Chapter 6

MINIMAL SUBMANIFOLDS

6.1. Some General Notions for Minimal Surfaces

Let $\phi : M^m \looparrowright \mathbb{R}^n$ with $m < n$ be a connected m-dimensional manifold immersed in $\mathbb{R}^n$. We endow M the metric g induced from standard Euclidean metric $\langle\, , \,\rangle$ on $\mathbb{R}^n$. Let us first observe the following formula of mean curvature for manifold isometrically immersed into Euclidean spaces.

Lemma 6.1. *Let $\phi : M^m \looparrowright \mathbb{R}^n$ be the immersion of M into $\mathbb{R}^n$. The mean curvature $\overrightarrow{H}$ of M is given by*

$$\overrightarrow{H} = \triangle_{LB}\phi := (\triangle_{LB}\phi^1, \ldots, \triangle_{LB}\phi^n).$$

where $\triangle_{LB} = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j)$ is the Laplace–Beltrami operator defined in exercise 4.4.

PROOF. First, observe that if $e_1, \ldots, e_m$ is an orthonormal frame of TM in a neighborhood U of $p \in M$, then $\triangle_{LB}$ can be written as

$$\triangle_{LB} f = \sum_{k=1}^{m} e_k(e_k f) - \nabla_{e_k} e_k(f) \text{ on } U.$$

Since ϕ is an immersion, we identify e_k with $\phi_* e_k \in \mathbb{R}^n$. Hence, $e_k(\phi) = \phi_*(e_k) = e_k$ and $e_k(e_k\phi) = \overline{\nabla}_{e_k} e_k$, where $\overline{\nabla}$ is the trivial connection on $\mathbb{R}^n$. Therefore, we have

$$\begin{aligned}\triangle_{LB}\phi &= \sum_k (e_k e_k \phi - \nabla_{e_k} e_k(\phi)) = \sum_k \overline{\nabla}_{e_k} e_k(\phi) - \nabla_{e_k} e_k \\ &= \sum_k (\overline{\nabla}_{e_k} e_k)^N = \sum_k B(e_k, e_k) = \overrightarrow{H}.\end{aligned}$$

$\square$

In particular, M is minimal if and only if $\triangle_{LB}\phi = 0$. If we expand the mean curvature for M,

$$\overrightarrow{H} = \triangle_{LB}\phi = g^{ij}\partial_i\partial_j\phi + \frac{1}{\sqrt{g}}\partial_j\phi\partial_i(\sqrt{g}g^{ij}) = g^{ij}\partial_i\partial_j\phi + \partial_j\phi\triangle_{LB}x_j.$$

Note that for $h \in C^\infty(\Omega)$, $\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j h) = \operatorname{div}(\nabla h) = \triangle_{LB}h$. Hence, for fixed j,

$$\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}) = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j(x_j)) = \triangle_{LB}x_j,$$

where x_j is the coordinate function for on M. Thus, we can write $\overrightarrow{H} = g^{ij}\partial_i\partial_j\phi + \partial_j\phi\triangle_{LB}x_j$. Since $\partial_k\phi$ is perpendicular to $\overrightarrow{H}$,

$$0 = \partial_k\phi \cdot \overrightarrow{H} = \partial_k\phi \cdot \overrightarrow{S} + g_{kj}\triangle_{LB}x_j,$$

where $\overrightarrow{S} = g^{ij}\partial_i\partial_j\phi$. Equivalently, we have:

$$\triangle_{LB}x_j = -g^{jk}(\phi_k \cdot \overrightarrow{S}).$$

Thus, $\overrightarrow{H} = \overrightarrow{S} - \phi_j g^{jk}(\phi_k \cdot \overrightarrow{S})$.

To further simplify the equation, let us first assume that $m = 2$ and $\phi : \Omega \to \mathbb{R}^n$ is a graph [1] of a function $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^{n-2}$. In other words, we can find $f = (f_3, \ldots, f_n)$ such that $\phi(x,y) = (x, y, f_3(x,y), \ldots, f_n(x,y))$. The Riemannian metric g_{ij} on Ω is then given by

$$\begin{cases} g_{11} = \partial_x\phi \cdot \partial_x\phi = 1 + |f_x|^2 \\ g_{12} = \partial_x\phi \cdot \partial_y\phi = f_x \cdot f_y \\ g_{22} = \partial_y\phi \cdot \partial_y\phi = 1 + |f_y|^2. \end{cases}$$

The inverse g^{ij} is then given by

$$g^{ij} = \frac{1}{g}\begin{pmatrix} 1 + |\partial_y f|^2 & -f_x \cdot f_y \\ -f_x \cdot f_y & 1 + |\partial_x f|^2 \end{pmatrix},$$

where $g = \det(g_{ij}) = (1 + |f_x|^2)(1 + |f_y|^2) - (f_x \cdot f_y)^2$. By previous computation, the equation $\triangle_{LB}\phi = 0$ is equivalent to the following

[1] Notice that any immersion $\phi : M \looparrowright \mathbb{R}^n$ of surface can be locally expressed as graph.

system of equations

$$g^{ij}\partial_i\partial_j\phi^l + \partial_j\phi^l \triangle_{LB} x_j = 0$$

or

$$\begin{cases} \triangle_{LB} f = \sum_{i,j} g^{ij}\partial_i\partial_j f + \partial_j f \triangle_{LB} x_j = 0; \\ \triangle_{LB} x_j = \frac{1}{\sqrt{g}} \sum_i \partial_i(\sqrt{g} g^{ij}) = 0, \quad j = 1, 2. \end{cases}$$

Claim 6.2. If $\sum_{i,j} g^{ij}\partial_i\partial_j f = 0$, then

$$\triangle_{LB} x = 0 = \triangle_{LB} y = 0 \Longleftrightarrow \triangle_{LB} f = 0$$

PROOF. Let $p := f_x$, $q := f_y$, $r := f_{xx}$, $s := f_{xy}$, $t := f_{yy}$,

$$S := \sum_{i,j} g^{ij}\partial_i\partial_j f = (1 + |q|^2)r - 2(p \cdot q)s + (1 + |p|^2)t.$$

Hence, $\triangle_{LB} f = S + p\triangle_{LB} x + q\triangle_{LB} y$. By lemma 6.1 ,the mean curvature is given by $\overrightarrow{H} = \triangle_{LB}\phi = (\triangle_{LB} x, \triangle_{LB} y, \triangle_{LB} f)$. Taking inner product with $\phi_x = (1, 0, p)$, $\phi_y = (0, 1, q)$, we get a system of linear equations:

$$\begin{cases} \triangle_{LB} x + p \cdot S + |p|^2 \triangle_{LB} x + p \cdot q \triangle_{LB} y = 0 \\ \triangle_{LB} y + q \cdot S + p \cdot q \triangle_{LB} x + |q|^2 \triangle_{LB} y = 0 \end{cases}$$

or

$$\begin{pmatrix} 1 + |p|^2 & p \cdot q \\ p \cdot q & 1 + |q|^2 \end{pmatrix} \begin{pmatrix} \triangle_{LB} x \\ \triangle_{LB} y \end{pmatrix} = - \begin{pmatrix} p \cdot S \\ q \cdot S \end{pmatrix}$$

Since the determinant of the matrix is equal to $g > 0$. Hence, if $S = 0$, then $\triangle_{LB} x = 0 = \triangle_{LB} y$. The converse is trivial. □

Thus, we see that the condition for the graph of $f : \Omega \to \mathbb{R}^{n-2}$ to be minimal is given by

$$\sum_{i,j} g^{ij}\partial_i\partial_j f_l = 0, \quad l = 3, \dots, n. \tag{6.1}$$

For the case of hypersurface graph, the minimal surface equation can be even simplified. Let $M^m \looparrowright \mathbb{R}^{m+1}$ be a hypersurface. For a local coordinate $\mathbf{x}$ on M, we denote $\partial_i = \mathbb{X}_i = \frac{\partial}{\partial x^i}$. In example 3.56, we have seen that the second fundamental form is given by

$$B(\partial_i, \partial_j) = (\nabla_{\partial_i}\partial_j)^N = (\frac{\partial}{\partial x^i}\mathbb{X}_j)^N = \mathbb{X}_{ij}^N.$$

Let μ be a unit normal vector field for M in $\mathbb{R}^{m+1}$. Then $B_{ij} = \mathbb{X}_{ij} \cdot \mu$. Therefore, the mean curvature is given by

$$H = \mathrm{tr}B = g^{ij}B_{ij} = g^{ij}\mathbb{X}_{ij} \cdot \mu = -g^{ij}\mathbb{X}_i \cdot \partial_j \mu = -d\mathbb{X} \cdot d\mu,$$

where $d\mathbb{X}, d\mu$ are vector–valued 1-forms. By written into component, we see that

$$H = -d\mathbb{X} \cdot d\mu = -\sum_{\alpha=1}^{m+1} \partial_\alpha \mu^\alpha,$$

where ∂_α is the partial derivative with respect to α-th coordinate on $\mathbb{R}^{m+1}$. Particularly, when M is given by the graph Γ_f of a smooth function $f : \Omega \to \mathbb{R}$ over a domain $\Omega \subset \mathbb{R}^m$. The unit normal vector field μ can be chosen as

$$\mu = \frac{1}{\sqrt{1+|\nabla f|^2}}(-\partial_1 f, -\partial_2 f, \ldots, -\partial_m f, 1).$$

Thus, the mean curvature for minimal hypersurface graph is just

$$H = -\sum_{\alpha=1}^{m} \mu_\alpha^\alpha = \sum_{\alpha=1}^{m} \partial_\alpha \left(\frac{\partial_\alpha f}{\sqrt{1+|\nabla f|^2}} \right). \tag{6.2}$$

Exercise 6.1. Let $\Omega \subset \mathbb{R}^{n-1}$ be a domain, $f : \Omega \to \mathbb{R}$ be a smooth function.

(1) Derive the (6.2) by considering the first variation of volume functional $A(t) := A(f + th) = \int_\Omega (1 + |\nabla f + t\nabla h|^2)^{1/2}$.
(2) Derive the second variation for $A(t)$ and show that hypersurfaces graphs are local minimum for $A(t)$[2].

[2]Alternatively, in problem 6.17, we will show that the minimal hypersurface graph Γ_f are actually volume minimizer among the submanifolds in $\Omega \times \mathbb{R}$ which is homologous to Γ_f

6.2. Gauss Map and Weierstrass Representation

Let $\phi : M^2 \looparrowright \mathbb{R}^n$ be an immersion. We endow M a Riemannian metric by $g = \phi^*\langle\cdot,\cdot\rangle$ and assume that M is oriented. Before proceeding, let us state a theorem concerning existence of *isothermal coordinate* on any surfaces.

Theorem 6.3 (Existence of Isothermal Coordinates)**.** *Any 2-dimensional Riemannian manifold (M, g), for each $p \in M$, there exists a local coordinate $(U, \mathbf{u} = (u_1, u_2))$ at p, called isothermal coordinate of M near p, such that*

$$g = e^\rho(du_1^2 + du_2^2),$$

where $\rho(\mathbf{u}) \in C^\infty(U)$.

We refer the proof of the theorem[3] to an elegant paper of Chern [Che55].

With this result, we are able to give (M, g) a *Riemann surface*[4] structure on M. First, we observe that

Exercise 6.2. Let $\mathbf{u} = (u_1, u_2)$ be an isothermal coordinate at p.

(1) Show that if $\mathbf{x} = (x_1, x_2)$ is another coordinate near p, then $\mathbf{x}$ is also an isothermal coordinate if and only if

$$\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 = \left(\frac{\partial u_1}{\partial x_2}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2; \quad \frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2} = 0.$$

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[3] In general, we define two Riemannian metrics g and g_1 on M is *conformal equivalent* if there exists $f \in C^\infty(M)$ such that $g_1 = e^{2f}g$ and (M, g) is *conformally flat* if for $p \in M$, there exists a neighborhood U of p and $f \in C^\infty(U)$ such that $(U, e^{2f}g)$ is flat. Hence, the theorem 6.3 means that any 2-dimensional Riemannian manifold is conformally flat. For more on conformal equivalence of Riemannian metric, see problem 6.2. For the proof for the special case when M is a minimal surface in Euclidean spaces, the existence of isothermal coordinates is a direct consequence of minimal surface equations, see problem 6.1

[4] Recall that a surface M is a Riemann surface if it admits an atlas whose local coordinates values in $\mathbb{C} \cong \mathbb{R}^2$ and transition functions are all holomorphic.

(2) Assume that $\mathbf{x}$, and $\mathbf{u}$ are both isothermal coordinates and that they have the same orientation, show that
$$\frac{\partial u_1}{\partial x_1} + i\frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_2} - i\frac{\partial u_2}{\partial x_1}.$$

Given two isothermal coordinates $\mathbf{x}$, $\mathbf{u}$ near p, we now introduce complex coordinates $z = x_1 + ix_2$, $w = u_1 + iu_2$ and differential operators
$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right); \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right),$$
then from exercise 6.2 we deduce that M has a Riemann surface structure since
$$\frac{\partial w}{\partial \bar{z}} = 0.$$
Now, back to the case M is isometrically immersed into $\mathbb{R}^n$. Applying the existence of isothermal coordinates on (M, g) shows that $g = e^\rho(dx_1^2 + dx_2^2) = e^\rho dz \otimes d\bar{z}$ and ϕ is *conformal*. Moreover, in the isothermal coordinates, the Laplace-Beltrami operator is just
$$\triangle_{LB} = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j) = e^{-\rho}\sum_{i=1}^{2}\partial_i^2 = 4e^{-\rho}\frac{\partial^2}{\partial\bar{z}\partial z}.$$
Combining with lemma 6.1, we see that

Proposition 6.4. *$\phi : M \looparrowright \mathbb{R}^n$ is minimal if and only if the complex functions $\phi_z^j := \frac{\partial \phi^j}{\partial z}$ are holomorphic. Moreover, we have*
$$\sum_{j=1}^{n}(\phi_z^j)^2 = 0; \quad |\phi_z|^2 = \frac{e^\rho}{2}.$$

PROOF. It remains to prove the last two identities. We write $\phi_z^j = \frac{1}{2}(\frac{\partial \phi^j}{\partial x_1} - i\frac{\partial \phi^j}{\partial x_2})$ and we have
$$\begin{aligned}\sum_{j=1}^{n}(\phi_z^j)^2 &= \frac{1}{4}\sum_{j=1}^{n}\left(\frac{\partial\phi^j}{\partial x_1}\frac{\partial\phi^j}{\partial x_1} - 2i\frac{\partial\phi^j}{\partial x_1}\frac{\partial\phi^j}{\partial x_2} - \frac{\partial\phi^j}{\partial x_2}\frac{\partial\phi^j}{\partial x_2}\right)\\ &= \frac{1}{4}\left(\left|\frac{\partial\phi}{\partial x_1}\right|^2 - 2i\left\langle\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}\right\rangle - \left|\frac{\partial\phi}{\partial x_2}\right|^2\right)\\ &= \frac{1}{4}(e^\rho - e^\rho) = 0.\end{aligned}$$

Similarly, $|\phi_z|^2 = \frac{1}{4}(|\partial_{x_1}\phi|^2 + |\partial_{x_2}\phi|^2) = \frac{e^\rho}{2}$. $\square$

If M has a global complex coordinate z (e.g., $M = \mathbb{C}$ or $\mathbb{D}$), then

$$\phi_z : M \longrightarrow \left(\sum_{i=1}^{n} z_i^2 = 0\right) \subset \mathbb{C}^n \setminus \{0\}$$

is a holomorphic map. For general case, one observes that if $w = w(z)$ is another isothermal coordinate, then $\phi_w = \phi_z \frac{\partial z}{\partial w}$ and $\frac{\partial z}{\partial w} \neq 0$. Therefore, we can define a well-defined holomorphic map into a quadric hypersurface Q_{n-2} in $\mathbb{CP}^{n-1}$:

$$\begin{aligned}\Phi : &M \to Q_{n-2} = (\sum_{j=1}^{n} Z_j^2 = 0) \subset \mathbb{CP}^{n-1} \\ &p \mapsto [\phi_z(p)].\end{aligned}$$

This map is called *Gauss map*[5]. In conclusion, we obtain another characterization for $\phi : M \looparrowright \mathbb{R}^n$ to be minimal.

