Chapter 4

HODGE THEOREM

We have seen in variations of geodesics and submanifolds that a geometric object with the minimizing property leads to strong constraint on the global geometry. However, it is in general difficult to find a minimizing object due to the hard non-linear analysis involved. For geodesics we bypass the difficulty using the completeness assumption. For minimal surfaces this is already subtle and will only be discussed in later chapters.

There is nevertheless a situation where the analysis involved is entirely within linear elliptic PDE and a satisfactory solution is achieved. This is *Hodge's theory of harmonic forms on a compact oriented Riemannian manifold* (M^m, g): every cohomology class $[\omega] \in H^p_{dR}(M, \mathbb{R})$ admits a *unique harmonic representative* which is also the *absolute minimizer* of the energy functional $E(\omega) = \int_M \omega \wedge *\omega$ within the class. Indeed what Hodge proved is the famous decomposition theorem named after him

$$A^p(M) = \operatorname{Ker} \triangle \overset{\perp}{\oplus} \operatorname{Im} \triangle = \mathbb{H}^p \overset{\perp}{\oplus} dA^{p-1} \overset{\perp}{\oplus} d^* A^{p+1}$$

for the Laplace operator $\triangle = dd^* + d^*d$, where $d^* = (-1)^{mp+1} * d *$ is the formal adjoint of d. The essential part is to show that the harmonic space $\mathbb{H}^p = \operatorname{Ker} \triangle$ is finite dimensional and consists of C^∞ p-forms.

Hodge developed his theory around 1930s. One has to go to Hilbert space completions (generalized functions) to start the discussion. Nowadays a nice approach is to use Fourier transforms on L^2 spaces and Sobolev spaces H_s to encode the degree of differentiability. The key result is the regularity theorem called Garding's inequality. Historically it was Hodge's theorem which laid the foundation of general linear elliptic PDEs.

We present the details in this chapter. We also give applications of harmonic forms under the Bochner principle which describes the difference between $\triangle$ and the connection Laplacian $\operatorname{tr} \nabla^2$ in terms of the curvature tensor. This leads to strong topological constraint when $\operatorname{Ric} \geq 0$.

1. Hodge $*$ operator

We need some preparation on inner product spaces in linear algebra. On an inner product space $(V, \langle\ ,\ \rangle)$ with $\dim_{\mathbb{R}} V = m$, we define the *Hodge $*$ operator* on $\Lambda^p(V)$ (the space of p-vectors)

$$* : \Lambda^p(V) \longrightarrow \Lambda^{n-p}(V)$$

as follows: for an O.N.B. $e_1, \ldots, e_m \in V$, we consider the "orthogonal complement"

$$e_1 \wedge \ldots \wedge e_p \overset{*}{\longmapsto} e_{p+1} \wedge \ldots \wedge e_m.$$

By taking into account the sign, for $\alpha = e_{i_1} \wedge \ldots \wedge e_{i_p} \in \Lambda^p(V)$, $*\alpha \in \Lambda^{n-p}(V)$ is the wedge of complemented e_j's with

$$\alpha \wedge *\alpha = e_1 \wedge \ldots \wedge e_m.$$

For general $\alpha \in \Lambda^p(V)$, $*$ is then defined by linear extensions.

For general p-vectors $\alpha, \beta \in \Lambda^p(V)$, $\alpha = \sum_{|I|=p} a_I e_{i_1} \wedge \ldots \wedge e_{i_p}$, $\beta = \sum_{|J|=p} b_J e_{j_1} \wedge \ldots \wedge e_{j_p}$,

$$\begin{aligned} \alpha \wedge (*\beta) &= \sum_{|I|,|J|=p} a_I b_J (e_{i_1} \wedge \ldots \wedge e_{i_p}) \wedge *(e_{j_1} \wedge \ldots \wedge e_{j_p}) \\ &= \langle \alpha, \beta \rangle\, e_1 \wedge \ldots \wedge e_n \end{aligned}$$

where $\langle \alpha, \beta \rangle$ is the induced inner product on $\Lambda^p(V)$: in the notations in Exercise 3.1, for $P_v = P(v_1 \ldots v_p)$, $P_w = P(w_1 \ldots w_p)$,

$$\langle v_1 \wedge \ldots \wedge v_m, w_1 \wedge \ldots \wedge w_m \rangle := \langle P_v, P_w \rangle = \det(\mathbf{v}^t \mathbf{w}).$$

Exercise 4.1. Show that $*$ is independent of the choices of orthonormal basis *with the same orientation* and $*^2 = (-1)^{p(n-p)}$.

For a general *oriented* Riemannian manifold (M^m, g), we define Hodge $*$ operator $* : A^p(M) \to A^{m-p}(M)$ by applying the above definition on each T_p^*M.

Exercise 4.2. Let $\omega = f\, dx^1 \wedge \ldots \wedge dx^p$ in a chart $(U, \mathbf{x})$. Give the explicit formula of $*\omega$ using the metric tensor g_{ij}.

In order to make use of the exterior algebra and $*$ operator in its full strength, we recall that vector fields are associated with their corresponding 1-forms by the "Riesz representations":

$$
\begin{aligned}
T_pM &\xrightarrow{\;\sim\;} T_p^*M \\
v &\longmapsto \tilde{v} : w \mapsto \langle v, w \rangle, \\
\tilde{\phi} &\longleftarrow\!\!\dashv \phi \in T_p^*M.
\end{aligned}
$$

It is also customary to write $\tilde{v}$ as $v^\flat$ since it is a one form $v_i = g_{ij}\, v^j$ and $\tilde{\phi}$ as $\phi^\#$ since it is a vector field $\phi^i = g^{ij}\, \phi_j$.

The advantage to consider forms rather than vector fields is that we may add or wedge forms in a transparent manner.

For example, in the Euclidean space $M = \mathbb{R}^3$, $g_{ij} = \delta_{ij}$, a vector field $\mathbf{F} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$ has $\tilde{\mathbf{F}} = P\,dx + Q\,dy + R\,dz$ and

$$
\begin{aligned}
\nabla f \equiv \operatorname{grad} f &= \widetilde{df}, \\
\operatorname{div} \mathbf{F} &= *\,d*\tilde{\mathbf{F}}, \\
\operatorname{curl} \mathbf{F} &= \widetilde{*d\,\tilde{\mathbf{F}}}.
\end{aligned}
$$

Thus we can put everything we need in vector calculus in the framework of differential forms. Moreover, the above intrinsic (coordinate-free) expressions extend the definitions of ∇f, $\operatorname{div} \mathbf{F}$ and $\operatorname{curl} \mathbf{F}$ to arbitrary Riemannian manifolds (M, g).

