DG-II, intro to geom. flows. 5/12, 2021

Motivations: $\Big\langle$ Solving Einstein eq'n
Geometrization of Manifolds

Recall Hilbert-Einstein functional $\mathcal{H}(g) = \int_M R\, d\mu$ e.g. M cpt

$$\delta\mathcal{H}(g)\, h = \int_M \left(-2R_{ij} + \frac{2}{n} R g_{ij}\right) h^{ij}\, d\mu$$

seems to be natural to consider evolution eq'n "$\cdot = \frac{\partial}{\partial t}$"

$$\dot g = \frac{2}{n} R g - 2\mathrm{Ric} \qquad \text{s.t. Einstein metric} \equiv \text{steady state sol.}$$

<u>Hamilton 1982</u> JDG: The existence fails even for short time!

Rescue: use $\dot g = \frac{2}{n} r g - 2\mathrm{Ric}$ instead, where $r := ⨍ R = \frac{\int R\, d\mu}{\int d\mu}$.

NRF

Thm (H): Equiv to $\underline{\dot g = -2\mathrm{Ric}}$; short time $\exists$, also for M^3, $\mathrm{Ric} > 0$
RF
$\Rightarrow$ long time $\exists$ & $g(t)$ "$\longrightarrow$" const $K \Rightarrow M \cong S^3$.

BASICS:

I, $\dot g_{ij} = -2R_{ij}$. $\tilde g = \psi g$ $\psi = \psi(t)$ ~~normalization factor~~

$\Rightarrow \underline{\tilde R_{ij} = R_{ij}}$, $\tilde R = \frac{1}{\psi} R$, $\underline{\tilde r = \frac{1}{\psi} r}$; $\qquad$ $\underline{\text{s.t. } \int d\tilde\mu = 1}$.

Prop: $\qquad$ i.e. $\underline{\int d\mu = \psi^{-n/2}}$ choose

$\Rightarrow \underline{\frac{\partial}{\partial \tilde t}\tilde g_{ij}} \overset{\downarrow}{=} \dot g_{ij} + \frac{d}{dt}(\log\psi)\, g_{ij}$ $\qquad$ $\frac{\frac{d}{dt}\int d\mu}{\int d\mu} = -\frac{n}{2}\frac{d}{dt}\log\psi$

$\uparrow$ $\qquad -2R_{ij} \qquad \frac{2}{n} r g_{ij}$ * $\qquad$ had seen $\frac{d}{dt}\log\sqrt{\det g_{ij}} = -R$

get normalized form.

~~need to use~~ new time scale $\qquad$ Lemma:

$$\boxed{\tilde t := \int_0^t \psi(t)\, dt} \qquad \boxed{\Rightarrow \frac{d}{dt}\log\psi = \frac{2}{n} r} \quad (\text{Since } \dot g = ~~-2\mathrm{Ric}~~)$$

pf: $\frac{\partial}{\partial\tilde t}\tilde g = \frac{\partial}{\partial t}(\psi g) \Big/ \frac{\partial\tilde t}{\partial t}$

$$= \frac{1}{\psi}(\dot\psi g + \psi\dot g) = (\log\psi)^{\cdot} g + \dot g = \frac{2}{n} r g - 2\mathrm{Ric} \quad *$$

$$= \frac{2}{n}\tilde r\tilde g - 2\widetilde{\mathrm{Ric}}.$$

NRF: $\dot g = \frac{2}{n} r g - 2\mathrm{Ric}$.

$\big(\Rightarrow V(t)$ is const easily: $\dot V(t) = \frac{d}{dt}\int_M \sqrt{\det g_{ij}}\, dx = \int_M \frac{g^{ij}}{2}\left(\frac{2}{n} r g_{ij} - 2R_{ij}\right) d\mu$

$$= \int_M (r - R)\, d\mu = 0. \big)$$

<u>**Claim:**</u> RF is a parabolic eq'n. It is not strictly parabolic, but the kernel of its symbol all come from the contracted 2nd Bianchi identity, hence from Diffeom(M). So it can be fixed using "certain "

Examples of Ricci flow : M not nec. cpt .

1. Einstein metrics ($n \neq 2$). If (M, g_0) has $\mathrm{Ric}(g_0) = \rho g_0$ ($R/n = \rho$ = const if $\underline{n \geq 3}$)

then $g(t) := (1 - 2\rho t) g_0$ satisfies $\dot{g} = -2\rho g_0 = -2\mathrm{Ric}(g_0)$

since Ric is scaling inv. (if $\underline{n \neq 2}$) $= -2\mathrm{Ric}(g)$

Rmk : in the RF $\mapsto$ NRF , get $\psi(t) = \frac{1}{1-2\rho t}$ back ! ($\tilde{t}$ is not important.)

2. Ricci Solitons : Def'n : It is (M, g_0) st $\mathrm{Ric}(g_0) + \frac{1}{2} L_\xi g_0 = \rho g_0$

for some vf ξ and const ρ. <u>gradient</u> soliton if $\xi = \nabla f$.

Let φ_t be the flow generated $\frac{1}{1-2\rho t} \xi$ then

$g(t) := (1-2\rho t) \varphi_t^* g_0$ satisfies

$\dot{g} = -2\rho \varphi_t^* g_0 + (1-2\rho t) \cdot \frac{1}{1-2\rho t} L_\xi g_0 = -2\mathrm{Ric}(g_0) - \cancel{L_\xi g_0} + \cancel{L_\xi g_0} = -2\mathrm{Ric}(g(t))$

High Point : RS = fixed pt of RF on space of metrics / diffeom + scaling

called shrinking ($\rho > 0$) , steady ($\rho = 0$) , expanding ($\rho < 0$) .

3. Hamilton's Cigar soliton on $\mathbb{R}^2$: ($\rho = 0$)

$g_{ij}(t) := \frac{4}{e^t + |x|^2} \delta_{ij}$ Ex. Show that $R = \frac{e^t}{e^t + |x|^2}$.

since dim $= 2$, $\Rightarrow$ $\dot{g} = -\frac{4e^t}{(e^t + |x|^2)^2} = -2\,\mathrm{Ric}$.

In fact , let $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ via $\varphi_t(x) = e^{-t/2} x$ (flow)

then $\varphi_t^* g_0 = \sum \frac{4}{1 + |e^{-t/2} x|^2} d(e^{-t/2} x_i) \otimes d(e^{-t/2} x_i) = g(t)$

Rmk : Will see at the end of lectures series it is important to ruled out cigar $C \times \mathbb{R}$ in doing surgery in 3D RF .

4. Compact example on S^2 (Rosenau's example) :

Ex. Show that for $t \in (-\infty, 0)$, (ie. ancient solution)

$$g_{ij}(t) := \frac{8\sinh(-t)}{1 + 2\cosh(-t)|x|^2 + |x|^4} \delta_{ij} .$$

1) extends to a smooth metric on S^2

2) $R(t) = \frac{\cosh(-t)}{\sinh(-t)} - \frac{2\sinh(-t)}{1 + 2\cosh(-t)|x|^2 + |x|^4}$.

$\Rightarrow \dot{g} = -R = -2\,\mathrm{Ric}$.

Rmk : It is not a soliton, but can be regarded as gluing of "2 cigars" !
A complete classification of 2d ancient sol is known [Hamilton, Sesum, Chu] .

II. $\frac{\partial}{\partial t} g_{ij} = -2R_{ij} =: E(g_{ij})$ $\qquad R_{jk} = R_{ij}{}^{i}{}_{k}$ $\quad (R^{\,i}{}_{kij})$,

variation $\tilde{g}_{ij} \mapsto \tilde{R}_{ijk}{}^{h} = \partial_i \tilde{\Gamma}^h_{jk} - \partial_j \tilde{\Gamma}^h_{ik}$ (n) 3rd place

$E'(g_{jk})\tilde{g}_{jk} = -2\tilde{R}_{jk} = g^{hi}(\tilde{g}_{jk,hi} - \tilde{g}_{hj,ik} - \tilde{g}_{ik,hj} + \tilde{g}_{hi,jk}) + \cdots$

$\sigma E'(g_{jk})(\xi)\tilde{g}_{jk} = g^{hi}(\tilde{g}_{jk}\xi_h\xi_i - \tilde{g}_{hj}\xi_i\xi_k - \tilde{g}_{ik}\xi_h\xi_j + \tilde{g}_{hi}\xi_j\xi_k)$

(*) at a pt p. let $g_{jk} = \delta_{jk}$, $\xi = (1,0,\cdots,0)$

always can rotate to such position

$\Rightarrow \quad \sigma E(g)(\xi)T$ at
$(j,k) = T_{jk}$ if $j \neq 1, k \neq 1$;
$(1,k) = 0$ if $k \neq 1$ $\rightarrow$ zero eigenvalue!
(Hamilton) $(1,1) = T_{22} + T_{33} + \cdots + T_{nn}$ not strictly elliptic

Lemma: All comes from "contracted Bianchi identity".
i.e. Diffeo. gp.

Ex. $\frac{2}{n}Rg_{ij} - 2R_{ij}$ has + eigenvalue!

"pf:" $g^{ij}D_iR_{jk} = \frac{1}{2}\partial_k R$. To use it, consider

$L(g): C^\infty(M, \mathrm{Sym}^2T^*M) \to A^1(M)$:

$(L(g)T)_k := \sum_{ij} g^{ij}(D_iT_{jk} - \frac{1}{2}D_kT_{ij})$ $\quad$ so $L(g)\cdot E(g) = 0$

1st variation $\Rightarrow$ $(L(g)\circ E(g))'(\tilde{g}) = \underline{L(g)}\,\underline{E'(g)}\,\tilde{g} + L'(g)E(g)\tilde{g}$ $\quad$ no contri in σ

$\deg = 1 + 2 = 3$ $\qquad$ deg 1 in $\tilde{g}$

Q: $\mathrm{im}(\sigma E'(g))$ "$\subset$" $\ker(\sigma L(g))$? It's true if the 1st var does come from $g(t)$. but we do not know yet!

Now $(\sigma L(g)(\xi)T)_k = \sum_{ij}(g^{ij}\xi_iT_{jk} - \frac{1}{2}\xi_kT_{ij})$

Normalized g_{jk}, ξ_i as in (*) $\Rightarrow$ $= \begin{cases} T_{1k} & \text{if } k \neq 1 \\ \frac{1}{2}(T_{11} - T_{22} \cdots - T_{nn}) & \text{if } k = 1. \end{cases}$

To prove lemma, remains to notice Q holds:

$$T \overset{\sigma E'(g)(\xi)}{\longmapsto} \begin{pmatrix} \sum_{i\neq 1} T_{ii} & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \overset{\sigma L(g)(\xi)}{\longmapsto} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

In fact, "=" holds! # $\qquad \tilde{T}$ since $\tilde{T}_{11} = \sum_{i\neq 1} T_{ii}$, $\tilde{T}_{ii} = T_{ii}$ $(i \neq 1)$

A better way to kill (handle) the degeneracy was soon discovered by DeTurk 1983: $\dot{\tilde{g}} = -2\,\mathrm{Ric}(\tilde{g}) - L_{\xi_t}\tilde{g}$ (DF) coming from diffeom(M)

where $\xi_t := \Delta_{\tilde{g}(t),h}\,\mathrm{id}_M \in TM$, h any fixed metric.

Ex. Show that DF is strictly parabolic, and for flow $\dot{\varphi}_t = \xi_t$, $g(t) := \varphi_t^*\tilde{g}(t)$ is a sol to RF, and vice versa.

$$B_{ijkl} := R_{\cdot i \cdot j} R_{\cdot k \cdot l} = B_{jilk} = B_{klij}$$

but not the others.

Theorem:

$$\Rightarrow \dot{R}_{ijkl} = \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk})$$

$$- (R_{\cdot jkl} R_i^{\cdot} + R_{i\cdot kl} R_j^{\cdot} + R_{ij\cdot l} R_k^{\cdot} + R_{ijk\cdot} R_l^{\cdot}) \quad \#$$

Pf: In general, $\dot{R}_{ijkl} = g_{hk} \dot{R}_{ijl}^{h} + \dot{g}_{hk} R_{ijl}^{h}$, since

$\dot{R}_{ijl}^{h} = \partial_i \dot{\Gamma}_{jl}^{h} - \partial_j \dot{\Gamma}_{il}^{h}$ with $\dot{\Gamma}_{ij}^{h} = \frac{1}{2} g^{hl}(\partial_i \dot{g}_{jl} + \partial_j \dot{g}_{il} - \partial_l \dot{g}_{ij})$,

plug in $\dot{g}_{ij} = -2R_{ij}$ $\Rightarrow$ Thm follows from the following lemma #

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Lemma: $\Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl})$

$$= \nabla_i \nabla_k R_{jl} - \nabla_i \nabla_l R_{jk} - \nabla_j \nabla_k R_{il} + \nabla_j \nabla_l R_{ik}$$

$$+ R_{\cdot jkl} R_{\cdot i} + R_{i\cdot kl} R_{\cdot j}$$

Pf: $\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0$

$$\Delta R_{ijkl} = g^{pq} \nabla_p \underline{\nabla_q R_{ijkl}} = g^{pq}(\nabla_p \underbrace{\nabla_i R_{qjkl}}_{①} - \nabla_p \underbrace{\nabla_j R_{qikl}}_{②})$$

$$① : g^{pq} \nabla_p \nabla_i R_{qjkl} - g^{pq} \underset{⊛}{\nabla_i \nabla_p} R_{qjkl}$$

$$= g^{pq} (R_{ipq}^{\;\;\;m} R_{mjkl} + R_{ipj}^{\;\;\;m} R_{qmkl} + R_{ipk}^{\;\;\;m} R_{qjml} + R_{ipl}^{\;\;\;m} R_{qjkm})$$ 3

"$-mq$" switch sign | use "pi" in Hamilton.