Corollary 6.5. *An immersion $\phi : M \looparrowright \mathbb{R}^n$ is minimal if and only if the Gauss map $\Phi : M \to Q_{n-2}$ is holomorphic.*

In fact, Φ is the complex conjugate of the differential geometric Gauss map. To see this, we make use a simple fact that $Q_{n-2} \cong \tilde{G}_{n,2}(\mathbb{R})$ (cf. Exercise 6.3 below). Recall that in the case of $n = 3$, the differential geometric Gauss map of a surface $\phi : M \looparrowright \mathbb{R}^3$ assigns each point $p \in M$ the unit normal vector

$$N(p) = \frac{\partial_1\phi \wedge \partial_2\phi}{|\partial_1\phi \wedge \partial_2\phi|}(p) \in S^2 \subset \mathbb{R}^3. \tag{6.3}$$

Now, we identify $S^2 = \tilde{G}_{3,1}(\mathbb{R}) \cong \tilde{G}_{3,2}(\mathbb{R})$ via $e_3 \in S^2 \mapsto [e_1 \wedge e_2]$ such that (e_1, e_2, e_3) is an oriented orthonormal basis of $\mathbb{R}^3$. In this way, we see that the differential geometric Gauss map $N : M \to Q_1$ is given by

$$N(p) = [\partial_1\phi(p) + i\partial_2\phi(p)] = \left[\frac{\partial\phi}{\partial\bar{z}}(p)\right] = \overline{\left[\frac{\partial\phi}{\partial z}(p)\right]},$$

[5]Since $\alpha = (\alpha_1 = \phi_z^1 dz, \ldots, \alpha_n = \phi_z^n dz)$ is a system of holomorphic 1-forms. Hence, in the terminology of algebraic geometry, Φ is just the sub-linear system of canonical divisor $|K_M|$

which indeed is the complex conjugate of Φ. Thus, we have

Corollary 6.6. *A surface $M \looparrowright \mathbb{R}^3$ is minimal if and only if its differential geometric Guass map N is anti-holomorphic.*

Exercise 6.3. Show that Q_{n-2} is diffeomorphic to the oriented Grassmannian $\tilde{G}_{n,2}(\mathbb{R}) := SO(n)/(SO(2) \times SO(n-2))$. Also, show that $\tilde{G}_{n,2}(\mathbb{R})$ is a double covering of $G_{n,2}(\mathbb{R})$ and is oriented.

From proposition 6.4, we see that for any minimal surface $\phi : M \looparrowright \mathbb{R}^n$, we associate a system of holomorphic 1-form $\alpha = (\alpha^1 = \phi^1_z dz, \ldots, \alpha^n = \phi^n_z dz)$. Hence, the surface itself can be recaptured by

$$\phi^k(z) := 2\mathrm{Re} \int_{z_0}^{z} \alpha_k, \quad k = 1, \ldots, n.$$

Conversely, when M is a simply-connected domain with global complex coordinate (e.g. $M = \mathbb{C}, \mathbb{D}$), it is easy to construct the minimal immersion by the following recipe:

(1) Find a system of holomorphic 1-forms $\alpha = (\alpha^1, \ldots, \alpha^n)$ on M such that $\sum_{k=1}^n (\alpha^i)^2 = 0$, $\sum_{k=1}^n |\alpha^i|^2 > 0$.
(2) Pick any $z_0 \in M$, we define $\phi := 2\mathrm{Re} \int_{z_0}^{z} \alpha : M \to \mathbb{R}^n$

In the case of $n = 3$, it is easy to describe all solutions of the equation $\alpha_1^2 + \alpha_2^2 + \alpha_3^3 = 0$. We first write the equation into

$$(\alpha_1 + i\alpha_2)(\alpha_1 - i\alpha_2) = -\alpha_3^2,$$

and we define

$$g := \frac{\alpha_3}{\alpha_1 - i\alpha_2}; \quad W := \alpha_1 - i\alpha_2.$$

Hence, g is a meromorphic function and W is a holomorphic 1-form on M. We can then express the equation into

$$\begin{cases} \alpha_1 + i\alpha_2 = -g\alpha_3 = -g^2 W \\ \alpha_1 - i\alpha_2 = W. \end{cases}$$

We can then express α_i's into g and W to get the **Weierstrass–Enneper representation** for minimal surfaces in $\mathbb{R}^3$:

$$\begin{cases} \alpha_1 = \frac{1}{2}(1-g^2)W \\ \alpha_2 = \frac{i}{2}(1+g^2)W \\ \alpha_3 = gW. \end{cases} \tag{W–E}$$

Now, we write $W = f dz$, then conformal factor e^ρ is given by

$$\frac{e^\rho}{2} = |\phi_z|^2 = \frac{1}{4}|1-g^2|^2|f|^2 + \frac{1}{4}|1+g^2|^2|f|^2 + |g|^2|f|^2$$
$$= \frac{1}{2}|f|^2(1+|g|^2)^2.$$

Conversely, given a simply-connected domain[6]. $M \subset \mathbb{C}$, a meromorphic function g on M and a holomorphic 1-form W such that W has zeros at the poles of g with twice order, (W–E) give a system of holomorphic 1-forms $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ on M. Thus, α defines a minimal immersion $\phi : M \looparrowright \mathbb{R}^3$.

Remark 6.7. Notice that we have to rule out the trivial solution $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore, $W = f dz$ can only have zeros exactly at the poles of g.

In fact, the meromorphic function g has a geometric interpretation as following.

Proposition 6.8. *Let $N : M \to S^2$ be the differential geometric Gauss map defined in* (6.3)*, $p : S^2 \to \mathbb{C} \cup \{\infty\}$ be the stereographic projection. Then $g = p \circ N$.*

[6]More generally, given a Riemann surface M, a holomorphic 1-form W, and a meromorphic function g, the minimal immersion $\phi(z) = 2\mathrm{Re} \int_{z_0}^{z} \alpha$ depends on the choice of based point $z_0 \in M$ and the choice of path joining z_0 and z. The change of z_0 is just translate ϕ by a constant. However, the map ϕ depends on the choice of path in general, and thus is not well-defined. For ϕ to be well-defined, we need to ensure that W has no real period, that is, Re $\int_\gamma W = 0$, for any closed loop γ in M

PROOF. We first write $\phi_z = (\alpha_1, \alpha_2, \alpha_3) = \frac{1}{2}(\phi_x - i\phi_y)$ with $\alpha_j = a_j dz = (u_j + iv_j)dz$. One can easily show that

$$\begin{aligned}\phi_x \times \phi_y &= 4\mathrm{Im}(a_2\overline{a_3}, a_3\overline{a_1}, a_1\overline{a_2}) \\ &= (1+|g|^2)|f|^2(2\mathrm{Re}(g), 2\mathrm{Im}(g), |g|^2 - 1).\end{aligned}$$

Hence, the Gauss map can be expressed in terms of g and $W = f dz$ as

$$N = \frac{1}{1+|g|^2}(2\mathrm{Re}(g), 2\mathrm{Im}(g), |g|^2 - 1) \tag{6.4}$$

Recall that the stereographic projection $p(x,y,z) = \frac{x}{1-z} + i\frac{y}{1-z}$ and thus

$$p \circ N = \mathrm{Re}(g) + i\mathrm{Im}(g) = g.$$

□

Exercise 6.4. Complete the details in the proof of Proposition 6.8.

6.3. Applications for Weierstrass Representation

We use Weierstrass representation to present some classical examples of minimal surfaces in $\mathbb{R}^3$.

Example 6.9 (Catenoid). *Catenoid* is given by the Weierstrass data

$$M = \mathbb{C} \setminus \{0\}, g(z) = z; f(z) = \frac{1}{z^2}.$$

Following (W–E), we then set

$$\alpha_1 = \frac{1-z^2}{2z^2}dz; \quad \alpha_2 = \frac{i(1+z^2)}{2z^2}dz; \quad \alpha_3 = \frac{dz}{z} \tag{6.5}$$

We pick $1 \in M$ as based point and $\phi(1) = (-2, 0, 0)$. Thus, the component of minimal immersion $\phi = (x, y, z)$ is given by:

$$\begin{aligned} x &= -2 + \mathrm{Re}\int_1^w \frac{1-z^2}{z^2}dz = \mathrm{Re}(-\frac{1}{w} - w) \\ y &= -\mathrm{Im}\int_1^w \frac{1+z^2}{z^2}dz = \mathrm{Im}(\frac{1}{w} - w) \\ z &= 2\mathrm{Re}\int_1^w \frac{dz}{z} = \log|w|^2 \end{aligned}$$

Hence, we have

$$x - iy = \frac{-1}{w} - \bar{w}$$

and

$$x^2 + y^2 = |x - iy|^2 = \frac{1}{|w|^2} + |w|^2 + 2 = (e^{z/2} + e^{-z/2})^2 = 4\cosh^2(z/2).$$

Thus, this shows that the catenoid is a surface of revolution. In fact, it is the only surface of revolution that is minimal other than the plane (cf. Problem 6.5). In terms of parametrization $w = u + iv$, we see that $\phi : M \to \mathbb{R}^3$ is given by

$$\phi(u, v) = \left(\frac{-u}{u^2 + v^2} - u, \frac{-v}{u^2 + v^2} - v, \log(u^2 + v^2) \right). \tag{6.6}$$

If we pass to universal cover $w = e^{\xi}$, $\xi = \alpha + i\beta \in \tilde{M} := \mathbb{C}$, then the catenoid has the parametrization

$$(-2\cosh\alpha\cos\beta, -2\cosh\alpha\sin\beta, 2\alpha). \tag{6.7}$$

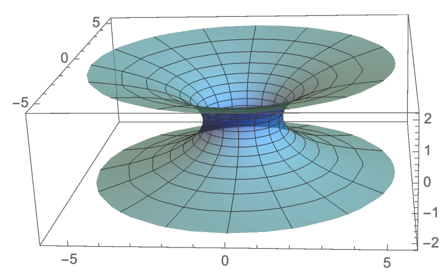


FIGURE 1. Catenoid

Before giving the next example, we first introduce the isometric deformation of minimal surfaces. Given Weierstrass data $(M, g, W = f dz)$. From (W–E), we then obtain a system of holomorphic 1-forms $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. For $\theta \in [0, \pi]$, we consider the deformation of immersion by θ:

$$\phi_\theta := 2\mathrm{Re}\left(e^{i\theta} \int \alpha \right). \tag{6.8}$$

Observe that $2|(\phi_\theta)_z|^2 = 2|e^{i\theta}\alpha|^2 = 2|\alpha|^2 = e^{\rho}$. Hence, for any θ, ϕ_θ is a minimal immersion and is isometric to $\phi = \phi_0$. Therefore, we then call ϕ_θ the *isometric deformation* of minimal surfaces ϕ. In particular, when $\theta = \pi/2$, we call $\phi_{\pi/2}$ the *conjugate* of ϕ since ϕ and $\phi_{\pi/2}$ are conjugate harmonic. We now give an example of conjugate minimal surface.

Example 6.10 (Helicoid). The helicoid can be defined as the conjugate of the catenoid. If we consider the conjugate of (6.5), we get

$$\alpha_1 = \frac{i(1-z^2)}{2z^2}dz; \quad \alpha_2 = -\frac{1+z^2}{2z^2}dz; \quad \alpha_3 = \frac{idz}{z}.$$

If we repeat the similar calculation as above, then we get

$$\phi(w) = \left(\mathrm{Im}\left(\frac{1}{w} - w\right), -\mathrm{Re}\left(\frac{1}{w} + w\right), -2\arg(w)\right)$$

However, ϕ^3 is not well–defined. Thus, let us pass to the universal cover $\tilde{M} = \mathbb{C}$ of $M = \mathbb{C} \setminus \{0\}$ by $w = e^{\xi}$, for $\xi \in \mathbb{C}$. Then $dw = e^{\xi}d\xi$ and

$$g(\xi) = e^{\xi}; \quad W(\xi) = e^{-\xi}d\xi.$$

Thus, the holomorphic 1-forms α^i on $\mathbb{C}$ is given by

$$\begin{aligned}
\alpha_1 &= \frac{i}{2}(1 - e^{2\xi})e^{-\xi}d\xi = i\sinh(\xi)d\xi, \\
\alpha_2 &= \frac{-1}{2}(1 + e^{2\xi})e^{-\xi}d\xi = -\cosh(\xi)d\xi, \\
\alpha_3 &= id\xi.
\end{aligned}$$

Thus, if we write $\xi = u + iv$, then

$$\begin{aligned}
\phi(u,v) &= (2\mathrm{Im}(\cosh(\xi), -2\mathrm{Re}(\sinh(\xi)), -2\mathrm{Im}(\xi)) \\
&= (2\sin v \sinh u, -2\cos v \sinh u, -2v)
\end{aligned} \tag{6.9}$$

This gives a parametric form for the helicoid. One can show that helicoid is an embedded ruled surface. In fact, it is only ruled surface that is minimal (cf Problem 6.6).

Exercise 6.5. Verify directly from parametrization (6.6) or (6.7) and (6.9) that both of them are minimal surfaces. Verify two parametrizations (6.7) and (6.9) are isometric.

Next, we give an example of minimal surface which is not embedded.

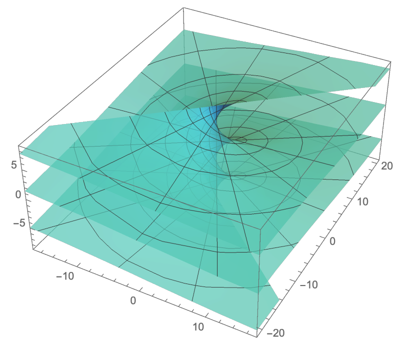


FIGURE 2. Helicoid

Example 6.11 (Enneper's Surfaces). Enneper's surface is given by the Weierstrass data $M = \mathbb{C}$, $g(z) = z$, $W = dz$. We can carry out the similar calculation to get the parametrization

$$\phi(u, v) = \left(u - \frac{u^3}{3} + uv^2, -v + \frac{v^3}{3} - u^2 v, u^2 - v^2\right). \tag{6.10}$$

Enneper's surface is a complete minimal surface yet it has self–intersection. Thus, it is not an embedded surface.

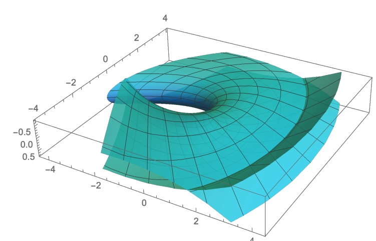


FIGURE 3. Enneper's Surface

Exercise 6.6. Carry out the details to get the parametrization (6.10) and verify directly that Enneper's surface is minimal. Also, specify the region where Enneper's surface self–intersects.

Example 6.12 (Scherk's Doubly Periodic Surface). The catenoid and helicoid were discovered by Meusinier in 1776. It was not until 1835 that Scherk discovered the next example of minimal surface. Scherk's example is given by the $z = \log(\cos(y)/\cos(x))$. One can show that the Weierstrass representation for Scherk's surface is given by

$$M = \mathbb{C} \setminus \{\pm 1, \pm i\}, g = z, W = \frac{2dz}{1 - z^4}. \tag{6.11}$$

Exercise 6.7. Verify that (6.11) is indeed the Weierstrass representation for Scherk's surface.