Exercise 4.3. Let (M, g) be a Riemannian manifold, ω be its volume form.

(1) For $f \in C^\infty(M)$, find the expression for ∇f in local coordinates $(U, \mathbf{x})$.
(2) For $X \in C^\infty(TM)$, show that $(\operatorname{div} X)\omega = \mathcal{L}_X(\omega)$. Deduce the expression for $\operatorname{div} X$ in local coordinate $(U, \mathbf{x})$ from it.

Exercise 4.4. Let (M, g) be a Riemannian manifold, $f \in C^\infty(M)$, we define the *Laplace–Beltrami operator* Δ_{LB} acting on $C^\infty(M)$ by

$$\triangle_{LB} f = \operatorname{div} \nabla f$$

Show that in local coordinate $(U, \mathbf{x})$,

$$\triangle_{LB} f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^i} \right),$$

where $g = \det(g_{ij})$. Deduce that the highest order term of $\triangle_{LB}$ is just the usual Laplacian and the lower order term vanishes at $x = 0$ if we take normal coordinate.

Particularly, this shows that $\triangle_{LB}$ is the usual Laplacian when (M, g) is the standard Euclidean space. In the next section, we will generalize Laplacian to differential forms by Hodge Laplacian.

Example 4.1. The *Maxwell equations* describing the electric field $\mathbf{E} = (E_1, E_2, E_3)$ and the magnetic field $\mathbf{B} = (B_1, B_2, B_3)$ read as

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \tag{4.1}$$

$$\operatorname{div} \mathbf{B} = 0, \tag{4.2}$$

$$\operatorname{div} \mathbf{E} = 4\pi\rho, \tag{4.3}$$

$$\operatorname{curl} \mathbf{B} = -\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \vec{j}, \tag{4.4}$$

where ρ is the charge density and $\vec{j}$ is the current density.

Consider the Minkowski space(-time) $\mathbb{R}^{3,1}$ with $x^0 = ct$ and define the energy–momentum tensor by the two form

$$\begin{aligned} F &= \sum F_{ij}\, dx^i \wedge dx^j \\ &:= \sum_{i=1}^{3} E_i\, dx^i \wedge dx^0 + B_1\, dx^2 \wedge dx^3 + B_2\, dx^3 \wedge dx^1 + B_3\, dx^1 \wedge dx^2. \end{aligned}$$

Or in matrix form:

$$(F_{ij}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}.$$

Then we check easily that (4.1) and (4.2) are equivalent to

$$dF = 0,$$

while (4.3) and (4.4) are equivalent to

$$* d * F = -\frac{4\pi}{c} \tilde{\mathbf{j}}.$$

Here $\mathbf{j} = (c\rho, \vec{j})$ is the density vector. The two equations can be further simplified to one equation using concepts in the next section.

Exercise 4.5. Carry out the details in Example 4.1.

2. Harmonic forms

Given a closed (i.e. compact without boundary) oriented manifold M^m with a Riemannian metric g, we define an inner product on the space $A^p(M)$ of smooth p-form on M by

$$(\alpha, \beta) := \int_M \langle \alpha, \beta \rangle \, dV = \int_M \alpha \wedge *\beta,$$

where $* : \Lambda^p \to \Lambda^{m-p}$ is the Hodge $*$ operator with $*^2 = (-1)^{(m-p)p}$.

Given a cohomology class $[\alpha] \in H^p_{dR}(M)$, a closed p-form α' represents $[\alpha]$ if and only if $\alpha' = \alpha + d\beta$ for some $\beta \in A^{p-1}(M)$. An important observation made by Hodge is that we can obtain a canonical representation of the class $[\alpha]$ by requiring the norm $\|\alpha + d\beta\|$ to be minimal.

Let d^* be the *formal adjoint*[1] of $d : A^p(M) \to A^{p+1}(M)$. Namely

$$\begin{aligned}(\alpha, d^*\beta) := (d\alpha, \beta) &= \int_M d\alpha \wedge *\beta \\ &= \int_M d(\alpha \wedge *\beta) - (-1)^p \alpha \wedge d(*\beta) \\ &= (-1)^{mp+1} \int_M \alpha \wedge *(*d * \beta).\end{aligned}$$

Since this holds for all α, we get

$$d^* = (-1)^{mp+1} * d* : A^{p+1}(M) \to A^p(M).$$

[1] So far $(A^p(M), (,))$ is only a pre-Hilbert space. We call d^* the formal adjoint of d to distinguish with the adjoint of a linear operator on Hilbert spaces

Definition 4.2. Let (M^m, g) be a Riemannian manifold, compact or not, the formal adjoint of d is defined by $d^* := (-1)^{mp+m+1} * d*$ on $A^p(M)$.

Exercise 4.6. Give the formula of d^* in local coordinates.

We will consider only compact manifolds in this chapter unless specified otherwise. Let α be a closed p-form which has "minimal norm" in $[\alpha] \in H^p_{dR}(M)$. Then for any $t \in \mathbb{R}$, $\beta \in A^{p-1}(M)$,

$$\begin{aligned}&(\alpha + td\beta, \alpha + td\beta)\\&= (\alpha,\alpha) + 2(\alpha, d\beta)t + (d\beta, d\beta)t^2\\&\leq (\alpha,\alpha)\end{aligned}$$

This happens if and only if that

$$0 = (\alpha, d\beta) = (d^*\alpha, \beta), \qquad \forall\, \beta \in A^{p-1}(M)$$

which is equivalent to $d^*\alpha = 0$. The discussion implies that

Lemma 4.3. *A closed C^∞ p-form α has minimal norm $\|\alpha\|$ in its de Rham cohomology class if and only if that $d^*\alpha = 0$. It is unique if it exists.*

Definition 4.4. The *Laplace operator* on $A^p(M)$ (or called the *Hodge Laplacian*) is defined by

$$\triangle = (d + d^*)^2 = dd^* + d^*d : A^p(M) \to A^p(M).$$

We call $\alpha \in A^p(M)$ is a *harmonic form* if $\triangle\alpha = 0$.

Lemma 4.5. *For a compact Riemannian manifold (M, g), $\triangle\alpha = 0$ if and only if $d\alpha = 0$ and $d^*\alpha = 0$.*

This follows easily from

$$(\triangle\alpha, \alpha) = (dd^*\alpha + d^*d\alpha, \alpha) = (d^*\alpha, d^*\alpha) + (d\alpha, d\alpha).$$

Thus, instead of looking for closed forms with minimal norm[2], Hodge's observation is to look for the *harmonic representative* of the cohomology class $[\alpha] \in H^p_{dR}(M)$.

