Chapter 5

BASIC LIE THEORY

1. Categories of Lie groups and Lie algebras

A C^∞ manifold G is a *Lie group* if G has a group structure and the group law

$$G \times G \to G; \quad (g,h) \mapsto gh^{-1} \text{ are } C^\infty.$$

For $g \in G$, we denote the left multiplication map $h \mapsto gh$ by L_g and right multiplication $h \mapsto hg$ by R_g. We have the induced map on tangent spaces:

$$dL_g : T_hG \to T_{gh}G; \quad dR_g : T_hG \to T_{hg}G.$$

A vector field $X \in C^\infty(TG)$ is *left invariant* if $X_{gh} = dL_g X_h$ for all $g, h \in G$. The *Lie algebra* $\mathfrak{g} = \operatorname{Lie} G$ of G is the vector space of all left invariant vector fields (l.i.v.f.) under bracket operation. Namely, as differential operators, for $f \in C^\infty(G)$:

$$[X,Y]f := X(Yf) - Y(Xf).$$

Since a l.i.v.f. X is determined by its value X_e at the identity $e \in G$, we identify

$$\mathfrak{g} \cong T_eG.$$

Abstractly, a vector space L over a field F (with $\operatorname{char} F \neq 2$) with an F-bilinear map $[\,,] : L \times L \to L$ is called a *Lie algebra* (over F) if $[x,y] = -[y,x]$ and

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0. \quad \text{(Jacobi identity)}\,.$$

It is clear that the bracket of vector fields has this property.

Example 5.1. Consider the *general linear group*

$$G = \mathrm{GL}(n,\mathbb{R}) = \{\, g \in M_{n\times n}(\mathbb{R}) \mid \det g \neq 0 \,\}.$$

From Cramer's rule, we see that $g \mapsto g^{-1}$ is C^∞ hence that G is a Lie group.

As an open subset of $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$, we have $T_eG = M_{n\times n}(\mathbb{R})$. The matrix algebra has a natural Lie algebra structure $\mathfrak{gl}(n,\mathbb{R})$ defined by

$$[A,B] := AB - BA.$$

Theorem 5.2. $\mathfrak{gl}(n,\mathbb{R})$ *coincides with* Lie G.

Proof. From $(gh(t))' = gh'(t)$, we see that $(L_g)_*A = gA$ for $g \in G$, $A \in T_eG$. Thus if $\tilde{A}$ is the l.i.v.f. with $\tilde{A}_e = A$, then $\tilde{A}_g = gA$. Let $G \hookrightarrow \mathbb{R}^{n^2}$ with coordinates $(x_{ij})_{i,j=1}^n$ being the entries of the corresponding matrix g. Then a tangent vector $A = (a_{ij}) \in T_eG$ and $\tilde{A}$ are equivalent to

$$A = \sum_{i,j} a_{ij} \frac{\partial}{\partial x_{ij}}\Big|_e \quad \text{and} \quad \tilde{A}_g = \sum_{i,j} (gA)_{ij} \frac{\partial}{\partial x_{ij}}\Big|_g$$

respectively. From

$$\sum_m \frac{\partial}{\partial x_{ij}}(x_{km}b_{ml}) = \sum_m \delta_{ki}\delta_{mj}b_{ml} = \delta_{ki}b_{jl},$$

we compute

$$\begin{aligned}[\tilde{A},\tilde{B}]_e &= \sum_{i,j,k,l} \left(a_{ij}\frac{\partial}{\partial x_{ij}}((gB)_{kl})\frac{\partial}{\partial x_{kl}} - b_{ij}\frac{\partial}{\partial x_{ij}}((gA)_{kl})\frac{\partial}{\partial x_{kl}} \right)\Big|_{g=e} \\ &= \sum_{i,j,l} a_{ij}b_{jl}\frac{\partial}{\partial x_{il}}\Big|_e - b_{ij}a_{jl}\frac{\partial}{\partial x_{il}}\Big|_e = \sum_{i,l}(AB-BA)_{il}\frac{\partial}{\partial x_{il}}\Big|_e.\end{aligned}$$

This corresponds to $AB - BA$ precisely. □

A *Lie subgroup* $H < G$ is itself a Lie group such that H is both a subgroup and an *immersion*. We allow $H \subset G$ to be *not closed*.

Example 5.3. Subgroups of matrix groups are the main sources of Lie groups.

(i) Let $\mathrm{SL}(n,\mathbb{R}) = \{ g \in \mathrm{GL}(n,\mathbb{R}) \mid \det g = 1 \}$ be the *special linear group*. Consider a smooth curve $t \mapsto g(t)$ with $g(0) = e$ and $\det g(t) = 1$. Then we compute $\operatorname{tr} g'(0) = 0$. So its Lie algebra is given by $\mathfrak{sl}(n,\mathbb{R}) = \{ A \in M_{n\times n}(\mathbb{R}) \mid \operatorname{tr} A = 0 \}$.

(ii) Let $\mathrm{O}(n,\mathbb{R}) = \{\, g \in \mathrm{GL}(n,\mathbb{R}) \,|\, g^T g = e \,\}$ be the *orthogonal group*. Consider a smooth curve $t \mapsto g(t)$ with $g(0) = e$ and $g(t)^T g(t) = e$. Then we compute $g'(0)^T + g'(0) = 0$. So its Lie algebra is given by:
$\mathfrak{o}(n,\mathbb{R}) = \{\, A \in M_{n\times n}(\mathbb{R}) \,|\, A^T = -A \,\}$.

(iii) Let $\mathrm{SO}(n,\mathbb{R}) = \{\, g \in \mathrm{O}(n,\mathbb{R}) \,|\, \det g = 1 \,\}$ be the *special orthogonal group* . It is clear that $\mathrm{O}(n,\mathbb{R})$ has two connected components and $\mathrm{SO}(n,\mathbb{R})$ is the identity component, so
$\mathfrak{so}(n,\mathbb{R}) = \mathfrak{o}(n,\mathbb{R})$.

(iv) Let $\mathrm{Sp}(2n,\mathbb{R}) = \{\, g \in M_{2n\times 2n}(\mathbb{R}) \,|\, g^T J g = J \,\}$ be the *symplectic group*, where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Its Lie algebra $\mathrm{sp}(2n,\mathbb{R})$ is given by:
$\{\, A \in M_{2n\times 2n}(\mathbb{R}) \,|\, A^T J = -JA \,\}$.

(v) We have similar *complex Lie groups* $\mathrm{GL}(n,\mathbb{C}), \mathrm{SL}(n,\mathbb{C}), \mathrm{O}(n,\mathbb{C})$, $\mathrm{SO}(n,\mathbb{C})$ and $\mathrm{Sp}(n,\mathbb{C})$. Indeed they are defined by algebraic equations with integer coefficient, so they can take values in any field. The corresponding Lie algebras $\mathfrak{gl}(n,\mathbb{C})$, $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$ and $\mathfrak{sp}(n,\mathbb{C})$ are complex Lie algebras.

(vi) Let $\mathrm{U}(n) = \{\, g \in \mathrm{GL}(n,\mathbb{C}) \,|\, g^* g = e \,\}$ be the *unitary group*. Consider a smooth curve $t \mapsto g(t)$ with $g(0) = e$ and $g(t)^* g(t) = e$. Then we compute $g'(0)^* + g'(0) = 0$. So its Lie algebra is given by $\mathfrak{u}(n) = \{\, A \in M_{n\times n}(\mathbb{C}) \,|\, A^* = -A \,\}$. Notice that $\mathfrak{u}(n)$ is a real Lie algebra.

(vii) Let $\mathrm{SU}(n) = \{\, g \in \mathrm{U}(n) \,|\, \det g = 1 \,\}$ be *special unitary group*.
$\mathfrak{su}(n) = \mathfrak{sl}(n,\mathbb{C}) \cap \mathfrak{u}(n)$.

All these subgroups can be realized as the subgroup preserving certain additional structure. For "S", g preserves volume. For "O", g preserves the Euclidean inner product. For "Sp", g preserves the non-degenerate symplectic form

$$x^T J y = (x_1 y_{n+1} - x_{n+1} y_1) + \cdots + (x_n y_{2n} - x_{2n} y_n).$$

And for "U", g preserves the Hermitian inner product.

$\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(2n, \mathbb{C})$, $\mathfrak{sp}(n, \mathbb{C})$ and $\mathfrak{so}(2n+1, \mathbb{C})$ are known as *classical complex semi-simple Lie algebras* of type A_n, B_n, C_n and D_n respectively. (To be explained later).

Theorem 5.4. *Given a Lie group G. There is an one to one correspondence between connected Lie subgroups of G and Lie subalgebras of $\mathfrak{g}$.*

PROOF. This follows from the *Frobenius Theorem* (cf. theorem 1.40). For a basis X_i of a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we defined a subspace distribution $\mathcal{H}_g$ which is spanned by X_{i_g} for all $g \in G$. The distribution $\mathcal{H} = \bigcup_{g \in G} \mathcal{H}_g$ is integrable. Indeed, for any two C^∞ vector fields $V = \sum f_i X_i$ and $W = \sum g_i X_i$, we compute

$$[V, W] = \sum f_i g_j [X_i, X_j] + \sum f_i (X_i g_j) X_j - \sum g_j (X_j f_i) X_i \in \mathcal{H}.$$

We then take H to be the maximal integral submanifold passing through $e \in G$.

To check that H is a group, let $g \in H$. The map L_g maps the manifold H to gH. The left invariance says that $dL_g \mathcal{H}_h = \mathcal{H}_{gh}$, hence gH is also an integral submanifold. Now H and gH both contain the element g, hence the maximality (uniqueness) implies that $H = gH$. This implies that H is closed under multiplication and also $g^{-1} \in H$ (since $e \in H$). So H is a subgroup of G.

Finally, H is a Lie groups simply because the map $H \times H \to H$ sending (g, h) to gh^{-1} is the restriction of the given C^∞ map $G \times G \to G$. □

Remark 5.5. For any Lie group G, the tangent bundle TG is a trivial vector bundle with global frame given by any basis of $\mathfrak{g}$.