Remark 6.13. The examples given in this section are classical examples of minimal surfaces discovered before 20th century. For more examples and recent development on minimal surfaces, one can consult Matthias Weber's *Museum of Minimal Surfaces*[7], which comprises figures and descriptions about 150 minimal surfaces, [CM11], and [MP12].

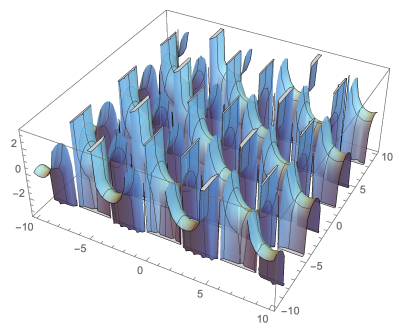


FIGURE 4. Scherk's Doubly Periodic Surface

We end this section by proving a theorem due to Osserman.

Theorem 6.14 (Osserman, 1959)**.** *Let $\phi : M \looparrowright \mathbb{R}^3$ be a complete minimal surface. If the differential geometric Gauss map $N : M \to S^2$ is not dense, then M is a plane.*

PROOF. We first pass to the universal cover $\pi : \widetilde{M} \to M$ of M, then there exists $\tilde{\phi} : \widetilde{M} \to \mathbb{R}^3$ such that $\tilde{\pi} = \pi \circ \phi$. With the induced metric on $\widetilde{M}$ from M, $\widetilde{M}$ is still a complete surface and $\tilde{\phi}$ is still a minimal immersion. Also, the Gauss map $\widetilde{N}$ of $\widetilde{M}$ factors into $\pi \circ N$.

[7] https://minimal.sitehost.iu.edu/archive

Thus, the same hypothesis holds for $\widetilde{N}$. By a rotation on S^2, we may assume that $(1,0,0) \in S^2$ is not the image of N. Since N is not dense, there exists $\eta > 0$ such that $N(p) = (N_1, N_2, N_3)$ with $N_3 \le \eta < 1$. Same is true for $\widetilde{N}$.

Next, we recall the *uniformization theorem*[8] of Riemann surface.

Fact 6.15 (Uniformization Theorem). Every simply connected Riemann surface $\widetilde{M}$ is conformal equivalent to S^2, $\mathbb{C}$, or $\mathbb{D}^2$.

(1) The case $\widetilde{M} = S^2$ is excluded due to the easy fact that a minimal surface in Euclidean space cannot be compact (cf. Problem 6.3).
(2) If $\widetilde{M} = \mathbb{D}^2$, then (6.4) shows that $|g| \le \sqrt{\frac{\eta+1}{1-\eta}} = M < \infty$. Thus, the meromorphic function g on $\widetilde{M}$ is in fact holomorphic. Hence, from remark 6.7, $W = f dz$ is nowhere vanishing. We then define $F : \mathbb{D}^2 \to \mathbb{C}$ by
$$w = F(z) = \int_0^z f(\xi) d\xi.$$
Thus, we can pick the largest disk $|w| < R$ such that $G = F^{-1}$ is defined (such $R < \infty$ by applying Liouville theorem on G). Let $w_0 \in \partial B_R(0)$ be a non-extendable point for $G(w)$ and let l be the line segment $w = t w_0$, for $t \in [0, 1)$. We denote $c = G(l)$.

Claim 6.16. C is a divergent curve (cf. Problem 3.13).

PROOF OF CLAIM. Suppose otherwise, there exists a sequence $t_n \to 1$ such that $z_n \in C$ such that $z_n \to z_0 \in \mathbb{D}^2$. But then $F(z_0) = w_0$ and since $F'(z_0) = f(z_0) \neq 0$. By open mapping theorem, $G(w)$ would be extendable to a neighborhood of w_0, a contradiction. $\square$

Since $\widetilde{M}$ is complete, by the claim and problem 3.13, we will get contradiction if we show $L(c) < \infty$. We already

[8]For the proof of Koebe's uniformization theorem, one can consult for instance [Gam01], p.439.

knew that $|g| \leq M < \infty$, Therefore,

$$L(c) = \int_C \sqrt{e^\rho/2}|dz| = \frac{1}{\sqrt{2}} \int_C |f|(1+|g|^2)|dz| \leq \frac{1+M^2}{\sqrt{2}} \int_C |f||dz|$$
$$= \frac{1+M^2}{\sqrt{2}} \int_C |F'||dz| = \frac{1+M^2}{\sqrt{2}} \int_l |dw| = \frac{(1+M^2)R}{\sqrt{2}} < \infty.$$

(3) If $\widetilde{M} = \mathbb{C}$, then $g : \mathbb{C} \to \mathbb{C}$ is an entire function. By Liouville theorem, we know that g is constant, for $|g| \leq M < \infty$ is bounded, Hence, $\widetilde{N}$ and thus N are both constant maps. This shows that $M \subset \mathbb{C}$. Since M is complete, $M = \mathbb{C}$.

□

Theorem 6.14 has an immediate corollary.

Corollary 6.17 (Classical Bernstein's Theorem)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$. If $M := \{(x, y, f(x, y))\} = 0$ is a minimal hypersurface, then f must be linear.*

PROOF. Since the Gauss map of a graph omits a hemisphere, M must be a plane and thus f is linear. □

Thus, theorem 6.14 is often called *Osserman's generalized Bernstein theorem*. Here are some further generalizations of Osserman's reuslt.

Remark 6.18.
(1) Osserman showed that for $\phi : M^2 \looparrowright \mathbb{R}^n$, similar lines of proof shows that if $\phi(M)$ is not a plne, then the (holomorphic) Gauss map $\Phi : M \to \mathbb{CP}^{n-1}$ meets a dense set of hyperplane (see [Oss69], Theorem 12.1).
(2) For minimal hypersurface graph M in $\mathbb{R}^{n+1}$ defined on whole $\mathbb{R}^n$ (namely, the higher dimension analogue of classical Bernstein theorem), Jim Simons showed in 1968 that for $n \leq 7$, M must be a hypersurface. However, Bombieri et al. showed in 1969 that the statement is false for $n \geq 8$.
(3) Xavier–Fujimoto Theorem: let $M \looparrowright \mathbb{R}^3$ be a complete minimal surface. If M is not planar, Xavier proved in 1981 that if the gauss map N cannot omit more than six points. In 1988, Fujimoto improved the result to five points. In fact, the result is optimal since Gauss map for Scherk's surface

(cf. Example 6.12) omits exactly 4 points. Also, for $k \leq 4$, the omitted k points can be presribed on S^2 (see [Oss69], Theorem 8.3).

(4) Osserman proved that any complete minimal surface [9] M in Euclidean spaces with $\int_M |K| dA < \infty$ is isometric to a punctured compact Riemann surface $\overline{M} \setminus \{p_1, \ldots, p_r\}$ (cf. [Oss69], Theorem 9.1).

6.4. Douglas' Solution to the Classical Plateau Problems

In this section, we discuss Plateau problem. Naively speaking, the statement of Plateau problem is the following: given a Jordan curve $\Gamma \subset \mathbb{R}^n$, can one find a surface Σ_0 which minimizes the area among the surfaces Σ with $\partial\Sigma = \Gamma$? The problem is in fact very subtle and difficult if we do not restrict the topolgical type for the surface Σ in consideration. For example, Flemming discovered an pathological example in 1955 that a Jordan curve which bounds a minimal surface with infinite genus.

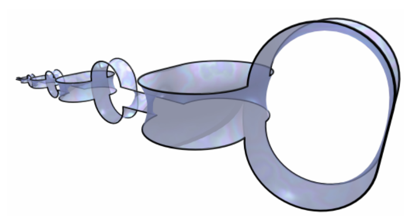


FIGURE 5. Flemming's Example of minimal surface with infinite genus
credit: Harrison, Pugh [HP16], p,276

Thus, for simplicity, we restrict to the problem for finding the surface of simplest topological type, namely a disk Δ. Also, instead of requiring the parametrization to be immersion, we may enlarge the

[9] In fact, his result works for any complete 2-dimensional Riemannian manifold (M, g) with non-positive Gaussian curvature and $\int_M |K| dA < \infty$. Since minimal surface in Euclidean space has non–positive Gaussian curvature (cf. Problem 6.4), the result applies.

class the parametrization to have branched point or even singular on finitely many C^1 curves on the disk.

Here is the precise formulation for classical Plateau problem.

(1) Let $\Gamma \subset \mathbb{R}^n$ be a given Jordan curve.
(2) $\Delta \subset \mathbb{R}^2$ be the closed unit disk.
(3) A *piecewise* C^1 map $\phi : \Delta \to \mathbb{R}^n$ in the sense that
   (a) ϕ is continuous, and
   (b) ϕ is a C^1 map outside $\partial\Delta$ and finite numbers of points and C^1-curves on Δ.
(4) On the boundary, $\phi\Big|_{\Delta} : \partial\Delta = S^1 \to \Gamma$ which is *monotone* in the sense that $\phi^{-1}(p)$ is connected, for any $p \in \Gamma$.

We then define the *competing class* (class of surfaces in consideration)

$$X_\Gamma := \{\phi : \Delta \longrightarrow \mathbb{R}^n : \phi \text{ satisfies (1)–(4) above}\},$$

and the *area functional* $A : X_\Gamma \to [0, \infty]$ by

$$A(\phi) := \int_\Delta \left|\partial_x \phi \wedge \partial_y \phi\right| dxdy.$$

Assume[10] that $G_\Gamma := \inf_{\phi \in X_\Gamma} A(\phi) < \infty$, we then wish to find $\phi \in X_\Gamma$ such that $A(\phi) = G_\Gamma$. One may try to tackle this problem via *direct method in calculus of variation*. That is, we first take a minimizing sequence $\phi_n \in X_\Gamma$ such that $\lim_{n\to\infty} A(\phi_n) = G_\Gamma$, and then we try to prove there exists a convergent subsequence whose limit is still in X_Γ. The second step amounts to prove the "compactness" of X_Γ in a suitable sense. In general, such approach will not work since the infinite-dimensional diffeomorphism group $\mathrm{Diff}(\Delta)$ acts on X_Γ by $\phi \mapsto \phi \circ \sigma$ and $A(\phi)$ is invariant under the action[11]. The presence of symmetric group in a variational problem which results in the redundancy on the configuration is known as *gauge theory* in modern mathematics and physics.

To remedy this, we need to control the parametrization of our minimizing sequence. One of the key in the proof of Douglas is to

[10] In [Law80], p.58–59, he constructs a Jordan curve Γ with $G_\Gamma = \infty$.
[11] See [Law80], p.60 for an illustration for this.

consider instead the minimizing problem for the Dirichlet integral (energy functional)[12]

$$D(\phi) := \int_{\Delta} (|\partial_x \phi||^2 + |\partial_y \phi|^2) dxdy. \tag{6.12}$$

From $|v \wedge w|^2 = |v|^2|w|^2 - |v \cdot w|^2 \leq \left[\frac{1}{2}(|v|^2 + |w|^2)\right]^2$, we get $A(\phi) \leq \frac{1}{2}D(\phi)$. Moreover, the equality holds if and only if

$$|\phi_x| = |\phi_y|; \quad \phi_x \cdot \phi_y = 0. \tag{6.13}$$

If $\phi : \Delta \to \mathbb{R}^n$ satisfies (6.13), then we say ϕ is *almost conformal*. When $|\phi_x| > 0$, we see that ϕ is in fact conformal since it induces a metric g on Δ given by

$$g = \lambda(dx^2 + dy^2),$$

where $\lambda = |\phi_x| = |\phi_y|$. Thus, we employ the following fact which is equivalent to theorem 6.3.

Fact 6.19. For any continuous map $\phi : \Delta \to \mathbb{R}^n$ with $\phi\Big|_{\Delta^\circ}$ being an C^1-immersion , there exists a hoemeomorphism $\sigma : \Delta \to \Delta$ with $\sigma\Big|_{\Delta^\circ}$ is C^1-diffeomorphism such that $\widetilde{\phi} = \phi \circ \sigma$ is conformal.

Granting this fact, for each Jordan curve $\Gamma \subset \mathbb{R}^n$, we now show:

Proposition 6.20. *$G_\Gamma = \frac{1}{2}d_\Gamma$, where $d_\Gamma := \inf_{\phi \in X_\Gamma} D(\phi)$*

PROOF. Clearly, we already have $G_\Gamma \leq \frac{1}{2}d_\Gamma$. For the reverse inequality, let $\{\phi_n\}_{n=1}^\infty$ be a sequence in X_Γ such that $A(\phi_n) \searrow G_\Gamma$. We may assume each $\phi_n \in C^1(\Delta^\circ)$ since we may approximate the piecewise C^1 sequence by C^1 uniformly on Δ. To apply fact 6.19, we need to ensure that ϕ_n are all immersion. To see this, let us consider $\phi_{n,r} : \Delta \to \mathbb{R}^{n+2}$ given by $\phi_{n,r}(x,y) = (\phi_n(x,y), rx, ry)$. For $r > 0$, though $\phi_{n,r} \notin X_\Gamma$ of course, it is still an immersion.

[12]In the case of one–dimension, we see that geodesics not only minimize the length (energy) functional but also have parametrization proportional to arc-length.

Hence, by fact 6.19, we can find a reparametrization $\sigma(x,y) = (u(x,y), v(x,y))$ such that

$$\tilde{\phi}_{n.r}(x,y) = \phi_{n,r} \circ \sigma(x,y) = (\tilde{\phi}(x,y), ru(x,y), rv(x,y)) \text{ is conformal,}$$

where $\tilde{\phi} = \phi \circ \sigma$. This implies $A(\tilde{\phi}_{n,r}) = \frac{1}{2}D(\tilde{\phi}_{n,r})$. Clearly, $A(\phi_{n,r})$ is continuous in r, we can choose $\epsilon > 0$ such that $|A(\phi_{n,\epsilon}) - A(\phi_n)| < 1/n$. Thus, we have the following estimation:

$$\frac{1}{2}D(\tilde{\phi}_n) \leq \frac{1}{2}D(\tilde{\phi}_{n,\epsilon}) = A(\tilde{\phi}_{n,\epsilon}) = A(\phi_{n,\epsilon}) \leq A(\phi_n) + 1/n.$$

□

In conclusion, we replace the minimizing problem for $A(\phi)$ to the minimizing problem for $D(\phi)$. Another good feature of $D(\phi)$ is that the Euler–Lagrange equation for it is given by $\triangle\phi = 0$, where $\triangle = \partial_x^2 + \partial_y^2$ is the standard Laplacian. We can then employ the classical theory of harmonic function to solve minimizing problem for $D(\phi)$, known as *Dirichlet principle*.

Fact 6.21 (Dirichlet Principle). Let $b : \partial\Delta \to \mathbb{R}^n$ be a continuous map, and define the admissible class

$$X_b := \{\psi : \Delta \to \mathbb{R}^n : \psi \text{ is piecewise } C^1, \quad \psi\big|_{\partial\Delta} = b\}.$$

Assume that $d_b = \inf_{\psi \in X_b} D(\psi) < \infty$, then there exists unique $\psi_b \in X_b$ with $D(\psi_b) = d_b$. Moreover, ψ_b solves the Dirichlet problem

$$\triangle\psi_b = 0; \quad \psi_b\big|_{\Delta} = b.$$

We only remark an ingredient in the proof for later use and leave the proof of fact 6.21 as exercise for readers. This is called *Harnack's principle*: if u_n is a sequence of harmonic functions which converge uniformly on any compact set of Δ° to a harmonic function u, then

$$D(u) \leq \liminf_{n\to\infty} D(u_n). \tag{6.14}$$

Exercise 6.8. Prove the Dirichlet principle.[13]

[13] One can consult [Law80] p.64–65 for a sketch of proof and [Ahl78], [GT01] for results on harmonic functions.