First of all, the operator $\triangle$ is self-adjoint, i.e. $(\triangle\alpha, \beta) = (\alpha, \triangle\beta)$ or $\triangle = \triangle^*$. Furthermore, the truly crucial property is that $\triangle$ *is elliptic*—a notion to be explained soon. It turns out that we may and should pose more general questions:

(1) When is the equation $\triangle\alpha = \beta$ solvable?
(2) How to solve it if it is solvable?

The answer is provided by *Hodge decomposition theorem* whose precise form is given in the next section. Let $\mathbb{H} = \{\alpha \mid \triangle\alpha = 0\}$ be the space of harmonic forms. Then Hodge decomposition theorem asserts that $\Delta\alpha = \beta$ is solvable if and only if $\beta \in \mathbb{H}^\perp$, the orthogonal complement of harmonic forms.

The "only if part" is easy. For any $\gamma \in \mathbb{H}$,

$$(\beta, \gamma) = (\triangle\alpha, \gamma) = (\alpha, \triangle\gamma) = 0 \Longrightarrow \ \beta \in \mathbb{H}^\perp.$$

For example, since M is compact, the only harmonic functions are constants. Let $\beta \in A^0(M)$ (a function). If $\Delta\alpha = \beta$ is solvable, then $\beta \in \mathbb{H}^\perp = \mathbb{R}^\perp$, i.e. $\int_M \beta \, dV = 0$.

The "if part" is non-trivial even for $p = 0$. The proof requires a, by now standard, formalism in functional analysis. Here we give a sketch of it and leave the details to the next section.

Let $L : \mathcal{H} \to \mathcal{H}$ be a continuous (i.e. with bounded norm $\|L\|$) linear operator on a pre-Hilbert space $\mathcal{H}$ with adjoint operator L^*. Consider the equation $Lw = \beta$. For any $\phi \in \mathcal{H}$,

$$(w, L^*\phi) = (Lw, \phi) = (\beta, \phi).$$

Thus any solution w defines a linear functional ℓ such that

$$\ell(L^*\phi) = (\beta, \phi) \qquad \forall\, \phi \in \mathcal{H}. \tag{4.5}$$

[2]This amounts to prove that a minimizing sequence α_i has a smooth limit after passing to a subsequence, known as the direct method of calculus of variations. However in practice it relies on technical estimates on various norms which are eventually equivalent to proving the regularity theorem in the related PDE.

If we know that ℓ is bounded, which is the hard part and requires that $\beta \in \mathbb{H}^{\perp}$ in the case we want to apply to, then the domain of definition of ℓ can be extended from $\operatorname{Im} L^*$ to $\mathcal{H}$, while keeping the same norm $\|\ell\|$, by the Hahn–Banach theorem.

Definition 4.6. A bounded linear functional ℓ on $\mathcal{H}$ satisfying (4.5) is called a *weak solution* to $Lw = \beta$.

If $\mathcal{H}$ is furthermore a Hilbert space (i.e. complete), then by Riesz representation theorem, such an ℓ comes from some $\alpha \in \mathcal{H}$ with

$$\ell(L^*\phi) = (\alpha, L^*\phi) = (L\alpha, \phi).$$

Hence $(L\alpha - \beta, \phi) = 0$ for all $\phi \in \mathcal{H}$ and then $L\alpha = \beta$. However, the space of C^∞ forms $A^p(M)$ is never complete and one must pass to certain completion to apply the above formalism. As a result, the solution α is a priorily only a certain L^2 form and we need a "regularity theorem" to conclude that α is indeed a C^∞ form.

3. Elliptic operators and Hodge decomposition

Consider the following typical situation:

$$\begin{tikzcd} E \arrow[d] & & \text{a } C^\infty \text{ vector bundle of rank } r, \\ M^m \arrow[u, bend right, "f\text{: a section}"'] & & \text{a } C^\infty \text{ manifold,} \end{tikzcd}$$

and $\mathcal{H} = C^\infty(M, E)$ be the vector space of all C^∞ sections. E.g. $E = \Lambda^p(T^*M)$, $r = C^m_p$, and $\mathcal{H} = A^p(M)$.

Definition 4.7. A $\mathbb{R}$-linear map $L : C^\infty(M, E) \to C^\infty(M, E)$ is a *linear differential operator of order* $d \in \mathbb{N}$ if under any local trivialization $U \subset \mathbb{R}^m$, $E|_U \cong U \times \mathbb{R}^r$, $C^\infty(U, E\big|_U) \cong C^\infty(U)^r$, L is represented by a differnetial operator of order d:

$$L : C^\infty(U)^r \longrightarrow C^\infty(U)^r.$$

That is, under the multi-index notations,

$$Lf = \sum_{|\alpha| \le d} A_\alpha(x)\, D^\alpha f, \qquad x \in U,$$

where $f = (f^1, \ldots, f^r)^t$ and $A_\alpha \in M_{r\times r}(C^\infty(U))$.

Write

$$L = \sum_{|\alpha|=d} A_\alpha(x) D^\alpha + \sum_{|\alpha|<d} A_\alpha(x) D^\alpha$$

with non-trivial top order term $L_d := \sum_{|\alpha|=d} A_\alpha\, D^\alpha \not\equiv 0$.

For $\xi = (\xi_1, \ldots, \xi_m)^t \in \mathbb{R}^m$, we define the polynomial in ξ_i's by

$$p_L(x,\xi) := \sum_\alpha A_\alpha(x)\, \xi^\alpha$$

called the symbol of L over $E|_U$. In general it depends on the trivialization and is not globally defined. Nevertheless we have

Exercise 4.7. Show that the top order term

$$\sigma_L(x,\xi) := p_{L_d}(x,\xi)$$

forms a tensor $\sigma_L \in C^\infty(\mathrm{Sym}^d(T^*M) \otimes \mathrm{End}\, E)$, called the principal symbol of L.

Definition 4.8. Let L be a differential operator of order d on $E \to M$.

(1) L is *elliptic* over an open subset $U \subset M$ if

$$\sigma_L(x,\xi) \in \mathrm{End}\, E$$

is invertible for all $x \in U$ and $\xi \in T_xM \setminus \{0\}$.

(2) Suppose that $E \to M$ is equipped with a C^∞ bundle metric. Then L is *uniform elliptic* over a subset $S \subset M$ if

$$|\sigma_L(x,\xi)\, v| \geq C|\xi|^d|v|, \quad \forall\, x \in S, \xi \neq 0,$$

for some C independent of $x \in S$.

In particular, ellipticity on U implies uniform ellipticity over any compact subsets $S \subset U$.