More generally, a *Lie group homomorphism* $\rho : G \to H$ is a C^∞ map which is also a group homomorphism. The tangent map $d\rho : TG \to TH$ is compatible with l.i.v.f.'s. To see this, $\rho(gg') = \rho(g)\rho(g')$ means $\rho \circ L_g = L_{\rho(g)} \circ \rho$, so

$$d\rho \circ dL_g = dL_{\rho(g)} \circ d\rho.$$

Thus $d\rho : \mathfrak{g} \to \mathfrak{h}$. $d\rho$ is indeed a Lie algebra homomorphism in the sense that $d\rho[X,Y] = [d\rho(X), d\rho(Y)]$, which is easily verified from the definitions.

2. Exponential map

We call a *nontrivial* Lie group homomorphism $\mathbb{R} \to G$ a *one parameter subgroup*, even though it may not be injective. The *exponential map* links Lie algebras with Lie groups through the consideration of all one parameter subgroups. Before treating the abstract setting, we look at the case for matrix groups.

Example 5.6. For $A \in M_{n\times n}(\mathbb{C})$, $t \in \mathbb{C}$, we define the absolutely convergent series

$$e^{tA} = 1 + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^k}{k!}A^k + \cdots .$$

It is easily checked that if $AB = BA$ then $e^A e^B = e^{A+B}$. Hence e^A has inverse e^{-A} and so $e^A \in \mathrm{GL}(n,\mathbb{C})$. Moreover $\gamma(t) = e^{tA}$ is the one parameter subgroup with

$$\gamma'(t) = e^{tA}A = dL_{\gamma(t)}A = A_{\gamma(t)}.$$

That is, e^{tA} is the integral curve of the l.i.v.f. determined by $A \in \mathfrak{gl}(n,\mathbb{C})$.

The discussion works for $\mathbb{C}$ being replaces by $\mathbb{R}$. Also if we take A be in a Lie subalgebra, the e^A lies in the corresponding Lie subgroup. This follows from the previous theorem. But we can also see how it works explicitly: For example,

$$\operatorname{tr} A = 0 \quad \Longrightarrow \quad \det e^A = e^{\operatorname{tr} A} = 1.$$

Also

$$A^* = -A \quad \Longrightarrow \quad (e^A)^* e^A = e^{A^*} e^A = e^{-A} e^A = I_n.$$

Now we turn to a general Lie group G. Let $X \in \mathfrak{g}$. Since $\mathbb{R}X < \mathfrak{g}$ is a one dimensional Lie subalgebra, by the previous theorem its integral curve is a one dimensional subgroup H. By taking the universal cover $\mathbb{R} \to H$ if necessary, we get a one parameter subgroup which we denote by $t \mapsto \exp tX$. We shall give a direct proof of this with stronger conclusions.

Let ϕ_t be *the flow generated by* X. That is, $\phi_t(g)$ is the curve with $\phi_0(g) = g$ and

$$\frac{d}{dt}\phi_t(g) = X_{\phi_t(g)}.$$

Theorem 5.7. *The range of t is $\mathbb{R}$ for all $g \in G$. Moreover, $\phi_t : G \to G$ is a one-parameter group of diffeomorphisms as right translations $\phi_t = R_{\phi_t(e)}$.*

PROOF. Consider the curve $g\phi_t(e)$. Since $g\phi_0(e) = g$ and

$$\begin{aligned}\frac{d}{dt}\big(g\phi_t(e)\big) &= dL_g\big(dL_{\phi_t(e)}X_e\big)\\ &= dL_{g\phi_t(e)}X_e\\ &= X_{g\phi_t(e)},\end{aligned}$$

we conclude that $\phi_t(g) = g\phi_t(e) = R_{\phi_t(e)}g$.

By substituting $g = \phi_s(e)$ we find $\phi_s(e)\phi_t(e) = \phi_t(\phi_s(e)) = \phi_{t+s}(e)$. This shows that for $g = e$, the range of t can be extended to all $\mathbb{R}$ and $\phi_t(e)$ is a one parameter subgroup. The theorem is proved by using the relation $\phi_t(g) = g\phi_t(e)$ again. □

Now we define the exponential map

$$\exp : \mathfrak{g} \to G$$

by $\exp tX = \phi_t(e)$ where ϕ_t is the flow generated by X. Since

$$(d\exp)_0(X) = \frac{d}{dt}\bigg|_{t=0} \exp tX = X,$$

we get $(d\exp)_0 = \mathrm{Id}_{\mathfrak{g}}$ and exp is invertible near $0 \in \mathfrak{g}$.

Corollary 5.8. *If $H < G$ is a Lie subgroup, then H is generated by* $\exp\mathfrak{h}$*.*

However, exp is not necessarily surjective, hence $\exp\mathfrak{g}$ is not necessarily a group.

Exercise 5.1. Let $X \in \mathfrak{sl}(2,\mathbb{R})$ and $d = \sqrt{|\det X|}$. Then

(i) $e^X = (\cosh d)I_2 + \frac{1}{d}(\sinh d)X$ if $\det X < 0$.

(ii) $e^X = (\cos d)I_2 + \frac{1}{d}(\sin d)X$ if $\det X > 0$.

(iii) $e^X = I_2 + X$ if $\det X = 0$.

Let $g_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$. Then g_a lies in a unique one parameter subgroup if $a > 0$. g_a lies in infinitely many one parameter subgroup if $a = -1$. If $a \neq -1$ and $a < 0$, then $g_a \notin \exp \mathfrak{sl}(2,\mathbb{R})$.

3. Adjoint representation

3.0.1. *Three adjoints I_g, Ad_g and ad_X.* For $g \in G$, let $I_g : G \to G$ be the inner automorphism $I_g(h) = L_g R_{g^{-1}}(h) = R_{g^{-1}} L_g(h) = ghg^{-1}$. Since $I_g(e) = e$, we get its differential

$$\mathrm{Ad}_g := dI_g : \mathfrak{g} \to \mathfrak{g}$$

as a Lie algebra automorphism. From $dI_{gg'} = d(I_g \circ I_{g'}) = dI_g \circ dI_{g'}$, we get the *adjoint representations of Lie group G*

$$\mathrm{Ad} : G \to \mathrm{Aut}\,\mathfrak{g}$$

and the *adjoint representation of Lie algebra $\mathfrak{g}$*

$$\mathrm{ad} := d(\mathrm{Ad}) : \mathfrak{g} \to \mathrm{End}\,\mathfrak{g}.$$

For G a matrix group, $\mathfrak{g}$ is a matrix Lie algebra and it is clear that $\mathrm{Ad}_g(Y) = gYg^{-1}$. For $g(t)$ a curve with $g(0) = e$ and $g'(0) = X$ we then compute

$$\mathrm{ad}_X(Y) = (g(t)Yg(t)^{-1})'(0) = XY - YX = [X,Y].$$

This property holds true in general:

Theorem 5.9. *For $X, Y \in \mathfrak{g}$,*

$$\mathrm{ad}_X Y = [X,Y].$$

PROOF. Let $f \in C^\infty(G)$ and ϕ, ψ be the flows generated by X, Y. Then

$$\begin{aligned}
(\mathrm{ad}_X Y)f &= \frac{d}{dt}\Big|_{t=0} (\mathrm{Ad}_{\exp tX} Y)f \\
&= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} f(I_{\exp tX}(\exp sY)) \\
&= \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} f(\exp tX \cdot \exp sY \cdot \exp(-tX)) \\
&= \frac{d}{ds}\frac{d}{dt}(f \circ \phi_{-t} \circ \psi_s \circ \phi_t(e))(0,0) \\
&= \frac{d}{ds} df(-X_{\psi_s(e)}) + d(f \circ \psi_s)X_e \Big|_{s=0} \\
&= -\frac{d}{ds}\Big|_{s=0} X_{\psi_s(e)} f + X_e \left(\frac{d}{ds}\Big|_{s=0} f \circ \psi_s \right) \\
&= -\frac{d}{ds}\Big|_{s=0} (Xf) \circ \psi_s(e) + X_e Y f \\
&= -Y_e X f + X_e Y f = [X,Y]_e f.
\end{aligned}$$

□

Remark 5.10. Readers with experience in differential geometry may observe that the proof is identical with the one for *Lie derivative* $L_X Y = [X, Y]$. Indeed,

$$\begin{aligned}
\mathrm{ad}_X Y &= \frac{d}{dt}\Big|_{t=0} (\mathrm{Ad}_{\exp tX} Y) \\
&= \frac{d}{dt}\Big|_{t=0} dR_{\exp(-tX)} dL_{\exp tX} Y \\
&= \frac{d}{dt}\Big|_{t=0} d\phi_{-t} Y = L_X Y
\end{aligned}$$

by the left invariance of Y and the definition of $L_X Y$.

It is harder to get explicit formula for Ad_g in the abstract setting. We have such a formula in two special cases, both are based on the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow[\rho]{} & H \\ {\scriptstyle\exp}\uparrow & & \uparrow{\scriptstyle\exp} \\ \mathfrak{g} & \xrightarrow[d\rho]{} & \mathfrak{h} \end{array}$$

To see this, simply notice that $\rho \exp tX$ and $\exp d\rho(tX)$ are both one parameter subgroups in H with the same tangent vector $d\rho(X)$ at $t = 0$.

By applying the diagram to $\rho = I_g$, we get:

$$\exp(\mathrm{Ad}_g X) = g(\exp X)g^{-1}.$$

(For matrix groups this is obvious).

By applying the diagram to $H = \mathrm{Aut}\,\mathfrak{g}$, $\rho = \mathrm{Ad}$ and $g = \exp X$, we get

$$\mathrm{Ad}_{\exp X} Y = e^{\mathrm{ad}_X} Y.$$

With these preparation, we give some applications of the adjoint representation:

3.0.2. *Center of a Lie group.* A Lie algebra is called *abelian* if $[X, Y] = 0$ for all X, Y. We denote $Z(G)$ by the center of G.

Proposition 5.11. *Let G be a connected Lie group, then $Z(G) = \mathrm{Ker}\,\mathrm{Ad}$. In particular, G is abelian if and only if $\mathfrak{g}$ is abelian.*

PROOF. If g is in the center, then for all $t \in \mathbb{R}$ and $X \in \mathfrak{g}$,

$$\exp tX = g(\exp tX)g^{-1} = \exp \mathrm{Ad}_g tX = \exp t\mathrm{Ad}_g X.$$

Hence $X = \mathrm{Ad}_g X$ for all X. That is, $\mathrm{Ad}_g = \mathrm{id}_{\mathfrak{g}}$.