Even with fact 6.21, we still have not solved the Plateau problem for only the existence of minimizer $\psi_b \in X_b \subset X_\Sigma$ is guaranteed. However, for different parametrization b of Γ, we will in general have different values of d_b. Hence, it remains to find a parametrization b of Γ such that $d_b = d_\Gamma$. In contrast with the situation in $A(\phi)$, notice that we have the following observations.

Exercise 6.9. Let Δ be the (closed) unit disk.

(1) $D(\phi)$ is invariant only under conformal group $\mathrm{Conf}(\Delta)$.
(2) The conformal changes $\mathrm{Conf}(\Delta)$ are all given by Möbius transform $e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$, for some $\theta \in \mathbb{R}$, $\alpha \in \Delta$.
(3) A Möbius transform is uniquely determined by its values on three distinct points on $\partial\Delta$.

Hence, all parametrizations $\phi : \partial_\Delta \to \Gamma$ differ by the "gauge group" $\mathrm{Conf}(\Delta)$ which leaves $D(\phi)$ invariant and we can normalize the parametrization ("gauge fixing") by prescribing values for three distinct points on $\partial\Delta$. We pick three distinct points $p_1, p_2, p_3 \in \Gamma$ and $z_1, z_2, z_3 \in \partial\Delta$ and consider the subclass

$$X'_\Gamma := \{\phi \in X_\Gamma : \phi(z_k) = p_k, \quad k = 1,2,3\} \subset X_\Gamma.$$

By above exercise, we still have $d_\Gamma = \inf_{\phi \in X'_\Gamma} D(\phi)$. The final key is the "compactness theorem" for X'_Γ.

Theorem 6.22 (Courant–Lebesuge). *Let $M > d_\Gamma$, then the family*

$$\mathfrak{F} := \{\phi\big|_{\partial\Delta} : \phi \in X'_\Gamma, D(\phi) \le M\}$$

is equicontinuous on $\partial\Delta$.

Before we prove the theorem, we need a key estimate.

Lemma 6.23. *For $z \in \partial\Delta$, $r > 0$, denote $C_r(z)$ by $\partial B_r(z) \cap \Delta$. For any $\delta \in (0,1)$, any $\phi \in X'_\Gamma$ with $D(\phi) \le M$, there exists $\rho = \rho(\phi)$ with $\delta \le \rho \le \sqrt{\delta}$ such that*

$$l(C_\rho)^2 \le 2\pi\epsilon(\delta),$$

where $\epsilon(\delta) = \frac{2M}{\log(1/\delta)}$ and $l(C_\rho)$ is the length of the curve $\phi\big|_{C_\rho(z)}$.

PROOF. Since the arc–length s for a circle C_r of radius r is given by $r\theta$, $ds = \theta dr + r d\theta$. We then write $D(\phi)$ as

$$D(\phi) = \int_\Delta \left(|\phi_x|^2 + |\phi_y|^2\right) dxdy = \int_\Delta \left(|\phi_x|^2 + |\phi_y|^2\right) dr \wedge ds.$$

Also, from $\begin{cases} x = r\cos\theta = r\cos(s/r) \\ y = r\sin\theta = r\sin(s/r) \end{cases}$, the chain rule implies

$$\phi_s = \phi_x \frac{\partial x}{\partial s} + \phi_y \frac{\partial y}{\partial s} = -\sin(s/r)\phi_x + \cos(s/r)\phi_y.$$

Hence, we have

$$\begin{aligned} |\phi_s|^2 &= |\phi_x|^2 \sin^2(s/r) + |\phi_y|^2 \cos^2(s/r) - 2\phi_x \cdot \phi_y \sin(s/r)\cos(s/r) \\ &\le |\phi_x|^2 + |\phi_y|^2. \end{aligned}$$

We then see that the integral

$$I := \int_\delta^{\sqrt{\delta}} \int_{C_r(z)} |\phi_s|^2 ds dr \le D(\phi) \le M.$$

On the other hand,

$$I = \int_\delta^{\sqrt{\delta}} \left(r \int_{C_r(z)} |\phi_s|^2 ds\right) \frac{dr}{r} = \int_\delta^{\sqrt{\delta}} \left(r \int_{C_r(z)} |\phi_s|^2 ds\right) d(\log r).$$

Since $\log r \nearrow$ and $p(r) := r\int_{C_r(z)} |\phi_s|^2 ds$ is continuous in r even if ϕ is only piecewise C^1, mean-value theorem for Riemann–Stieljes integral implies that $\exists \rho$ with $\delta \le \rho \le \sqrt{\delta}$ such that

$$I = p(\rho) \int_\delta^{\sqrt{\delta}} d\log(r) = p(\rho) \frac{\log(1/\delta)}{2} \le M.$$

Finally, by Cauchy–Schwartz inequality,

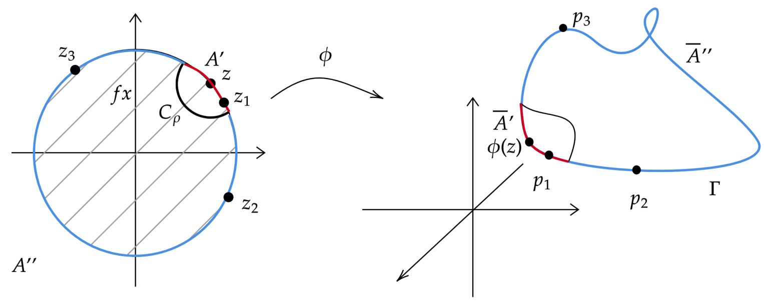
$$\begin{aligned} l(C_\rho)^2 = \left(\int_{C_\rho(z)} |\phi_s| ds\right)^2 &\le \int_{C_\rho(z)} |\phi_s|^2 ds \int_{C_\rho(z)} ds \\ &\le 2\pi\rho \int_{C_\rho(z)} |\phi_s|^2 ds \le 2\pi\epsilon(\delta). \end{aligned}$$

$\square$

PROOF OF THEOREM 6.22. First of all, we observe that

(a) given $e > 0$, there exists $d > 0$ such that for any $p, p' \in \Gamma$ with $0 < |p - p'| < d$, one of the two components $\Gamma \setminus \{p, p'\}$ has diameter$< e$ (since Γ is a continuous image of circle). Particularly, if we choose $e < \min_{i \neq j} |p_i - p_j|$, the component can only contain at most one of p_i's.

(b) We choose $\delta < 1$ such that $\sqrt{2\pi\epsilon(\delta)} < d$ and such that for any $z \in \partial\Delta$, $|z - z_i| > \sqrt{\delta}$ for at least two z_i's.



Now, for any given $\phi \in \mathfrak{F}$, for any $z \in \partial\Delta$, lemma 6.23 shows that $\exists\rho = \rho(\phi)$ with $\delta < \rho < \sqrt{\delta}$ such that $l(C_\rho(z)) < d$. Let A' and A'' be the components of $\partial\Delta$ divided by $C_\rho(z)$ with A' containing z, $\overline{A}'$ and $\overline{A}''$ be their images. By (a), one of $\overline{A'}$ or $\overline{A}''$ has diameter$< e$ for $l(C_\rho(z)) < d$. By the construction in (b), we know that $\operatorname{diam}(A') < d$. In other words, for $|z' - z| < \delta < \rho$, we get

$$|\phi(z) - \phi(z')| < e.$$

However, since δ is independent of z and ϕ, this proves the equicontinuity of $\mathfrak{F}$. $\square$

We then arrive Douglas' theorem on Plateau's problem.

Corollary 6.24. *Let Γ be a Jordan curve in $\mathbb{R}^n$ such that $G_\Gamma < \infty$. Then there exists a continuous map $\phi : \Delta \to \mathbb{R}^n$ such that*

(1) *$\phi\big|_{\partial\Delta}$ maps $\partial\Delta$ monotonically onto Γ.*
(2) *$\phi\big|_{\Delta^\circ}$ is harmonic and almost conformal.*
(3) *$D(\phi) = d_\Gamma$ and $A(\phi) = G_\Gamma$.*

PROOF. Let b_n be a sequence in $\mathfrak{F}$ such that $\lim_{n\to} b_n = d_\Gamma$. By theorem 6.22 and Arzela–Ascoli theorem, there exists a subsequence

$\{b_{n_k}\}_{k=1}^{\infty}$ which converges uniformly to some $b \in \mathfrak{F}$. From Harnack's principle (6.14), we have

$$D(\phi_b) \leq \liminf_{j\to\infty} D(\phi_{b_{n_j}}) = d_\Gamma.$$

Hence, $D(\phi_b) = d_\Gamma$. $\square$

Exercise 6.10. Show that in the case of $\Gamma \subset \mathbb{C}$ (i.e. $n = 2$), this gives a proof of Riemann mapping theorem for domain Ω bounded by Γ.

We emphasis again that Douglas' solution is the area minimizer among the competing class, which may not has minial area among surfaces of all topological type (Exercise!). Particularly, this shows that Dirichlet problem for minimal surfaces equation is not equivalent to Plateau's problem. This also shows non–uniqueness for the solution of Plateau's problem[14]. Furthermore,the Dirichlet problem for minimal hypersurface equation (cf. (6.2)) may not be even solvable. The solvability of the Dirichlet problem is characterized by:

Theorem 6.25 (Jenkins–Serrin, 1968, see [GT01], Theorem 14.4). *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$-domain, then the Dirichlet problem for any $\phi \in C^{2,\alpha}(\overline{\Omega})$,*

$$\begin{cases} \sum_i \partial_i \left(\frac{\partial_i f}{\sqrt{1+|\nabla f|^2}} \right) = 0 \\ f|_{\partial\Omega} = \phi|_{\partial\Omega} \end{cases}$$

is solvable if and only if the mean curvature $H_{\partial\Omega} \geq 0$[15] (In fact, we need only Ω is C^2 and $\phi \in C^0(\partial\Omega)$ by a limiting process).

More generally, we can consider the prescribing mean curvature equation:

$$H = \frac{M_u}{(1+|\nabla u|^2)^{3/2}}, \quad \text{or } M_u = H(1+|\nabla u|^2)^{3/2}.$$

where $M_u := (1+|\nabla u|^2)\triangle u_i - u_j u_{ij}$. In this case, we have

[14]For more thorough discussion on various generalization, regularity, and uniqueness of Plateau problem, we refer to [Law80]. For an overview on development and open problems on the subject, one can consult [HP16].

[15]For $n = 2$, this is equivalent to Ω is convex.

Theorem 6.26 ([GT01], Theorem 16.10,11)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^0-domain, $\phi \in C^0(\partial\Omega)$. For $H \in C^1(\overline{\Omega})$ with "some suitable conditions", the Diriechlet problem*

$$\begin{cases} M_u = H(1+|\nabla u|^2)^{3/2} \\ u\big|_{\partial\Omega} = \phi \end{cases}$$

is uniquely solvable. Particularly, if H is constant, the "suitable condition" is equivalent to $H_{\partial\Omega} \geq H$.

Exercise 6.11. Find an example of Jordan curve in $\mathbb{R}^3$ in which the Douglas' solution is not area minimizer.

6.5. Complex Manifolds and Almost Complex Structure

In the remaining of this chapter, we study another important class of minimal submanifolds, the complex submanifolds of Kähler manifolds. For the completeness, we shall start with the notion of an almost complex manifold and a complex manifold.

Definition 6.27. Let M be a smooth manifold of dimension $2n$. A smooth atlas $\{U_i, \phi_i\}_{i\in I}$ on M is called a *holomorphic atlas* if the transition function

$$\phi_{ij} := \phi_j \circ \phi_i^{-1} : \phi_i(U_i) \subset \mathbb{C}^n \to \phi_j(U_j) \subset \mathbb{C}^n$$

is a bi-holomorphic map[16], for any $i, j \in I$. Each (U_i, ϕ_i) is called a *holomorphic chart* and a maximal holomorphic atlas is called a *complex structure* on M. A *complex manifold* of (complex) dimension n is a smooth manifold of dimension $2n$ together with a complex structure.

Clearly, if M is a complex manifold of dimension n, then $\forall p \in M$, $V := T_pM \cong \mathbb{C}^n$. The multiplication by i gives an endomorphism $i : V \to V$ which satisfies $i^2 = -I_V$. Conversely, given any $V \cong \mathbb{R}^{2n}$

[16]For open subset $U \subset \mathbb{C}^n$, a function $f : U \to \mathbb{C}$ is *holomorphic* if it is holomorphic in each variable. A non-trivial theorem of Hartog (cf. [Hör90] Theorem 2.2.8) asserts that f is holomorphic if and only if it is (complex) analytic. Similarly, a map $f : U \to \mathbb{C}^m$ is holomorphic if each component is holomorphic.

with $J : V \to V$ satisfying $J^2 = -I_V$, then V has a $\mathbb{C}$-vector space structure given by

$$(a+bi)\cdot v := av + bJv, \quad \forall v \in V, a, b \in \mathbb{R}.$$

This motivates the following definition.

Definition 6.28. Let M be a smooth manifold of dimension n. A tensor J of type $(1,1)$, regarded as $J \in C^\infty(\mathrm{End}(TM))$, such that for any $p \in M$, $J_p : T_pM \to T_pM$ such that $J_p^2 = -\mathrm{Id}_{T_pM}$ is called a *almost complex structure* on M. We call (M, J) a *almost complex manifold*.

The existence of almost complex structure has the following immediate consequence.

Exercise 6.12. Show that if M admits an almost complex structure, then M must be orientable and have even dimension[17].

Obviously, if M is a complex manifold, then holomorphic coordinate $\mathbf{z} = (z^1, \ldots, z^n)$ with $z^j = x^j + iy^j$ gives a natural choice of almost complex structure J by multiplication by i:

$$J(\partial/\partial x^j) = \partial/\partial y^j; \quad J(\partial/\partial y^j) = -\partial/\partial x^j. \tag{6.15}$$

Therefore, $(M, J = i)$ is of course an almost complex manifold.

In general, we say an almost complex structure J on M is *integrable*[18] if it is induced from a complex structure on M as above. The

[17]The converse is more difficult. For instance, Ehresmann and Hopf proved that S^4 does not admit any almost complex structure. The existence of almost complex structure is a *topological question* in nature. One way to prove the reulst of Ehresmann and Hopf is by characteristic classes and Hirzerbruch signature theorem discussed in later chapters, cf. Problem 7.4

[18]The integrability of almost complex structure is a highly non–trivial problem. If (M, g) is Riemannian 2-manifold, then theorem 6.3 establishes the existences of complex structure on surfaces. In four dimension, using deeper theory in complex surface, one can show that there exists compact almost complex manifold which admits no complex structure. For higher dimension, a well–known open problem is that whether S^6 admits integrable complex structure. More generally, for $m \geq 3$, it is not known whether there exists an almost complex manifold M^{2m} which admits no integrable complex structure.

integrability of an almost complex structure J on M is closely related to a $(1,2)$-tensor N_J known as *Nijenhuis tensor*, which is defined by

$$N_J(Y,Z) := J([JY,JZ]) + [JY,Z] + [Y,JZ] - J([Y,Z]).$$

Lemma 6.29. *If M is a complex manifold and J is the induced complex structure via* (6.15)*, then $N_J \equiv 0$.*

Exercise 6.13. Prove lemma 6.29.

The converse of lemma 6.29 is a deep theorem in PDE. We refer the proof to the original source [NN57] or the monograph [Hör90].