Remark 4.9. In a similar manner we define differential operators $L : C^\infty(E) \to C^\infty(F)$ of order d between vector bundles over M. The principal symbol is a section $\sigma_L \in C^\infty(\mathrm{Sym}^d(T^*M) \otimes \mathrm{Hom}(E,F))$ and L is elliptic if $\sigma_L(x,\xi)$ is invertible for all $x \in M$, $\xi \in T^*M \setminus \{0\}$. In particular $\mathrm{rk}\, E = \mathrm{rk}\, F$. But E and F need not be isomorphic. For our current purpose it is enough to consider the case $E = F$.

Exercise 4.8. Show that $\sigma_{\triangle}(x,\xi) = |\xi|^2\,\mathrm{Id}$ for the Hodge Laplacian $\triangle$ on $A^p(M)$, hence it is uniformly elliptic on M.

Now we state two fundamental results in elliptic PDE.

Let $Lw = \beta$ be a linear elliptic PDE on $E \to M$ over a compact manifold M. In order to apply the Hilbert space formalism we assume that (M,g) is Riemannian and E has a bundle metric h. If E is a complex vector bundle we require that the metric h is hermitian. Denote by $L : \mathcal{H} \to \mathcal{H}$ with $\mathcal{H} = C^\infty(E)$. $\mathcal{H}$ is a pre-Hilbert space under $(f,g) := \int_M h(f,g)\,dV$. Denote $\|f\| = (f,f)^{1/2}$. Then

Theorem 4.10 (Regularity theorem)**.** *Any weak solution of $Lw = \beta$ is automatically smooth.*

Theorem 4.11 (Compactness theorem)**.** *For a sequence $\alpha_n \in \mathcal{H}$, if $\|\alpha_n\| \le C$ and $\|L\alpha_n\| \le C$ are both bounded, then $\{\alpha_n\}$ has a Cauchy subsequence.*

Remark 4.12. We will show later that both theorems are consequences of the Gårding inequality: for $f \in \mathcal{H}$,

$$\|f\|_{s+d} \le C(\|Lf\|_s + \|f\|_s),$$

where d is the order of L, and the norms are Sobolev norms to be defined and studied in section 5.

Now we return to the case $L = \Delta$, $\mathcal{H} = A^p = A^p(M)$. By assuming the above two PDE theorems, we prove

Theorem 4.13 (Hodge decomposition theorem)**.** *Let $\mathbb{H} = \mathbb{H}^p := \{\alpha \in A^p(M) \mid \triangle\,\alpha = 0\}$ be the space of harmonic p-forms.*

(1) $\dim \mathbb{H} < \infty$, *and*
(2) $A^p = \mathbb{H} \oplus^\perp \triangle\, A^p$. *That is,* $\mathrm{Im}\,\triangle = \mathbb{H}^\perp$.

PROOF. (1) If $\dim \mathbb{H} = \infty$, we select

$$u_1, u_2, \ldots \in \mathbb{H}, \quad \|u_i\| = 1, \quad u_i \perp u_j \quad \forall\, i \ne j$$

satisfying $\Delta u_i = 0$. It is clear that $\{u_i\}$ has no Cauchy subsequence hence contradicts to the compactness theorem (Theorem 4.11). Thus

$l := \dim \mathbb{H} < \infty$ and $\mathbb{H} \subset \mathcal{H}$ is a closed subspace. In particular $\mathbb{H}^\perp$ is also a closed subspace and $A^p = \mathbb{H} \oplus \mathbb{H}^\perp$. Indeed, pick an O.N.B. $\{w_1, w_2, \ldots, w_l\}$ of $\mathbb{H}$. For any $\alpha \in A^p$,

$$\alpha = \beta + \sum_{i=1}^{l} (\alpha, w_i)\, w_i =: \beta + H(\alpha)$$

where $\beta \in \mathbb{H}^\perp$ and $H(\alpha)$ is the harmonic projection of α.

(2) We need to show $\mathbb{H}^\perp = \triangle A^p$. The direction "$\supset$" is clear:

$$(\triangle\alpha, \gamma) = (\alpha, \triangle\gamma) = 0 \quad \forall\, \gamma \in \mathbb{H}.$$

For the other direction we need the following

Claim 4.14. There exists $c > 0$ such that $\|\beta\| \leq c\|\triangle\beta\|$ for all $\beta \in \mathbb{H}^\perp$.

Let $\alpha \in \mathbb{H}^\perp$ and we define a linear functional ℓ on $\operatorname{Im} \triangle$ by

$$\ell(\triangle\phi) := (\alpha, \phi).$$

It is well defined: if $\triangle\phi_1 = \triangle\phi$ then $\phi_1 - \phi \in \mathbb{H}$ and then $(\alpha, \phi_1) = (\alpha, \phi)$. Also ℓ is a bounded linear functional on $\operatorname{Im} \triangle$: let $\beta = \phi - H(\phi) \in \mathbb{H}^\perp$, then

$$\begin{aligned} \|\ell(\triangle\phi)\| &= \|\ell(\triangle\beta)\| = \|(\alpha, \beta)\| \leq \|\alpha\| \cdot \|\beta\| \\ &\leq c\|\alpha\| \cdot \|\triangle\beta\| = (c\|\alpha\|) \cdot \|\triangle\phi\|. \end{aligned}$$

By Hahn–Banach theorem, ℓ can be extended to a bounded linear function on $\mathcal{H} = A^p$. That is, ℓ is a weak solution to $\triangle w = \alpha$.

By the regularity theorem (Theorem 4.10), there exists a smooth $w \in A^p$ such that $\triangle w = \alpha$. So $\mathbb{H}^\perp \subset \triangle A^p$ follows. □

PROOF OF CLAIM 4.14. Suppose the contrary, then there exists a sequence $\beta_j \in \mathbb{H}^\perp$ with $\|\beta_j\| = 1$ and $\|\triangle\beta_j\| \to 0$.

By the compactness theorem (Theorem 4.11), we may assume that $\{\beta_j\}$ is itself a Cauchy sequence. For $\psi \in A^p$, define

$$\ell(\psi) := \lim_{j\to\infty} (\beta_j, \psi).$$

The linear functional ℓ is clearly bounded: $\|\ell\| \leq 1$, and

$$\ell(\triangle\psi) = \lim_{j\to\infty} (\beta_j, \triangle\psi) = \lim_{j\to\infty} (\triangle\beta_j, \psi) = 0$$

by Cauchy's inequality, i.e. ℓ is a weak solution of $\triangle\beta = 0$.

By the regularity theorem (Theorem 4.10), there exists a $\beta \in A^p$ such that $\ell(\psi) = (\beta, \psi)$ and $\triangle\beta = 0$, i.e. $\beta \in \mathbb{H}$.

