A_t}_{k-1})dt,$$

where $A_t = A_0 + t\alpha$, $t \in [0, 1]$.

Let $T_\phi(A_0, A_1) := k \int_0^1 \phi(\alpha, \underbrace{A_t, \ldots, A_t}_{k-1})dt$, called the *transgression* of ϕ from A_0 to A_1. Then $[\phi(\Omega)]$ is independent of the choice of A. Thus, one can define a ring homomorphism $w : I(G) \to H^*(M)$ by $w(\phi) = [\phi(\Omega)]$, called the *Chern–Weil homomorphism*.

([Che95], Appendix §2)

7.21. In this problem, we show that $\pi_1(SO(n)) = \mathbb{Z}_2$, for all $n \geq 3$. For $n = 3$, this is already done in problem 5.8.

(1) Show that $SO(n+1)$ is a principal $SO(n)$-bundle over S^n.
(2) Use fact 7.41 to show that $\pi_1(SO(n+1)) \cong \pi_1(SO(n))$, for $n \geq 3$.

Remark 7.53. By problem 5.10, the 2-folded covering $\mathrm{Spin}(n)$ of $SO(n)$ is also a Lie group, called the *spin group*. The spin group is closely related to Clifford algebra discussed in the next chapter.

A Digression on Morse Theory.
From problem 7.22 to problem 7.24 , we establish the *fundamental theorem of Morse theory*:

Theorem 7.54 (Fundamental theorem of Morse theory)**.** *Let M be a compact smooth manifold, f be a Morse function (cf. definition 1.46) on M with $f^{-1}((-\infty, a])$ is compact for any $a \in \mathbb{R}$. Then M has a homotopy type of a finite CW complex with each λ-cells for each critical point of index λ.*

The existence of such Morse function is proved in problem 1.13.

7.22. Let M be a smooth manifold, $f \in C^\infty(M, \mathbb{R})$. For $a \in \mathbb{R}$, let $M^a := \{x \in M : f(x) \leq a\} = f^{-1}((-\infty, a])$. If for $a < b$ and suppose that $f^{-1}([a,b])$ is compact and contains no critical points of f. Choosing a Riemannian metric on M, let $X \in C^\infty(M, TM)$ be the vector field defined by

$$X(p) = \rho(p)\text{grad} f(p),$$

where $\rho(p) \in C_0^\infty(M, \mathbb{R})$ equals to $|\text{grad} f|^{-2}$ on $f^{-1}([a,b])$ and vanishes outside a compact neighborhood of $f^{-1}([a,b])$.

(1) Show that X is a complete vector field (i.e., the the flow ϕ_t generated by X has domain on whole $\mathbb{R}$).
(2) Use the flow ϕ_t generated by X to show that M_a is diffeomorphic to M_b and construct $r : [0,1] \times M^b \to M^b$ such that $r_0 = id_{M^b}$ and r_1 is a retraction from M_b to M_a.

In conclusion, if there is no critical values among $[a,b]$, the sublevel set M^b is a deformation retraction of M^a and diffeomorphic to M^a.

([Mil63], §3)

7.23 (Reeb)**.** Let M be a closed n-manifold, f be a Morse function on M with exactly two critical points p, q. We may assume p is the maximum and q is the minimum of f. By scaling, we may assume $f(p) = 1$, $f(q) = 0$.

(1) Use lemma of Morse (cf. problem 1.12) to show that for sufficiently small ϵ, $f^{-1}([0,\epsilon])$ and $f^{-1}([1-\epsilon, 1])$ is diffeomorphic to the closed n-cell.
(2) Use problem 7.22 to show $M^{1-\epsilon}$ is diffeomorphic a n-disk as well. Deduce that M is union of two disks glueing along their boundary.
(3) Construct explicitly a homeomorphism from standard sphere S^n to M.

([Mil63], §4)

Remark 7.55. One should remark that M may not diffeomorphic to S^n with standard diffeomorphic structure. In fact, this is how Milnor proved there exists smooth manifold homeomorphic to S^7 but with non–diffeomorphic differentiable structure. We will return this in chapter 8.20

Next, we study the change of topology when we pass through a critical point. Let M be a smooth manifold, $f : M \to \mathbb{R}$, p be a non–degenerate critical point with index λ, $f(p) = c$. Assume there exists $\epsilon > 0$ such that $f^{-1}([c-\epsilon, c+\epsilon])$ is compact and p is the only critical point on it. In the next problem, we will show that for sufficiently small $\epsilon > 0$, $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with λ-cell e^λ attached.