Theorem 6.30 (Newlander–Nirenberg)**.** *An almost complex structure J is integrable if and only if $N_J \equiv 0$.*

Similar to the real case, a map $f : M^m \to N^n$ between two complex manifolds is called a *holomorphic map* if

$$\mathbf{y} \circ f \circ \mathbf{x}^{-1} : \mathbf{x}(f^{-1}(V) \cap U) \subset \mathbb{C}^m \to \mathbf{y}(V) \subset \mathbb{C}^n$$

is *holomorphic* for any choice of holomorphic charts $(U, \mathbf{x})$ on M and $(V, \mathbf{y})$ on N. Particularly, a holomorphic map $f : M \to \mathbb{C}$ is called a *holomorphic function*. We denote $\mathcal{O}(U)$ by the space of holomorphic function over open set $U \subset M$. For almost complex manifolds, a C^∞ map of almost complex manifolds $f : (M, J_M) \to (N, J_N)$ is *(pseudo) holomorphic* or J-holomorphic if

$$df \circ J_M = J_N \circ df.$$

Exercise 6.14. If M, N are complex manifolds, holomorphic maps and pseudo-holomorphic maps are equivalent. Deduce that an integrable almost complex structure J must induced by a unique complex structure.

For an almost complex manifold (M, J), we can extend J to an endomorphism on the complexified tangent bundle $TM \otimes \mathbb{C}$, which we still denote by J. Since $J^2 = -id_{TM\otimes\mathbb{C}}$, we have the decomposition

$$TM \otimes \mathbb{C} = T^{1,0}M \bigoplus T^{0,1}M, \tag{6.16}$$

where $T^{1,0}M$ is the i-eigenspace of J and $T^{0,1}M$ is the $(-i)$-eigenspace of J. We call $T^{1,0}M$ the *holomorphic tangent bundle* and $T^{0,1}M$ the *anti-holomorphic tangent bundle*. Dually, on the complexified cotangent bundle $T^*M \otimes \mathbb{C}$ J induces an endomorphism $J^t \in \mathrm{End}(T^*M \otimes \mathbb{C})$ with $(J^t)^2 = -id_{T^*M\otimes\mathbb{C}}$. Likewise, we have the eigenspace decomposition into holomorphic/anti-holomorphic cotangent bundle:

$$T^*M \otimes \mathbb{C} = T^{*1,0}M \bigoplus T^{*0,1}M$$

and the bidegree decomposition on the exterior bundles

$$\bigwedge^k T^*M \otimes \mathbb{C} = \bigoplus_{p+q=k} \bigwedge^{p,q} T^*M, \tag{6.17}$$

where $\bigwedge^{p,q} T^*M = \bigwedge^p(T^{*1,0}M) \otimes \bigwedge^q(T^{*0,1}M)$. Moreover, from (6.17), the space $A^k(M,\mathbb{C})$ of $\mathbb{C}$-valued smooth k-forms on M decomposes into

$$A^k(M,\mathbb{C}) \otimes \mathbb{C} = \bigoplus_{p+q=k} A^{p,q}(M), \tag{6.18}$$

where $A^{p,q}(M) = C^\infty(\bigwedge^{p,q} T^*M)$. We denote $\pi_{p,q} : A^k(M) \to A^{p,q}(M)$ be the projection.

Exercise 6.15. Let (M,J) be an almost complex manifold.

(1) Let $I : TM \otimes \mathbb{C} \to TM \otimes \mathbb{C}$ be the fiberwise conjugation. Show that $I : T^{1,0} \to T^{0,1}$ is an anti-linear isomorphism.
(2) Show that $T^*M \otimes \mathbb{C} = (TM \otimes \mathbb{C})^* := \mathrm{Hom}_{\mathbb{C}}(TM,\mathbb{C})$. Similarly, $T^{*1,0}M = (T^{1,0}M)^*$ and $T^{*0,1}M = (T^{0,1}M)^*$.

Recall that in Chapter 2, we define Cartan's exterior operator $d : A^k(M) \to A^{k+1}(M)$, we extend d to $\mathbb{C}$-valued differential forms on M. With (6.18), we define

$$\partial = \pi_{p+1,q} \circ d : A^k(M) \to A^{p+1,q}(M)$$
$$\bar\partial := \pi_{p,q+1} \circ d : A^k(M) \to A^{p,q+1}(M).$$

In general, $d = \bigoplus_{p+q=k} \pi_{p,q} \circ d \neq \partial + \bar\partial$. In fact, $d = \partial + \bar\partial$ is equivalent to the integrability of J (cf. problem 6.7).

Now, we investigate the case when M is a complex manifold. In this case, J is given by the (6.15). One can check readily that the complex coordinate[19] $z^j = x^j + iy^j$its conjugate $\bar{z}^j = x^j - iy^j$ gives a local frame $dz^j = dx^j + idy^j$ for $T^{*1,0}M$ and $d\bar{z}^j = dx^j - idy^j$ for $T^{*0,1}M$, respectively. Moreover, the dual frame of dz^j and $d\bar{z}^j$ is given by

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right); \quad \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right).$$

Exercise 6.16. Let M be a complex manifold.

(1) Verify that $\partial/\partial z^j$ is the dual frame of dz^j and $\partial/\partial z^j$ forms a local frame for $T^{1,0}M$. Similarly, check for $\partial/\partial\bar{z}^j$.

(2) Show that $TM \cong T^{1,0}M$ as a real vector bundle.

Remark 6.31. From above exercise, one can show that $TM \cong T^{1,0}M$ as a real vector bundle and even isomorphic as a complex vector bundle if we identify J as i. Yet, We still regard them differently via (6.16).

More generally, on any holomorphic local chart $(U, \mathbf{z})$ of M, $\{dz^I \wedge d\bar{z}^J : |I| = p, |J| = q\}$ is a basis for $\bigwedge^{p,q} T^*M\big|_U$. Thus, we have

$$d(f dz^I \wedge d\bar{z}^J) = \left(\frac{\partial f}{\partial z^j} dz^j + \frac{\partial f}{\partial \bar{z}^j} d\bar{z}^j\right) \wedge dz^I \wedge d\bar{z}^J.$$

Hence, we see that $d = \partial + \bar{\partial}$. From $d^2 = 0 = (\partial + \bar{\partial})^2 = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2$ and bidegree decomposition, we see that

$$\partial^2 = 0 = \bar{\partial}^2; \quad \partial\bar{\partial} = -\bar{\partial}\partial.$$

Clearly, a C^∞ function $f : U \to \mathbb{C}$ is holomorphic if $\bar{\partial} f = 0$.

6.6. Hermitian Metrics and Kähler Manifolds

Let (M^{2n}, J) be an almost complex manifold. Let g be a Riemannian metric on M. We say g is a *hermitian metric* if $\forall p \in M$

$$g_p(J_p v, J_p w) = g_p(v, w), \forall v, w \in T_p M.$$

[19]We will interchangeably denote the complex coordinate by $z^j = x^j + y^j$ and by $z^j = x^{2j-1} + x^{2j}$.

Notice that such metric always exists for given any Riemannian metric g, we can define a hermitian metric h by $h(v,w) := g(v,w) + g(Jv, Jw)$.

Let ∇ be the Levi-Civita connection with respect to a hermitian metric g. Since g is J–compatible, it is natural to ask whether J is parallel with respect to ∇, that is, whether $\nabla J = 0$? From $\nabla_V(JW) = (\nabla_V J)(W) + J\nabla_V W$, we see that $\nabla J = 0$ is equivalent to the commutativity of J and covariant derivative: $\nabla_V(JW) = J\nabla_V W$. We will see shortly that this is a non–trivial condition which in fact implies J is integrable.

Definition 6.32. (M, J, g) is called a **Kähler manifold** if $\nabla J = 0$.

Another point of view on Kähler condition is called the *symplectic* viewpoint.

Definition 6.33. A C^∞ manifold M is called a **symplectic manifold** if there exists $\omega \in A^2(M)$ such that $\forall p \in M$, ω_p is a non–degenerate skew-symmetric bilinear form on T_pM and $d\omega = 0$. The closed 2-form ω is called a *symplectic form* on M.

Given (M, J, g), the *fundamental* 2-*form* ω associated to (M, J, g) is given by

$$\omega(v,w) := g(Jv, w).$$

Since $\omega(w,v) = g(Jw,v) = g(J^2w, Jv) = -g(Jv,w) = -\omega(v,w)$, we have $\omega \in A^2(M)$. A fundamental theorem of Kähler manifolds states the following.

Theorem 6.34. *$\nabla J = 0$ if and only if $d\omega = 0$ and J is integrable.*

PROOF. The heart of proof lies in the following:

Claim 6.35.

(a) $d\omega(X,Y,Z) = g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) - g((\nabla_Z J)Y, X)$.
(b) $d\omega(X,Y,Z) - d\omega(X, JY, JZ) = 2g((\nabla_X J)Y, Z) + g(N_J(Y,Z), X)$.

It is easy to see that the theorem follows from the identities (a) and (b). It remains to establish the claim.

For (a), from theorem 2.13, we have

$$\begin{aligned} d\omega(X,Y,Z) = & X(\omega(Y,Z)) - Y(\omega(X,Z)) + Z(\omega(X,Y)) \\ & -\omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X). \end{aligned}$$

We locally extend tangent vectors X, Y, Z to a vector fields and write $[X,Y] = (\nabla_X Y - \nabla_Y X)$. Moreover, it suffices to prove for X, Y, Z to be coordinate vectors of normal coordinate for (M, g) at p. Thus, we have for instance $(\nabla_X Y)(p) = 0$ and $[X,Y](p) = 0$. Observe that we then have $g((\nabla_X J)Y, Z) = -g((\nabla_X J)Z, Y)$ (cf. exercise 6.17 below). From this, we then compute

$$\begin{aligned} d\omega(X,Y,Z) =& X(g(JY,Z)) - Y(g(JX,Z)) + Z(g(JX,Y)) \\ =& g((\nabla_X J)Y,Z) - g((\nabla_Y J)X,Z) + g((\nabla_Z J)X,Y) \\ =& g((\nabla_X J)Y,Z) + g((\nabla_Y J)Z,X) - g((\nabla_Z J)Y,X) \\ =& g((\nabla_X J)Y,Z) + g(\nabla_Y(JZ),X) - g(J(\nabla_Y Z),X) \\ & -g(\nabla_Z(JY),X) + g(J(\nabla_Z Y),X). \end{aligned}$$

Notice that $J^2 = -id$ implies for any vector field X,

$$(\nabla_X J)J + J(\nabla_X J) = 0 \quad \text{as endomorphism on } TM.$$

Thus, we have

$$\begin{aligned} d\omega(X,JY,JZ) &= g((\nabla_X J)JY,JZ) + g((\nabla_{JY} J)JZ,X) - g((\nabla_{JZ} J)JY,X) \\ &= -g(J(\nabla_X J)Y,JZ) + g((\nabla_{JY} J)JZ,X) - g((\nabla_{JZ} J)JY,X) \\ &= -g((\nabla_X J)Y,Z) + g((\nabla_{JY} J)JZ,X) - g((\nabla_{JZ} J)JY,X) \\ &= -g(\nabla_X J)Y,Z) - g(\nabla_{JY} Z,X) - g(J(\nabla_{JY} JZ),X) \\ & \quad + g(\nabla_{JZ} Y,X) + g(J(\nabla_{JZ} JY),X). \end{aligned}$$

Thus, we obtain (b):

$$\begin{aligned}
&d\omega(X,Y,Z) - d\omega(X,JY,JZ)\\
=\,& 2g(\nabla_X J)Y,Z) + g(\nabla_Y(JZ),X) - g(\nabla_{JZ}Y,X)\\
&- g(J(\nabla_Y Z),X) + g(J(\nabla_Z Y),X)\\
&- g(\nabla_Z(JY),X) + g(\nabla_{JY}Z,X)\\
&+ g(J(\nabla_{JY}JZ),X) - g(J(\nabla_{JZ}JY),X)\\
=\,& 2g(\nabla_X J)Y,Z) + g(N_J(Y,Z),X).
\end{aligned}$$

□

Exercise 6.17. Show that $g((\nabla_X J)Y,Z) = -g((\nabla_X J)Z,Y)$.

Combining theorem 6.34 and Newlander–Nirenberg theorem, we see that Kähler manifolds are all complex manifolds. Hence, a Kähler manifold equips with the structure of complex, symplectic, and Riemannian manifold.

Remark 6.36. In the proof of the theorem, we see that the condition $\nabla J = 0$ is stronger than $N_J = 0$. It is an interesting to know whether one can prove the integrability of almost complex structure J without invoking Newlander–Nirenberg theorem.

Now, for any Riemannian metric g on a complex manifold M, we can then extend g to a $\mathbb{C}$-bilinear form $\widetilde{g}$ on $TM \otimes \mathbb{C}$ by:

$$\widetilde{g}(v+iv', w+iw') = g(v,w) - g(w,w') + i(g(v',w) + g(v',w)).$$

Hence, in holomorphic coordinate $\mathbf{z}$, we can write

$$\widetilde{g} = \widetilde{g}_{\alpha\beta} dz^\alpha \otimes dz^\beta + \widetilde{g}_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta + \tilde{g}_{\bar{\alpha}\beta} d\bar{z}^\alpha \otimes dz^\beta + \tilde{g}_{\bar{\alpha}\bar{\beta}} d\bar{z}^\alpha \otimes d\bar{z}^\beta.$$

Notice that for $\tilde{g}$, we always have

$$\begin{aligned}
\widetilde{g}_{\alpha\bar{\beta}} &= \frac{1}{4}\left[g(\partial_{x^\alpha},\partial_{x^\beta}) + g(\partial_{y^\alpha},\partial_{y^\beta}) + i(g(\partial_{x^\alpha},\partial_{y^\beta}) + g(\partial_{x^\alpha},\partial_{y^\beta}))\right]\\
&= \frac{1}{2}\left(g_{\alpha\beta} + i g_{\alpha,n+\beta}\right) = \overline{\widetilde{g}_{\bar{\alpha}\beta}},
\end{aligned}$$

since $g_{\alpha,n+\beta} = -g_{\beta,n+\alpha}$. Here, we denote $g_{\alpha\beta} = g(\partial_{x^\alpha},\partial_{x^\beta})$ and we identify ∂_{y^β} as $\partial_{x^{\beta+n}}$.

In case of g is hermitian[20], we have:

Lemma 6.37. *g is hermitian if and only if $\widetilde{g}_{\alpha\beta} = 0 = \widetilde{g}_{\bar{\alpha}\bar{\beta}}$.*

PROOF. We denote $\partial/\partial z^\alpha$ and $\partial/\partial z^\beta$ by ∂_α and ∂_β. By definition,

$$\widetilde{g}_{\alpha\beta} = \widetilde{g}(\partial_\alpha, \partial_\beta) = -\widetilde{g}(i\partial_\alpha, i\partial_\beta) = -\widetilde{g}(J(\partial_\alpha), J(\partial_\beta)) = -\widetilde{g}(\partial_\alpha, \partial_\beta).$$

Hence, $\widetilde{g}_{\alpha\beta} = 0$. Similar proof shows $\widetilde{g}_{\bar{\alpha}\bar{\beta}} = 0$. Next, for $\widetilde{g}_{\alpha\bar{\beta}}$,

$$\begin{aligned}\widetilde{g}(\partial_\alpha, \partial_{\bar{\beta}}) &= \frac{1}{4}\left(g(\partial_{x^\alpha} - i\partial_{y^\alpha}, \partial_{x^\beta} + i\partial_{y^\beta})\right) \\ &= \frac{1}{4}\left[g(\partial_{x^\alpha}, \partial_{x^\beta}) + g(\partial_{y^\alpha}, \partial_{y^\beta}) + i(g(\partial_{x^\alpha}, \partial_{y^\beta}) - g(\partial_{y^\alpha}, \partial_{x^\beta}))\right]\end{aligned}$$

□

With $\mathbb{C}$-extension $\widetilde{g}$ on $TM \otimes \mathbb{C}$, we can define the *usual hermitian metric h* on holomorphic tangent bundle $T^{1,0}M$ via

$$h(v, w) := \widetilde{g}(v, \overline{w}), \quad \text{for } v, w \in T^{1,0}_p M.$$

Hence, in local holomorphic coordinate $\mathbf{z}$, $h = h_{\alpha\beta} dz^\alpha \otimes d\bar{z}^\beta$, where $h_{\alpha\beta} = \widetilde{g}_{\alpha\bar{\beta}}$. For the fundamental 2-form ω, we extend it $\mathbb{C}$ by $\omega(u, v) = \widetilde{g}(Ju, v)$. In local holomorphic coordinate $\mathbf{z}$, $\partial_\alpha := \partial/\partial z^\alpha$,

$$\begin{aligned}\omega(\partial_\alpha, \partial_\beta) &= \widetilde{g}(J\partial_\alpha, \partial_\beta) = i\widetilde{g}(\partial_\alpha, \partial_\beta) = 0 \\ \omega(\partial_\alpha, \partial_{\bar{\beta}}) &= i\widetilde{g}(\partial_\alpha, \partial_{\bar{\beta}}) = i\widetilde{g}_{\alpha\bar{\beta}}.\end{aligned}$$

Hence, we have the local expression for fundamental 2-form

$$\omega = i\widetilde{g}_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta. \tag{6.19}$$

From lemma 6.37 and (6.19), we see that given any one of ω, $\widetilde{g}$, and g, one can determine the others. We call the fundamental 2-form ω a *hermitian form* on the complex manifold M or a *Kähler form* if M is furthermore Kähler.