Conversely, $g \in \mathrm{Ker}\,\mathrm{Ad}$ implies that $\exp X = g(\exp X)g^{-1}$. Hence g commutes with all elements in a neighborhood of e in G. By the connectedness of G we conclude that g commutes with every elements in G. □

Corollary 5.12. $[X, Y] = 0$ *implies that* $\exp X \cdot \exp Y = \exp(X + Y)$.

PROOF. Let $\mathfrak{h}$ be the two dimensional abelian Lie subalgebra of $\mathfrak{g}$ spanned by X and Y. Consider the Lie group H generated by $\exp \mathfrak{h}$. The proposition show that H is abelian and so the curve $\gamma(t) = \exp tX \cdot \exp tY$ is an one parameter subgroup. Since $\gamma'(0) = X + Y$, we conclude that $\exp tX \cdot \exp tY = \exp t(X+Y)$. ☐

Corollary 5.13. *If G is a connected Lie groups with trivial center, then*

$$\mathrm{Ad} : G \hookrightarrow \mathrm{Aut}\,\mathfrak{g} = \mathrm{GL}(\mathfrak{g})$$

is a faithful representation. In particular, G is a matrix subgroup.

3.0.3. *Normal Lie subgroups.* A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is a *Lie ideal* if $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$. In this case we denote by $\mathfrak{h} \lhd \mathfrak{g}$. It is clear that $\mathfrak{h}$ is at least a subalgebra.

Proposition 5.14. *Let $H < G$ be a connected Lie subgroup of a connected Lie group. Then*

$$H \lhd G \iff \mathfrak{h} := \mathrm{Lie}\, H \ \lhd\ \mathfrak{g}.$$

PROOF. Let $g = \exp X$ with $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$,
If $\mathfrak{h}$ is a Lie ideal of $\mathfrak{g}$, then

$$\begin{aligned} g(\exp Y)g^{-1} &= \exp \mathrm{Ad}_g Y \\ &= \exp(e^{\mathrm{ad}_X} Y) \\ &= \exp\left[\left(I + \mathrm{ad}_X + \frac{1}{2!}\mathrm{ad}_X^2 + \cdots\right)Y\right] \\ &\in \exp \mathfrak{h} \subset H. \end{aligned}$$

Since H is generated by $\mathfrak{h}$, this proves that H is normal.

Conversely, if H is normal, then the above computation shows that

$$\gamma(t) := \exp(e^{\mathrm{ad}_{tX}} Y) \in H.$$

Hence $\mathfrak{h} \ni \gamma'(0) = \mathrm{ad}_X Y = [X, Y]$ and $\mathfrak{h}$ is a Lie ideal. ☐

3.1. Fundamental correspondences.

3.1.1. *Equivalence of categories.*

Theorem 5.15. *Let G and H be connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, If G is simply connected, then there is a one to one correspondence between Lie group homomorphisms $G \to H$ and Lie algebra homomorphisms $\mathfrak{g} \to \mathfrak{h}$.*

IDEA OF PROOF. The is proved by exploring the Frobenius theorem on the product group $G \times H$ in a manner similar to the subgroup case.

Indeed a morphism $\rho : G \to H$ is equivalent to a subgroup $\Gamma \subset G \times H$ (graph of ρ) such that $\pi_G : G \times H \to G$ maps Γ onto G bijectively.

The given map $\mathfrak{g} \to \mathfrak{h}$ gives rise to a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$ and by the subgroup case we have proved, the corresponding Lie subgroup exists. The remaining problem is to prove the bijectivity of Γ onto G when G is simply connected. □

Exercise 5.2. Complete the remaining problem of Theorem 5.15.

3.1.2. *Ado's imbedding theorem.*

Theorem 5.16. *Every (finite dimensional) Lie algebra can be regarded as a Lie subalgebra of some $\mathfrak{gl}(n, \mathbb{R})$. Hence every simpley connected Lie group is a subgroup of $\mathrm{GL}(n, \mathbb{R})$. Moreover, every compact Lie group can be imbedded as a closed subgroup of some $\mathrm{O}(n, \mathbb{R})$.*

For a proof, see [Bou98], chapter I.

4. Differential geometry on Lie groups

4.1. Levi-Civita connection. Any inner product $\langle\, , \rangle_e$ on $T_eG = \mathfrak{g}$ uniquely determined a left invariant (Riemannian) metric on G by left translations. Namely for $v, w \in T_gG$,

$$\langle v, w \rangle_g := \langle dL_{g^{-1}}v, dL_{g^{-1}}w \rangle_e.$$

A *bi-invariant metric* is a metric which is both left and right invariant. We will shortly determine all Lie groups which admit bi-invariant metrics.

Proposition 5.17. (i) *For any left invariant metric $\langle\,,\rangle$ on G, and $X, Y \in \mathfrak{g}$, the Levi-Civita connection is given by*

$$\nabla_X Y = \tfrac{1}{2}([X,Y] - \mathrm{ad}_X^* Y - \mathrm{ad}_Y^* X).$$

(ii) *If $\langle\,,\rangle$ is bi-invariant, then $\langle \mathrm{ad}_Z X, Y\rangle + \langle X, \mathrm{ad}_Z Y\rangle = 0$ for $X, Y, Z \in \mathfrak{g}$. In particular, $\nabla_X Y = \frac{1}{2}[X,Y]$.*

Moreover, $R(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$ and $R(X,Y,X,Y) = \frac{1}{4}|[X,Y]|^2 \geq 0$.

PROOF. Recall that the *Levi-Civita connection* is the unique first order differential operator $\nabla_X : C^\infty(TM) \to C^\infty(TM)$ with $\nabla_X Y - \nabla_Y X = [X,Y]$ (torsion free) and $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle X, \nabla_Y Z\rangle$ (metrical). For any three vector fields $X, Y, Z \in C^\infty(TM)$, a cyclic computation leads to

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ - \langle Z, [Y,X]\rangle - \langle Y, [X,Z]\rangle - \langle X, [Y,Z]\rangle. \end{aligned}$$

If $X, Y, Z \in \mathfrak{g}$, all the inner products are constant in G. This leads to (i).

For (ii), the bi-invariance implies in particular that for $X, Y, Z \in \mathfrak{g}$,

$$\langle \mathrm{Ad}_{\exp tZ} X, \mathrm{Ad}_{\exp tZ} Y\rangle = \langle X, Y\rangle.$$

Take differentiation at $t = 0$ leads to $\langle \mathrm{ad}_Z X, Y\rangle + \langle X, \mathrm{ad}_Z Y\rangle = 0$. In the above formula, only the term $-\langle Z, [Y,X]\rangle$ is left, hence $\nabla_X Y = \frac{1}{2}[X,Y]$.

By the definition of the Riemann curvature operator,

$$\begin{aligned} R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \\ &= \tfrac{1}{4}[X,[Y,Z]] - \tfrac{1}{4}[Y,[X,Z]] - \tfrac{1}{2}[[X,Y],Z] = -\tfrac{1}{4}[[X,Y],Z], \end{aligned}$$

where the Jacobi identity is used to rewrite the second term. Finally,

$$\begin{aligned} R(X,Y,Z,W) &:= \langle R(X,Y)W, Z\rangle = -\tfrac{1}{4}\langle [[X,Y],W], Z\rangle \\ &= \tfrac{1}{4}\langle [W,[X,Y]], Z\rangle = -\tfrac{1}{4}\langle [X,Y],[W,Z]\rangle = \tfrac{1}{4}\langle [X,Y],[Z,W]\rangle, \end{aligned}$$

where the ad_W invariance of $\langle\,,\rangle$ is used. □

It is also straightforward to deduce from (i):

Corollary 5.18. *For left invariant metrics,*

$$R(X,Y,X,Y) = |\mathrm{ad}_X^* Y + \mathrm{ad}_Y^* X|^2 - \langle \mathrm{ad}_X^* X, \mathrm{ad}_Y^* Y\rangle \\ - \tfrac{3}{4}|[X,Y]|^2 - \tfrac{1}{2}\langle [[X,Y],Y],X\rangle - \tfrac{1}{2}\langle [[Y,X],X],Y\rangle.$$

Exercise 5.3. Show the Corollary 5.18 by Proposition 5.17(i).

4.1.1. *Lie groups with bi-invariant metrics.*

Theorem 5.19. *A connected Lie group G with a bi-invariant metric is complete, the exponential map is surjective and its one parameter subgroups coincides with geodesics through $e \in G$.*

Proof. By Proposition 5.17, for any l.i.v.f. X, $\nabla_X X = \frac{1}{2}[X,X] = 0$. Hence one parameter subgroups are the same as geodesics through $e \in G$. This implies that geodesics through e can be extended infinitely, so G is complete by the *Hopf-Rinow theorem*. In particular, the two exponential maps exp and $\exp_e$ (in Riemannian geometry) coincide and are surjective. □

Corollary 5.20. *If G has a bi-invariant metric, then any Lie group immersion $H \to G$ is totally geodesic.*

Corollary 5.21. *There is no bi-invariant metrics on* $\mathrm{SL}(2,\mathbb{R})$.

Exercise 5.4. When G is compact, the bi-invariant metrics always exist. For example, for $G \subset \mathrm{O}(n,\mathbb{R}) \subset S^{n^2-1}(\sqrt{n})$, the Euclidean metric $\langle A, B\rangle = \mathrm{tr}\, AB^T$ is bi-invariant.

Example 5.22. The Euclidean metric on $\mathbb{R}^n$ is clearly bi-invariant.

These examples turns out to be basically all the examples:

Theorem 5.23. *A simply connected Lie group G which admits a bi-invariant metric takes the form $G = \mathbb{R}^n \times H$ for H compact and $n \in \mathbb{Z}_{\geq 0}$.*

PROOF. Let $\mathfrak{z} \lhd \mathfrak{g}$ be the center, which is clearly an ideal. Then $\mathfrak{h} := \mathfrak{z}^\perp < \mathfrak{g}$ is also an ideal: For $a \in \mathfrak{z}^\perp$, $b \in \mathfrak{g}$, and $c \in \mathfrak{z}$,

$$\langle [b,a], c\rangle = -\langle a, [b,c]\rangle = 0 \Longrightarrow [b,a] \in \mathfrak{z}^\perp.$$

(This holds true for any ideal $\mathfrak{z}$.) Since G is simply connected, the decomposition $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{h}$ leads to $G = Z \times H$ with $\operatorname{Lie} Z = \mathfrak{z}$ and $\operatorname{Lie} H = \mathfrak{h}$.