7.24. By Morse lemma, there exists a neighborhood U of p and a local coordinate $(x^1, \ldots, x^n)$ on U with $x^i(p) = 0$ such that

$$f\big|_U(x) = c - (x^1)^2 - \cdots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \cdots + (x^n)^2.$$

We choose smaller $\epsilon > 0$ such that $D := \{\sum_{k=1}^n (x^k)^2 \leq 2\epsilon\} \subset U$.

Let $x_- := (x^1)^2 + \cdots + (x^\lambda)^2$, $x_+ := (x^{\lambda+1})^2 + \cdots + (x^n)^2$, and $\mu : [0, \infty) \to \mathbb{R}$ be a smooth function such that $\mu(0) > \epsilon$, $\operatorname{supp}\mu \subset [0, 2\epsilon]$, and $-1 < \mu' < 0$. Then $f\big|_U = c - x_- + x_+$ and we modify f by $F := c - x_- + x_+ - \mu(x_- + 2x_+)$.

(1) Show that F satisfies the following:
   (a) $F = f$ outside D and $F \leq f$.
   (b) F is a Morse function with the same critical points as f, $F(p) < c - \epsilon$, and $F(q) = f(q)$, for any critical point q of f other than p.
   (c) $\{F \leq c + \epsilon\} = \{f \leq c + \epsilon\}$.
(2) Show that $F^{-1}([c - \epsilon, c + \epsilon])$ is compact and contains no critical point.

By problem 7.22, we know that $M^{c+\epsilon} = \{f \leq c + \epsilon\} = \{F \leq c + \epsilon\}$ deformation retracts to $\{F \leq c - \epsilon\}$. We now denote $\{F \leq c - \epsilon\}$ by $M^{c-\epsilon} \cup H$, where $H = \overline{\{f \leq c - \epsilon\} \setminus \{F \leq c - \epsilon\}} = \{F \leq c - \epsilon\} \cap \{f \geq c - \epsilon\}$.

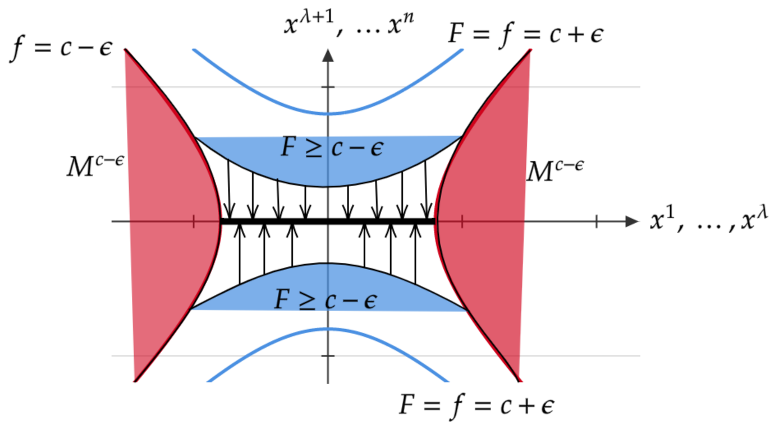
(3) Show that H is defined by the system of inequalities

$$\begin{cases} x_+ + x_- \leq 2\epsilon \\ f = -x_- + x_+ \geq -\epsilon \\ F = -x_- + x_+ + \mu(x_- + 2x_+) \leq -\epsilon. \end{cases}$$

and contains a λ-cell $e^\lambda := \{x_- \leq \epsilon, x_+ = 0\}$.

The situation is illustrated in above figure for $n = 2$, $\lambda = 1$. The area H is the area with vertical arrow and e^λ is the horizontal dark line.

(4) Show that $M^{c-\epsilon} \cup e^\lambda$ is a deformation retraction of $M^{c-\epsilon} \cup H$ by defining the deformation retract in three cases:
   (a) $x_- \leq \epsilon$
   (b) $\epsilon \leq x_- \leq x_+ + \epsilon$
   (c) $x_+ + \epsilon \leq x_-$.



Hence, we have shown that $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with λ-cell e^λ attached.

(5) Deduce theorem 7.54.

In fact, one can show that any smooth manifold is homotopy equivalent to a CW complex, cf. [Mil63], §3.