Exercise 6.18.

(1) Show that a $(1,1)$-form ω is a hermitian form if and only if it is a real 2-form and positive in the sense that for any $p \in M$,

$$-i\omega(u, \bar{u}) > 0, \quad \forall u \in T^{1,0}_p M.$$

[20] We call a complex manifold M with a hermitian metric g a *hermitian manifold*.

(2) Show that on any hermitian manifold (M, ω), the Riemannian volume form of M^m is given by $\omega^m/m!$.

Here are some simplest examples for Kähler manifold.

Example 6.38. Let $M = \mathbb{C}^n$, $g = \sum_{j=1}^{2n} dx_i^2 = \sum_{i=1}^{n} dx_i^2 + dy_i^2$. The standard complex structure on $\mathbb{C}^n$ is given by rotating each copies of $\mathbb{C}$ by $\pi/2$ (i.e. multiplication by i). Thus, we have

$$J(\partial/\partial x^j) = \partial/\partial y^j; J(\partial/\partial y^j) = -\partial/\partial y^j.$$

Thus, the fundamental 2-form is given by

$$\omega(\partial_{x^j}, \partial_{x^j}) = g(J\partial_{x^j}, \partial_{x^j}) = g(\partial_{y^j}, \partial_{x^j}) = 0; \quad \omega(\partial_{x^j}, \partial_{y^j}) = g(\partial_{y^j}, \partial_{y^j}) = \delta_{ij}.$$

Hence, $\omega = \sum_{j=1}^{n} dx_j \wedge dy_j$. Clearly, $d\omega = 0$ and thus $\mathbb{C}^n$ is Kähler.

In contrast of C^∞ case where we have Whitney embedding theorem, we have:

Exercise 6.19. Any complex submanifold M of $\mathbb{C}^n$ has to be non–compact unless it is finite points.

Example 6.39. Recall in chapter 5, we have defined the projective space $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times$, which can be regarded as the set of all lines through 0 in $\mathbb{C}^{n+1}$. The *homoegenous coordinate* $[Z_0 : \cdots : Z_n]$ is the equivalence class representing $(Z_0, \ldots, Z_n)$. $\mathbb{CP}^n$ has a standard chart $U_i := \{[Z_0 : \cdots : Z_n] \in \mathbb{P}^n : Z_i \neq 0\}$ with

$$U_i \to \mathbb{C}^n$$
$$[Z_0 : Z_1 : \cdots : Z_n] \mapsto (z_1, \ldots, z_n) = (Z_0/Z_i, \ldots, \widehat{Z_i/Z_i}, \ldots, Z_n/Z_i)$$

One can show that $\mathbb{CP}^n$ is a compact complex manifold. On $\mathbb{CP}^n$, we have the *Fubini–Study metric*, which is usually given by its fundamental 2-form:

$$\begin{aligned}\omega_{FS} := &\frac{i}{2\pi}\partial\bar{\partial}\log|Z|^2 \quad \text{where } |Z|^2 = \sum_{j=0}^{n} Z_j\bar{Z}_j \\ = &\frac{i}{2\pi}\partial\bar{\partial}\log(1+|w|^2) quad \text{ in a coordinate system } (U_i, \mathbf{w}) \\ = &\frac{i}{2\pi}\partial\frac{w_j d\bar{w}^j}{1+|w|^2} \\ = &\frac{i}{2\pi}\frac{(\delta_{ij}(1+|w|^2) - \bar{w}_i w_j) dw^i \wedge d\bar{w}^j}{(1+|w|^2)^2}.\end{aligned}$$

Clearly, $d\omega_{FS} = (\partial + \bar{\partial})\omega_{FS} = 0$. Thus, $\mathbb{CP}^n$ is Kähler provided that ω_{FS} defines a hermitian structure on M (cf. Exercise below).

Exercise 6.20.

(1) Show that $\mathbb{CP}^n$ is compact complex manifold.
(2) Show that ω_{FS} indeed defines a hermitian structure on M.
(3) Show that $\int_{\mathbb{CP}^n} \omega_{FS}^n = 1$.

Most known examples of Kähler manifolds are constructed based on this fact.

Lemma 6.40. *Let $M \looparrowright N$ be a holomorphic immersion of N. If N is Kähler, then M is Kähler.*

PROOF. Let g, ω be Kähler metric and Kähler form on N. Then the induced metric i^*g has a fundamental 2-form $\widetilde{\omega} = i^*\omega$. Hence, $d\widetilde{\omega} = di^*\omega = i^*d\omega = 0$ is closed, and thus M is Kähler. $\square$

Using above lemma, we can construct Kähler manifolds which are closely related to algebraic geometry.

Example 6.41. Given k holomorphic functions $f_1, \ldots, f_k$ on $\mathbb{C}^n$, the zero locus $M = \{z \in \mathbb{C}^n : f_j(z) = 0, \quad i = 1, \ldots, k\}$ is a Kähler manifold if M is smooth. If $f_j \in \mathbb{C}[z_1, \ldots, z_n]$, we call M an *affine variety*.

Example 6.42. Let $f_1, \ldots, f_k \in \mathbb{C}[z_0, \ldots, z_n]$ be homogenous polynomials. Though $f_1, \ldots, f_k$ are not functions on $\mathbb{CP}^n$, the zero locus

$$M := V(f_1, \ldots, f_k) := \{[Z] \in \mathbb{CP}^n : f_j(Z) = 0, \quad j = 1, \ldots, k$$

is still well–defined. M is called a *projective variety*. If M is a manifold, then we call M a *projective manifold*. Therefore, any projective manifold is compact Kähler[21].

On the other hand, we now show a simple topological constraint for compact Kähler manifolds. Recall that $\omega^m / m!$ is a canonical volume form on a Kähler manifold (M^m, ω). Therefore, if M is compact Kähler manifold and $[\omega^k] = 0 \in H^{2k}_{\mathrm{dR}}(M)$ for some $k = 1, \ldots, m$,then $[\omega] = 0 \in H^2_{\mathrm{dR}}(M)$. By Stokes' theorem, we have

$$\mathrm{vol}(M) = \int_M \frac{\omega^n}{n!} = 0,$$

which leads to a contradiction. Thus, we have proved:

Corollary 6.43. *If M^m is a compact Kähler manifold[22], then betti number $b_{2k} > 0$, for any $k = 1, \ldots, m$.*

In fact, there exists complex manifold with $b_2 = 0$ which must be not a Kähler manifold. In problem 6.14 and 6.16, we investigate another topological constraint on Kähler manifolds and another example for non–Kähler complex manifold,

6.7. Minimality and Calibration

Let $i : M^m \looparrowright \overline{M}^{m+k}$ be a holomorphic immersion. Then there exists an orthonormal basis $\{e_1, Je_1, \ldots, e_m, Je_m\}$ for T_pM such that it

[21] We mention two fundamental theorems in complex geometry. First of all, a theorem of Chow (cf. [GH94], p.167) asserts that any complex submanifold of $\mathbb{CP}^n$ is in fact a projective manifold. Secondly, Kodaira's embedding theorem (cf. [GH94], p.181) states that a compact Kähler manifold (M, ω) which can be embedded into $\mathbb{CP}^n$ for some $n \in \mathbb{N}$ if and only if the de Rham class of Kähler form $[\omega]$ lies in $H^2(M, \mathbb{Q})$.

[22] In fact, as seen from the proof, the statement holds for any compact symplectic manifold.

completes to a basis $\{e_1, Je_1, \ldots, e_{m+k}, Je_{m+k}\}$ of $T_p\overline{M}$. By definition of the second fundamental form,

$$B(e_i, e_i) + B(Je_i, Je_i) = \left(\nabla_{e_i} e_i + \nabla_{Je_i} Je_i\right)^N, \quad i = 1, \ldots, m.$$

Notice that if $\overline{M}$ is Kähler, then M is also Kähler. Also, $\nabla_J = 0$ and J is compatible with metrics. We then have

$$B(u, Jv) = (\nabla_u(Jv))^N = (J\nabla_u v)^N = J((\nabla_u v)^N) = J(B(u, v)).$$

Therefore, B is J–bilinear, for B is symmetric in u and v. This implies that $B(u, v) + B(Ju, Jv) = 0$. As a result, the mean curvature

$$\overrightarrow{H} = \sum_{k=1}^{m} [B(e_k, e_k) + B(Je_k, Je_k)] = 0.$$

This shows that M is minimal. In fact, the preceding argument shows that we do not really need $\overline{M}$ to be Kähler. Instead, we need only the immersion satisfying

$$B(u, v) + B(Ju, Jv) = 0, \quad u, v \in T_pM. \tag{6.20}$$

Definition 6.44. Let M be a Kähler manifold with complex structure J, $\overline{M}$ be a Riemannian manifold. A smooth immersion $i : M \looparrowright \overline{M}$ is a *pluri-harmonic map* if it satisfies (6.20).

Exercise 6.21. Show that if $i : M \looparrowright \overline{M}$ is a pluriharmonic map, then it has *curvature decreasing property* of the following sense:

$$\begin{aligned}&\overline{R}(e, u, e, v) + \overline{R}(Je, u, Je, v)\\ =&R(e, u, e, v) + R(Je, u, Je, v) + 2\langle B(e, u), B(e, v)\rangle\end{aligned}$$

Next, we show that complex submanifolds of Kähler manifolds are not only minimal but actually volume minimizer.

Theorem 6.45. *Any complex submanifold M^m of a Kähler manifold $\overline{M}$ is a stable minimal submanifold which minimize volume in its homology class $[M] \in H_{2m}(\overline{M}, \mathbb{Z})$ in the sense that any smooth submanifold $M' \subset \overline{M}$, $M = M'$ outside a compact set and $[M] = [M'] \in H_{2m}(\overline{M}, \mathbb{Z})$, $\mathrm{vol}(M) \leq \mathrm{vol}(M')$ and equality holds if and only if M' is also a complex submanifold of $\overline{M}$.*

The proof of the theorem 6.45 is based on the following elementary yet important inequality of Wirtinger.

Lemma 6.46 (Wirtinger). *Let M^{2m} be a real oriented submanifold of a Kähler manifold $\overline{M}$ with Käher form ω. Let dV be the induced volume form of ω on M. Then for any $p \in M$,*

$$\frac{\omega^m}{m!}\Big|_{T_pM} \le dV_p,$$

and the equality holds if and only if the subspace $T_pM \subset T_p\overline{M}$ is J-invariant.

PROOF. For any unit vector $u, v \in T_p\overline{M}$,

$$\omega(u,v)^2 = \langle Ju, v\rangle^2 \le |Ju|^2|v|^2 = |u|^2|v|^2,$$

and the equality holds if and only if $Ju = \pm v$. That is, Span$\{u, v\}$ is a complex 1-dimension subspace of $T_p\overline{M}$ (but may have reverse orientation with $T_p\overline{M}$[23]). We now consider $\omega' := \omega\big|_{T_pM}$. Since ω' is a skew–symmetric bilinear form on T_pM, there exists an oriented orthonormal basis $e_1, \ldots, e_{2m}$ of T_pM such that ω' is represented by

$$\begin{pmatrix} 0 & \lambda_1 & & & & & \\ -\lambda_1 & 0 & & & & & \\ & & 0 & \lambda_2 & & & \\ & & -\lambda_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \lambda_m \\ & & & & & -\lambda_m & 0 \end{pmatrix},$$

where $\lambda_k = \omega(e_{2k-1}, e_{2k}) = g(Je_{2k-1}, e_{2k})$ and $|\lambda_k| \le 1$, for $k = 1, \ldots, m$. Equivalently, if $\theta_1, \ldots, \theta_{2m}$ are dual basis of $\{e_i\}$ on T_p^*M, then

$$\omega' = \sum_{k=1}^m \lambda_k \theta_{2k-1} \wedge \theta_{2k}.$$

[23]If the orientation on M is compatible with the orientation on $\overline{M}$, then the case -1 in the proof will not occur. In this case, T_pM is a complex vector subspace of

Thus, the m-fold exterior product ω' is given by

$$(\omega')^m = m!\lambda_1 \dots \lambda_m \theta_1 \wedge \dots \wedge \theta_{2m} = m!\lambda_1 \dots \lambda_m dV_p.$$

We then obtain $\left|\frac{(\omega')^m}{m!}\right| \leq dV_p$ and the equality holds if and only if $|\lambda_i| = 1$, for all $k = 1, \dots, m$. By above argument, we know that $|\lambda_k| = 1$ if and only if $J(e_{2k-1}) = \pm e_{2k}$. □

Now, we can give theorem 6.45 the famous one line proof.

PROOF OF THEOREM 6.45. If M' is homologous to M, then by Wirtinger's inequality and

$$\text{vol}(M) = \int_M \frac{\omega^m}{m!} = \int_{M'} \frac{\omega^m}{m!} \leq \int_{M'} dV_{M'} = \text{vol}(M'), \tag{6.21}$$

and the equality holds if and only if M' is also a complex manifold. □

We end this chapter by a brief discussion on the holonomy groups and calibrations. The general reference for this topic is Joyce's monograph [Joy00] and a paper of Harvey and Lawson [HL82]. We have seen the notion of holonomy in the proof of Synge theorem (theorem 3.39). Let us first give the precise definition of holonomy groups.

Definition 6.47. Let (M^n, g) be a Riemannian manifold, ∇ be the Levi-Civita connection on M. For $p \in M$, a loop[24] γ based at p then the parallel transport $P_\gamma : T_pM \to T_pM$ is a linear isomorphism since $P_\gamma^{-1} = P_{\gamma^{-1}}$, where $\gamma^{-1}(t) = \gamma(1-t)$. The *holonomy group* [25] is defined by

$$\text{Hol}_p(M, g) := \{P_\gamma : \gamma \text{ is a loop based at } p\} \subset GL(T_pM).$$

Since $\nabla g = 0$, $\text{Hol}_p(M, g) \subset O(T_pM, g_p) \cong O(n)$. Also, for a fixed M, we denote $\text{Hol}(g)$ by $\text{Hol}_p(M.g)$ since it is clear that $\text{Hol}_p(M, g)$ is independent of the choice of base point p up to conjugation. If M is orientable, then $\text{Hol}(g) \subset SO(n)$. For simply connected M (hence orientable), one expects that for "generic metric",

[24]That is, $\gamma : [0,1] \to M$ is a piecewise C^1-map with $\gamma(0) = \gamma(1) = p$.