The center $Z \lhd G$ is simply connected and abelian, hence $Z \cong \mathbb{R}^n$ for some n. Let $e_1, \dots, e_h \in \mathfrak{h}$ be an orthonormal basis. For any $X \in \mathfrak{h}$, the group H with the induced bi-invariant metric has Ricci curvature

$$\operatorname{Ric}(X,X) = \tfrac{1}{4}\sum_{i=1}^{h} |[X,e_i]|^2 > 0.$$

By translation, this show that the Ricci curvature has a positive lower bound on H. Hence by the theorem of Bonnet-Meyer H must be compact. □

5. Homogeneous spaces

5.1. General homogeneous spaces. Let $H < G$ be a *closed* Lie subgroup. Then the coset space $G/H = \{\, gH \mid g \in G \,\}$ has a natural C^∞ manifold structure such that the projection map $\pi : G \to G/H$ is C^∞. G acts transitively on G/H by left translations. Also the stabilizer (also called isotropy subgroup) $G_{[gH]} \cong H$ at each point $[gH]$. Conversely, given a transitive C^∞ action $G \times M \to M$ on a C^∞ manifold M. Let $H = G_{m_0}$ for some $m_0 \in M$. Then $G/H \cong M$. A space of the form G/H is called a *homogeneous space*. If $H \lhd G$ then G/H is a also Lie group.

Example 5.24. Here are some standard examples:

(i) $\mathrm{O}(n) \times S^{n-1} \to S^{n-1}$ is transitive and $\mathrm{O}(n)_{e_n} \cong \mathrm{O}(n-1)$. So $S^{n-1} \cong \mathrm{O}(n)/\mathrm{O}(n-1)$.

(ii) $\mathrm{U}(n) \times S^{2n-1} \to S^{2n-1}$ is transitive and $\mathrm{U}(n)_{e_n} \cong \mathrm{U}(n-1)$. So $S^{2n-1} \cong \mathrm{U}(n)/\mathrm{U}(n-1)$. Similarly, $S^{2n-1} \cong \mathrm{SU}(n)/\mathrm{SU}(n-1)$. In particular, $S^1 \cong \mathrm{U}(1)$ and $S^3 \cong \mathrm{SU}(2)$ are Lie groups.

(iii) *Real projective space*: $\mathbb{R}P^{n-1} = S^{n-1}/\{\pm 1\}$. So

$$\mathbb{R}P^{n-1} \cong \mathrm{O}(n)/\mathrm{O}(n-1) \times \{\pm 1\} \cong \mathrm{SO}(n)/\mathrm{O}(n-1).$$

(iv) *Complex projective space*: $\mathbb{C}P^{n-1} = (\mathbb{C}^n \backslash \{0\})/\mathbb{C}^\times$. So

$$\mathbb{C}P^{n-1} \cong S^{2n-1}/S^1 \cong \mathrm{U}(n)/\mathrm{U}(n-1) \times \mathrm{U}(1) \cong \mathrm{SU}(n)/\mathrm{U}(n-1).$$

(v) *Stiefel manifold of k-frames*: $\mathrm{GL}(n, \mathbb{R}) \times \tilde{V}_{n,k} \to \tilde{V}_{n,k}$ is transitive where $\tilde{V}_{n,k}$ is the set of all k frames in $\mathbb{R}^n$. For $S = \{e_1, \dots, e_k\}$,

$$G_S = \left\{ \begin{pmatrix} I & A \\ 0 & B \end{pmatrix} \in \mathrm{GL}(n, \mathbb{R}) \right\}.$$

So $\tilde{V}_{n,k} \cong \mathrm{GL}(n, \mathbb{R})/G_S$. For $V_{n,k}$ the set of all orthonormal k-frames,

$$V_{n,k} \cong \mathrm{O}(n)/\mathrm{O}(n-k) \cong \mathrm{SO}(n)/\mathrm{SO}(n-k).$$

For complex Stiefel manifold $V^{\mathbb{C}}_{n,k}$ of k-frames in $\mathbb{C}^n$,

$$V^{\mathbb{C}}_{n,k} \cong \mathrm{U}(n)/\mathrm{U}(n-k) \cong \mathrm{SU}(n)/\mathrm{SU}(n-k).$$

(vi) *Grassmannian manifolds*: Let $G_{n,k}$ be the set of all k-dimensional subspaces in $\mathbb{R}^n$, then $G_{n,k} \cong V_{n.k}/\mathrm{O}(k) \cong \mathrm{O}(n)/\mathrm{O}(n-k) \times \mathrm{O}(k)$ and $\dim G_{n,k} = k(n-k)$. Similarly for the complex Grassmannian

$$G^{\mathbb{C}}_{n,k} \cong V^{\mathbb{C}}_{n.k}/\mathrm{U}(k) \cong \mathrm{U}(n)/\mathrm{U}(n-k) \times \mathrm{U}(k).$$

It is a complex manifold with $\dim_{\mathbb{C}} G^{\mathbb{C}}_{n,k} = k(n-k)$. Grassmannians generalizes projective spaces. They are very important for the study of vector bundles.

(vii) *Poincaré's upper half plane*: Let $\mathbf{H} = \{ z \in \mathbb{C} \,|\, \mathrm{Im}\, z > 0 \}$. $\mathrm{SL}(2, \mathbb{R})$ acts on $\mathbf{H}$ transitively by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d},$$

The stabilizer at i is $\mathrm{SO}(2, \mathbb{R})$, so $\mathbf{H} \cong \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2, \mathbb{R})$. $\mathbf{H}$ is non-compact and analytically isomorphic to the unit disk, an example of the *bounded symmetric domains*. The double coset space

$$\Gamma\backslash \mathrm{H} \cong \Gamma\backslash \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$$

with $\Gamma < \mathrm{SL}(2,\mathbb{R})$ contains all Riemann surfaces of genus $g \geq 2$ (uniformization theorem). If $\Gamma < \mathrm{SL}(2,\mathbb{Z})$ is an *arithmetic subgroup*, then it represents certain moduli spaces of elliptic curves.

5.2. Riemannian homogeneous spaces. For further study, we need notions and results from differential geometry. Let (M, ds^2) be a Riemannian manifold. That is, ds^2 is a family of inner products $\langle\, , \rangle_x$ on T_xM varying smoothly in $x \in M$. An isometry $g : M \to M$ is a C^∞ map such that $g^*ds^2 = ds^2$. Equivalently, $\langle dg(v), dg(w)\rangle_{g(x)} = \langle v, w\rangle_x$ for all $v, w \in T_xM$. It is known that the *full isometry group*

$$G \equiv \mathrm{O}(M, ds^2) := \{\, g \in C^\infty(M, M) \mid g^*ds^2 = ds^2 \,\}$$

is a Lie group. For each $x \in M$, G_x induces a linear representation $\rho : G_x \to \mathrm{O}(T_xM)$. Since an isometry maps geodesics to geodesics, $\rho(h)$ determines h through the *geodesic exponential map* $\exp_x : U \subset T_xM \to M$ and thus ρ is injective. In particular, each isotropy group G_x is *compact*.

A connected Riemannian manifold (M, ds^2) is *Riemannian homogeneous* if for any two points $x, y \in M$, there exists an isometry g such that $g(x) = y$. In this case, we have a transitive action $G \times M \to M$ and $M \cong G/G_x$. In particular, M is homogeneous with compact isotropy.

Proposition 5.25. *A Riemannian homogeneous space is complete.*

Exercise 5.5. Prove the Proposition 5.25

A natural question arises: *When is a general homogeneous space $M \cong G/H$ Riemannian homogeneous? That is we are searching for metrics on G/H such that G acts on it as isometries.* Such a metric is called a *G-invariant metric*, which may not always exist. Also there could be different ways to represent M as a group quotient. Thus we need to clarify these issues first.

In considering the homogeneous structure we may assume that G acts on G/H *effectively* in the sense that any $g \in G\backslash\{e\}$ acts non-trivially. Indeed,

$$g[kH] = [kH] \iff k^{-1}gk \in H \iff g \in kHk^{-1}.$$

Hence g acts trivially if and only if $g \in \bigcap_{k\in G} kHk^{-1} =: H_0$. It is clear that H_0 is the largest subgroup of H with $H_0 \lhd G$. Thus

$$G/H \cong \frac{G/H_0}{H/H_0} =: G_1/H_1$$

has an effective G_1 action.

Denote $G \to G/H$ by $g \mapsto \bar{g} := gH$. There is a natural identification $T_{\bar{e}}\, G/H = \mathfrak{g}/\mathfrak{h}$. Since Ad_H and $\mathrm{ad}_{\mathfrak{h}}$ act on $\mathfrak{g}$ and leave the subspace $\mathfrak{h}$ invariant, we get the natural adjoint actions on $\mathfrak{g}/\mathfrak{h}$ induced from $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$.

Lemma 5.26. *For $h \in H$, $dL_h \equiv \mathrm{Ad}_h$ modulo $\mathfrak{h}$ on $T_{\bar{e}}\, G/H$.*

PROOF. Differentiate the equation $h\exp(tX)H = h\exp(tX)h^{-1}H$. □

Proposition 5.27. *A G-invariant metric on the homogeneous space $M = G/H$ is equivalent to an inner product $\langle\, ,\rangle$ on $\mathfrak{g}/\mathfrak{h} \cong T_{\bar{e}}M$ which is Ad_H-invariant. If H is connected, this is equivalent to "$\mathrm{ad}_{\mathfrak{h}}$-invariance": Namely, for $A \in \mathfrak{h}$, $X, Y \in \mathfrak{g}/\mathfrak{h}$,*

$$\langle \mathrm{ad}_A X, Y\rangle + \langle X, \mathrm{ad}_A Y\rangle = 0.$$

PROOF. The necessity of Ad_H invariance on $\langle\, ,\rangle$ follows from the above lemma. To see its sufficiency, we simply define for $v, w \in T_{\bar{g}}\, G/H$

$$\langle v, w\rangle_{\bar{g}} := \langle dL_{g^{-1}}v, dL_{g^{-1}}w\rangle.$$

Then $\langle v, w\rangle_{\overline{gh}} = \langle dL_{h^{-1}}dL_{g^{-1}}v, dL_{h^{-1}}dL_{g^{-1}}w\rangle = \langle dL_{g^{-1}}v, dL_{g^{-1}}w\rangle = \langle v, w\rangle_{\bar{g}}$. Hence the left invariant metric on G/H is well defined.