$R_{i*\cdot j} R_{*\cdot kl}$

1st Bianchi

$-R_{*kl\cdot} - R_{*lk\cdot}$

$$= R_i^{\;h} R_{hjkl} - B_{ijkl} + B_{ijlk} - B_{ikjl} + B_{iljk}$$ 4

Also ⊛ : $g^{pq} \nabla_p R_{qjkl} = \nabla_k R_{jl} - \nabla_l R_{jk}$

contracted 2nd Bianchi

$$\Rightarrow ① = \nabla_i \nabla_k R_{jl} - \nabla_i \nabla_l R_{jk} - (B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) + R_i^{\;h} R_{hjkl}$$

② simply $(i,j) \leftrightarrow (j,i)$. Sum $\Rightarrow$ Lemma # $\Rightarrow$ Thm.

Cor. (a) $\dot{R}_{ik} = \Delta R_{ik} + 2 R_{*ik*} R_{**} - 2 R_{i*} R_{k*}$ 6

(b) $\dot{R} = \Delta R + |Ric|^2$.

Pf: for (b): $R = g^{ik} R_{ik}$

$\Rightarrow \dot{R} = \underline{\dot{g}^{ik}} R_{ik} + g^{ik} \dot{R}_{ik} = 2 g^{ip} g^{qk} R_{ik} R_{pq} + g^{ik} \Delta R_{ik} + \cancel{2|Ric|^2} - \cancel{2|Ric|^2}$ 7

$\dot{g}^{ik} = - g^{ip} \dot{g}_{pq} g^{qk}$ $\qquad 2 g^{ip} g^{qk} R_{ik} R_{pq} = 2|Ric|^2$. ✱

(Need $g^{jl} B_{ijkl} = 2 g^{jl} B_{ijlk}$.)

Cor: if $R > 0$ at $t=0$ then it holds always.

(need strong max. principle. Ex.) 9

Ex. In fact, $\dot{R} \geq \Delta R + \frac{2}{n} R^2$. Let $A = \inf R(g_0) > 0$

Show that $R(g(t)) - \frac{1}{A^{-1} - \frac{2}{n} t} \geq 0$ on $[0, T)$, $T \leq \frac{n}{2A}$.

if exists

Uhlenbeck's trick:

Let $(V, h) \overset{u}{\underset{\sim}{\rightarrow}} TM$, "arbitrary" but isometric at $t=0$.

fixed metric $\qquad$ i.e. $h = g_0$ always.

$u = (u^i_a)$ $\qquad$ frame F_a $\qquad \frac{\partial}{\partial x^i}$

$v^a F_a \mapsto w^i \frac{\partial}{\partial x^i}$.

$w^i = \sum_a u^i_a v^a$. on frame $F_a = \sum u^i_a \frac{\partial}{\partial x^i}$

Evolution: $\boxed{\frac{\partial}{\partial t} u^i_a = g^{ij} R_{jk} u^k_a}$

So $\underline{h_{ab}} = h(F_a, F_b) = u^i_a u^j_b \, h(\partial_i, \partial_j) = \sum_{i,j} g_{ij} u^i_a u^j_b$

$\underline{R_{abcd}} = R_{ijkl} u^i_a u^j_b u^k_c u^l_d$.

Γ^l_{ik} on TM pull back to Δ^b_{ic} on V

st. $D_i v^a = \frac{\partial}{\partial x^i} v^a + \Delta^a_{ib} v^b$ $\qquad \forall t$.

$\Rightarrow D_i u^j_a = 0$, $D_i h_{ab} = 0$. (why?)

Ex. Show that in this frame, the last 4 terms in Thm vanish!

Ex. Evolution of curvature : DG-II 5/19, 2021

$$\dot{R}_{abcd} = \Delta R + 2\underset{①}{(B_{abcd} - B_{abdc})} + 2\underset{②}{(B_{acbd} - B_{adbc})}$$

$$B_{abcd} = R_{aebf} R_{cedf}$$

$$B_{abdc} = R_{aebf} R_{decf}$$

($\Rightarrow$ ① $= 2 R_{aebf} R_{ecdf}$? not a good choice !)

Def^n: View $R_{abcd} = M_{\alpha\beta}\, \phi^\alpha_{ab}\, \phi^\beta_{cd}$

where $\phi^\alpha = \sum_{a<b} \phi^\alpha_{ab}\, e^a \wedge e^b$ ONF of Λ^2 curv. operator

Then $R_{abef} R_{cdef} = - R_{aefb} R_{cdef} - R_{afbe} R_{cdef}$

$$= R_{aefb} R_{cefd} + R_{aefb} R_{cfde} + R_{afbe} R_{cefd} + R_{afbe} R_{cfde} = ①$$

df cf

• so(n) structure on Λ^2 : $[\phi, \psi]_{ab} = (\phi \circ \psi - \psi \circ \phi)_{ab}$

$= \phi_{ac}\psi_{bc} - \psi_{ac}\phi_{bc}$

Let $[\phi^\alpha, \phi^\beta] = C^{\alpha\beta}_\gamma\, \phi^\gamma$, Since $\psi_{cb} = -\psi_{bc}$.

define $(M \# N)_{\alpha\beta} = C^{\gamma\eta}_\alpha\, C^{\delta\theta}_\beta\, M_{\gamma\delta}\, N_{\eta\theta}$, $M^\# := M \# M$ (or $M^{\#2}$)

$M, N \in Sym^2 \Lambda^2$ Lie alg. square.

Lemma: Then $R^\#_{abcd} := M^\#_{\alpha\beta}\, \phi^\alpha_{ab}\, \phi^\beta_{cd}$ corr. to $R^\#_{abcd} =$ ②

Pf: $2(B_{acbd} - B_{adbc}) = 2(R_{aecf} R_{bedf} - R_{aedf} R_{becf})$

$$= 2\left(M_{\gamma\delta}\, \phi^\gamma_{ae}\, \phi^\delta_{cf} \cdot M_{\eta\theta}\, \phi^\eta_{be}\, \phi^\theta_{df} - M_{\gamma\theta}\, \phi^\gamma_{ae}\, \phi^\theta_{df} \cdot M_{\eta\delta}\, \phi^\eta_{be}\, \phi^\delta_{cf}\right)$$

$$= 2 M_{\gamma\delta} M_{\eta\theta} \boxed{C^{\delta\theta}_\alpha \phi^\alpha_{cd}}\, \phi^\gamma_{ae}\, \phi^\eta_{be} = \cdots$$

* Ex. complete the proof.

Example $n=3$. θ^a $a=1,2,3$ ONB of T_p^*M

$$\varphi^1 = \frac{1}{\sqrt2}\,\theta_1\wedge\theta_2$$

get ONB: $\varphi^2 = \frac{1}{\sqrt2}\,\theta_2\wedge\theta_3$ $\qquad \varphi^\alpha = (\varphi^\alpha_{ab})$ skew

of Λ^2. $\varphi^3 = \frac{1}{\sqrt2}\,\theta_3\wedge\theta_1$ $\qquad = \sum_{a<b} \varphi^\alpha_{ab}\,\theta^a\wedge\theta^b$

$$\varphi^1 = \frac{1}{\sqrt2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}, \quad \varphi^2 = \frac{1}{\sqrt2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0\end{pmatrix}, \quad \varphi^3 = \frac{1}{\sqrt2}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0\end{pmatrix}$$

e.g. $[\varphi^1,\varphi^2] = \underbrace{\left(\frac{-1}{\sqrt2}\right)}_{= c^{12}_{\;\;3}}\varphi^3$, i.e. $c^{\alpha\beta\gamma} = \langle[\varphi^\alpha,\varphi^\beta],\varphi^\gamma\rangle$

$= \pm\frac{1}{\sqrt2}$ for $\alpha\neq\beta\neq\gamma$.

Then $M^\#_{\alpha\beta} = \underbrace{c_\alpha^{\gamma\eta}\,c_\beta^{\delta\theta}}_{= 0 \text{ or } \pm\frac12;} M_{\gamma\delta}M_{\eta\theta}$ $\quad (\gamma\eta,\delta\theta)\leftrightarrow(\eta\gamma,\theta\delta)$ same term. $\quad$ Do not need M to be symmetric!

$$= (\operatorname{adj} M)_{\alpha\beta} = (M^t)^{-1}\cdot\det M.$$

Example $n=4$. $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ since $*^2 = id$.
$so(4) = so(3)\oplus so(3)$.

ONB are Λ^2_+: $\varphi^1 = \frac12(\theta_{12}+\theta_{34})$, $\varphi^2 = \frac12(\theta_{13}+\theta_{42})$, $\varphi^3 = \frac12(\theta_{14}+\theta_{23})$

Λ^2_-: $\varphi^\alpha = \ldots$ $\quad -$ $\quad -$ $\quad -$

Block decomp gives $M = \begin{pmatrix} A & B \\ B^t & C\end{pmatrix}$,

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$\varphi^1 = \frac12\left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right)$, $\varphi^2 = \frac12\left(\begin{array}{cc|cc} & & 1 & \\ & & & -1 \\ \hline -1 & & & \\ & 1 & & \end{array}\right)$, $\varphi^3 = \frac12\left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & -1 & & \\ -1 & & & \end{array}\right)$

$\Rightarrow [\varphi^1,\varphi^2] = (-)\varphi^3$.

The above $\Rightarrow M^\# = 2\begin{pmatrix} A^\# & B^\# \\ (B^t)^\# & C^\#\end{pmatrix}$

B might not be sym.

$\Rightarrow \dot A = \Delta A + (A^2 + BB^t) + 2A^\#$ etc.

Ex. (a) Show that $A_{11} = R_{1212} + R_{3434} + 2R_{1234}$ etc.
and $\operatorname{tr} A = \operatorname{tr} C \iff$ Bianchi id.

(b) $B_{11} = R_{1212} - R_{3434}$
$B_{12} = R_{1213} + R_{3413} - R_{1242} - R_{3442}$ etc.

Also $B_{11} = \frac12(R_{11}+R_{22}-R_{33}-R_{44})$, $B_{12} = R_{23} - R_{14}$.

Rmk: Difficulty for $n\geq 5$ due to $so(n)$ is simple!

~~Hamilton's~~ strong max principle for tensors (Suggested by Hirsch)

(V, h) v.b. $A(t)$ conn. comp. with h

$\downarrow$

(M, g) cpt. $g(t)$ $\quad U \subseteq V$ open $\quad \phi$: v.f. on U tangent to fibers

Let $X \subset U$ closed, inv. under parallel translation in $A(t)$, $\forall t$ also each fiber X_x is convex. Then

Theorem. ODE : $\dot f = \phi(f)$ remains in X

$\Rightarrow$ PDE : $\dot f = \Delta f + \phi(f)$ remains in X.

Apply to: $P \times_G E$ $\quad X = P \times_G Z \quad Z \subset E_{G\text{-v.s}} \quad \phi$: G-inv v.f.

Z convex, closed. inv. under ϕ.

(pf is left for reading).

Def'n: $Z \subset E$ = v.s. of sym bi-linear forms on $so(n)$ closed, convex, $O(n)$-inv. inv. under $\dot M = M^2 + M^\#$

Then Z is a pinching set if $|\tilde M| \le C |M|^{1-\delta}$ $\quad M - (\mathrm{tr}\, M)\, Id.$

for some $C, \delta > 0$, $\forall M \in Z$.

open $U \subset so(n)$ is a pinching set if every cpt $K \subset U$ is contained in some Z.

Prop: If U is pinching and $\mathrm{tr}\, M > 0$ $\forall M \in U$

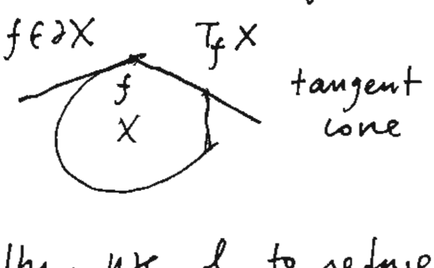
then given g_0 with $Rm_0 \in U$ $\xrightarrow[t \to \infty]{}$ g_∞ const. $K > 0$

idea of Pf: (of Prop) M cpt $\Rightarrow$ Z exists. $Rm_0 = M \in X = P \times_G Z$.