([Mil63], §3)

Clutching Functions and Vector Bundles over Spheres. In section 7.5, we states that the the set of isomorphism classes of oriented vector bundle can be identified as the homotopy class from M to universal oriented Grassmannian. From problem 7.25 to problem 7.26, we will discuss the case of $M = S^n$, which can be treated elementary. The method here is usually known as *clutching construction*.

7.25. Let S^k be a k-sphere, $D^k_\pm$ be the upper and lower hemi-spheres of S^k, and S^{k-1} be identified as the equator $D^k_+ \cap D^k_-$.

(1) Given an oriented vector bundle $E \to S^k$ of rank r, show that $E\big|_{D^k_\pm}$ is trivial.
(2) Let $h_\pm : E\big|_{D^k_\pm} \to D^k_\pm \times \mathbb{R}^r$ be trivializations, $g := h_+ \circ h_- : S^{k-1} \to SO(r)$ be the transition functions. Show that the homotopy class $[g] \in [S^{k-1}, SO(r)]$ is independent of the choice of the trivializations $h_\pm$.

7.26. Conversely, given a map $g : S^{k-1} \to SO(r)$, then one can define

$$E_g := (D^k_+ \times \mathbb{R}^r) \coprod (D^k_- \times \mathbb{R}^r) / \sim,$$

where $(x, v) \sim (y, w)$ if and only if $x = y \in S^{k-1}$ and $w = g(x)(v)$.

(1) Show that E_g is an oriented rank r bundle over S^k.
(2) Show that if $f, g : S^{k-1} \to SO(r)$ be homotoic maps, then $E_f \cong E_g$ (Hint: Use problem 2.19).

In conclusion, we have proved: $\text{Vect}_r^+(S^k, \mathbb{R}) = [S^{k-1}, SO(r)] = \pi_{k-1}(SO(r))$ and a map $g : S^{k-1} \to SO(r)$ is called a *clutching function*.

K-Groups and Chern Character.

7.27 (Topological K-ring)**.** Let X be a compact Hausdorff space. We denote $\text{Vect}(X) := \text{Vect}(X, \mathbb{C})$ be the set of isomorphism classes of complex vector bundle over X. Then $\text{Vect}(X)$ has a structure of abelian semigroup given by direct sum.

(1) Show that if A is an abelian semi-group, then there exists a unique abelian group $K(A)$ and a semi-group homormophism $\alpha : A \to K(A)$ satisfying the universal property: if G is any group and $f : A \to G$ be a semigroup homomorphism, then there exists a unique group homomorphism $\tilde{f} : K(A) \to G$ such that $\tilde{f} \circ \alpha = f$.
(2) We denote $K(X)$ by the abelian group associated to $\text{Vect}(X)$ as in (1), called the *topological K-group* of X. Show that every element of $K(X)$ is of the form $[V] - [W]$, for $[V], [W] \in \text{Vect}(X)$.
(3) Show that $[V] = [W]$ in $K(X)$ if and only if there exists $n \in \mathbb{N}$ such that $V \oplus \underline{\mathbb{C}}_X^n \cong W \oplus \underline{\mathbb{C}}_X^n$, where $\underline{\mathbb{C}}_X^n$ is the trivial rank n bundle over X.
(4) Show that the tensor product on $\text{Vect}(X)$ induces a ring structure on $K(X)$.

The ring $K(X)$ is called the (topological) K-ring of X. If we replace $\text{Vect}(X, \mathbb{C})$ by $\text{Vect}(X, \mathbb{R})$, then the corresponding ring is denoted by $KO(X)$, called the *real K-ring*.

7.28 (Basic Operations on K-ring)**.**

(1) Let $f : X \to Y$ be a continuous map. Show that pull-back bundle $f^* : \text{Vect}(Y) \to \text{Vect}(X)$ induces a ring homomorphism $f^* : K(Y) \to K(X)$.
(2) Given compact spaces X, Y, we can define a map $\mu : K(X) \otimes K(Y) \to K(X \times Y)$ given by $\mu(a \otimes b) = p^*(a)q^*(b)$, where p, q are projections from $X \times Y$ onto X and Y. Show that μ is a ring homomorphism.

(3) We can define the *reduced K-groups* $\tilde{K}(X)$ by the kernel of $K(X) \to K(pt)$. Show that $[V] = [W]$ in $\tilde{K}(X)$ if and only if $V \oplus \underline{\mathbb{C}}_X^n \cong W \oplus \underline{\mathbb{C}}_X^m$, for some $m, n \in \mathbb{N}$.