The remaining statement on $\mathrm{ad}_{\mathfrak{h}}$ is left as an exercise. □

Exercise 5.6. Show the remaining statement on $\mathrm{ad}_{\mathfrak{h}}$ in Proposition 5.27.s

Theorem 5.28. *Assume that G acts on $M = G/H$ effectively. Then M admits a G invariant metric if and only if $\mathrm{Ad}_H \subset \mathrm{GL}(\mathfrak{g})$ has compact closure.*

Moreover, G invariant metrics on G/H are precisely left invariant metrics on G which is also H bi-invariant.

PROOF. ($\Rightarrow$) Write $G/H = G^*/H^*$ with $G^* = O(M, ds^2)$, $H^* = G^*_{\bar{e}}$. Then $G \to G^*$, and hence $\mathfrak{g} \to \mathfrak{g}^*$, is injective. We know that $\operatorname{im} \mathrm{Ad}_{H^*} \subset GL(\mathfrak{g}^*)$ is compact since H^* is. To realize it inside the orthogonal group we simply pick an arbitrary inner product on $\mathfrak{g}^*$ and average it by this compact image so that the resulting inner product $\langle\,,\rangle^*$ on $\mathfrak{g}^*$ is Ad_{H^*}-invariant. (This is the same procedure to construct bi-invariant metrics on a compact Lie group.) Let $\langle\,,\rangle = \langle\,,\rangle|^*_{\mathfrak{g}}$. Then it is clear that the image $\mathrm{Ad}_H \subset O(\mathfrak{g}, \langle\,,\rangle)$.

($\Leftarrow$) If Ad_H has compact closure $K \subset \mathrm{GL}(\mathfrak{g})$, starting with any inner product on $\mathfrak{g}$ the averaging procedure over K again produces an Ad_H-invariant inner product $\langle\,,\rangle$ on $\mathfrak{g}$. Let $\mathfrak{p} := \mathfrak{h}^\perp$ which is isomorphic onto $\mathfrak{g}/\mathfrak{h}$ under π. It is clear that $\mathrm{Ad}_H(\mathfrak{p}) \subset \mathfrak{p}$ since $\langle \mathrm{Ad}_H(\mathfrak{h}^\perp), \mathfrak{h}\rangle = \langle \mathfrak{h}^\perp, \mathrm{Ad}_H \mathfrak{h}\rangle = 0$. Thus $\langle\,,\rangle|_{\mathfrak{p}}$ defines the desired Ad_H-invariant inner product on $\mathfrak{g}/\mathfrak{h}$. □

6. Symmetric spaces

6.1. Local and global symmetric spaces. A connected Riemannian manifold (M, ds^2) is a *symmetric space* if for all $x \in M$ there is an isometry $s_x : M \to M$ such that x is an isolated fixed point of s_x and $ds_x : T_xM \to T_xM$ sends $v \to -v$. It is *locally symmetric* if s_x exists only locally.

To construct local isometry, consider the map $\hat{s}_x$ which reverses geodesics γ with $\gamma(0) = x$:

$$\hat{s}_x(\gamma(t)) = \gamma(-t).$$

This coincides with s_x when (M, ds^2) is locally symmetric, because local isometry maps geodesics to geodesics and geodesics are determined by initial conditions $\gamma(0)$ and $\gamma'(0)$.

Proposition 5.29. *Symmetric spaces are Riemannian homogeneous.*

PROOF. Let $G = \mathrm{O}(M, ds^2)$. In particular it contains the subgroup generated by the symmetries s_x, $x \in M$. We only need to show that G acts on M transitively. For any x, y which are joined by a geodesic γ with $\gamma(0) = x$, $\gamma(T) = y$, let s_z be the isometry with $z = \gamma(T/2)$. Clearly $s_z(x) = y$.

In general, x and y can be joined by a sequence of broken geodesics γ_i. Then we take the isometry to be the composite of those s_{z_i}'s. □

Proposition 5.30. *In terms of curvature, (M, ds^2) is locally symmetric if and only if that $\nabla R = 0$, that is the curvature tensor is parallel.*

PROOF. Indeed, "⇒" is easy: For any tensor T of even degree, ∇T is of odd degree. Since s_x is a local isometry, we get

$$\nabla T = s_x^*(\nabla T) = -\nabla T,$$

hence $\nabla T = 0$. "⇐" is a consequence of the *Cartan Theorem* (cf. theorem 3.47). □

Corollary 5.31. *Simply connected locally symmetric spaces are symmetric.*

This follows from $\nabla R = 0$ and the *Cartan-Ambrose-Hicks Theorem* (cf. theorem 3.48).

Theorem 5.32. *A connected Lie group G with a bi-invariant metric, e.g., for G compact times Euclidean, is a $G \times G$ symmetric space.*

PROOF. Let $G \times G$ act on G by $(g, h)\alpha = g\alpha h^{-1}$. Then $G \cong G \times G/G$, with the stabilizer at $e \in G$ being the diagonal group isomorphic to G. We claim that the map

$$s_g : h \mapsto gh^{-1}g$$

defined the symmetry at g.

We check this for $s_e : h \mapsto h^{-1}$ first. Indeed, near $e \in G$ the map s_e is given by $\exp X \mapsto \exp(-X)$. From this we see that s_e reverses one parameter subgroups and $ds_e = -\mathrm{Id}_{T_eG}$.

To show that s_e is an isometry, consider any point $g \in G$ and a vector $v = dL_g X \in T_g G$ with $X \in T_e G$. Then $v = \gamma'(0)$ where $\gamma(t) = g \exp tX$. Then $s_e \gamma(t) = \exp(-tX) g^{-1} = R_{g^{-1}} \exp(-tX)$. Hence

$$(ds_e)_g\, v = (ds_e)_g\, \gamma'(0) = -dR_{g^{-1}} X.$$

With $w = dL_g Y$, we compute by using bi-invariance of the metric that

$$\left\langle (ds_e)_g\, v, (ds_e)_g\, w \right\rangle = \left\langle -dR_{g^{-1}} X, -dR_{g^{-1}} Y \right\rangle = \langle X, Y \rangle = \langle v, w \rangle .$$

For general $g \in G$, $s_g = L_g R_g s_e$ is the composite of three isometries, hence s_g is also an isometry.

It remains to check that $(ds_g)_g = -\mathrm{Id}_{T_gG}$. As before let $v = dL_g X \in T_g G$. $\gamma(t) = g \exp tX$. Then $s_g \gamma(t) = g \exp(-tX) g^{-1} g = g \exp(-tX)$. Hence

$$(ds_g)_g\, v = (ds_g)_g\, \gamma'(0) = -dL_g X = -v.$$

This completes the proof that G is symmetric. □

6.2. Symmetric spaces via Lie algebras. When is a homogeneous space $M = G/H$ symmetric? This will be reduced to a problem on Lie algebras. Recall that $\sigma \in \mathrm{Aut}\, G$ ia an *involution* if $\sigma \neq \mathrm{Id}_G$ and $\sigma^2 = \mathrm{Id}_G$.

Theorem 5.33. (Basic structure theorem for symmetric spaces).

(a) *Let $M = G/H$ be a symmetric space with $G = \mathrm{O}(M, ds^2)$, then*

$$\sigma : G \to G; \qquad g \mapsto \sigma(g) = s_x g s_x$$

is an involution of G and $K = G^\sigma$ is a closed subgroup containing H such that $K^\circ = H^\circ$. H contains no non-trivial normal subgroup of G.

(b) *Conversely, let G be a Lie group with an involution σ. Let $K = G^\sigma$ and fix a G-invariant metric $\langle\, ,\rangle$ on $M = G/K$. Let $\bar{\sigma}$ be the* *diffeomorphism on M induced from σ. If $\langle\,,\rangle$ is $\bar{\sigma}$-invariant then M is symmetric.*

(c) *A simply connected Lie group G with an involution σ is equivalent to a $\mathbb{Z}_2$ graded decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ in the sense that*

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}.$$

Given σ, the subalgebra $\mathfrak{h}$ and the subspace $\mathfrak{p}$ are the ± 1 eigenspace of $d\sigma : \mathfrak{g} \to \mathfrak{g}$ respectively.

PROOF. For (a), σ is an involution since

$$\sigma(gh) = s_x g h s_x = (s_x g s_x)(s_x h s_x) = \sigma(g)\sigma(h)$$

and $\sigma^2(g) = \sigma(s_x g s_x) = s_x(s_x g s_x)s_x = g$. One can check that $K \cap H$ is open and closed in K, hence $F^\circ = H^\circ$. H contains no non-trivial normal subgroup of G since otherwise the action of G on M is not effective.

For (b), $\langle\,,\rangle$ is $\bar{\sigma}$-invariant means that $\bar{\sigma}$ is an isometry on M. Since $(d\sigma_e)^2 = \mathrm{id}_{\mathfrak{g}}$, we have the ± 1 eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $T_{\bar{e}}M \cong \mathfrak{p}$. So $d\bar{\sigma}_{\bar{e}} = -\mathrm{id}_{T_{\bar{e}}M}$. Thus $s_{\bar{e}} := \bar{\sigma}$ is the symmetry at $\bar{e}$. We noticed that a Riemannian homogeneous space which is symmetric at one point is then symmetric everywhere. Indeed, the symmetry at $\bar{g}$ is given by

$$s_{\bar{g}} := L_g \circ \bar{\sigma} \circ L_{g^{-1}} = L_g \circ \sigma \circ L_{g^{-1}} \pmod{K}.$$

It is clear that $s_{\bar{g}}$ is well defined, $s_{\bar{g}}(\bar{g}) = \bar{g}$, $s_{\bar{g}}^2 = \mathrm{id}_M$ and $s_{\bar{g}}$ is an isometry. The property $(ds_{\bar{g}})_{\bar{g}} = -\mathrm{id}$ can be easily checked as in the Lie group case.