$\Rightarrow |\tilde M| \le C |M|^{1-\delta}$. the remaining is a direct generalization of Hamilton's 3d pf. back to normalized eq'n #

idea of pf: (of Theorem) consider only the Euclidean case $\phi : U \subset \mathbb{R}^k \to \mathbb{R}^k$

$S_f X \ni \ell$ support fcn at $f \in \partial X$, $|\ell| = 1$, $\ell(f) \ge \ell(k)$ $\forall k \in X$.



Fact : $\phi(f) \in T_f X \Leftrightarrow \ell(\phi(f)) \le 0 \quad \forall \ell \in S_f X$.

$\Rightarrow \left[\ \dot f = \phi(f) \text{ stays in } X \Leftrightarrow \phi(f) \in T_f X \quad \forall f \in \partial X.\ \right]$

This is Lemma 4.1 in [H]

Finally, use ℓ to reduce V.B. PDE to scalar PDE + strong max. principle #

Example: 3d case: ODE $\dot{M} = M^2 + M^\#$

start with $M = \mathrm{diag}(m_1, m_2, m_3)$, $m_1 \le m_2 \le m_3$,

then M^2, $M^\#$ also diag, and M remains so.

$\frac{d}{dt} \log f = g \ge c \Rightarrow f \ge e^{ct+d} > 0$

$\dot{m}_1 = m_1^2 + m_2 m_3$

$\dot{m}_2 = m_2^2 + m_1 m_3$

$\dot{m}_3 = m_3^2 + m_1 m_2$

$\frac{d}{dt}(m_2 - m_1) = (m_2 - m_1)(m_2 + m_1 - m_3)$,

$\Rightarrow m_1 \le m_2$ preserved. Similarly $m_2 \le m_3$

$\frac{d}{dt}(m_1 + m_2) = m_1^2 + m_2^2 + (m_1 + m_2) m_3 \ge 0$ preserved.

$K \ge 0 \Leftrightarrow m_1 \ge 0$. preserved, $Ric \ge 0 \Leftrightarrow m_1 + m_2 \ge 0$.

Ex. Show above (only for 3d), also $R = m_1 + m_2 + m_3$

in fact, all trivial: only $\underset{m_1}{R_{1212}}$, $\underset{m_2}{R_{2323}}$, $\underset{m_3}{R_{1313}}$

Thm. $\forall C$, $\exists \delta > 0$ st $\forall K$ the following closed convex set is preserved

(a) $m_1 + m_2 \ge 0$

(b) $m_2 + m_3 \le C(m_1 + m_2)$

(c) $m_3 - m_1 \le K(m_1 + m_2 + m_3)^{1-\delta}$

Rmk: (b) $\Rightarrow$ $m_3 \le 2C m_2$

$m_3 + m_1 - m_2 \le 6C m_2$ trivially!

$|\tilde{M}| \le C_1 |M|^{1-\delta}$

$\Uparrow$ same C_1.

Any cpt set in M with $m_1 + m_2 > 0$ is contained in such.

pf: $\frac{d}{dt} \log(m_1 + m_2) = \frac{\dot{m}_1 + \dot{m}_2}{m_1 + m_2} = \frac{m_1^2 + m_2^2}{m_1 + m_2} + m_3 \ge m_1 + m_3$

$\frac{d}{dt} \log(m_2 + m_3) = \frac{\dot{m}_2 + \dot{m}_3}{m_2 + m_3} = m_1 + \frac{m_2^2 + m_3^2}{m_2 + m_3} \le m_1 + m_3$

$\Rightarrow \frac{d}{dt} \log \frac{m_1 + m_2}{m_2 + m_3} \ge 0 \Rightarrow \frac{m_1 + m_2}{m_2 + m_3} \nearrow \Rightarrow$ (b).

For (c): $\frac{d}{dt} \log(m_3 - m_1) = m_3 + m_1 - m_2$

$\frac{d}{dt} \log(m_1 + m_2 + m_3) \underset{\uparrow}{\ge} m_3 + m_1 - m_2 + \frac{m_2^2}{m_1 + m_2 + m_3}$

direct expansion

for $\varepsilon = 1/36C^2$, by Rmk:

$\ge (1+\varepsilon)(m_3 + m_1 - m_2) \Rightarrow \frac{(m_1 + m_2 + m_3)^{\frac{1}{1+\varepsilon} =: 1-\delta}}{m_3 - m_1} \nearrow$ #

DG-II 5/21, 2021

Thm (gradient estimate)

$\forall \eta > 0$, $\exists C(\eta, n, g_0)$ st. $|\nabla R|^2 \le \eta R^3 + C(\eta)$ on $t \in [0,T)$, $T < \infty$.

Lemma: Let $S = |Ric|^2$, then (-1) $\partial_t R^2 = \Delta R^2 \underline{- 2|\nabla R|^2} + 4RS$

(0) $\partial_t |\nabla R|^2 = \Delta |\nabla R|^2 - 2|\nabla^2 R|^2 + 4\, \nabla R \cdot \nabla S$

(1) $\partial_t \frac{|\nabla R|^2}{R} = \Delta \frac{|\nabla R|^2}{R} - \frac{2}{R^3} | R \nabla_i \nabla_j R - \nabla_i R \nabla_j R |^2 + \underline{\frac{4}{R} \nabla R \cdot \nabla S} - \underline{\frac{2S}{R^2} |\nabla R|^2}$

(2) $\partial_t S = \Delta S - 2|\nabla Ric|^2 + 4 R_{ij} R_{k\ell} R^{ikj\ell}$

(-1) + (2) $\Rightarrow$ $\partial_t (S - \frac{1}{n} R^2) = \Delta (S - \frac{1}{n}R^2) - 2\underbrace{\left(|\nabla Ric|^2 - \frac{1}{n}|\nabla R|^2\right)}_{*} + 4 \underbrace{\mathring{R}_{ij} R_{k\ell} R^{ikj\ell}}_{**}$

** Ex. also the last term $\le 4R(S - \frac{1}{n}R^2)$ for $n = 3$. (≥ 0 is useless!)

* Lemma: $|\nabla Ric|^2 \ge \frac{3n-2}{2(n-1)(n+2)} |\nabla R|^2$, so $|\nabla Ric|^2 - \frac{1}{n}|\nabla R|^2 \ge \frac{(n-2)^2}{2n(n-1)(n+2)} |\nabla R|^2$.

Pf: We have orthogonal decomposition into irred comp.

$\partial_i R_{jk} = E_{ijk} + F_{ijk}$, $E_{ijk} = \frac{n-2}{2(n-1)(n+2)} (g_{ik} \partial_j R + g_{ij} \partial_k R) + \frac{n}{(n-1)(n-2)} \partial_{jk} \partial_i R$

st. F_{ijk} is totally trace-free (= sum of 3 traces parts)

$\underline{g^{ij} \partial_i R_{jk} = \frac{1}{2} \partial_k R}$ $\Rightarrow$ $\langle E, F \rangle = 0$, and $|E|^2 = \frac{3n-2}{2(n-1)(n+2)} |\nabla R|^2$ trivially ✱

Let $F = \frac{|\nabla R|^2}{R} - \eta R^2 + N(S - \frac{1}{n}R^2)$ for $N = N_n$ large ($N_3 = 168$)

for $\eta \le \frac{1}{n}$, $\frac{\partial}{\partial t}\left(\frac{|\nabla R|^2}{R} - \eta R^2\right) \le \Delta \left(\frac{|\nabla R|^2}{R} - \eta R^2\right) - \frac{4}{n} \eta R^3 + \underline{8\sqrt{n}\, |\nabla Ric|^2}$

since $\nabla R \cdot \nabla S = 2\langle \nabla R \cdot R_{jk}, \nabla R_{jk} \rangle \le 2 |\nabla R| |Ric| |\nabla Ric|$

Pick N to kill $8\sqrt{n}\, |\nabla Ric|^2$ via Lemma $\le 2\sqrt{n}\, |\nabla Ric| \cdot R \cdot |\nabla Ric|$

$$\Rightarrow \frac{\partial F}{\partial t} \le \Delta F + \left(4NR \underbrace{(S - \tfrac{1}{n}R^2)}_{\le C R^{2-\delta}} - \frac{4}{n} \eta R^3 \right) \le \Delta F + C(\eta).$$

by pinching estimate

Lemma: If $R \ge \rho$ at $t = 0$ then $T \le n/2\rho$.

Pf: $\dot{R} = \Delta R + 2S \ge \Delta R + \frac{2}{n} R^2$. Let $f = R_{min}$ then $\dot{f} \ge \frac{2}{n} f^2$

$\Rightarrow$ $\frac{-1}{f} + \frac{1}{\rho} \ge \frac{2t}{n}$ $\Rightarrow$ $\frac{1}{\rho} - \frac{2t}{n} \ge \frac{1}{f} \ge 0$ $\Rightarrow$ $t \le n/2\rho$ ✱

proof of Thm: $\max F_t \le \max F_0 + C(\eta) t \le C_1(\eta)$.

$$\Rightarrow |\nabla R|^2 \le \eta R^3 + C_1(\eta) R \le 2\eta R^3 + \tilde{C}(\eta).$$

Ex. We have assumed $0 < \eta \le \frac{1}{n}$. Show that the conclusion holds $\forall \eta > 0$.

RmK. It's trivial to see

Thm (Long time existence) X^n cpt

$\dot g = -2\,Ric$ have a unique sol on $[0,T)$. If $T<\infty$ then $\max_X |R_{ijk\ell}| \to \infty$ as $t \to T$.

Lemma ([H] §13+12) Under $\dot g$ $-2Ric$,

(a) $\frac{\partial}{\partial t} D^n Rm = \Delta D^n Rm + \sum_{i+j=n} D^i Rm * D^j Rm$

(b) $\frac{d}{dt}\int_X \underline{|D^n Rm|^2} + 2\int_X \underline{|D^{n+1} Rm|^2} \le C \cdot \max_X |Rm| \cdot \int_X |D^n Rm|^2$

measured in $g(t)$ $\qquad C = C(n, \dim X)$.

idea of pf of Thm: If $T<\infty$ and $|Rm| \le C$ on $[0,T)$ then

(b) $\Rightarrow \int_X |D^n Rm|^2 \le C_n$ since $\frac{df}{dt} \le Cf$ is bounded by $f(0)$.

To get sup norm estimate. by interpolation, can show

$\int_X |D^n Rm|^p \le C_{n,p} \quad \forall n,\ p<\infty \qquad \tilde C_{n,p}$ 2 ways to look at it !

Hence by $\underline{\text{Sobolev for functions}}$, for $p>n$ $\quad \max |D^n Rm|^{2} \le \underline{C(t)} \int_X \left(|D^n Rm|^{2p} + \left(\nabla |D^n Rm|^2\right)^p\right)$

① but not on derivatives of $g_{ij}(t)$

② Also ∇f for functions is ind. of conn. $\Gamma_{ij}^k \Rightarrow C(t)$ is bounded as $t \to T$.

③ since $g(t)$ are all equiv. via $\int_0^T \max_X |\dot g_{ij}|\,dt \le C < \infty$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -2R_{ij}$

$\Rightarrow |D^n Rm| \le C_n \Rightarrow g(t)$ has all t-derivatives bounded $\Rightarrow \lim_{t\to T} g(t) = g(T)$ in C^∞ ※

Cor/Thm: for X^3, $Ric>0$, $T<\infty$ and $\lim_{t\to T} R_{max}/R_{min} = 1$.

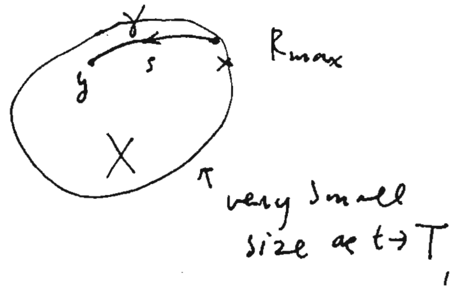
pf: In dim 3, R_{ij} det $R_{ijk\ell}$ (Ex. cf. Thm 8.1)

$\Rightarrow \max |R_{ij}| \to \infty \Rightarrow \max R \to \infty$ since $|Ric|^2 \le R^2$.

Let $|DR| \le \frac12 \eta^2 R^{3/2} + C(\eta)$ on $[0,T)$. $\exists\,\theta$ st. $C(\eta) \le \frac12 \eta^2 R_{max}^{3/2}$ on $[\theta,T)$.

$\Rightarrow |DR| \le \eta^2 R_{max}^{3/2}$ on $[\theta,T) \Rightarrow R \ge (1-\eta) R_{max}$

along geod γ of $s \le \frac{1}{\eta R_{max}^{1/2}}$



ind of t

But now $R_{ij} \ge \epsilon R g_{ij}$ for some $\epsilon>0$, Myers' thm

$\Rightarrow$ such pts y covers M for η small.

$\Rightarrow R_{min} \ge (1-\eta) R_{max} \Rightarrow R_{min}/R_{max} \to 1$ as $t \to T$ ※

Final Step: In the normalized eq'n $\frac{\partial}{\partial \tilde t} \tilde g_{ij} = \frac23 \tilde r \tilde g_{ij} - 2\tilde R_{ij}$

$\left(\begin{array}{l} \tilde R_{max}/\tilde R_{min} \to 1 \text{ as } \hat t \to \tilde T \\ \tilde R_{ij} \ge \epsilon \tilde R \tilde g_{ij} \text{ some } \epsilon>0 \end{array}\right) \Rightarrow \tilde R_{\substack{min\\max}} \le C \Rightarrow \tilde T = \infty \Longrightarrow \tilde M \simeq S^3$

reading (§16, 17).