Then $(\beta_j, \psi) \to (\beta, \psi)$ and in particular $0 = (\beta_j, \beta) \to \|\beta\|^2$ and thus $\beta = 0$. On the other hand, the Cauchy sequence $\{\beta_j\}$ has limit β implies that $\|\beta\| = \lim_{j\to\infty} \|\beta_j\| = 1$, which is a contradiction. $\square$

The proof actually applies to more general cases:

Exercise 4.9. Extend the Hodge decomposition for any elliptic operator $L : C^\infty(E) \to C^\infty(F)$ between two vector bundles E, F over a compact M. (Assuming regalarity and compactness theorems.)

For the Hodge Laplacian $\triangle$, a lot more can be said.

Definition 4.15. Since $A^p = \mathbb{H} \oplus (\triangle A^p)$, we define the *Green operator*

$$G := \begin{cases} 0 & \text{on } \mathbb{H}, \\ \Delta^{-1} & \text{on } \mathbb{H}^\perp. \end{cases} \qquad \text{i.e. } I = H + \Delta G,$$

where $H : A^p \to \mathbb{H}$ is the *harmonic projection*. Hence $G\alpha$ is the unique solution in $\mathbb{H}^\perp$ for $\Delta w = \alpha - H(\alpha)$.

Recall that a bounded linear operator is compact if the image of any bounded subset has compact closure.

Exercise 4.10. Show that

(1) G commutes with any operator T with $T\triangle = \triangle T$.
(2) G is a bounded, self-adjoint, compact operator.

Notice that from $\triangle = dd^* + d^*d$ we see immediately that $d\triangle = dd^*d = \triangle d$ and $d^*\triangle = d^*dd^* = \triangle d^*$. Hence

$$[d, G] = [d^*, G] = [\Delta, G] = 0.$$

The Hodge decomposition can be refined to

$$A^p = \mathbb{H} \oplus \operatorname{Im} \triangle = \mathbb{H} \oplus \operatorname{Im} d \oplus \operatorname{Im} d^*,$$

which is still an orthogonal decomposition. A precise formula could be given in terms of the Green operator G. Indeed we can rewrite the identity $I = H + \Delta G$ as

$$\begin{aligned}\alpha &= H(\alpha) + (dd^* + d^*d)G\alpha \\ &= H(\alpha) + d(d^*G\alpha) + d^*(dG\alpha).\end{aligned}$$

In particular we have:

Proposition 4.16. *If $d\alpha = 0$ then $\alpha = H(\alpha) + d(d^*G\alpha)$.*

Therefore, the harmonic representation of a cohomology class is unique: if $\alpha_1 - \alpha_2 = d\beta$ then $H(\alpha_1) - H(\alpha_2) = H(d\beta) = 0$.

Here is another simple yet important application of the Hodge decomposition theorem:

Theorem 4.17 (Poincaré duality)**.** *Let M be a C^∞ compact oriented manifold of dimension m. Then the natural pairing*

$$H^p_{dR}(M) \otimes H^{m-p}_{dR}(M) \longrightarrow H^m_{dR}(M) \cong \mathbb{R} : \quad (\omega, \eta) \mapsto \int_M \omega \wedge \eta,$$

is a perfect pairing. In particular,

$$H^p_{dR}(M) \cong H^{m-p}_{dR}(M)^*.$$

PROOF. Let g be a metric on M. It suffices to prove the theorem using harmonic representatives of de Rham cohomology.

Since $*\triangle = \triangle *$, we see that $* : \mathbb{H}^p \to \mathbb{H}^{m-p}$ which maps harmonic forms to harmonic forms. From $*^2 = (-1)^{p(n-p)}$ we conclude that $* : \mathbb{H}^p \cong \mathbb{H}^{n-p}$. This isomorphism depends on g and is not the natural one stated in the theorem.

Nevertheless it shows that $\mathbb{H}^n$ is spanned by $*1 = dV_g$ and hence $H^m_{dR}(M) \cong \mathbb{R}$ under $[\Omega] \mapsto \int_M \Omega$ (the trace map). Moreover, it also implies that the natural pairing is perfect by noticing that

$$(\omega, *\omega) \mapsto \int_M \omega \wedge *\omega = \|\omega\|^2 \neq 0$$

if $[\omega] \neq 0$. This completes the proof. $\square$

Exercise 4.11. (1) Let $M = \mathbb{R}^m/\Lambda$ where $\Lambda \subset \mathbb{R}^m$ is a lattice generated by m linearly independent vectors. Let g be the flat metric induced from $\mathbb{R}^m$. Determine $\mathbb{H}^p(M)$ and show that the wedge of harmonic forms is still harmonic.

(2) Give an example (M, g) and two harmonic forms ω, η so that $\omega \wedge \eta$ is not harmonic.

4. Bochner Principle

Given any C^∞ manifold M, let ∇ be an affine connection on TM. We denote the induced connection on T^*M and $\bigwedge^p T^*M$ again by ∇. For $\omega \in A^p(M)$, $\nabla\omega(X_0, X_1, \ldots, X_p) \in A^p(M) \otimes A^1(M)$ is given by

$$(\nabla\omega)(X_0, \ldots, X_p) = (\nabla_{X_0}\omega)(X_1, \ldots, X_p).$$

We define the anti-symmetrization $(\nabla\omega)^{\text{alt}} \in A^{p+1}(M)$ by

$$(\nabla\omega)^{\text{alt}}(X_0, \ldots, X_p) = \sum_{j=0}^{p} (-1)^j (\nabla_{X_j}\omega)(X_0, \ldots, \widehat{X_j}, \ldots, X_p).$$

Exercise 4.12. Show that ∇ is torsion-free if and only if $(\nabla\omega)^{\text{alt}} = d\omega$.

Now, we assume ∇ is the Levi-Civita connection on (M, g).

Definition 4.18. For $X, Y \in C^\infty(TM)$, we define $\nabla^2_{X,Y}$ acting on any tensor fields T by

$$\nabla^2_{X,Y}T = \nabla_X\nabla_Y T - \nabla_{\nabla_X Y}T.$$

The *connection Laplace* acting on any tensor fields T is defined by the trace $\operatorname{tr}(\nabla^2 T)$ with respect to g.

Exercise 4.13. For $f \in C^\infty(M)$, show that $\triangle_{LB} f = \operatorname{tr}(\nabla^2 f)$, where $\triangle_{LB}$ is the Laplace–Beltrami operator. Also, we have $\triangle = -\triangle_{LB}$.

In general, $\triangle$ and $\operatorname{tr}\nabla^2$ are related by

Proposition 4.19 (Bochner formula)**.** *Let M be compact.*

$$\triangle = -\operatorname{tr}\nabla^2 - \sum_{i,j} \eta^i \wedge \iota_{e_j} R(e_i, e_j),$$

where $\{e_j\}$ is an orthonormal local frame of TM, $\{\eta^i\}$ is its dual frame.