For (c), let $v \in \mathfrak{h}$ and $w \in \mathfrak{p}$. Then

$$d\sigma[v, w] = [d\sigma(v), d\sigma(w)] = [+v, -w] = -[v, w].$$

Hence $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$. The proofs of the other two inclusions are similar.

Conversely, given $\mathbb{Z}_2$ graded decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, define a Lie algebra morphism $T : \mathfrak{g} \to \mathfrak{g}$ with $T|_{\mathfrak{h}} = \mathrm{id}$ and $T|_{\mathfrak{p}} = -\mathrm{id}$. Since G is simply connected, this gives rise to a Lie group morphism

$\sigma : G \to G$. Since $d(\sigma^2) = d\sigma \circ d\sigma = T \circ T = \mathrm{id}_{\mathfrak{g}}$, we conclude that $\sigma^2 = \mathrm{id}_G$ by the unique correspondence between morphisms. □

Exercise 5.7. Show that in $K \cap H$ in Theorem 5.33 is open.

So the problem on constructing symmetric spaces is reduced to finding a $\mathbb{Z}_2$ decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ with compatible inner product $\langle \, , \rangle$ on $\mathfrak{p}$. Combining with Proposition 5.27 and Theorem 5.28, the corresponding metric on G/H can be constructed from a left invariant metric on G which is bi-invariant on H. Examples are provided by the *semi-simple Lie groups*.

6.3. Examples via semi-simple Lie groups. Let $\mathfrak{g}$ be a Lie algebra over $F = \mathbb{R}$ or $\mathbb{C}$. Define the *Killing form*

$$B(X, Y) = \mathrm{tr}(\mathrm{ad}\, X \, \mathrm{ad}\, Y); \quad B : \mathfrak{g} \times \mathfrak{g} \to F.$$

It is the main source to provide adjoint invariant quadratic forms:

Lemma 5.34 (Exercise). *B is ad-invariant: $B(\mathrm{ad}_Z X, Y) + B(X, \mathrm{ad}_Z Y) = 0$.*

Exercise 5.8. Show Lemma 5.34

We say that $\mathfrak{g}$ is *semi-simple* if B is non-degenerate, $\mathfrak{g}$ is *simple*if $\mathfrak{g}$ is not abelian and $\mathfrak{g}$ contains no proper Lie ideals.

Theorem 5.35. *$\mathfrak{g}$ is semi-simple if and only if $\mathfrak{g}$ is a direct sum of simple ideals.*

The proof for the "only if" part is similar to the proof of Theorem 5.23 by using B in place of the bi-invariant metric. The "if" part follows from the *Killing-Cartan criterion* which will not be presented here.

We say G is semi-simple (simple) if $\mathfrak{g}$ is semi-simple (simple). For G simple, every bi-invariant metric $\langle \, , \rangle$ is determined by its value at e and proportional to the Killing form.

Example 5.36. We give two main series of examples of symmetric spaces G/H that arise from semi-simple Lie groups G. Notice that we had seen that H may always be assumed to be compact.

(i) Type I: G is compact and B is negative definite. E.g.

$$\begin{aligned} \mathrm{SO}(2n)/\mathrm{U}(n), &\quad \mathrm{SO}(p+q)/\mathrm{SO}(p)\times\mathrm{SO}(q),\\ \mathrm{SU}(2n)/\mathrm{SO}(n), &\quad \mathrm{SU}(p+q)/\mathrm{SU}(p)\times\mathrm{U}(q),\\ \mathrm{Sp}(n)/\mathrm{U}(n), &\quad \mathrm{Sp}(p+q)/\mathrm{Sp}(p)\times\mathrm{Sp}(q). \end{aligned}$$

These includes spheres, projective spaces and Grassmannians.

(ii) Type II: G is non-compact and B is indefinite.

In this case there is a maximal compact subalgebra $\mathfrak{h}$ and $\mathbb{Z}_2$ decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ (the Cartan decomposition). Moreover, B is *negative definite* on $\mathfrak{h}$ and *positive definite* on $\mathfrak{p}$. E.g.

$$\mathrm{SO}(p,q)/\mathrm{SO}(p)\times\mathrm{SO}(q),$$

with respect to the indefinite inner product $\sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2$. For $q = 1$, we get the *Poincaré upper half space* $\mathbf{H}^p$.

Similarly

$$\mathrm{SU}(p,q)/\mathrm{U}(p)\times\mathrm{SU}(q),$$

with respect to the indefinite Hermitian inner product $\sum_{i=1}^{p} |z_i|^2 - \sum_{j=p+1}^{p+q} |z_j|^2$. For $q = 1$, we get the unit ball in $\mathbb{C}^p$. Other examples are $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$ (for $n = 2$ we had seen that this gives Poincaré upper half plane), $\mathrm{SO}(n,\mathbb{C})/\mathrm{SO}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n)$.

The main theory of Cartan says that any simply connected symmetric space may be decomposed into a product of three factors

$$M = M_0 \times M_+ \times M_-,$$

where M_0 is a Euclidean space, M_+ is of compact type and M_- is of non-compact type. Both M_+ and M_- may be further decomposed into irreducible factors and each factor can be constructed from certain semi-simple Lie algebras in a way similar to the above examples. The details can be found in Helgason's classic text.

7. Curvature for symmetric spaces

7.1. Riemannian submersion. A map $f : (\bar{M}, \bar{g}) \to (M, g)$ is a *Riemannian submersion* if it is a C^∞ submersion $f : \bar{M} \to M$ and $df : T^h\bar{M} \to TM$ is an isometry, where $T\bar{M} = T^v\bar{M} \oplus T^h\bar{M}$ is the orthogonal decomposition defined by: $T^v_{\bar{p}}\bar{M} := \ker df_{\bar{p}}$ is the *vertical tangent space* which is also the tangent space of the fiber submanifold $\bar{M}_p := f^{-1}(p)$ with $p = f(\bar{p})$, and $T^h_{\bar{p}}\bar{M} := (T^v_{\bar{p}}\bar{M})^\perp$ is the *horizontal tangent space*.

If $f(\bar{p}) = p$ and $X \in T_pM$, then there is a unique *horizontal lift* $\bar{X} \in T_{\bar{p}}\bar{M}$ such that $df_{\bar{p}}\bar{X} = X$. Under such a lifting, one may relate the Levi-Civita connection and Riemannian curvature tensor on M in terms of those on $\bar{M}$. This is particularly useful in dealing with Riemannian homogeneous spaces or symmetric spaces of the form $G \to M = G/H$. The following simple relations, due to O'Neill, can be found in most textbook in Riemannian geometry. The proofs are left as exercises.

Theorem 5.37. *Let $f : \bar{M} \to M$ be a Riemannian submersion. Then*

(a) $\bar{\nabla}_{\bar{X}}\bar{Y} = \overline{\nabla_X Y} + \frac{1}{2}[\bar{X}, \bar{Y}]^v$ *for any vector fields X, Y on M and any lifts $\bar{X}, \bar{Y}$. The vertical component $[\bar{X}, \bar{Y}]^v$ is tensorial in X and Y.*

(b) *For any $X, Y, Z, W \in T_pM$,*

$$\begin{aligned} R(X, Y, Z, W) = \bar{R}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) + \tfrac{1}{2}\langle[\bar{X}, \bar{Y}]^v, [\bar{Z}, \bar{W}]^v\rangle \\ + \tfrac{1}{4}\langle[\bar{X}, \bar{Z}]^v, [\bar{Y}, \bar{W}]^v\rangle - \tfrac{1}{4}\langle[\bar{X}, \bar{W}]^v, [\bar{Y}, \bar{Z}]^v\rangle. \end{aligned}$$

(c) $R(X, Y, X, Y) = \bar{R}(\bar{X}, \bar{Y}, \bar{X}, \bar{Y}) + \frac{3}{4}|[\bar{X}, \bar{Y}]^v|^2.$

Combining with the curvature formula for Lie groups, we may achieve

Theorem 5.38. (a) *Let G be a compact semi-simple Lie group with an involution σ ($\sigma^2 = \mathrm{id}$). Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be the $\pm$ eigenspace decomposition. Then $-B$ defines a bi-invariant metric on G and G/H is a symmetric space with curvature*

$$R(X, Y, X, Y) = |[X, Y]|^2.$$

(b) *Let G be a non-compact semi-simple Lie group and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be a $\mathbb{Z}_2$ decomposition as in Example 5.36 (ii). Then $B|_{\mathfrak{p}}$ defines an invariant metric on G/H and make it a symmetric space with curvature*

$$R(X,Y,X,Y) = -|[X,Y]|^2.$$

PROOF. We only give the proof for (b). The proof for (a) is similar and easier.

For the Riemannian submersion $G \to G/H$ with the left invariant metric on G defined by

$$\langle\, ,\rangle|_{\mathfrak{h}} = -B|_{\mathfrak{h}}, \qquad \langle\, ,\rangle|_{\mathfrak{p}} = B|_{\mathfrak{p}},$$

we have $T_e^v G = \mathfrak{h}$ and $T_e^h G = \mathfrak{p}$. Let $X, Y, Z \in T_{[H]}G/H \cong \mathfrak{p}$. By Corollary 5.18 and Theorem 5.37 (c), we get

$$\begin{aligned} R(X,Y,X,Y) &= |\mathrm{ad}_X^* Y + \mathrm{ad}_Y^* X|^2 - \langle \mathrm{ad}_X^* X, \mathrm{ad}_Y^* Y\rangle \\ &\quad - \tfrac{3}{4}|[X,Y]^{\mathfrak{p}}|^2 - \tfrac{1}{2}\langle [[X,Y],Y],X\rangle - \tfrac{1}{2}\langle [[Y,X],X],Y\rangle. \end{aligned}$$

Since $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{h}$, $[X,Y]^{\mathfrak{p}} = 0$. Also $[X,Z] \in \mathfrak{h}$ implies that $\langle \mathrm{ad}_X^* Y, Z\rangle = \langle Y, [X,Z]\rangle = 0$, hence $\mathrm{ad}_X^* Y \in \mathfrak{h}$. Now for $T \in \mathfrak{h}$, the left invariant metric $\langle\, ,\rangle$ on G is also right invariant under H (i.e. ad-invariance of B) says that

$$\langle [T,X],Y\rangle + \langle X,[T,Y]\rangle = 0,$$

which is equivalent to $\langle \mathrm{ad}_X^* Y + \mathrm{ad}_Y^* X, T\rangle = 0$, hence $\mathrm{ad}_X^* Y + \mathrm{ad}_Y^* X = 0$. By setting $X = Y$ we get also $\mathrm{ad}_X^* X = 0$. So only the last two terms remained in the curvature formula.