[25]The definition works for any vector bundle $E \to M$ and a connection ∇^E on E. We will stick to the case $E = TM$ and ∇^E is the Levi-Civita connection on (M, g).

$\mathrm{Hol}(g) = SO(n)$. A natural question is that when the holonomy group becomes "smaller"? We first quote the following theorem of de Rham.

Fact 6.48 (de Rham Decomposition Theorem,1952). Let (M, g) be a complete, simply-connected Riemannian manifold. If $G = \mathrm{Hol}(g) = G_1 \times G_2$, then there exist simply-connected Riemannian manifolds (M_i, g_i) such that (M, g) is isometric to $(M_1 \times M_2, g_1 \times g_2)$ and $G_i = \mathrm{Hol}(M_i, g_i)$, for $i = 1, 2$.

We call a Riemannian manifold (M, g) *irreducible* if it cannot be isometric to a product. Another complication for holonomy groups arise when M is a symmetric space (cf. section 5.6). In fact, for a symmetric space $M = G/H$, the holonomy group of M is exactly H (cf. [Joy00], Proposition 3.3.5). Thus, the study of holonomy groups for Riemannian symmetric spaces are consequence for Cartan's classification on Riemannian symmetric spaces. Finally, we have the Berger's list for holonomy groups the remaining case.

Fact 6.49. Let (M^n, g) be a simply connected, irreducible Riemannian manifold (M, g) which is not a symmetric space. Then $\mathrm{Hol}(g)$ is one of the following

(1) $SO(n)$
(2) $U(m)$, for $n = 2m$ with $m \geq 2$.
(3) $SU(m)$, for $n = 2m$ with $m \geq 2$.
(4) $Sp(m)Sp(1)$, for $n = 4m$ with $m \geq 2$.
(5) $Sp(m)$, for $n = 4m$ with $m \geq 2$.
(6) G_2, for $n = 7$.
(7) $\mathrm{Spin}(7)$, for $m = 8$.

Here is a discussion on *geometry of special holonomy*. The philosophy is that if the holonomy group is smaller, then some tensors must be fixed by holonomy group. Thus, it carries an extra structure.

(1) $SO(n)$ is the holonomy group of genetic metrics.
(2) $\mathrm{Hol}(g) \subset \mathrm{U}(m)$ if and only if (M, g) is a Kähler manifold. This corresponds to $\nabla J = 0$.

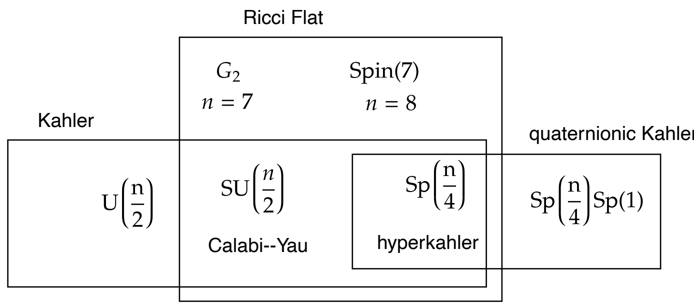
(3) If $\mathrm{Hol}(g) \subset \mathrm{SU}(m)$, then (M, g) is called a *Calabi–Yau manifolds*. The condition det is parallel corresponds to $\nabla\Omega = 0$, where $\Omega = f dz^1 \dots dz^n$ is a holomorphic n-form on M. Equivalently, this means that the *canonical line bundle* $K_M := \bigwedge^{n,0} T^*M$ is trivial. Such a metric is Ricci-flat. Yau's theorem (cf. [Yau78]) on Calabi conjecture which asserts that if K_M is trivial, then there exists such a metric g. Hence, any smooth hypersurface of degree $(n+1)$ in $\mathbb{CP}^n$ is a compact Calabi–Yau manifold.

(4) If $\mathrm{Hol}(g) \subset \mathrm{Sp}(m)\mathrm{Sp}(1)$, then (M, g) is called a *quaternionic Kähler manifold*. Quaternionic Kähler manifolds are not Kähler in general. They are Einstein but not Ricci-flat. It is an open problem whether there exists compact, non–symmetric quaternionic Kähler manifolds with positive scalar curvature.

(5) If $\mathrm{Hol}(g) \subset Sp(m)$, then (M, g) is called a *hyperkähler manifold*. As $\mathrm{Sp}(m) \subset \mathrm{SU}(2m) \subset \mathrm{U}(2m)$, hyperkähler manifolds are Ricci flat Kähler manifolds. Yau's theorem can also be used to construct compact examples for hyperkähler manifolds.

(6) and (7) G_2 is the exceptional Lie group in the Cartan's classification for compact simple Lie groups, which is a compact, simply-connected Lie group of dimension 14. On the other hand, $\mathrm{Spin}(7)$ is the double covering for $SO(8)$, which is a compact, simply-connected Lie group of dimension 21 (cf. remark 7.53). A way to describe them is through the *Cayley's octonions* $\mathbb{O} \cong \mathbb{R}^8$. We splits $\mathbb{O} \cong \mathbb{R} \oplus \mathrm{Im}\mathbb{O}$, where $\mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$ is the the imaginary octonions. Then $\mathrm{Spin}(7) = \mathrm{Aut}(\mathbb{O})$ and $G_2 = \mathrm{Aut}(\mathrm{Im}\mathbb{O})$. Any Riemannian manifold with holonomy groups lying inside G_2 or $\mathrm{Spin}(8)$ is Ricci flat. The first compact examples for manifolds with holonomy groups G_2 and $\mathrm{Spin}(7)$ are constructed by Joyce in 1996.

The discussion above can be summarized in the following illustration:



The connection between special holonomy and minimal submanifolds is the following.

Definition 6.50. Let (M, g) be a Riemannian manifold , a k-form ϕ is called a *calibration* on M if it satisfies

(1) $d\phi = 0$.
(2) For any $p \in \overline{M}$, any oriented k-plane V of $T_p\overline{M}$, $\phi|_V = \lambda \mathrm{vol}_V$ with $\lambda \leq 1$, where vol_V is the induced volume form by metric g on $\overline{M}$.

An oriented k-dimensional submanifold $\Sigma \subset M$ is called *calibrated by* ϕ if $\phi|_{T_p\Sigma} = \mathrm{vol}_{T_p\Sigma}$, for all $p \in \Sigma$.

Thus, the one line argument (6.21) again shows that Σ is the volume minimizer within the homology class $[\Sigma] \in H_p(M, \mathbb{Z})$. The notion of calibrations was introduced in the paper of Harvey and Lawson [HL82] in 1982.

Exercise 6.22. Let $G \subset SO(n)$ be a holonomy group of a Riemannian manifold (M, g) which is on the Berger's list (2)–(7). Construct a calibration ϕ on M.

The above exercise gives a method to construct calibrations on manifolds with special holonomy. We end our discussion with some examples.

Example 6.51. In the case of $U(m)$, which corresponds to (M, g) is a Kähler manifold. Then $\nabla J = 0$ is equivalent to $\nabla\omega = 0$. Since $d\omega = (\nabla\omega)^{\mathrm{alt}}$ (cf. exercise 4.12), this shows that $d\omega = 0$. Also, from Wirtinger's inequality, we know that $\omega^k/k!$ is a calibration on M for

$1 \leq k \leq m$, and any $\omega^k/k!$-calibrated submanifolds are complex k-dimensional submanifold of M.

Example 6.52. In the case of $SU(m)$, which corresponds to (M, g) is a Calabi–Yau manifold. The non–vanishing holomorphic top form Ω is parallel. Again, this implies $d\Omega = 0$. Also, we write $\Omega = \mathrm{Re}\Omega + i\mathrm{Im}\Omega$, then one can show that $\Omega_1 := \mathrm{Re}(\Omega)$ is a calibration on M if and only if for any oriented m-plane of T_pM, $\mathrm{Im}(\Omega)|_V = 0$. In this case, Ω_1-calibrated submanifolds are called *special Lagrangian submanifolds*, which is a submanifold of real dimension m. Special Lagrangian submanifolds plays a central role in Strominger–Yau–Zaslow proposal for mirror symmetry, known as *SYZ conjecture*. It turns out that it is extremely difficult to construct non–hyperkähler special Lagrangian submanifolds.

Example 6.53. In the case of $G_2 = \mathrm{Aut}(\mathrm{Im}(\mathbb{O})$, it acts on $\mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$. Another characterization for G_2 is that it is the subgroup of $\mathrm{GL}_7(\mathbb{R})$ preserving the 3-forms ϕ_0 on $\mathbb{R}^7$ given by

$$\phi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356}, \tag{6.22}$$

where $dx^{ijk} := dx^i \wedge dx^j \wedge dx^k$. Since $G_2 \subset O(7)$, it also fixes the Hodge dual 4-form $*\phi$ with respect to $g_0 = \sum_{i=1}^{7}(dx^i)^2$ on $\mathbb{R}^7$, which is given by

$$*\phi_0 = dx^{4567} + dx^{2367} + dx^{2345} + dx^{1357} - dx^{1346} - dx^{1256} - dx^{1247}.$$

A 3-form ϕ on an oriented 7-fold M is called *positive* if for any $p \in M$, there exists an oriented isomorphism between T_pM and $\mathbb{R}^7$ which identify ϕ_p with ϕ_0 in (6.22). One can similar define a 4-form ψ to be *positive* if there exists an identification between ψ_p and $*\phi_0$ above for any $p \in M$. A choice of positive 3-form ϕ is called a G_2*-structure* on M. Given such ϕ, we can associate a unique positive 4-form ψ and a Riemannian metric g on M such that $\psi = *\phi$ and ϕ, $*\phi$, g are identified with ϕ_0, $*\phi_0$, g_0 under an oriented isomorphism $T_pM \cong \mathbb{R}^7$, for any $p \in M$.

One can then show that (cf. [Joy00] Proposition 10.1.3) if M is an oriented 7-fold with a G_2-structure ϕ and the induced metric g, then $\mathrm{Hol}(M, g) \subset G_2$ if and only if $\nabla \phi = 0$. Hence, if $\nabla \phi = 0$, then $\nabla(*\phi) = 0$. This shows that both ϕ and $*\phi$ are closed forms. We then call the ϕ-calibrated 3-submanifolds by *associative submanifolds* and $*\phi$-calibrated 4-submanifolds by *coassociative submanifolds*.

Example 6.54. For $\mathrm{Spin}(7) \subset \mathrm{GL}(8, \mathbb{R})$, it can also be regarded as the subgroup of $\mathrm{GL}(8, \mathbb{R})$ preserving the 4-form $\Omega_0 \in \bigwedge^4 \mathbb{R}^8$ given by

$$\begin{aligned} \Omega_0 = {} & dx^{1234} + dx^{1256} + dx^{1278} + dx^{1357} - dx^{1368} \\ & - dx^{1458} - dx^{1467} - dx^{2358} - dx^{2367} - dx^{2457} \\ & + dx^{2468} + dx^{3456} + dx^{3478} + dx^{5678}. \end{aligned} \tag{6.23}$$

With respect to the standard metric $g_0 = \sum_{i=1}^{8} (dx^i)^2$ on $\mathbb{R}^8$, $\Omega_0 = *\Omega_0$. Similar to the case of G_2, a 4-form Ω on an oriented 8-manifold M is called *admissible* if for any $p \in M$, there exists an oriented isomorphism $T_pM \cong \mathbb{R}^8$ which identifies Ω_p with Ω_0 in (6.23). Hence, given such Ω, it associates a unique metric g on M such that both Ω and g are identified as Ω_0 and g_0 on $\mathbb{R}^8$ simultaneously at each point $p \in M$. We call such a choice of Ω a $\mathrm{Spin}(7)$-*structure* on M.

Again, one can show that (cf. [Joy00], Proposition 10.5.3) if M is an oriented 8-fold with a $\mathrm{Spin}(7)$-structure Ω and the induced metric g, then $\mathrm{Hol}(M, g) \subset \mathrm{Spin}(7)$ if and only if $\nabla \Omega = 0$. If so, then $d\Omega = 0$. The Ω-calibrated 4-submanifolds are called *Cayley submanifolds*.

6.8. Problems

6.1 (Isothermal Coordinates on Minimal Surfaces)**.** Let $\Omega \subset \mathbb{R}^2$ be a domain, $f : \Omega \to \mathbb{R}^{n-2}$ be a smooth function.

(1) Show that if the graph $\phi(x, y) = (x, y, f(x, y))$ is minimal, then the following equations are satisfied:

$$\partial_x \left(\frac{1 + |f_y|^2}{\sqrt{g}} \right) = \partial_y \left(\frac{f_y \cdot f_x}{\sqrt{g}} \right) \tag{i}$$

$$\partial_x \left(\frac{f_x \cdot f_y}{\sqrt{g}} \right) = \partial_y \left(\frac{1 + |f_x|^2}{\sqrt{g}} \right). \tag{ii}$$

(2) Let $U(x,y)$, $V(x,y)$ be functions satisfying

$$\frac{\partial U}{\partial x}=\left(\frac{1+|f_y|^2}{\sqrt{g}}\right);\frac{\partial U}{\partial y}=\frac{\partial V}{\partial x}=\left(\frac{f_x\cdot f_y}{\sqrt{g}}\right);\frac{\partial V}{\partial y}=\left(\frac{1+|f_x|^2}{\sqrt{g}}\right).$$

in a neighborhood of $p\in\Omega$ Show that $u=x+U$, $v=y+V$ defines a coordinate on some smaller neighborhood of p and (u,v) is an isothermal coordinate[26].

6.2 (Conformal Change of Riemannian metric)**.** Suppose two Riemannian metrics g and g_1 on M is conformal equivalent, i,e, there exists $f\in C^\infty(M)$ such that $g_1=e^{2f}g$. In the following, we denote the quantity with a subscript 1 by the quantity under the metric g_1. Show that we have the following identities.

(1) $\mathrm{Ric}_1=\mathrm{Ric}-(n-2)(\nabla(df)-df\otimes df)-(\triangle f+(n-2)|df|^2)g$.
(2) $s_1=e^{-2f}\left[s+2(n-1))\triangle f-(n-2)(n-1)|df|^2\right]$.

In the following, we assume M is oriented.

(3) $*_1=e^{(n-2p)f}*_g$ on $A^p(M)$.
(4) $d_1^*=e^{-2f}(d^*-(n-2p)\iota_{\nabla f}$.
(5) $\triangle_1=e^{-2f}(\triangle-(n-2p)d(\iota_{\nabla f})-(n-2p-2)\iota_{\nabla f}d+2(n-2p)df\wedge\iota_{\nabla f}-2df\wedge d^*)$.

6.3. Show that a minimal surface M in $\mathbb{R}^n$ cannot be compact.

6.4. Let (M,g) be an oriented surface with a Riemannian metric.

(1) Show that the sectional curvature (i.e., Gaussian curvature) of any (M,g) in isothermal coordinate $\mathbf{u}=(u_1,u_2)$ or $z=u_1+iu_2$ is given by

$$K(\mathbf{u})=-\frac{1}{2}\triangle_{LB}\rho=-\frac{1}{2}\triangle_{LB}\log\lambda=-\frac{2}{\lambda}\frac{\partial^2\log\lambda}{\partial\bar{z}\partial z}$$

where $g=e^\rho(du_1^2+du_2^2)=e^\rho dz\otimes d\bar{z}$ and $\lambda=e^\rho$.