In the literature this is also known as the *Bochner–Lichnerwicz–Witezenböck formula*.[3].

Exercise 4.14. With the notations in proposition 4.19, for $\omega \in A^p(M)$, prove that

$$d\omega = \sum_{i=1}^{n} \eta^i \wedge \nabla_{e_i}\omega; \qquad d^*\omega = -\sum_{i=1}^{n} \iota_{e_i}\nabla_{e_i}\omega. \tag{4.6}$$

Exercise 4.15. (1) Prove the Bochner formula 4.19 using (4.6).
(2) Denote by ∇^* the formal adjoint of ∇, show that

$$\nabla^*\nabla = -\mathrm{tr}\nabla^2.$$

Hence, another common form of Bochner formula is given by

$$\triangle = \nabla^*\nabla - \sum_{i,j} \eta^i \wedge \iota_{e_j} R(e_i, e_j).$$

Now, following Bochner, we use Proposition 4.19 to derive the topological constraints given by the Ricci curvature conditions.

Corollary 4.20 (Bochner)**.** *Let (M, g) be a closed Riemannian manifold.*

1. *If $\mathrm{Ric} > 0$ then $b_1 = h^1(M) = 0$. More precisely,*
2. *If $\mathrm{Ric} \geq 0$ then $h^1(M) \leq m = \dim(M)$. If furthermore $\mathrm{Ric} > 0$ at some point, then $h^1(M) = 0$. The equality holds if and only if M is a flat torus, i.e. $M \cong \mathbb{R}^n/\Gamma$ where $\Gamma \cong \mathbb{Z}^n$ is a lattice.*

PROOF. Before proving the statements, we first prove an identity also due to Bochner.

---

[3]In general, an identity expressing difference between two second–order elliptic operator with the same principle symbols in terms of curvatures is known as *Weitzenböck formula* or *Bochner formula*. Such formula was first indicated by Witzenböck in 1925, yet it was Bochner who first used the formula to relate topology and curvature estimates on compact manifolds (cf. Corollary 4.20) in 1948. There are many variants for formula of such type in different contexts. For instance, in 1963, Lichnerwicz developed an analogous formula for Dirac operators on spin bundles, which we will discuss in later chapter.

Claim 4.21. For any $\theta \in A^1(M)$,

$$\langle \triangle\theta, \theta\rangle = \frac{1}{2}\triangle|\theta|^2 + |\nabla\theta|^2 + \mathrm{Ric}(\tilde{\theta}, \tilde{\theta}),$$

where $\tilde{\theta} \in C^\infty(TM)$ is the metric dual of θ (cf. section 1).

Given $p \in M$, assume that $\{\eta^i\}_{i=1}^m$ is the dual frame of a local orthonormal frame $\{e_i\}_{i=1}^m$ of TM around p. We write $\theta = \sum_{i=1}^m a_i\eta^i$.

Since $\{e_i\}$ is an orthonormal frame, $g(e_i, e_j) = \delta_{ij}$ and hence $\nabla_{e_i}e_i = 0$. The connection Laplace of θ is then given by

$$\mathrm{tr}\nabla^2\theta = \sum_{i=1}^m \nabla^2_{e_i,e_i}\theta = \sum_{i=1}^m \nabla_{e_i}\nabla_{e_i}\theta.$$

Therefore, from Proposition 4.19 we deduce

$$\langle \triangle\theta, \theta\rangle = \underbrace{-\sum_i\langle \nabla_{e_i}\nabla_{e_i}\theta, \theta\rangle}_{(I)} \underbrace{- \langle \sum_{i,j}\eta^i \wedge \iota_{e_j}R(e_i, e_j)\theta, \theta\rangle}_{(II)}.$$

By direct calculation,

$$(I) = \sum_i e_i\langle \nabla_{e_i}\theta, \theta\rangle - |\nabla_{e_i}\theta|^2 = -\frac{1}{2}\triangle|\theta|^2 - |\nabla\theta|^2,$$

and

$$(II) = \sum_{i,j}(R(e_i, e_j)\theta)(e_j)\langle \eta^i, \theta\rangle$$

Notice that for $X, Y, Z \in C^\infty(M)$, $\omega \in A^1(M)$, a direct computation shows $(R(X, Y)\omega)(Z) = -\omega(R(X, Y)Z)$. Therefore,

$$\begin{aligned}(II) &= -\sum_{i,j}\theta(R(e_i, e_j)e_j)\langle \eta^i, \theta\rangle \\ &= -\sum_{i,j,k,l} a_k a_i\langle R^l_{jij}e_l, e_k\rangle = -\sum_{i,j,k} a_k a_i R_{kjij} \\ &= -\mathrm{Ric}(\sum_k a_k e_k, \sum_i a_i e_i) = -\mathrm{Ric}(\tilde{\theta}, \tilde{\theta}).\end{aligned}$$

As a result, we have proved:

$$\langle \triangle\theta, \theta\rangle = \frac{1}{2}\triangle|\theta|^2 + |\nabla\theta|^2 + \mathrm{Ric}(\tilde{\theta}, \tilde{\theta}).$$

We now prove (1) and (2) by contradiction. If $h^1(M) \neq 0$, we take a non-zero cohomology class $[\theta] \in H^1_{dR}(M)$. By Hodge decomposition (cf. 4.13), we choose $\theta \neq 0$ to be the harmonic 1-form representing $[\theta] \neq 0$.

For (1), if Ric > 0, choose $p \in M$ such that $|\theta(p)| > 0$ achieves maximal. Then $(\triangle|\theta|^2)(p) \geq 0$ by second derivative test. However, this contradicts to $\mathrm{Ric}(\tilde{\theta}, \tilde{\theta}) > 0$.

For (2), if Ric ≥ 0 , by taking integration on both sides of 4.21,

$$0 = \frac{1}{2}\int_M \triangle|\theta|^2 + \int_M |\nabla\theta|^2 + \int_M \mathrm{Ric}(\tilde{\theta}, \tilde{\theta}).$$

By Stokes' theorem, since $\partial M = \emptyset$, we have

$$0 = \int_M |\nabla\theta|^2 + \int_M \mathrm{Ric}(\tilde{\theta}, \tilde{\theta}).$$

Since Ric ≥ 0, we must conclude that $\nabla\theta \equiv 0$, i.e. θ is parallel and is determined by $\theta_q \in T_q^*M$, $\forall q \in M$. Hence $h^1 \leq \dim T_q^*(M) = m$. Also, $\nabla\theta \equiv 0$ and Ric ≥ 0 in turn implies $\mathrm{Ric}(\tilde{\theta}, \tilde{\theta}) \equiv 0$. Now, if Ric $>$ 0 at one point p and $\theta \neq 0$, we have $\mathrm{Ric}(\tilde{\theta}, \tilde{\theta}) > 0$, a contradiction.