Since $[[X,Y],Y], [[Y,X],X] \in \mathfrak{p}$, we compute by ad-invariance of B

$$\begin{aligned} R(X,Y,X,Y) &= -\tfrac{1}{2}B([[X,Y],Y],X) - \tfrac{1}{2}B([[Y,X],X],Y) \\ &= \tfrac{1}{2}B([X,Y],[X,Y]) + \tfrac{1}{2}B([Y,X],[Y,X]). \end{aligned}$$

Since $[X,Y] \in \mathfrak{h}$, this gives $-|[X,Y]|^2$. The proof is complete. □

8. Topology of Lie groups and symmetric spaces

For a group action G on a manifold M, a differential form $\omega \in \Lambda^p(M)$ is an invariant form if $g^*\omega = \omega$ for all $g \in G$.

Theorem 5.39. *Let $M = G/H$ be a symmetric space with compact G, Then*

$$H^*(M, \mathbb{R}) \cong A^*_{\text{inv}}(M) = \mathbf{H}^*(M).$$

PROOF. By the de Rham theorem $H^*(M, \mathbb{R}) \cong H^*_{\text{dR}}(M, \mathbb{R})$, hence we need to show that every invariant form is also closed and every closed differential form is equivalent to an unique invariant form.

Step 1. $\omega \in A^p_{\text{inv}}(M) \Rightarrow d\omega = 0$. We show first that $\hat{\omega} := s_x^*\omega \in A^p_{\text{inv}}(M)$ for any $x \in M$. For this, recall $s_x g = \sigma(g) s_x$, where σ is the involution. So

$$g^*\hat{\omega} = g^* s_x^* \omega = s_x^* \sigma(g)^* \omega = s_x^* \omega = \hat{\omega}.$$

From $ds_x = -\text{Id}$ on $T_x M$ we get $\hat{\omega}|_x = (-1)^p \omega|_x$. Together with the invariance of ω and $\hat{\omega}$, this implies that $\hat{\omega} = (-1)^p \omega$.

Now $d\omega$ and $d\hat{\omega}$ are also invariant forms (since d commutes with g^*) and $s_x^* d\omega = d(s_x^*\omega) = d\hat{\omega} \in A^{p+1}_{\text{inv}}(M)$. So similarly $d\hat{\omega} = (-1)^{p+1} d\omega$. But we also have $d\hat{\omega} = (-1)^p d\omega$, hence we conclude $d\omega = 0$.

Step 2. $d\omega = 0 \Rightarrow \omega \sim \tilde{\omega} \in A^p_{\text{inv}}(M)$. We prove this step for any homogeneous space with compact G. On a Lie group G, pick up any left invariant metric, its volume form give rise to invariant measure $d\mu$ which can be normalized to have total volume 1 if G is compact.

For any $g \in G$, $g^*\omega \sim \omega$ since the map $g : M \to M$ is homotopic to identity. This holds true for any affine linear combination: $\sum \mu_i g_i^* \omega \sim \omega$ with $\sum \mu_i = 1$. Taking limits (by definition of Riemann sum) we find that

$$\tilde{\omega} := \int_G g^* \omega \, d\mu_g \sim \omega.$$

$\tilde{\omega} \in \Lambda^p_{\text{inv}}(M)$ since for any $h \in G$,

$$h^*\omega = h^* \int_G g^* \omega \, d\mu_g = \int_G h^* g^* \omega \, h^* d\mu_g = \int_G (gh)^* \omega \, d\mu_{gh} = \tilde{\omega}.$$

Step 3. We show that an exact invariant form must be zero. Fix a G invariant metric on M. We recall the Hodge star operator $* : \Lambda^p T_x^* M \to \Lambda^{n-p} T_x^* M$. The G invariance implies that $*g^* = g^* *$ for any $g \in G$. Hence $\omega \in A^p_{\text{inv}}(M) \Rightarrow *\omega \in A^p_{\text{inv}}(M)$. In particular $d(*\omega) = 0$ by step 1.

For $\omega, \eta \in A^p(M)$,

$$\langle \omega, \eta \rangle = \int_M \omega \wedge *\eta$$

is a inner product on $A^p(M)$. Now for $\omega = d\eta$ an exact invariant p form,

$$\langle \omega, \omega \rangle = \int_M d\eta \wedge (*\omega) = \int_M d(\eta \wedge (*\omega)) - (-1)^p \int_M \eta \wedge d(*\omega) = 0$$

by Stokes theorem and $d(*\omega) = 0$. Hence $\omega = 0$ as desired.

Step 4. It remains to show that invariant forms are precisely harmonic forms. If $\omega \in A^p_{\text{inv}}(M)$, we have just seen that

$$d\omega = 0 \quad \text{and} \quad d^*\omega = (-1)^{p(n-p)} *d*\omega = 0.$$

Thus $\triangle\omega = (dd^* + d^*d)\omega = 0$. If we assume the *Hodge theorem* which says that $H^*_{dR}(M, \mathbb{R}) \cong \mathbf{H}^*(M)$, then harmonic forms must be invariant forms.

We would like to give a direct proof: First notice that $\triangle\eta = 0$ if and only if that $d\eta = 0$ and $d^*\eta = 0$, which is seen from the identity

$$\langle \triangle\eta, \eta \rangle = \|d\eta\|^2 + \|d^*\eta\|^2.$$

Let $\triangle\omega = 0$. For any $X \in \mathfrak{g}$, we compute by *Cartan's homotopy formula*

$$L_X\omega = \iota_X d\omega + d\iota_X\omega = d\iota_X\omega.$$

The invariance of the metric implies that $\triangle$ commutes with L_X, hence $L_X\omega$ is also harmonic. So $\langle L_X\omega, L_X\omega \rangle = \langle L_X, d\iota_X\omega \rangle = \langle d^* L_X\omega, \iota_X\omega \rangle = 0$ and then $L_X\omega = 0$. By definition of Lie derivatives this implies that ω is an invariant form. The proof is completed. $\square$

Corollary 5.40. *For a connected compact Lie group G, viewed as a $G \times G$ symmetric space, the de Rham cohomology are given by bi-invariant forms, which are precisely harmonic forms:*

$$H^*(G, \mathbb{R}) \cong A^*_{G \times G\text{-inv}}(G) \cong \Lambda^*_{\text{Ad-inv}}[\mathfrak{g}^*].$$

PROOF. Only the last equality requires explanation. The left invariant forms are uniquely determine by their values at $e \in G$, so we have

$$A^*_{\text{left-inv}}(G) \cong \Lambda^*[\mathfrak{g}^*].$$

A left invariant form is right invariant if and only if it is adjoint invariant:

$$R^*_{g^{-1}}\omega = R^*_{g^{-1}} \circ L^*_g \omega = I^*_g \omega.$$

The inner automorphism I_g induces the adjoint action Ad_g on $\mathfrak{g}$, hence on the dual space $\mathfrak{g}^*$. This gives the action I^*_g on left invariant forms. □

Corollary 5.41. *Let G be a compact Lie group and $\Omega(X, Y, Z) := B([X, Y], Z)$ where B is the Killing form. If $\Omega \neq 0$, say if G is semi-simple, then $H^3(G) \neq 0$.*

PROOF. The skew symmetry of Ω in X, Y is obvious. In Z, this is equivalent to adjoint invariance of B. The adjoint invariance of Ω follows from the adjoint invariance of B and the Jacobi identity. □

We conclude the discussion with two fundamental results in cohomology and homotopy theory of Lie groups without proof.

Theorem 5.42 (Hopf Theorem)**.** *For connected Lie group G, $H^*(G)$ is a finitely generated free exterior algebra $\Lambda[y_1, \ldots, y_n]$, with y_i being of odd degree. For example, $H^*(\text{U}(n), \mathbb{R}) \cong \Lambda[y_1, \ldots, y_{2n-1}]$, $\deg y_i = i$.*

Theorem 5.43 (Bott Periodicity Theorem)**.**

(i) *Unitary case: $\pi_{i-1}(\text{SU}(2m)) \cong \pi_{i+1}(\text{SU}(2m))$ for $1 \leq i \leq 2m$. Hence for $\text{U} := \varinjlim \text{U}(m)$, we have $\pi_{i-1}(\text{U}) \cong \pi_{i+1}(\text{U})$.*

(ii) *Orthogonal case: For* $\mathrm{O} := \varinjlim \mathrm{O}(n)$, *we have* $\pi_i(\mathrm{O}) \cong \pi_{i+8}(\mathrm{O})$. *The first eight values* $\pi_i(\mathrm{O})$ *for* $0 \leq i \leq 7$ *are* $\mathbb{Z}_2$, $\mathbb{Z}_2$, 0, $\mathbb{Z}$, 0, 0, 0, $\mathbb{Z}$ *respectively.*

9. Problems

5.1 ([War83] Ch.6 #20) (The Peter-Weyl Theorem)**.** The *representative ring* of a compact Lie group G is the ring generated over the complex numbers by the set of all continuous functions f for which there is a continuous homomorphism $\rho : G \to GL(n, \mathbb{C})$ for some n such that $f = \rho_{ij}$ for some choice of i and j. The Peter-Weyl theorem states that *representative ring is dense in the space of complex-valued continuous functions on G in the uniform norm*. That is, if g is a complex valued continuous function on G, and if $\epsilon > 0$ is given, then there is a function f in the representative ring such that $|f(\sigma) - g(\sigma)| < \epsilon$ for all $\sigma \in G$. We outline the proof of this theorem which is based on the uniform completeness of the eigenfunctions of the Laplacian. One can choose a Riemannian structure on G such that each of the diffeomorphisms ℓ_σ for $\sigma \in G$ (left translation by σ) is an isometry (that is, $\langle v, w\rangle_\tau = \langle d\ell_\sigma v, d\ell_\sigma w\rangle_{\sigma\tau}$ for all $\tau \in G$ and all $v, w \in G_\tau$). Since the C^∞ functions are dense in the space of continuous functions in the uniform norm, and since by result of above exercise ([War83] Ch.6 #16 (h)) the direct sum of the eigenspaces of the Laplacian is dense in the space of C^∞ functions in the uniform norm, it suffices for the Peter-Weyl theorem to prove that each eigenfunction of the Laplacian $\triangle : C^\infty(G) \to C^\infty(G)$ belongs to the representative ring.