Ricci flow 2D : normalized RF   PG-II. 5/26

Hamilton 1988 $\dot g = (r-R)g$ conformal change, since $R_{ij} = \frac{1}{2} R g_{ij}$

$2|Ric|^2 = R^2$

Ex 1. $\deg P := k$ if $\tilde P = \psi^k P$. If $\dot P = \Delta P + Q$ in UNRF (or just RF) then $\deg Q = k$ and $\dot{\tilde P} = \tilde\Delta \tilde P + \tilde Q + \frac{2}{4} k \tilde r \tilde P$ in NRF.

$\Rightarrow \dot R = \Delta R + R^2 - rR$ in NRF ($\deg R = -1$), GB $\Rightarrow r = \frac{4\pi\chi(M)}{V}$ const. in t.

$(R-r)R \quad \Rightarrow$ Both $R \geq 0$, $R \leq 0$ are preserved.

Let $R - r = \Delta\varphi$ (Hodge), $h := \Delta\varphi + |\nabla\varphi|^2 = R - r + |\nabla\varphi|^2$.

Ex 2. Show that in NRF $\dot\varphi = \Delta\varphi + r\varphi - ⨍|\nabla\varphi|^2$ call $b(t) := ⨍|\nabla\varphi|^2$.

$\dot h = \Delta h - 2|M|^2 + rh$; $M_{ij} = \nabla_i\nabla_j\varphi - \frac{1}{2}\Delta\varphi\, g_{ij}$.

$\Rightarrow R - r \leq c_1 e^{rt}$ (at max pt)

Also $R_{min} \geq (R_{min} - r)R_{min} \Rightarrow R \geq -c_2$

$\Big\} \Rightarrow \exists$ NRF on $[0,\infty)$.

Convergence : Uniformization of cpt surface :

1) $\chi(M) = 2-2g < 0$. ie $g \geq 2$, ie. $\underline{r<0}$. very nice case !

$\Rightarrow R_{min} \geq (R_{min} - r) r \Rightarrow R - r \geq -c_3 e^{rt} \Rightarrow R \xrightarrow[t\to\infty]{} r$ exp. conv.

$\forall v \in T_xM$, $\left| \frac{\partial}{\partial t}|v|_t^2 \right| = \left|(r-R)|v|_t^2\right| \leq C e^{rt}\, |v|_t^2 \Rightarrow g(t) \longrightarrow g(\infty)$. exp. conv.

2) $\chi(M) = 0$, ie. $g=1$, $r=0$. "Ricci flat case"

Have $c_1 \geq R \geq -c_2$. Let $\tilde\varphi = \varphi + \int_0^t b(s)ds \Rightarrow \dot{\tilde\varphi} = \Delta\tilde\varphi$.

$$\Rightarrow \begin{cases} \frac{\partial}{\partial t}|\nabla\tilde\varphi|^2 = \Delta|\nabla\tilde\varphi|^2 \underline{- 2|\nabla^2\tilde\varphi|^2} \\ \frac{\partial}{\partial t}\tilde\varphi^2 = \Delta\tilde\varphi^2 \underline{- 2|\nabla\tilde\varphi|^2} \end{cases} \underset{\text{trivially}}{\Rightarrow} \frac{\partial}{\partial t}\left(t|\nabla\tilde\varphi|^2 + \tilde\varphi^2\right) \leq \Delta\left(t|\nabla\tilde\varphi|^2 + \tilde\varphi^2\right)$$

$\Rightarrow$ Rough conclusion $|\nabla\tilde\varphi|^2 \leq \frac{c_3}{1+t}$ on $M \times [0,\infty)$

Now we have $R = \Delta\varphi = \Delta\tilde\varphi$ and $R^2 = |\Delta\tilde\varphi|^2 \leq 2|\nabla^2\tilde\varphi|^2$

Ex 3. Show that $\frac{\partial}{\partial t} t(R + 2|\nabla\tilde\varphi|^2) \leq \Delta t(R + 2|\nabla\tilde\varphi|^2)$ when $t(R+2|\nabla\tilde\varphi|^2) \geq 1 + 4c_3$.

This $\Rightarrow$ In any case $R + 2|\nabla\tilde\varphi|^2 \leq \frac{c_4}{1+t}$.

As in 1). but now we get only C^0 conv. $R \to 0$ and $g(t) \to g(\infty)$

3) $\chi(M) > 0$, ie. $g=0$, $r>0$. Fano case, the Difficult One !!

Need entropy estimate $E = \int_M R\log R$ if $R>0$ (Hamilton)
(extended by B. Chow (JDG 1991) get $R>0$ after finite time)

$\Rightarrow g(t) \to$ shrinking Ricci soliton on $S^2 \Rightarrow$ round metric

Will postpone it here. Since we will discuss more general method to deal with it after Perelman's work.

(93, DGA)

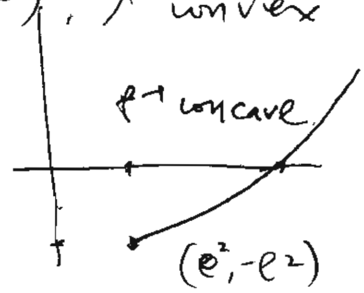
Thm. Hamilton–Ivey estimate for 3-fold:

If $M, g(t)$, $\dot g = -2\,\mathrm{Ric}$ complete curv. bdd $\forall t \geq 0$

$\lambda \geq \mu \geq \nu$ at $t=0$ (curv op), $\nu \geq -1 \Rightarrow R \geq (-\nu)\big(\log(-\nu) - 3\big)$ $\forall t$ if $\nu < 0$.

Pf: Let $y = f(x) = x\log x - 3x$, $x \in [e^2, \infty)$, $\nearrow$ convex

$\Rightarrow \lim\limits_{y\to\infty} \dfrac{f^{-1}(y)}{y} = \lim\limits_{x\to\infty} \dfrac{x}{x\log x - 3} = 0$.



Let K be the set of M st

(1) $\underbrace{\lambda + \mu + \nu}_{R} \geq -3$ (2) $\nu + f^{-1}(\underbrace{\lambda+\mu+\nu}_{R}) \geq 0$

K is closed (OK), convex since $\operatorname{tr} M$ is linear $\Rightarrow$ (1),

and both $\underline{\nu(M)}$, $f^{-1}\circ \operatorname{tr} M$ are concave functions $\Rightarrow$ (2).

i.e. first (smallest) eigenvalue **Ex 4**: $\lambda_1(M)$ is concave on $M \in \mathrm{Sym}^2$

The ODE is $\dot\lambda = \lambda^2 + \mu\nu \Rightarrow \dot R = \frac{1}{2}\big((\lambda+\mu)^2 + (\mu+\nu)^2 + (\nu+\mu)^2\big) \geq 0$

$\dot\mu = \mu^2 + \lambda\nu$

$\dot\nu = \nu^2 + \lambda\mu$ $\Rightarrow$ (1) is preserved under ODE/PDE

For (2), $\Leftrightarrow$ $\lambda + \mu + \nu \geq f(-\nu)$ if $\nu \leq -e^2$

i.e. $\lambda + \mu \geq n(\log n - 2)$ if $n := -\nu \geq e^2$ (so RHS ≥ 0)

Idea: only need to check pts on the boundary (i.e. $=$ holds)

Case (i) $\lambda \geq 0, \mu \geq 0$. $\lambda + \mu = n\,\underline{(\log n - 2)}$

$\dot\lambda + \dot\mu \overset{?}{\geq} n\,\underline{(\log n - 1)}$ i.e. $\lambda^2 + \mu\underline{\nu} + \mu^2 + \lambda\underline{\nu} \geq -(\underline{\nu}^2 + \lambda\mu)\left(\dfrac{\lambda+\mu}{\underline{n}} + 1\right)$

cancelling out $\Rightarrow$ which is of course trivial.

Case (ii) $\lambda > 0, \mu < 0$. Set $m := -\mu > 0$. $(\lambda \geq m)$ $\mu \geq \nu \Rightarrow m \leq n$

i.e. Ask if $\lambda^2 + m^2 \geq \lambda m \dfrac{\lambda - m}{n} - n^2 + \lambda m$

i.e. $(\lambda^2 + m^2)n - \lambda^2 m + \lambda m^2 + n^3 - \lambda m n \geq 0$?

$= n^3 + m^3 + (n-m)(\lambda^2 - \lambda m + m^2)$ yes! Magic! ✲

Ex 5 In fact $R \geq (-\nu)\big(\log(-\nu) + \log(1+t) - 3\big)$ if $\nu < 0$, $\forall t \geq 0$.

(Hint: Same f. But for K use (1) $\geq \dfrac{-3}{1+t}$ (2) $\nu(1+t) + \cdots$)

Def'n: Under normalized Ricci flow $\dot g = \frac{2}{n} r g - 2\,\mathrm{Ric}$

1) $g(t)$ is non-singular sol if $t \in [0, \infty)$ and $|\mathrm{Rm}| \leq C < \infty$

2) Moreover, $g(t)$ collapsed if $\hat\rho(t_i) \to 0$ ind. of t

for some $t_i \to \infty$ $\max\limits_{x\in M}$ of 'injectivity radius $\operatorname{inj}(x, g(t_i))$

(inj hull ?)

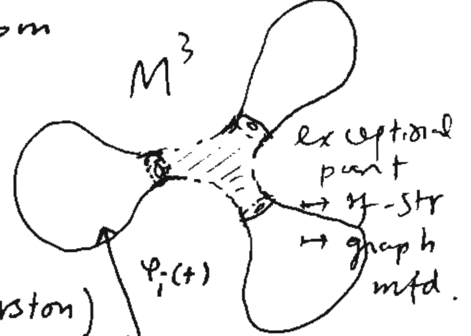
Rmk: Cheeger–Gromov–Fukaya have a theory of $\mathcal{F}$-str. on collapsed limits. (dimension reduction)

Thm (Hamilton 1999) Let $g(t)$, $t \in [0,\infty)$ be <u>non-singular</u> <u>non-collapsed</u> on cpt M^3. Then

i) $\exists\, t_k \to \infty$, $\varphi_k \in \mathrm{Diffeo}(M)$ st. $\varphi_k^* g(t_k) \xrightarrow{C^\infty} g_\infty$ of const. K, or

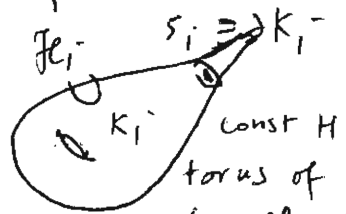
ii) $\exists$ complete non-cpt $\mathcal{H}_i = \mathbb{H}^3/\Gamma_i$ of finite vol st $\forall\, t \geq T_0 < \infty$, $i = 1, \dots, m$ $\exists$ cpt $K_i \subset \mathcal{H}_i$ and $\varphi_i(t) : K_i \hookrightarrow M$ diffeom st. $\varphi_i^* g(t) \xrightarrow{C^\infty} g_{K_i}$ as $t \to \infty$.

Moreover, Inj hull$(M \setminus \bigcup_i \varphi_i(t) K_i) \to 0$ and $\varphi_i : \pi_1(S_i) \hookrightarrow \pi_1(M)$. (Incompressible tori)



Cor. Non-sing sol, non-collapsed $\Rightarrow$ Geometrizable (Thurston)

Some ideas how the Hamilton–Ivey pinching is used to get convergence:



(eg. 2dim case)

For normalized Ricci flow, get

Ex1 $\Rightarrow$ $\dot{R} = \Delta R + 2|Ric|^2 - \frac{2}{n} r R$

$(n=3)$ $\quad = \Delta R + 2|\mathring{Ric}|^2 + \frac{2}{3} R(R - r)$

$\Rightarrow \dot{R}_{min} \geq \frac{2}{3} R_{min}(R_{min} - r)$

$\Rightarrow$ if $R_{min} \leq 0$ then $\nearrow$ (non-decreasing) in t
$R_{min} \geq 0$ then keep so in t. Get 3 cases:

(1) $R_{min}(t) > 0$ for some $t > 0$.

(2) $R_{min}(t) \leq 0$ $\forall\, t \in [0,\infty)$ and $\lim_{t\to\infty} R_{min}(t) = 0$

(3) As in (2), but $\lim_{t\to\infty} R_{min}(t) < 0$.

For (1): Unnormalized Ricci flow exists on $[0,T)$, $T < \infty$, $R \to \infty$ as $t \to T$.
Set $\lambda \geq \mu \geq \nu \geq -1$ at $t=0$. Get whenever $\nu < 0$:

$$R \geq (-\nu)(\log(-\nu) + \log(1+t) - 3)$$

$\Rightarrow |\nu| = -\nu$ is not compatible with λ

Normalized $\tilde{g} = \psi g$, $\psi(t) \nearrow \infty$. By assumption $|\widetilde{Rm}| \leq C \Rightarrow$ Any (rescaled) negative sect curvature $\to 0$

$\Rightarrow$ the non-collapsed limit has $K \geq 0$. Vol finite $\Rightarrow$ <u>manifold M unchanged</u>.