On the other hand, if the equality holds, $h^1 = \dim M = m$, then the universal cover $\tilde{M} \longrightarrow M$ has m parallel 1-forms. Therefore it has m parallel vector fields. We conclude that $\tilde{M} \cong \mathbb{R}^m$. □

5. Fourier Transform and Sobolev Spaces

Let us first introduce some standard notations.

(1) $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$.
(2) $x \cdot y = x_1y_1 + x_2y_2 + \cdots + x_my_m$, $|x| = (x \cdot x)^{1/2}$.
(3) $\alpha = (\alpha_1, \ldots, \alpha_m)$ multi-index, $\alpha_i \in \mathbb{N} \cup \{0\}$, $|\alpha| = \sum_{i=1}^m \alpha_i$.
(4) $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_m^{\alpha_m}$.
(5) $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_m^{\alpha_m}}$, $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$.

To prove the regularity theorem and compactness theorem (cf. theorem 4.10, 4.11), we need to define the concept of *Sobolev space*. To achieve this, we first need the some rudiments in Fourier analysis.

Definition 4.22 (Fourier transdorm). For $\xi \in \mathbb{R}^m$, $x \in \mathbb{R}^m$, let $f \in C_0^\infty(\mathbb{R}^m, \mathbb{C})$ be a smooth function with compact support. The Fourier transform of f is defined as

$$\hat{f}(\xi) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-ix\cdot\xi} f(x)\, dx.$$

Also, recall the *convolution* of two functions

$$f * g := \int_{\mathbb{R}^m} f(x-y)g(y)\, dy = \int_{\mathbb{R}^m} f(z)g(x-z)\, dz.$$

It is a standard trick to use convolution to construct smooth approximation of functions with inferior smothness. Let $f \in C_0^\infty(\mathbb{R}^m)$ with $\int_{\mathbb{R}^m} f = 1, f \geq 0, f(0) \neq 0$. Such a function is sometimes called a *molifier*. The existence of molifiers is evident from our construction on bump functions in chapter 1. For $u \in \mathbb{R}_+$, define $f_u(x) := \frac{1}{u^m} f\left(\frac{x}{u}\right)$.We can see that $\int_{\mathbb{R}^m} f_u = 1$. Such a sequence f_u is called a "δ-function", and we can write it as $\delta_0 = \lim\limits_{u\to 0} f_u$.

Proposition 4.23. *For $g \in C^0$, $f_u * g(x) \to g(x)$ as $u \to 0+$ is a smooth approximation of identity*[4].

Exercise 4.16.

(1) Prove the above proposition.
(2) Show that $C_0^\infty(\mathbb{R}^m)$ is dense in $L^2(\mathbb{R}^m)$ (with respect to L^2-norm).

Thus, we may extend the definition Fourier transform $\hat{f}$ to $f \in L^2(\mathbb{R}^m)$ by continuity. That is, we define $\hat{f} = \lim_{k\to} \widehat{f_k}$, where $f_k \in C_0^\infty(\mathbb{R}^m)$ and $f_k \to f$ in $L^2(\mathbb{R}^m)$. Recall that we also have *inverse Fourier transform*.

Definition 4.24 (Inverse Fourier Transform). For $g(\xi) \in C_0^\infty(\mathbb{R}^m)$, the inverse Fourier transform $\check{g}(x)$ is defined as[5]

$$\check{g}(x) := (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{i\xi\cdot x} g(\xi)\, d\xi.$$

[4]That is, $\delta_0 = \lim_{u\to 0} f_u$ is the identity element for $*$

[5]There are several conventions for Fourier transform and its inversion. Another common convention is that $\hat{f}(\xi) = \int_{\mathbb{R}^m} f(x)e^{-ix\cdot\xi}dx$ while the inversion is given by $\check{g}(x) := (2\pi)^{-m} \int_{\mathbb{R}^m} g(\xi)e^{ix\cdot\xi}d\xi$.

Similarly, we can define $\check{g}$ for $g \in L^2(\mathbb{R}^m)$ by continuity arguments. We list the basic properties and correspondence with convolutions in the following theorem and refer the proof to [Gil95], Chapter 1.1.

Theorem 4.25 (Basic Properties of Fourier Transform). *For $f \in C_0^\infty(\mathbb{R}^m)$, we have*

(1) $\check{\hat{f}} = f$.
(2) $D_\xi^\alpha \hat{f}(\xi) = \widehat{x^\alpha f(x)}$, $\widehat{D_x^\alpha f(x)} = \xi^\alpha \hat{f}(\xi)$.
(3) $\widehat{f * g} = \hat{f}\hat{g}$, $\hat{f} * \hat{g} = \widehat{f \cdot g}$.

Moreover, Fourier Transform on $L^2(\mathbb{R}^m)$ is an isometry onto itself[6], *i.e.*

$$L^2(\mathbb{R}^m) \underset{\check{}}{\overset{\hat{}}{\rightleftarrows}} L^2(\mathbb{R}^m) \ .$$

From the formula $\widehat{D_x^\alpha f} = \xi^\alpha \hat{f}$, we notice that derivatives of f will corresponds multiplications of $\hat{f}$ in the frequency (phase) space. This formula enlightens the idea of weak derivatives. First, for $k \in \mathbb{N}$, $f \in C_0^\infty(\mathbb{R}^m, \mathbb{C})$, any multi-index α with $|\alpha| = k$, we consider the norm to evaluate the L^2-norms of its derivatives:

$$\int_{\mathbb{R}^m} \sum_{|\alpha| \le k} |D^\alpha f(x)|^2 dx.$$

Thus, from $\widehat{D_x^\alpha f} = \xi^\alpha \hat{f}$, we have

$$\begin{aligned} |f|_k^2 &:= \sum_{|\alpha| \le k} \int_{\mathbb{R}^m} |D^\alpha f(x)|^2 dx \\ &= \sum_{|\alpha| \le k} \int_{\mathbb{R}^m} |\widehat{D^\alpha f}(\xi)|^2 d\xi = \sum_{|\alpha| \le k} \int_{\mathbb{R}^m} |\xi^\alpha|^2 |\hat{f}(\xi)|^2 d\xi. \end{aligned}$$

Observe that there exists constant $C_1, C_2 > 0$ such that

$$C_1(1 + |\xi|^2)^k \le \sum_{|\alpha| = k} |\xi^\alpha|^2 \le C_2(1 + |\xi|^2)^k.$$

[6]This is called *Plancherel theorem*

Hence, the norm is equivalent to the norm[7]

$$\int_{\mathbb{R}^m}(1+|\xi|^2)^k|\widehat{f}(\xi)|^2d\xi,$$

which is free of any differentiation. More generally, we have the following definition.