Now, G acts on the C^∞ functions on G by

$$\sigma(f) = f \circ \ell_\sigma \text{ for } \sigma \in G.$$

Prove that since the ℓ_σ are isometries, this action commutes with the Laplacian

$$\triangle(f \circ \ell_\sigma) = (\triangle f) \circ \ell_a, \text{ for } \sigma in G.$$

Let V_λ be the (finite dimensional) eigenspace associated with the eigenvalue λ of $\triangle : C^\infty(G) \to C^\infty(G)$. Prove that the action of G leaves V_λ invariant. Then let $\varphi_1, \ldots, \varphi_n$ be a basis of V_λ, and let

$$\sigma(\varphi_i) = \sum_j g_{ji}(\sigma)\varphi_j.$$

Then $\sigma \to \{g_{ji}(\sigma)\}$ is a homomorphism of $G \to GL(n, \mathbb{R})$. Prove that this homomorphism is continuous. Then observe that

$$\varphi_i(\sigma) = \varphi_i \circ \ell_\sigma(e) = \sum_j g_{ji}(\sigma)\varphi_j(e),$$

so that φ_i belongs to the representative ring.

5.2 (cf. [Car92] Ch.1 #7)**.** When G is compact, the bi-invariant metrics always exist. For example, for $G \subset \mathrm{O}(n, \mathbb{R}) \subset S^{n^2-1}(\sqrt{n})$, the Euclidean metric $\langle A, B \rangle = \operatorname{tr} AB^T$ is bi-invariant.

5.3 ([Car92] Ch.8 #8) (Riemannian submersions)**.** A differentiable mapping $f : \overline{M}^{n+k} \to M^n$ is called a *submersion* if f is surjective, and for all $\bar{p} \in \overline{M}$, $df_{\bar{p}} : T_{\bar{p}}\overline{M} \to T_{f(\bar{p})}M$ has rank n. In this case, for all $p \in M$, the *fiber* $f^{-1}(p) = F_p$ is a submanifold of $\overline{M}$ and a tangent vector of $\overline{M}$, tangent to some F_p, $p \in M$, is called a *vertical vector* of the submersion. If, in addition, $\overline{M}$ and M have Riemannian metrics, the submersion f is said to be *Riemannian* if, for all $p \in \overline{M}$, $df_p : T_p\overline{M} \to T_{f(p)}M$ preserves lengths of vectors orthogonal to F_p. Show that:

(a) If $M_1 \times M_2$ is the Riemannian product, then the natural projections $\pi_i : M_1 \times M_2 \to M_i$, $i = 1, 2$ are Riemannian submersions.

(b) Let the tangent bundle TM be given the Riemannian metric as:

$$\langle V, W\rangle_{(p,v)} = \langle d\pi(V), d\pi(W)\rangle_p + \langle \frac{Dv}{dt}(0), \frac{Dw}{ds}(0)\rangle_p$$

for $(p, v) \in TM$, V, W tangent vectors at (p, v) in TM where $V = \alpha'(0)$, $W = \beta'(0)$ for curves α, β chosen such that $\alpha(t) = (p(t), v(t))$, $\beta(t) = (q(s), w(s))$, $p(0) = q(0) = 0$, $v(0) = w(0) = v$ (cf. [Car92] Ch.3 #2). Show that the projection $\pi : TM \to M$ is a Riemannian submersion.

5.4 ([Car92] Ch.8 #9) (Conneciton of a Riemannian submersion)**.** Let $f : \overline{M} \to M$ be a Riemannian submersion. A vector $\bar{x} \in T_{\bar{p}}\overline{M}$ is *horizontal* if it is orthogonal to the fiber. The tangent space $T_{\bar{p}}\overline{M}$ then admits a decomposition $T_{\bar{p}}\overline{M} = (T_{\bar{p}}\overline{M})^h \oplus (T_{\bar{p}}\overline{M})^v$, where $(T_{\bar{p}}\overline{M})^h$ and $(T_{\bar{p}}\overline{M})^v$ denote the subspaces of horizontal and vertical vectors, respectively. If $X \in \mathcal{X}(M)$, the *horizontal lift* $\overline{X}$ of X is the horizontal field defined by $df_{\bar{p}}(\overline{X}(\bar{p})) = X(f(p))$.

(1) Show that $\overline{X}$ is differentiable.

(2) Let ∇ and $\overline{\nabla}$ be the Riemannian connections of M and $\overline{M}$ respectively. Show that

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla_X Y} + \frac{1}{2}[\overline{X},\overline{Y}]^v, \quad X, Y \in \mathcal{X}(M),$$

where Z^v is the vertical component of Z.

(3) $[\overline{X},\overline{Y}]^v(\bar{p})$ depends only on $\overline{X}(\bar{p})$ and $\overline{Y}(\bar{p})$.

5.5 ([Car92] Ch.8 #10) (Curvature of a Riemannian submersion)**.** Let $f : \overline{M} \to M$ be a Riemannian submersion. Let $X, Y, Z, W \in \mathcal{X}(M)$, $\overline{X}, \overline{Y}, \overline{Z}, \overline{W}$ be their horizontal lifts, and let R and $\overline{R}$ be the curvature tensors of M and $\overline{M}$ respectively. Prove that:

(1)

$$\begin{aligned}\left\langle \overline{R}(\overline{X},\overline{Y})\overline{Z},\overline{W}\right\rangle &= \langle R(X,Y)Z,W\rangle - \frac{1}{4}\left\langle [\overline{X},\overline{Z}]^v, [\overline{Y},\overline{W}]^v\right\rangle \\ &\quad + \frac{1}{4}\left\langle [\overline{Y},\overline{Z}]^v, [\overline{X},\overline{W}]^v\right\rangle - \frac{1}{2}\left\langle [\overline{Z},\overline{W}]^v, [\overline{X},\overline{Y}]^v\right\rangle\end{aligned}$$

(2) $K(\sigma) = \overline{K}(\overline{\sigma}) + \frac{3}{4}\left|[\overline{X},\overline{Y}]^v\right|^2 \geq \overline{K}(\overline{\sigma})$, where σ is the plane generated by the orthonormal vectors $X, Y \in \mathcal{X}(M)$ and $\overline{\sigma}$ is the plane generated by $\overline{X}, \overline{Y}$.

5.6. ([Car92] Ch.8 #11) [The complex projective space] Let

$$\mathbb{C}^{n+1} \setminus \{0\} = \{(z_0, \dots, z_n) = Z \neq 0 \mid z_j = x_j + iy_j, j = 0, \dots, n\}$$

be the set of all non-zero $(n+1)$-tuples of complex numbers z_j. Define equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$: $(z_0, \dots, z_n) \sim W = (w_0, \dots, w_n)$ if $z_j = \lambda w_j$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$. The equivalence class of Z will be denoted by $[Z]$ (the complex line passing through the origin and through Z). The set of such classes is called, by analogy with the real case, *the complex projective space* $\mathbb{P}^n(\mathbb{C})$ of complex dimension n.

(1) Show that $\mathbb{P}^n(\mathbb{C})$ has a differentiable structure of a manifold of real dimension $2n$ and that $\mathbb{P}^1(\mathbb{C})$ is diffeomorphic to $\mathbb{S}^2$.

(2) Let $(Z, W) = z_0\overline{w_0} + \cdots + z_n\overline{w_n}$ be the hermitian product on $\mathbb{C}^{n+1}$, where the bar denotes complex conjugation. Identify $\mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2}$ by putting $z_j = x_j + iy_j = (x_j, y_j)$. Show that

$$\mathbb{S}^{2n+1} = \{N \in \mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2} \mid (N, N) = 1\}$$

is the unit sphere in $\mathbb{R}^{2n+2}$.

(3) Show that the equivalence relation $\sim$ induces on $\mathbb{S}^{2n+1}$ the following equivalence relation: $Z \sim W$ if $e^{i\theta}Z = W$. Establish that there exists a differentiable map (the Hopf fibering) $f : \mathbb{S}^{2n+1} \to \mathbb{P}^n(\mathbb{C})$ such that

$$\begin{aligned} & f^{-1}([Z]) \\ & \quad = \{e^{i\theta}N \in \mathbb{S}^{2n+1} \mid N \in [Z] \cap \mathbb{S}^{2n+1}, 0 \leq \theta \leq 2\pi\} \\ & \quad = [Z] \cap \mathbb{S}^{2n+1}. \end{aligned}$$

(4) Show that f is a submersion.

5.7 ([Car92] Ch.8 #12) (Curvature of the complex projective space)**.** Define a Riemannian metric on $\mathbb{C}^{n+1} \setminus \{0\}$ in the following way: If $Z \in \mathbb{C}^{n+1} \setminus \{0\}$ and $V, W \in T_Z(\mathbb{C}^{n+1} \setminus \{0\})$,

$$\langle V, W \rangle_Z = \frac{\mathrm{Re}(V, W)}{(Z, Z)}.$$

Observe that the metric $\langle\ ,\ \rangle$ restricted to $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1} \setminus \{0\}$ coincides with the metric induced from $\mathbb{R}^{2n+2}$.

(1) Show that, for all $0 \leq \theta \leq 2\pi$, $e^{i\theta} : \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}$ is an isometry, and that, therefore, it is possible to define a Riemannian metric on $\mathbb{P}^n(\mathbb{C})$ in such a way that the submersion f is Riemannian.
(2) Show that, in this metric, the sectional curvature of $\mathbb{P}^n(\mathbb{C})$ is given by

$$K(\sigma) = 1 + 3\cos^2\varphi,$$

where σ is generated by the orthonormal pair X, Y, $\cos\varphi = \langle \overline{X}, i\overline{Y} \rangle$, and $\overline{X}, \overline{Y}$ are the horizontal lifts of X and Y, respectively. In particular, $1 \leq K(\sigma) \leq 4$.

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