Slight ext of Hamilton's 3 fold thm $\Rightarrow$ M = quot. of $\begin{cases} \mathbb{R}^3 & \text{(flat)} \\ \Sigma^2 \times \mathbb{R} & \text{top'ly on } \Sigma^2 \\ S^3 & \text{top'ly} \end{cases}$ split the flat part

- Thm of Schoen–Yau (last semester), $R > 0 \Rightarrow$ not $\mathbb{R}^3/\Gamma$
- Since normalized Ricci flow on $\Sigma^2 \cong S^2 \to S^2$ (Hamilton–Chow JDG 1991) but scaling factors are $\neq$ for 2d v.s. 3d $\Rightarrow$ $g(t)$ can't be non-sing.
- Thus the limit has $K > 0 \Rightarrow$ const Ric & $K \Rightarrow$ isom. to S^3/Γ.

DG – II 5/28, 2021

Singularity Model

Def'n (convergence)

Recall: Given $(X_j, O_j, \mathcal{F}_j, g_j)$, $j = 1, 2, 3 \cdots$ (metric; frame at $O_j \in X_j$) $\xrightarrow{C^\infty} (X, O, \mathcal{F}, g)$

(if $\exists\ \varphi_j : K_j \hookrightarrow X_j$, $K_j \subset X$ diffeom, $\varphi_j^* g_j \xrightarrow{C^\infty} g$ on cpt subset, $O \mapsto O_j$, $\mathcal{F} \mapsto \mathcal{F}_j$)

Fact ($\Rightarrow$) (a) $\forall$ radius s, $k \in \mathbb{N}$

$|D^k Rm_j| \le B(s,k)$ on $B_{O_j}(s) \subset (X_j, g_j)$

(b) $\exists\, b > 0$, st $inj(O_j, X_j) =: \rho_j \geq b$.

Thm ($\Leftarrow$) holds for a subsequence. (Green-Wu)

Hamilton:

for a sequence of RF solutions on (a, b), need only

$|Rm(t)| \le C(t)$ for $\forall t$ and inj radius bdd at $t = 0$

Def'n: $g(t)$ max sol of RF on X, $[0, T)$ satisfies inj rad bdd if

always assume X cpt or $g(t)$ complete with $|Rm(t)| \le C(t)$

$\rho(t) \geq \frac{c}{\sqrt{M(t)}}$ where $M(t) = \max|Rm(t)|$. There are 3 types:

Type I: $T < \infty$, $\sup (T-t) M(t) < \infty$

Type II(a): $T < \infty$, $\sup (T-t) M(t) = \infty$

II(b): $T = \infty$, $\sup t M(t) = \infty$

Type III: $T = \infty$, $\sup t M(t) < \infty$

$g(t)$ is a sing model if not flat,

Type I: $(-\infty, \Omega)$, $0 < \Omega < \infty$ $|Rm| \le \frac{\Omega}{\Omega - t}$, "=" at $t = 0$

II: $(-\infty, \infty)$, $|Rm| \le 1$, "=" at $t = 0$ (eternal solutions)

III: $(-A, \infty)$, $0 < A < \infty$, $|Rm| \le \frac{A}{A+t}$, "=" somewhere at $t = 0$

Thm: $g(t)$ as above, $\exists$ sequence of dilations of it $\longrightarrow$ sing. model of same type

(Hamilton 1995)

Pf: Do Type I, other for readings (if interested): (Exercise)

i.e. $\Omega = \limsup (T-t) M(t) < \infty$, $\Omega > \epsilon > 0$

$= \lim_{j \to \infty} (T - t_j) |Rm(p_j, t_j)|$, $\underbrace{\ }_{\text{max pt}} = M(t_j)$ i.e. $\lambda_j = \frac{1}{\sqrt{|Rm(p_j, t_j)|}}$

Let p_j be O_j, translate in time st $t_j = 0$

scale the metric $\rightarrow$ (dilate in space by λ_j st $|Rm(O_j, 0)| = 1$; in time by λ_j^2 st it is still a RF sol.

It exists as a sol on $[-A_j, \Omega_j)$ with ($\Omega_j = (T - t_j) M(t_j) \to \Omega$, $A_j = t_j M(t_j) \to \infty$)

the curv. bound is easily checked $\divideontimes$

- It is important to classify sing. model with ~~curv op~~ $M \geq 0$.

The main tool is (Li-Yau-)Hamilton's Harnack inequality:

If $Ric > 0$ too, then Hamilton: Type II (i.e. eternal) $\Rightarrow$ Steady Ricci Soliton

Chen-Zhu Type III $\Rightarrow$ homothetically expanding Ricci Soliton

Def'n (Recall): steady if $\lambda = 0$

i.e. $\underline{R_{ij} + \frac{1}{2} \nabla_j V_i + \frac{1}{2} \nabla_i V_j} - \lambda g_{ij} = 0$ if $\lambda < 0$ (shrinking if $\lambda > 0$)

In fact even simpler here: $\nabla_i V_j = R_{ij} + \frac{1}{2t} g_{ij}$, here $\nabla_i V_j = \nabla_j V_i$

If $V_j = \nabla_j f$ (eg. $\pi_1 = 0$) then get gradient Ricci solitons

Thm (Hamilton JDG 1993) $g(t)$ complete sol of $\dot{g} = -2Ric$ with $|Rm| \le C$ on $[0,T)$, curv. op ≥ 0.

Let $P_{ijk} := \nabla_i R_{jk} - \nabla_j R_{ik}$, $M_{ij} := \Delta R_{ij} - \frac{1}{2}\nabla_i\nabla_j R + 2R_{ikjl}R^{kl} - R_{ik}R_j^k + \frac{R_{ij}}{2t}$

$\Rightarrow \forall\, W_i, U_{ij} \in \Lambda^2 \; Z := M_{ij}W^iW^j + 2P_{ijk}U_{ij}W^k + R_{ijkl}U^{ij}U^{kl} \geq 0$

Rmk: "=" holds for exp. grad. soliton $\nabla_a\nabla_b f = R_{ab} + \frac{1}{2t}g_{ab}$

Ex. Cor. $\forall\, V_i$, take $U_{ij} = \frac{1}{2}(V_iW_j - V_jW_i)$ and <u>tracing over W</u> $\Rightarrow$

$Q := \dot{R} + \frac{R}{t} + 2\nabla R\cdot V + 2Ric(V,V) \geq 0$. In particular for $0 < t_1 < t_2$

$t_2 \cdot R(x_2,t_2) \geq t_1\cdot R(x_1,t_1)\cdot \exp\left(-\frac{d_{t_1}(x_1,x_2)}{2(t_2-t_1)}\right)$. Hint: integrate $(\dot{x}(t),t)$ in spacetime,

- Easy application: for type III / $Ric > 0$ $\Rightarrow$ tR has max at (t_0, x_0) (shift time to $(0,\infty)$) $\overset{\vee}{0}$
  may get $\frac{\partial}{\partial t}(tR)(t_0,x_0) = 0$
  but $V=0$ in $Q \Rightarrow \frac{\partial}{\partial t}(tR) \geq 0 \Rightarrow tR = \varphi$ $\forall t \geq t_0$ function on M only
  ? $\Rightarrow \nabla R = 0$ (otherwise $Q<0$ somewhere e.g. $V = -\nabla R\cdot\epsilon$ small) $\Rightarrow \varphi = \text{const} = C$.
  but this V may not work $\forall t$ since "∇"R dep on t!
  Now Z before taking trace in W is $\geq 0 \mapsto$ get R_{ij}!

Q: The remaining argument by Cao or Cao-Zhu seem to be problematic !!

Origin of the expression Z:

On $\nabla_a V_b = R_{ab} + \frac{1}{2t}g_{ab}$ $\quad (\nabla_a V_b = \nabla_b V_a)$

$\Rightarrow *\; \nabla_a R_{bc} - \nabla_b R_{ac} = R_{abcd}V_d$

$\Rightarrow \nabla_a\nabla_b R_{cd} - \nabla_a\nabla_c R_{bd} = \nabla_a(R_{bcde}V_e)$

$= (\nabla_a R_{bcde})V_e + R_{bcde}\underline{\nabla_a V_e} = (\nabla_a R_{bcde})V_e + R_{bcae}R_{ae} + \frac{1}{2t}R_{bcda}$

trace a,b get $M_{ab} + P_{cba}V_c = 0$.

also $* \Rightarrow P_{cab}V_c + \underline{R_{acbd}}V_cV_d = 0$

sum $\Rightarrow M_{ab} + (P_{cab} + P_{cba})V_c + R_{acbd}V_cV_d = 0$. Apply $W_aW_b \Rightarrow *$

V is special (for $=$), W is arbitrary. $U = \frac{1}{2}V\wedge W$? How much freedom?

Easy Corollary (Hamilton) for <u>ancient sol.</u> (eg. I, II)

if sol exists in (α, T), may replace t by $t-\alpha$ in M_{ij} (or Z)

take $\alpha \to -\infty$ may get rid of $\underline{R_{ij}/2t}$ term $\Rightarrow \dot{R} \geq 0$

ie $R(x,t)$ is pointwise non-decreasing in t.

changing V in Q leads to strong constraint!

$\to$ Perelman's ancient K-solutions for 3-folds

Perelman's 1st paper DG-II 6/2, 2021

Def'n : $\mathcal{F}(g,f) := \int_M (R+|\nabla f|^2)\, \underline{e^{-f}\, dV}$ M cpt, $f \in C^\infty(M)$

1st Var. : for $v_{ij} = \delta g_{ij}$, $h = \delta f$, let $v := g^{ij} v_{ij}$. then

$$\delta\mathcal{F}(v_{ij}, h) = \int_M e^{-f}\Big(\underbrace{-\Delta v + \nabla_i\nabla_j v_{ij} - R_{ij} v_{ij}}_{=\delta R \text{ (recall)}}$$

$$- v_{ij}\nabla_i f \nabla_j f + 2\nabla f\cdot\nabla h + (R+|\nabla f|^2)\underline{\left(\tfrac{v}{2} - h\right)}\Big)\, dV$$

Stokes' $= \int_M e^{-f}\big(\underline{\nabla_i f\, \nabla_j v_{ij} - v_{ij}\nabla_i f\nabla_j f}\big) - \underline{\nabla f\cdot\nabla v} - 2h\Delta f + 2h|\nabla f|^2$

$$- R_{ij} v_{ij} + (R+|\nabla f|^2)\left(\tfrac{v}{2} - h\right) \quad - v_{ij}\nabla_i\nabla_j f \quad v\Delta f - v|\nabla f|^2$$

$$= \int_M \left(-v_{ij}(R_{ij} + \nabla_i\nabla_j f) + \left(\tfrac{v}{2} - h\right)(2\Delta f - |\nabla f|^2 + R)\right) e^{-f}\, dV$$

Rmk: If $dm = e^{-f}dV$ is fixed then of course $\frac{v}{2} - h \equiv 0$.

for all measure m,

Prop: $\mathcal{F}^m := \int_M (R + |\nabla f|^2)\, dm$, where $f = \log dV/dm$

has $\underline{-(R_{ij} + \nabla_i\nabla_j f) \text{ as } L^2 \text{ gradient}}$. if its flow

$* \ \underline{\dot g = -2(R_{ij} + \nabla_i\nabla_j f)}$ exists (= Ricci flow up to diffeom)

for $t \in [0,T]$, then $\mathcal{F}^m \le \frac{n}{2T}\int_M dm$ at $t=0$.

Pf: Set $\int_M dm = 1$. $\Rightarrow \dot f = \frac{1}{2}(\det g_{ij})^{\cdot} = -R - \Delta f$

and $\frac{d}{dt}\mathcal{F}^m = 2\int_M |R_{ij} + \nabla_i\nabla_j f|^2\, dm$ ($v_{ij} = \delta g_{ij} := \dot g$)

Now $*$ is indeed a DeTurck flow for $\xi_t := \nabla^t f$: Tautological property of a gradient flow !

indeed for $\dot\varphi_t = \xi_t \Rightarrow \tilde g(t) := \varphi_{-t}^* g(t)$ has $\dot{\tilde g} = \varphi_{-t}^*\dot g + L_{\xi_t} g$

$$= -2\varphi_{-t}^* R_{ij} - 2\cancel{\nabla_i\nabla_j f} + 2\cancel{\nabla_i\nabla_j f} = -2R_{ij}(\tilde g(t))(\ldots) \leftrightarrow \nabla_i\xi^j + \nabla_j\xi^i = 2\nabla_i\nabla_j f$$

modified by φ_t^* get $\left(\begin{array}{l}\dot g = -2R_{ij} \\ \dot f = -\Delta f + |\nabla f|^2 - R \quad \text{(Easy Ex.)}\end{array}\right.$

(Monotonicity Formula)

"$\dot{\tilde f}$" $= (f\circ\varphi_t)' = \dot f\circ\varphi_t + df(\nabla f)$
$= -R - \Delta f + |\nabla f|^2$.

and $\underline{\mathcal{F}_t^{**} = 2\int_M |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f}\, dV.}$

$$\ge \frac{2}{n}\int (R+\Delta f)^2 e^{-f} \ge \frac{2}{n}\left(\int (R+\underline{\Delta f})\,\underline{e^{-f}}\right)^2$$ take care location of $\sim$.