(2) Now, we assume that $\phi:M\looparrowright\mathbb{R}^n$ is a minimal immersion. Let $\phi_z:=\frac{\partial\phi}{\partial z}$ and $\phi_{zz}:=\frac{\partial^2\phi}{\partial z^2}$. Show that the Gaussian curvature is given by

$$K=-\frac{|\phi_{zz}\wedge\phi_z|^2}{|\phi_z|^6},$$

where $|\phi_{zz}\wedge\phi_z|^2=|\phi_{zz}|^2|\phi_z|^2-|\langle\phi_z,\phi_{zz}\rangle|^2$.

[26] Since every surface can be locally written as graph of some functions, this establishes the existence of isothermal coordinates for minimal surfaces.

(3) Show that the Gaussian curvature K of a minimal surfaces in Euclidean spaces is non–positive. Moreover, $K = 0$ only at isolated points or $K \equiv 0$.
(4) Deduce that a flat minimal surface in $\mathbb{R}^n$ must be a plane[27].

6.5. Show a theorem of Euler: catenoid and plane are the only minimal surface of revolution (Hint: consider the area functional for surfaces of revolution and its Euler–Lagrange equation).

6.6 (Ruled Surfaces and Helicoid)**.** A surface $M \subset \mathbb{R}^3$ is *ruled* if it can be parametrized as $X(s,t) = \alpha(t) + sv(t)$, where $\alpha(t)$ is a smooth curve in $\mathbb{R}^3$ and $v(t)$ is a vector field along $\alpha(t)$. The line L_t passing $\alpha(t)$ and parallel to $v(t)$ is called the *ruling* of the surface and $\alpha(t)$ is called the *directrix*. We assume that $v'(t) \neq 0$ and $|v(t)| \equiv 1$.

(1) Show that one can reparametrize M uniqely such that $\langle \alpha'(t), v'(t)\rangle = 0$ and $|v(t)| = |v'(t)| = 1$.
(2) Show that the Gaussian curvature and mean curvature for ruled surface under parametrization in (1) is given by

$$K = -\frac{\lambda^2}{\lambda^2 + s^2}; \quad H = -\frac{1}{2(\lambda^2+s^2)^{3/2}}\left(Js^2 + \lambda' s + \lambda(\lambda J + F)\right),$$

where $F = \langle \alpha'(t), v(t)\rangle$, $\lambda = \det(\alpha', v, v')$, $J = \det(v, v', v'')$
(3) Show that helicoids are ruled surfaces.
(4) Use the formula in (2) to show that a theorem of Catalan: the only complete minimal ruled surfaces are plane and helicoid.

6.7. Let (M, J) be an almost complex manifold. A smooth section $X \in C^\infty(T^{1,0}M)$ is called a *vector field of* $(1,0)$*–type*. Show that the following are equivalent:

(1) $N_J \equiv 0$.
(2) If X, Y are vector fields of $(1,0)$–type, then $[X, Y]$ is still $(1,0)$–type.
(3) $d = \partial + \bar\partial$.
(4) $\bar\partial^2 = 0$.

[27] One can also use Gauss equation (Proposition 3.55) to prove this by showing that M must be totally geodesic. Hence, M is a plane. This argument also generalizes to flat minimal submanifolds of dimension k in $\mathbb{R}^n$.

6.8. A *complex Lie group* is a Lie group with complex structure such that group operations are all holomorphic maps.

(1) Show that $GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$ are complex Lie groups.
(2) For any complex Lie group G, show that the adjoint $Ad : G \to gl(\mathfrak{g})$ is holomorphic.
(3) Show that a compact connected complex Lie group must be abelian.

6.9. Let V be a complex vector space of complex dimension n, Λ be a lattice of rank $2n$, and $X := V/\Gamma$ be the quotient space.

(1) Show that X is a compact complex manifold and X is diffeomorphic to $(S^1)^{2n}$ as smooth manifolds. X is called the *complex tori*.
(2) We endow the metric g on X via the standard hermitian metric on $V \cong \mathbb{C}^n$. Show that (X, g) is Kähler.

6.10. Let (M, g) be a hermitian manifold.

(1) Show that the following are equivalent:
   (a) g is Kähler.
   (b) In any local holomorphic coordinate $\mathbf{w}$ of M, $\partial_k g_{i\bar{j}} = \partial_i g_{k\bar{j}}$.
   (c) For any $p \in M$, there exists a *holomorphic normal coordinate* $\mathbf{z}$ around p such that

$$g_{i\bar{j}}(p) = \delta_{ij}; \quad \partial_k g_{i\bar{j}}(p) = \partial_{\bar{k}} g_{i\bar{j}}(p) = 0.$$

(2) Derive the formula for Chirstoffel symbols and Riemannian curvature for Kähler manifolds.
(3) Show that the Ricci curvature is given by $\mathrm{Ric}_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det(g_{k\bar{l}})$.

6.11. Let (M, g) be a Kähler manifold.

(1) Show that g is the only Kähler metric in its conformal class (cf. Problem 6.2).
(2) Show that the set of Kähler metric, represented by its fundamental 2-form, forms an open convex cone in the space $\{\omega \in A^2(M; \mathbb{R}) \cap A^{1,1}(M) : d\omega = 0\}$. The cone is usually called the *Kähler cone*.

Problem 6.12 to problem 6.14 are dedicated to extension of Hodge theorem to compact hermitian manifolds and compact Kähler manifold.

6.12 (Hodge Laplacians on Hermitian Manifolds)**.** Let (M^m, g) be a hermitian manifold of complex dimension m, $\widetilde{g}$ be the $\mathbb{C}$-extension of g, ω be the associated fundamental 2-form. We extend $\mathbb{C}$-linearly the Hodge $*$-operator in section 4.1 to $* : A^k(M,\mathbb{C}) \to A^{2m-k}(M,\mathbb{C})$.

(1) Show that $\alpha \wedge *\bar{\beta} = \langle \alpha, \beta \rangle \frac{\omega^m}{m!}$, where $\alpha, \beta \in A^k(M,\mathbb{C})$ and $\langle \cdot, \cdot \rangle$ is the induced hermitian inner product on $A^k(M,\mathbb{C})$ induced by g.

(2) Show that with respect to the bidegree decomposition (6.18), $* : A^{p,q}(M) \to A^{m-p,m-q}(M)$.

(3) Show that the decomposition (6.18) is orthogonal with resepct to $\langle \cdot, \cdot \rangle$ on $A^k(M,\mathbb{C})$.

Now, assume M is compact. We define an hermitian inner product on $A^k(M,\mathbb{C})$ by

$$(\alpha, \beta) := \int_M \alpha \wedge *\bar{\beta}, \quad \alpha, \beta \in A^k(M,\mathbb{C}).$$

Let $d = \partial + \bar{\partial}$. We denote d^*, ∂^* and $\bar{\partial}^*$ are formal adjoints of d, ∂ and $\bar{\partial}$ with respect to $(\cdot, \cdot)$, respectively.

(4) Show that $\partial^* = - * \partial *$, $\bar{\partial}^* = - * \bar{\partial} *$, $d^* = \partial^* + \bar{\partial}^*$ and $(\partial^*)^2 = (\bar{\partial}^*)^2 = 0$.

We then define ∂-Laplacian and $\bar{\partial}$-Laplacian by

$$\triangle_\partial := \partial^* \partial + \partial \partial^*; \quad \triangle_{\bar{\partial}} := \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*.$$

(5) Compute the principal symbols for $\triangle_\partial$ and $\triangle_{\bar{\partial}}$ and show that they are elliptic operators.

6.13 (Hodge Theorem for Compact Hermitian Manifolds)**.** Let (M^m, g) be a compact hermitian manifold. From 6.12, we know that $\triangle_{\bar{\partial}}$ is an elliptic operator. Employing theorem 4.10 and 4.11 and intimating the arguments in section 4.3 to show the Hodge theorem for $\bar{\partial}$-Laplacian: let $\mathbb{H}^{p,q}_{\bar{\partial}} := \{\alpha \in A^{p,q}(M) : \triangle_{\bar{\partial}} \alpha = 0\}$ be the space of $\bar{\partial}$-harmonic (p,q)-forms.

(1) $\dim \mathbb{H}^{p,q}_{\bar{\partial}} < \infty$.

(2) We have the following orthogonal decomposition with respect to $(\cdot, \cdot)$ on $A^{p,q}(M)$:

$$A^{p,q}(M) = \mathbb{H}^{p,q}_{\bar{\partial}}(M) \bigoplus \mathrm{Im}(\bar{\partial}) \bigoplus \mathrm{Im}(\bar{\partial}^*).$$

6.14 (Kähler Identities and Hodge Decomposition for Compact Kähler Manifolds)**.** Let (M, g) be a Kähler manifold, ω be its Kähler form. We denote $L : A^{p,q}(M) \to A^{p+1,q+1}(M)$ by the *Lefschetz operator* $L(\alpha) = \alpha \wedge \omega$ and let $\Lambda := L^* : A^{p,q}(M) \to A^{p-1,q-1}(M)$ be its adjoint.

(1) Show that we have the following identities, known as *Kähler identities*:

$$[\Lambda, \bar{\partial}] = -i\partial^*; \quad [\Lambda, \partial] = i\bar{\partial}^*. \tag{6.24}$$

(Hint: Prove the case $M = \mathbb{C}^n$ first.)

(2) Deduce from (6.24) that $\triangle_d = 2\triangle_\partial = 2\triangle_{\bar{\partial}}$ and $[\triangle_d, \pi_{p,q}] = 0$.

We denote $H^{p,q}(M)$ by $\ker(d) \cap A^{p,q}(M)/(\mathrm{Im}(d) \cap A^{p,q}(M))$.

(3) Show that for compact Kähler manifold, the cohomology group $H^k(M, \mathbb{C}$ has the following decomposition:

$$H^k(M, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(M), \tag{6.25}$$

and $H^{p,q}(M) = \overline{H^{q,p}(M)}$.

(4) Deduce that for compact Kähler manifold, the betti number of odd degree must be even.

6.15. Let M is a compact Kähler manifold of complex dimension n. Suppose $N \subset M$ is a complex submanifold of M of complex dimension k, then its homology class $[N] \in H_{2k}(M, \mathbb{Z})$ is an integral class. By Poincaré duality $H_{2k}(M, \mathbb{Z}) \cong H^{2n-2k}(M, \mathbb{Z})$, N corresponds to a cohomology class $[\eta_N] \in H^{2n-2k}(M, \mathbb{Z})$. Show that the Poincaré dual η_N is of pure type, namely,

$$[\eta_N] \in H^{2n-2k}(M, \mathbb{Z}) \cap H^{n-k,n-k}(M).$$

(Hint: evaluate the integral $\int_N \iota^* \eta_N = 1$ near a point of N). In fact, the conclusion holds if N is an analytic subvariety of M (i.e., N is locally zero locus of holomorphic functions and may have singular points). Thus, we call a cohomology of this sort an *analytic cycle*.

Remark 6.55. The famous Hodge conjecture is a sort of converse for problem 6.15. The statement for Hodge conjecture is following.

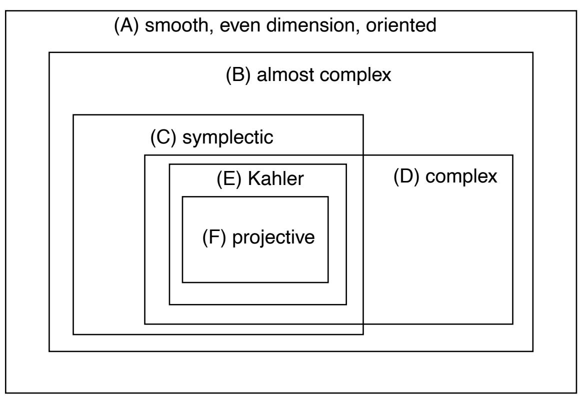
Conjecture 6.56 (Hodge Conjecture). *let M be a projective complex manifold (i.e., M is a complex submanifold of $\mathbb{CP}^N$, for some $N \in \mathbb{N}$). Every cohomology class $\gamma \in H^{k,k}(M) \cap H^{2k}(M, \mathbb{Q})$ is a $\mathbb{Q}$-linear combination of analytic cycles.*

The only known case in general is $k = 1$, called Lefschetz $(1,1)$-classes theorem, cf. [GH94], p.163. Also, by Chow's theorem (cf. footnote 21), every analytic cycle is in fact an algebraic cycle.

6.16 (Hopf Manifolds)**.** Given $\lambda \in \mathbb{C}^\times$, define a group action$\mathbb{Z}$ on $\mathbb{C}^n \setminus \{0\}$ by $k \cdot (z_1, \dots, z_n) = (\lambda^k z_1, \dots, \lambda^k z_n)$.

(1) Show that for $0 < |\lambda| < 1$, the action is free.
(2) Show that the quotient space $M = (\mathbb{C} \setminus \{0\})/\mathbb{Z}$ has a structure of complex manifold and is diffeomorphic to $S^1 \times S^{2n-1}$. M is known as **Hopf manifold**.
(3) For $n \geq 2$, show that Hopf manifold cannot be a Kähler manifold and it cannot have any symplectic structure (cf. remark below)

Remark 6.57. As we have seen, a complex manifold has a natural almost complex structure. If we start with a symplectic manifold (M, ω), then given any Riemannian metric g on M, one can induce an almost complex structure J so that $\omega = g(Ju, v)$. In theorem 6.34, we know that a Kähler manifold is a complex and symplectic manifold. Also, any complex projective manifold is a Kähler manifold. We can summarize above by the illustration:



We now give a brief summary[28] on the examples and non–examples on these categories.

(A) and (B) : In problem 7.4, we will show that S^4 admits no almost complex structure. Moreover, in problem 8.1, we will show that only S^{2n} admitting almost complex structure are S^2 and S^6.

(B) and (C) : S^6 has an almost complex structure (cf. problem 8.1). However, $H^2(S^6) = 0$. Hence, it cannot be symplectic (cf. footnote 22).

[28]The discussion follows da Silva's note on symplectic geometry:https://people.math.ethz.ch/~acannas/Papers/lsg.pdf, section 17.3

(C) and (D) : Hopf surface (cf. problem 6.16) cannot be a symplectic manifold.

(B) and (D) : See the footnote 18.

(B),(C), and (D) : The 4-manifold $\mathbb{CP}^2\#\mathbb{CP}^2\#\mathbb{CP}^2$ (cf. section 8.6 for definition of connected sum) has an almost complex structure (using characteristic classes). However, it is not a complex surface (by Kodaira classification for compact complex surface) and it is not symplectic (proved by Taubes [Tau94] using Seiberg–Witten invariant).

(C),(D),and (E) : For a complex manifold with a symplectic structure, it might not be a Kähler manifold. The first example was constructed by Thurston, cf. [Thu76].

(E) and (F) : In problem 6.9, we show that every complex tori $X^n = V/\Lambda$ has a Kähler manifold structure. For $n \geq 2$, if a complex tori is projective, there must exist a hermitian metric h on V satisfying the *Riemann condition* (cf. [GH94], section 2.6). Thus, generic complex tori with dimension > 1 are not projective.

6.17 (Calibration on Hypersurface)**.** Let $f : \Omega \subset \mathbb{R}^{n-1} \to \mathbb{R}$ be a smooth function with $|\nabla f| \neq 0$. Let μ be the unit normal vector field for the minimal hypersurface $\Gamma_f \subset \mathbb{R}^n$. We define an $(n-1)$-form on $\Omega \times \mathbb{R} \subset \mathbb{R}^n$ by $\omega := \iota_\mu dx^1 \wedge \cdots \wedge dx^n$, where $dx^1 \wedge \cdots \wedge dx^n$ is the volume form on $\mathbb{R}^n$. Show that ω is a calibration on $\Omega \times \mathbb{R}$ and the Γ_f is calibrated by ω. Particularly, we have shown that minimal graph are volume minimizer within the submanifold $\Sigma \subset \Omega \times \mathbb{R}$ such that $\partial\Sigma = \partial\Gamma_f$.