$= |\nabla f|^2 e^{-f}$ by Stokes'.

$$= \frac{2}{n}\mathcal{F}^2$$

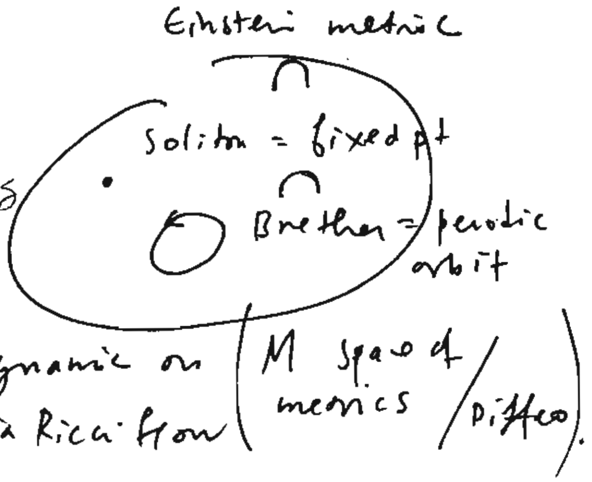
$\Rightarrow 0 \ge \frac{-1}{\mathcal{F}} \ge \frac{2t}{n} + \frac{-1}{\mathcal{F}_0} \Rightarrow \frac{1}{\mathcal{F}_0} \ge \frac{2T}{n}$ $*$ Cor. Always have $T < \infty$!

Also, $\mathcal{F}$ non-decreasing in t.

Def^{n}: A breather $g(t)$:= $\exists\, t_1 < t_2$, $\alpha > 0$

st $g(t_2) \sim \alpha\, g(t_1)$. $\alpha < 1$, $= 1$, > 1

shrinking, steady, expanding



Thm: Cpt case, Perelman ... Hamilton

Breather $\Rightarrow$ Soliton $\Rightarrow$ Einstein

$\forall \alpha$ ... if $\alpha \geq 1$

Pf: $P(\alpha = 1)$: Let $\lambda(g_{ij}) := \inf\limits_{\int_M e^{-f} dV = 1} \mathcal{F}(g_{ij}, f)$

Write $e^{-f} = u^2$ then $\mathcal{F} = \int_M (R u^2 + 4|\nabla u|^2)\, dV$

$\Rightarrow \lambda(g_{ij}) = \lambda_1(-4\Delta + R)$. $\underset{\text{by } **}{\Rightarrow}$ $\lambda(t) = \lambda(g_{ij}(t))$ non-decreasing in t

Also $\underline{\lambda(t_1) = \lambda(t_2)} \Rightarrow \forall t \in [t_1, t_2]$, $R_{ij} + \nabla_i \nabla_j f = 0$ for minimizer $f(\cdot, t)$

$P(\alpha > 1)$: if $g(t)$ exists. This applies to the steady case.

Let $\tilde{\lambda}(g_{ij}) := \lambda(g_{ij}) \cdot V(g_{ij})^{\frac{2}{n}}$ which is diffeo / scaling inv.

$\underline{\exists\, t}$ st $V'(t) > 0$ since $\int (R + |\nabla f|^2)\, e^{-f} dV$

but $-(\log V)' = -\frac{V'}{V} = \frac{1}{V}\int R\, dV \geq \lambda(g_{ij}(t))$ gives α^{-1} inv. by $\int e^{-f} dV = 1$

$\Rightarrow \tilde{\lambda}(g_{ij}(t)) < 0$. (Also $\lambda(t)$) $f = 0$ $\quad V(\alpha g)^{\frac{2}{n}} = \alpha V(g)$ $\quad e^{-f - \log \alpha^{n/2}} \cdot \alpha^{\frac{n}{2}} dV$

Claim: $\tilde{\lambda}(t)$ is non-decreasing at such t

$\tilde{\lambda}'(t) \geq 2V^{\frac{2}{n}} \int |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} + \frac{2}{n} \underline{V}^{\frac{2-n}{n}} \underline{\lambda} \underline{\int (-R)}$

barrier sense

f = minimizer of $\mathcal{F}$

$\geq 2V^{\frac{2}{n}} \Big(\int \big|R_{ij} + \nabla_i \nabla_j f - \frac{1}{n}(R + \Delta f)\, g_{ij}\big|^2 e^{-f}$

$+ \frac{1}{n}\int (R + \Delta f)^2 e^{-f} - \frac{1}{n}\underline{\left(\int (R + \Delta f)\, e^{-f}\right)^2} \Big) \geq 0$ $*$

"$=$" for $t \in [t_1, t_2] \Rightarrow R + \Delta f = C(t)$ $\quad$ since $V^{-1}\lambda \int -R = (-\lambda)\frac{\int R}{V} \geq (-\lambda)\int (R + |\nabla f|^2) e^{-f}$

$= -\left(\int (R + \Delta f)\, e^{-f}\right)^2$

But E-L eq^{n} of f is $-2\Delta f + |\nabla f|^2 - R = c_1$

$\delta \int (R + |\nabla f|^2)\, e^{-f} = \int 2\nabla f \cdot \nabla h\, e^{-f} - (R + |\nabla f|^2)\, e^{-f} h$

$= \int (-2\Delta f + 2|\nabla f|^2 - R - |\nabla f|^2)\, h$ subj. to $\int e^{-f} = 1$

$\Rightarrow \Delta f - |\nabla f|^2 = C_2 = 0$ since $\int (\Delta f - |\nabla f|^2)\, e^{-f} = 0 \Rightarrow f = \text{const}$.

$\Rightarrow H(\alpha \geq 1)$, i.e. get Einstein metrics! $\ast$

Ex. Give a direct pf to Hamilton's part. (cf. hint in [P, 2.5*]) or [CZ, Prop 1.1.1])

$P(\alpha < 1)$ is more difficult! Need:

Def^{n} (Perelman's W functional). For $\tau > 0$, subject to

$$W(g_{ij}, f, \tau) := \int_M \left[\tau(|\nabla f|^2 + R) + (f - n)\right] (4\pi\tau)^{-\frac{n}{2}} e^{-f}\, dV \quad \int = 1.$$

W is inv under parabolic scaling. Similar to $\mathcal{F}$, it has evolution eq'n:

$$\left(\begin{array}{l} \dot g = -2R_{ij} \\ \star\ \dot f = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau} \\ \dot\tau = -1 \end{array}\right.$$

(Monotonicity Formula) $\star\star$

and $\frac{d}{dt} W(g_{ij}(t), f(t), \tau(t)) = \int_M 2\tau \left| R_{ij} + \nabla_i\nabla_j f - \frac{g_{ij}}{2\tau}\right|^2 (4\pi\tau)^{-\frac{n}{2}} e^{-f}\, dV \geq 0$

Ex. Compute $\delta W(v_{ij}, h, \eta)$ (cf. CZ. P.206) and prove above formulas

PS. p.207 is NOT needed!

Let $\mu(g_{ij}, \tau) := \inf W(g_{ij}, f, \tau)$ over $\int (4\pi\tau)^{-\frac{n}{2}} e^{-f} = 1$

Now also " $\nu(g_{ij}) := \inf_{\tau>0} \mu(g_{ij}, \tau) \nearrow$ "? realized by sm f (Rothaus JFA 1981)

CZ replaced this by studying $\underline{\mu(g_{ij}(t), \tau - t)}$ along RF

Fix t_0, let $f_0 \in C^\infty(M)$ minimizes $\mu(g_{ij}(t_0), \tau - t_0)$

Since $\star$ is equiv. to $\dot F = -\Delta F + RF$ for $F := (4\pi\tau)^{-n/2} e^{-f}$

Can solve $f(t)$, $f(t_0) = f_0$ $\underline{\text{for } t \leq t_0}$ (since it is backward heat eq)

$\Rightarrow \mu(g_{ij}(t), \tau - t) \leq W(g_{ij}(t), f(t), \tau - t) \leq \mu(g_{ij}(t_0), \tau - t_0)$ for $t \leq t_0$

Also "=" holds iff $g_{ij}(t)$ is a shrinking gradient soliton.

Now we may prove $P(\alpha < 1)$: Let $g_{ij}(t_2) \sim \alpha g_{ij}(t_1)$, $t_1 < t_2, \alpha < 1$

for $\tau > 0$ to be determined,

$$\mu(g_{ij}(t_1), \tau - t_1) = \mu(\alpha g_{ij}(t_1), \alpha(\tau - t_1)) = \mu(g_{ij}(t_2), \alpha(\tau - t_1))$$

$$\overset{\star\star}{\wedge}$$

$\mu(g_{ij}(t_2), \tau - t_2)$ So let $\alpha(\tau - t_1) = \tau - t_2$ i.e. $\tau = \frac{t_2 - \alpha t_1}{1-\alpha} \Rightarrow$ "="

$\divideontimes$

Application to local non-collapsing (cpt case)

Def'n: Let $\dot g = -2\mathrm{Ric}$ on $[0, T)$. g locally collapses at T if

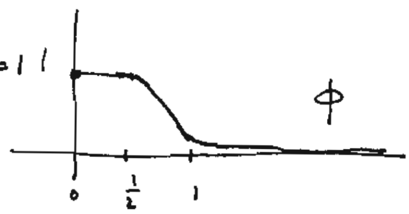
$\exists\ t_k \to T$, $B_k = B(p_k, r_k)$ at time t_k st. $r_k^2/t_k \leq C$, $|Rm(t_k)|_{B_k} \leq \frac{1}{r_k^2}$, $\frac{|B_k|}{r_k^n} \to 0$.

Thm (Perelman 2002). If M is cpt, $T < \infty$ then $g(t)$ is local non-collap.

Pf: If $\exists$ collapsing $B_k(p_k, r_k)$ at $t_k \to T$, take

$f_k(x) = -\log \phi(d_{t_k}(x, p_k)/r_k) + c_k$ st. $\int (4\pi\tau)^{-\frac{n}{2}} e^{-f} = 1$

so. $c_k \to -\infty$



with $\tau = t_k + r_k^2 \Rightarrow \mu(g_{ij}(0), t_k + r_k^2)$ as $\frac{|B_k|}{r_k^n} \to 0$

大考驗! use $\star\star$??? $\to -\infty$ $\divideontimes$

Def'n: g is κ-noncollapsed on scale ρ if $\forall B_r$, $r < \rho$,

$|Rm| \leq \frac{1}{r^2}$ on $B_r \Rightarrow V(B_r) \geq \kappa r^n$.

Cor. In Thm, if $\exists\ Q_k := |Rm|(p_k, t_k) \to \infty$ and $|Rm|(x, t) \leq C Q_k$ $\forall t < t_k$

Then the dilation limit of $g_{ij}(t_k)$ at p_k with factor Q_k is a complete ancient solution, κ-non-collapsed on all scales for some $\kappa > 0$.

(*)

$g_\tau = 2\,Ric$, M cpt (or closed), or complete with bdd curv.

for $\gamma(\tau)$ a curve, $0<\tau_1\le\tau\le\tau_2$, $\mathcal{L}(\gamma)$ "$\mathcal{L}$-length" is

$$\mathcal{L}(\gamma) := \int_{\tau_1}^{\tau_2}\sqrt{\tau}\left(R(\gamma(\tau)) + |\dot\gamma(\tau)|^2\right)d\tau \qquad \text{now } \dot{} = \tfrac{d}{d\tau}$$

Let $X(\tau) := \dot\gamma(\tau)$, $Y(\tau)$ any vf along $\gamma(\tau)$, then

1st var.

$$(\delta\mathcal{L}(\gamma))Y = \int_{\tau_1}^{\tau_2}\sqrt{\tau}\left(\nabla R\cdot Y + 2\underbrace{\nabla_Y X}_{=\nabla_X Y}\cdot X\right)d\tau \qquad 2\left(\tfrac{d}{d\tau}Y\cdot X - Y\cdot\nabla_X X - 2Ric(Y,X)\right)$$

$$= 2\sqrt{\tau}\,\langle X, Y\rangle\Big|_{\tau_1}^{\tau_2} + 2\int_{\tau_1}^{\tau_2}\sqrt{\tau}\,\langle Y, \nabla_X X - \tfrac12\nabla R + 2Ric(\cdot, X) + \tfrac{X}{2\tau}\rangle\,d\tau \qquad \tfrac{d}{d\tau}g$$

$\mathcal{L}$-geodesic equation $=0$

$$\Rightarrow \frac{d}{d\tau}(\sqrt{\tau}X) = \nabla_X(\sqrt{\tau}X) = \frac{\sqrt{\tau}}{2}\nabla R - 2Ric(\cdot, \sqrt{\tau}X)$$

Ex. fact: $\exists$, $\forall p, q\in M$, $0<\tau_1<\tau_2$

$\Rightarrow$ $v := \lim_{\tau\to0^+}\sqrt{\tau}X(\tau)$ exists and $\neq 0$ by ODE

so the above ext to $\tau_1 = 0$

$\Rightarrow$ $\mathcal{L}exp_\tau: v\mapsto\gamma(\tau)$ is defined!

Let $L(q,\bar\tau)$ = $\mathcal{L}$-length of the $\mathcal{L}$-shortest $\gamma(\tau)$, $0\le\tau\le\bar\tau$.

$$\Rightarrow \nabla L(q,\bar\tau) = 2\sqrt{\bar\tau}\,X(\bar\tau) \quad \text{via 1st var formula}$$

$$L_{\bar\tau}(q,\bar\tau) = \frac{d}{d\tau}L(\gamma(\tau),\tau)\Big|_{\tau=\bar\tau} - \nabla L\cdot X = \sqrt{\bar\tau}\left(R+|X|^2\right) - \underbrace{\nabla L\cdot X}_{=2\sqrt{\bar\tau}|X|^2}$$

$$= 2\sqrt{\bar\tau}R - \sqrt{\bar\tau}\left(R+|X|^2\right)$$

Now $\frac{d}{d\tau}\left(R+|X|^2\right)(\gamma(\tau)) = R_\tau + \nabla R\cdot X + 2\nabla_X X\cdot X + 2Ric(X,X)$

i.e. along γ — by geod eq'n

$$(**)\quad = R_\tau + 2\nabla R\cdot X - 2Ric(X,X) - \frac{1}{\tau}|X|^2$$

$$= -H(X) - \frac{1}{\tau}\left(R+|X|^2\right) \qquad H = \text{Hamilton's trace quadratic with } t = -\tau$$

multiply $\tau^{\frac32}$ and int by parts of LHS get

$$\tau^{\frac32}\left(R+|X|^2\right)\Big|_0^{\bar\tau} - \frac32\int_0^{\bar\tau}\tau^{1/2}\left(R+|X|^2\right)d\tau = -\int_0^{\bar\tau}\tau^{\frac32}H\,d\tau - \int_0^{\bar\tau}\tau^{\frac12}\left(R+|X|^2\right)$$

$$\bar\tau^{3/2}\left(R+|X|^2\right) - 0, \qquad = \tfrac32 L, \qquad = K(\gamma,\bar\tau), \qquad = L$$

notice the remaining $\tau^{1/2}$!!

i.e. $\bar\tau^{\frac32}\left(R+|X|^2\right) = -K + \frac12 L$ Ex. Caution: A naive ODE solving on $(**)$ will fail!

$\Rightarrow$ Lemma (grad estimate)

$$L_{\bar\tau} = 2\sqrt{\bar\tau}R - \frac{1}{2\bar\tau}L + \frac{1}{\bar\tau}K$$

$$|\nabla L|^2 = -4\bar\tau R + \frac{2}{\sqrt{\bar\tau}}L - \frac{4}{\sqrt{\bar\tau}}K.$$

Theorem (Perelman): Let $\ell(q,\tau) := \dfrac{L(q,\tau)}{2\sqrt{\tau}}$ (reduced distance)

Then the reduced vol $\tilde V(\tau) := \int_M \tau^{-\frac n2}e^{-\ell(q,\tau)}dq$ is non-decreasing in τ.

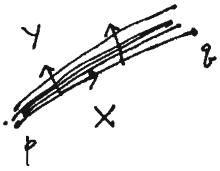
Rmk: [Kleiner-Lott v.5 2013] Notes on Perelman's papers are good supplements.

Ex. Work out the flat case $M = \mathbb{R}^n$, for $p = 0, q$, $\gamma(\tau) = \sqrt{\tau}\,\frac{q}{\sqrt{\bar\tau}} = 2\sqrt{\tau}\,v$

$\ell(q,\bar\tau) = |q|^2/4\bar\tau$ and $\tilde V(\tau)$ is constant in τ. (shrinking Gaussian Soliton)

To prove Thm, need 2nd var. (comp. geom.) along $\mathcal{L}$-geod. γ shortest one!

clearly $\delta_Y^2 \mathcal{L}(\gamma) = \int_0^{\bar\tau} \sqrt{\tau}\left(YYR + 2\langle D_Y \underline{D_Y X}, X\rangle + 2|\underline{D_Y X}|^2\right)d\tau$



" $S(\tau)$ dep on τ " $\qquad$ " $D_X Y$ " $\qquad$ " $D_X Y$ "

How to take out? $2\left(\langle D_X D_Y Y, X\rangle + \langle R(Y,X)Y, X\rangle\right)$

Now $\frac{d}{d\tau}\langle D_Y Y, X\rangle = \langle D_X D_Y Y, X\rangle + \langle D_Y Y, D_X X\rangle + 2Y\cdot Ric(Y,X) - X\cdot Ric(Y,Y)$!

since $g(D_Y Y, X) = Y g(Y,X) - g(Y, D_Y X)$, $D_Y X = D_X Y$

If $Y(0) = 0$, int. by parts

$\Rightarrow \delta_Y^2 \mathcal{L} = 2\sqrt{\bar\tau}\langle D_Y Y, X\rangle + \int_0^{\bar\tau}\sqrt{\tau}\Big(D_Y D_Y R + 2\langle R(Y,X)Y, X\rangle + 2|D_X Y|^2 \tfrac{1}{2}\, g(Y,Y)$

Index form $I(Y,Y) \to$ $\qquad -4 D_Y Ric(Y,X) + 2 D_X Ric(Y,Y)\Big)\,d\tau$

Ex. Show that changing $Y\cdot$, $X\cdot$ to D_Y, D_X produce terms $-2\langle D_Y Y, G\rangle$ where G is the $\mathcal{L}$-geod eq'n, hence $= 0$

To const. Y, fix $Y(\bar\tau)$ with $|Y(\bar\tau)| = 1$ and solve on $[0,\bar\tau]$ the ODE

$$D_X Y = -Ric(Y,\cdot) + \frac{1}{2\tau} Y \qquad \text{"}\mathcal{L}\text{-parallel transport?"}$$

$\Rightarrow \frac{d}{d\tau}|Y|^2 = 2Ric(Y,Y) + 2\langle D_X Y, Y\rangle = \frac{1}{\tau}|Y|^2 \Rightarrow |Y(\tau)|^2 = \frac{\tau}{\bar\tau} \Rightarrow Y(0) = 0$!

also $\langle Y_i(\tau), Y_j(\tau)\rangle = \frac{\tau}{\bar\tau}\langle Y_i(\bar\tau), Y_j(\bar\tau)\rangle$.

Get $\mathrm{Hess}_L(Y,Y) = YYL - D_Y Y\cdot L$

$\leq \delta_Y^2 L - 2\sqrt{\bar\tau}\langle D_Y Y, X\rangle = \int_0^{\bar\tau}\sqrt{\tau}\Big(D_Y D_Y R + 2\langle R(Y,X)Y, X\rangle - 4 D_Y Ric(Y,X) + 2 D_X Ric(Y,Y)$

$\qquad + 2|Ric(Y,\cdot)|^2 - \frac{2}{\tau}Ric(Y,Y) + \frac{1}{2\tau\bar\tau}\Big)\,d\tau \qquad ① + 2②$

Since $\frac{d}{d\tau} Ric(Y(\tau), Y(\tau)) = Ric_\tau(Y,Y) + \underline{D_X Ric(Y,Y)} + 2Ric(D_X Y, Y)$

$\qquad\qquad \text{"}\ \frac{1}{\tau}Ric(Y,Y) - 2\cdot|Ric(Y,\cdot)|^2$

$\Rightarrow \mathrm{Hess}_L(Y,Y) \leq \int_0^{\bar\tau}\sqrt{\tau}\Big[① + 4② - 2\Big(\frac{d}{d\tau}Ric(Y,Y) - Ric_\tau(Y,Y) + 2|Ric(Y,\cdot)|^2 - \frac{1}{\tau}Ric(Y,Y)\Big)\Big]$

Ex.
(int. by parts) $= \frac{1}{\sqrt{\bar\tau}} - 2\sqrt{\bar\tau}\,Ric(Y,Y) - \int_0^{\bar\tau}\sqrt{\tau}\,H(X,Y)\,d\tau$

Hamilton's quadratic with $t = -\tau$

Taking trace over $|Y_i(\bar\tau)| = 1$, ONB

$\Rightarrow \Delta L \leq -2\sqrt{\bar\tau}R + \frac{n}{\sqrt{\bar\tau}} - \frac{1}{\bar\tau}\int_0^{\bar\tau}\underline{\sqrt{\tau}^3 H(\tilde Y_i, \tilde Y_i)\,d\tau}$ since $\tilde Y_i = Y_i\left(\frac{\bar\tau}{\tau}\right)^{1/2}$ ONB at τ.

(in the barrier sense) $\qquad$ " K, now has no signs!

Def'n: $Y(\tau)$ is $\mathcal{L}$-Jacobi if it is variation field of $\mathcal{L}$-geodesics.

Ex. Fact: "=" holds in $\mathrm{Hess}_L(Y,Y) \leq \cdots \Leftrightarrow Y$ is an $\mathcal{L}$-Jacobi field.

Will see$^{(i)}$ $\Delta L \leq$, "=" holds $\Leftrightarrow R_{ij} + \frac{1}{2\sqrt{\bar\tau}}\nabla_i\nabla_j L \cdot \frac{1}{2\bar\tau} g_{ij}$ grad. shrinking soliton

and prove$^{(ii)}$ $\tilde V(\tau) \nearrow$ via $\mathcal{L}$-exp. & $^{(iii)}$get local non-collapsing.

Diff Geom II, 6/9, 2021

Let $\gamma : [0,\bar{\tau}] \to M$ be a $\mathcal{L}$-geod. Y $\mathcal{L}$-Jacobi on γ, $|Y(\bar{\tau})|=1$

let $D_X \tilde{Y} = -\mathrm{Ric}(\tilde{Y},\cdot) + \frac{1}{2\tau}\tilde{Y}$ with $\tilde{Y}(\bar{\tau}) = Y(\bar{\tau})$.

$\Rightarrow \frac{d}{d\tau}|Y|^2 = 2\mathrm{Ric}(Y,Y) + 2(D_X Y, Y)$

$$\text{at } \tau=\bar{\tau} \quad = 2\mathrm{Ric}(Y,Y) + \frac{1}{\sqrt{\bar{\tau}}}\mathrm{Hess}_L(Y,Y) \leq \frac{1}{\bar{\tau}} - \frac{1}{\bar{\tau}}\int_0^{\bar{\tau}} \sqrt{\tau}\, H(X, \underline{\tilde{Y}})$$

(*) "$D_Y X = D_Y\left(\frac{\nabla L(\tau)}{2\sqrt{\tau}}\right)$"

But ~~last Ex~~ $\Rightarrow$ "=" iff $\tilde{Y}$ is $\mathcal{L}$-Jacobi, i.e. $\tilde{Y} = Y$

hence $\frac{d}{d\tau}|Y|^2 = \frac{d}{d\tau}|\tilde{Y}|^2 = \frac{1}{\bar{\tau}}$.

Thm (i). Apply to $\frac{d}{d\tau}\langle Y_i, Y_j\rangle = 2\mathrm{Ric}(Y_i,Y_j) + \frac{1}{\sqrt{\bar{\tau}}}\mathrm{Hess}_L(Y_i,Y_j) = \frac{1}{\bar{\tau}}\delta_{ij}$

$\Rightarrow \Delta L = -2\sqrt{\bar{\tau}}R + \frac{n}{\sqrt{\bar{\tau}}} - \frac{1}{\bar{\tau}}K$ holds everywhere $\Leftrightarrow R_{ij} + \frac{1}{2\sqrt{\bar{\tau}}}\nabla_i\nabla_j L = \frac{1}{2\bar{\tau}}g_{ij}$

i.e. grad. shrinking soliton

Jacobian J of $\mathcal{L}\text{-exp}_{\bar{\tau}}(v) \equiv \mathcal{L}\exp_v(\bar{\tau})$

~~By def'n~~, Jacobi fields V_i is constr. by $V_i(\tau) := (d\mathcal{L}\exp_v(\tau))\, v_i$; choose $v_i \in T_pM$ s.t. ONB at $\bar{\tau}$ for $V_i(\tau)$.

standard Riem geom + (*) above (factor $\frac{1}{2\sqrt{\bar{\tau}}}$) + Ricci flow

$$\Rightarrow \frac{d}{d\tau}\log J(\bar{\tau}) = \sum_{i=1}^n \left(\langle D_X V_i, V_i\rangle(\bar{\tau}) + \frac{1}{2}\cdot 2\mathrm{Ric}(V_i,V_i)\right)$$

$$= \frac{1}{2\sqrt{\bar{\tau}}}\Delta L + R \leq \underline{\frac{n}{2\bar{\tau}} - \frac{1}{2}\bar{\tau}^{-3/2}K}$$

~~By def'n~~:

$$\frac{d}{d\tau}\ell(\tau) := \frac{d}{d\tau}\ell(\overset{\shortparallel}{q}{}_{\gamma(\tau)},\tau) = \frac{d}{d\tau}\left(\frac{1}{2\sqrt{\tau}}L(\overset{\shortparallel}{q}{}_{\gamma(\tau)},\tau)\right) = -\frac{1}{4}\tau^{-\frac{3}{2}} + \frac{d}{d\tau}L(\gamma(\tau),\tau)\cdot\frac{1}{2\sqrt{\tau}}$$

Along $\gamma(\tau)$:

$$= -\frac{1}{2\tau}\ell(\tau) + \frac{1}{2}(R+|X|^2)$$

Thm (ii). $\tilde{V}(\tau) = \int_M \tau^{-n/2} e^{-\ell(q,\tau)}\,dq \searrow$ in τ. ($\nearrow$ in t)

Pf: Recall $\tau^{-3/2}(R+|X|^2) = -K + L$, hence $\underline{\frac{d}{d\tau}\ell(\tau) = -\frac{1}{2}\tau^{-3/2}K}$

The above two $\Rightarrow \frac{d}{d\tau}\log J(\tau) \leq \frac{n}{2\tau} + \frac{d}{d\tau}\ell(\tau) \Rightarrow$ Thm (ii) if

$\int_M(\cdots) < \infty$. But at $\tau=0$ it is just $\int_{\mathbb{R}^n} e^{-|v|^2}dv$ ✱

RmK: The key point is before $\int_M$ we already have $\searrow$. So this may be applied to the non-cpt (complete case) later.

Ex / Prop: Indeed, (a) $\ell_{\bar{\tau}} - \Delta\ell + |\nabla\ell|^2 - R + \frac{n}{2\bar{\tau}} \geq 0$. grad esti $+ \Delta L \leq \ldots$

(b) $\bar{L}_\tau + \Delta\bar{L} \leq 2n$ for $\bar{L} = 2\sqrt{\tau}L = 4\tau\ell$.

(a) $\Rightarrow \left(\frac{d}{d\tau} - \Delta + R\right)(\bar{\tau}^{-n/2}e^{-\ell}) \leq 0 \Rightarrow \frac{d}{d\tau}\tilde{V} \leq \int_M \Delta(\bar{\tau}^{-n/2}e^{-\ell}) = 0$ if M is cpt.

i.e. Thm (ii).

(b) $\Rightarrow$ min of $\bar{L}(\cdot,\bar{\tau}) - 2n\bar{\tau} \searrow$, hence:

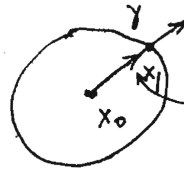
Cor. $\min\limits_{q\in M}\ell(q,\tau) \leq \frac{n}{2}$ if M cpt.

Def$^{\text{n}}$: $\frac{\partial g}{\partial t} = -2\,Ric$ is κ-collapsed (non) at (x_0, t_0) on scale $r > 0$ if
$(\forall)$ $|Rm|(x,t) \le \frac{1}{r^2}$ $\forall (x,t)$ st $d_{t_0}(x, x_0) < r$, $t \in [t_0 - r^2, t_0]$
but (then) $|B_{x_0}(r)| \underset{(\ge)}{<} \kappa r^n$ at t_0. price to pay for $\underline{\text{local version}}$!

Thm (No local Collapsing II, Perelman): $\forall A > 0$, $\exists \kappa(A) > 0$ st. for
$\dot g = -2Ric$ on $t \in [0, r_0^2]$ with $\left(\begin{array}{l} |Rm| \le \frac{1}{r_0^2} \text{ on } B_{0,x_0}(r_0) \times [0, r_0^2] \\ V(B_{0,x_0}(r_0)) \ge A^{-1} r_0^n \end{array} \right)$
$\Rightarrow$ $g(t)$ κ-non collapsed on all $r < r_0$
at every pt (x, r_0^2) with $d_{r_0^2}(x, x_0) \le A r_0$.

Lemma (int. version of Bonnet-Myer) At $t = t_0$:
(a) If $Ric \le (n-1)K$ on $\underline{B_{t_0,x_0}(r_0)}$ then $d(x,t) := dist_t(x, x_0)$ satisfies
$\frac{d}{dt} d_t - \Delta d \ge -(n-1)\left(\frac{2}{3} K r_0 + r_0^{-1}\right)$ $\underline{\text{outside it}}$.
(b) If on $B_{t_0,\underline{x_0}}(r_0) \cup B_{t_0,\underline{x_1}}(r_0)$ then
$$\frac{d}{dt} d_t(x_0, x_1) \ge -2(n-1)\left(\frac{2}{3} K r_0 + r_0^{-1}\right)$$
~~pf of Lemma (a)~~: the whole pt is $\frac{d}{dt}\, \underline{dist_t(x, x_0)} = \frac{d}{dt}\int_0^d \sqrt{g_{ij}\dot x^i \dot x^j}\, ds = \int_\gamma -Ric(X,X)\, ds$
joined by geod γ, $X = \dot\gamma$

$\Delta d \le \sum_{k=1}^{n-1} \delta''_{Y_k}(\gamma) \equiv \int_0^d \left(|\dot Y_k|^2 - R(X, Y_k, X, Y_k)\right) ds$

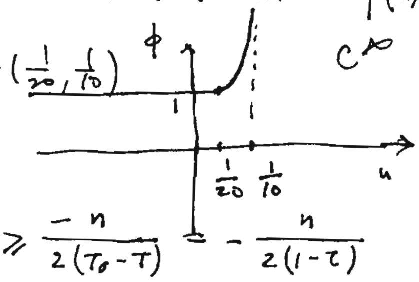


X parallel $\Rightarrow \dot Y_k = 0$; $= \int_0^{r_0} \frac{1}{r_0^2}\left(n-1 - s^2 Ric(X,X)\right) ds + \int_{\underline{r_0}}^d -Ric(X,X)\, ds$
linear, $x_1 = \gamma(r_0)$
$Y_k(x_0) = 0 \Rightarrow Y_k(s) = \frac{s}{r_0} e_i(s)$
$= \frac{d}{dt} d + \frac{n-1}{r_0} + \int_0^{r_0}\left(1 - \frac{s^2}{r_0^2}\right) \underset{\underset{(n-1)K}{\wedge}}{\underline{R(X,X)}}\, ds$
(b) is Similar. #

~~pf of Thm Sketched~~:
By scaling, set $r_0 = 1$, $d_1(x, x_0) = A$. Let $p = x$, $\tau(t) = 1 - t$
Prove by contradiction: If $\exists r < 1$, $|Rm|(y,t) \le \frac{1}{r^2}$ for $y \in B_{1,x}(r)$, $1 - r^2 \le t \le 1$
with $\epsilon := |B_{1,x}(r)|^{1/n}/r$ $\underline{\text{small}}$.

$*$ ~~Claim~~ A: $\tilde V(\epsilon r^2) \le 2\epsilon^{n/2}$ (using $\mathcal{L}\exp$).
But $\tilde V(1) \le \tilde V(\epsilon r^2)$. $\tilde V(1) = \int_M e^{-\ell(q,1)}\, dVol(q)$, will get $*$
if we bound $\ell(q,1)$ ~~above~~ on $B_{0,x_0}(1)$.
$\underline{\text{Claim B}}$: Enough to bound $\min \ell(q, \frac{1}{2})$ ~~above~~ on $B_{\frac{1}{2},x_0}(\frac{1}{10})$. ∞

The idea is to use $\bar L_\tau + \Delta \bar L \le 2n$ and localize by a radial function $\phi(u)$
$$2\frac{\phi'^2}{\phi} - \phi'' \ge (2A + 100n)\phi' - C(A)\phi, \quad u \in \left(\tfrac{1}{20}, \tfrac{1}{10}\right)$$



eg. $\phi(u) = 1/\left[e^{(2A+100n)(\frac{1}{10} - u)} - 1\right]$

Notice that $R_\tau = -\Delta R - 2|Ric|^2 \le -\Delta R - \frac{2}{n}R^2 \Rightarrow R \ge \frac{-n}{2(\tau_0 - \tau)} = -\frac{n}{2(1-\tau)}$

$\underline{\text{for } \bar\tau \in [0, \frac{1}{2}]}$ $\Rightarrow L(q, \bar\tau) \ge -\int_0^{\bar\tau} \sqrt{\tau}\, \frac{n}{2(1-\tau)}\, d\tau \ge -n\int_0^{\bar\tau} \sqrt{\tau}\, d\tau = -\frac{2}{3}\bar\tau^{3/2}$

$\Rightarrow \bar L(q, \bar\tau) = 2\sqrt{\bar\tau}\, L(q, \bar\tau) \ge -\frac{4n}{3}\bar\tau^2 \ge -\frac{n}{3} \Rightarrow \underline{\bar L + 2n + 1 \ge 1}$.
super rough!

Put $h(y,t) = \phi\left(d(y,t) - A\cdot(2t-1)\right)\left(\bar{L}(y,1-t)+2n+1\right)$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{d_t(y,x_0)}$

$h(y,t) \geq 0$ for $t \geq \frac{1}{2}$

$\min_y h(y,1) \leq h(x,1) = 2n+1$, $\min h(y,\frac{1}{2})$ is achieved for some $d(y,\frac{1}{2}) \leq \frac{1}{10}$.

For $\square := \frac{\partial}{\partial t} - \Delta$, we compute

$$\begin{cases} \square h = (\bar{L}+2n+1)\cdot\left(-\phi'' + (\square d - 2A)\phi'\right) - 2\nabla\phi\cdot\nabla\bar{L} + \phi\,\square\bar{L} \\ \nabla h = (\bar{L}+2n+1)\nabla\phi + \phi\nabla\bar{L} \end{cases}$$

At a min pt of h, $\nabla h = 0$ and get $\quad -(\bar{L}_\tau + \Delta L) \geq -2n$

$$\square h = (\bar{L}+2n+1)\left(-\phi'' + (\square d - 2A)\phi' + 2\frac{\phi'^2}{\phi}\right) + \phi\,\square\bar{L}$$

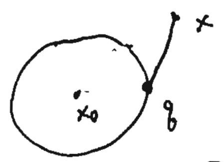
since $\phi' \neq 0 \Rightarrow d(y,t) \geq \frac{1}{20}$. also $Ric \leq n-1$ in $B(x_0,\frac{1}{20})$ $(|Rm| \leq 1)$

Lemma (a) $\Rightarrow$ rough estimate $\square d \geq -100(n-1)$

$\Rightarrow \square h \geq -(2n + C(A))\phi \geq -(2n+C(A))\,h$

$\Rightarrow \frac{d}{dt} h_{min}(t) \geq -(2n+C(A))\,h_{min}(t) \Rightarrow \min \bar{L}(y,\frac{1}{2}) + 2n+1 \leq (2n+1)\,e^{n+\frac{C(A)}{2}}$ #

Rmk on Claim B (and claim A): $\forall q \in B_{0,x_0}(1)$, chose $\gamma(\tau): [0,1] \to M$, $\gamma(0) = x$, $\gamma(1) = q$
$\gamma(\tau) \in B_{1-\tau,x_0}(1)$ for $\underline{\tau \in [\frac{1}{2},1]}$, $g(\tau)$ all equiv there



The point is to see $\gamma(\frac{1}{2}) \in B_{\frac{1}{2},x_0}(\frac{1}{10})$

Then use def^{n} of $\mathcal{L}(\gamma)|_{[\frac{1}{2},1]} = \int_{\frac{1}{2}}^{1}\sqrt{\tau}\,(R + |\dot{\gamma}|^2)\,d\tau$. ($\leq 1$ enough !)

This is how Cao-Zhu or Kleiner-Lott did, but I feel NOT ok.

* Perelman's original writing is elegant: " $\tilde{V}(1)$ can not be small unless $\min \ell(x,\frac{1}{2})$ over $d_{\frac{1}{2}}(x,x_0) \leq \frac{1}{10}$ is large" !

Thm: No local collapsing $\Rightarrow$ Little loop Lemma.

LLL of Hamilton says: $\dot{g} = -2Ric$ on cpt M, $[0,T)$, $T < \infty$
Then $\exists \rho > 0$ st. if $\exists (x_0,t_0)$ $|Rm| \leq \frac{1}{r^2}$ on $B_{t_0,x_0}(r)$
for some $r \leq \sqrt{T}$, then $inj(M,x_0,t_0) \geq \rho r$.

Cor. $\Rightarrow inj \geq \frac{c_2}{\sqrt{K_{max}}}$ (inj radius condition) $\Rightarrow$ Existence of sing models
in the sequence of almost max pt (x_k,t_k)

Pf of Thm: Combine with [Cheng-Li-Yau] (1981, AJM)

$$inj(M,x_0) \geq r\,\frac{|B_{x_0}(r)|}{|B_{x_0}(r)| + V_\lambda^n(2r)}$$

where $\lambda \leq K \leq \Lambda$ on $B_{x_0}(4r_0)$
$r \leq \min(r_0, \frac{\pi}{4\sqrt{\Lambda}})$ if $\Lambda > 0$ #