Definition 4.26.

(1) For $s \in \mathbb{R}$, which is regarded as order of L^2-derivatives , we define

Sobolev s-norm of a L^2-function f by

$$|f|_s^2 := \int_{\mathbb{R}^m}(1+|\xi|^2)^s|\hat{f}(\xi)|^2\,d\xi,$$

and the inner product $(\cdot,\cdot)_s$ by

$$(f,g)_s := \int_{\mathbb{R}^m}(1+|\xi|^2)^s\hat{f}(\xi)\hat{g}(\xi)\,d\xi.$$

2) The *Sobolev s-space* $H_s(\mathbb{R}^m)$ is defined to [8] be the completion of $C_0^\infty(\mathbb{R}^m)$ in $L^2(\mathbb{R}^m)$ (w.r.t the s-norm).

---

[7] Another common choice of weight is $(1+|\xi|)^{2k}$.

[8] Here is an alternative way to define Sobolev space. The notion of *weak derivative* can be defined more directly by "integration by part". That is, we say $v \in L^2(\mathbb{R}^m,\mathbb{C})$ is a α-th L^2*-weak derivative* for a locally integrable function $f \in L^1_{\text{loc}}(\mathbb{R}^m)$ if it satisfies

$$\int_{\mathbb{R}^m} v\varphi dx = (-1)^{|\alpha|}\int_{\mathbb{R}^m} f(\partial^\alpha\varphi)dx, \quad \forall\varphi\in C_0^\infty(\mathbb{R}^m,\mathbb{C}).$$

For $k \in \mathbb{N}$, if we assume that $f \in L^2(\mathbb{R}^m)$ having L^2-derivative $d^\alpha f$ for any $|\alpha| \le k$, we then define *Sobolev k-norm* by

$$|f|_k^2 := \int_{\mathbb{R}^m}\sum_{|\alpha|\le k}|d^\alpha f(x)|^2dx, \tag{4.7}$$

and define the Sobolev space $H^k(\mathbb{R}^m)$ by

$$H^k(\mathbb{R}^m) := \{u \in L^1_{\text{loc}}(\mathbb{R}^m) : \exists\partial^\alpha f \in L^2(\mathbb{R}^m), \quad \forall|\alpha|\le k\}. \tag{4.8}$$

The definition is in fact equivalent to our definition. More generally, for $1 \le p < \infty$, we can consider $W^{k,p}$ consisting of locally integrable function f having L^p-weak derivative up to order k. We can also define the Sobolev norm on $W^{k,p}$ by $|f|_{k,p} = \left(\sum_{|\alpha|\le k}|d^\alpha f|^p_{L^p}\right)^{1/p}$.

Then again by the formula $\widehat{D_x^\alpha f} = \xi^\alpha \hat{f}$,

$$D^\alpha : H_s \cap C_0^\infty \longrightarrow H_{s-|\alpha|},$$
$$|D_x^\alpha f|^2_{s-|\alpha|} = \int_{\mathbb{R}^m} (1+|\xi|^2)^{s-|\alpha|} |\xi^\alpha \hat{f}(\xi)|^2 \, d\xi.$$

The key question here is that: although we define the weak derivative by multiplication in frequency space, when is f is actually C^k-differentiable if $\hat{f}$ lies in some Sobolev space H_s, i.e. $\xi^\alpha \hat{f}(\xi)$ lies in L^2-space? This is answered by the following

Theorem 4.27 (key lemmas in Euclidean spaces)**.**

(1) (Sobolev lemma) *If $s > k + m/2$ and $f \in H_s$, then $f \in C^k$ and there exists a universal constant $C > 0$ depending only on s such that*
$$|f|_{C^k} \leq C|f|_s.$$
where $|f|_{C^k} = \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^m} |\partial^\alpha f(x)|$ is the supremum norm of C^k-functions.

(2) (Rellich lemma) *If $s > t$, $H_s \hookrightarrow H_t$ is a compact imbedding.*

(3 (interpolation inequality) *Let $s > t > u$. $\forall\, \epsilon > 0$, $\exists\, C(\epsilon)$ such that*
$$|f|_t \leq \epsilon |f|_s + C(\epsilon)|f|_u$$
for all $f \in C_0^\infty$.

PROOF.

(1) Let $k = 0$. First, we consider $f \in C_0^0(\mathbb{R}^m)$:

$$\begin{aligned}
|f(x)| &= \left| \int_{\mathbb{R}^m} e^{ix\xi} \hat{f}(\xi) \, d\xi \right| \\
&= \left| \int_{\mathbb{R}^m} \left[e^{ix\xi} \hat{f}(\xi)(1+|\xi|^2)^{s/2} \right] (1+|\xi|^2)^{-s/2} \, d\xi \right| \\
&\leq |f|_s \left(\int_{\mathbb{R}^m} \frac{1}{(1+|\xi|^2)^s} \, d\xi \right)^{1/2} \qquad \text{(Cauchy–Schwarz inequality)} \\
&\leq C|f|_s \,,
\end{aligned}$$

for some C depending on $s > m/2$. In other words, $|f|_{C^0} \leq C|f|_s$, for some constant $C > 0$ depending only on s.

Now, for any $f \in H_s(\mathbb{R}^m)$, we can choose a sequence $f_j \xrightarrow{H_s} f$ with $f_j \in C_0^0(\mathbb{R}^m)$. By the above result,

$$|f_i - f_j|_{C^0} \leq C|f_i - f_j|_s$$

infers that f_i forms a Cauchy sequence in C^0. Thus, $f_i \to f$ uniformly implies that $f \in C^0$. So

$$|f|_{C^0} \leq |f - f_i|_{C^0} + |f_i|_{C^0} \leq C(|f - f_i|_s + |f_i|_s)$$

and then $|f|_{C^0} \leq C|f|_s$ by taking $i \to \infty$.

For $k > 0$ and $s > k + m/2$, apply the same argument to $D^\alpha f$, for any multi-index α with $|\alpha| = k$:

$$|D^\alpha f|_{C^0} \leq C|D^\alpha f|_{s-k} \leq C|f|_s.$$

Hence, for any α with $|\alpha| = k$, $D^\alpha f \in C^0(\mathbb{R}^m)$, and thus $H^s \subset C^k$. Also, we obtain $|f|_{C^k} \leq C|f|_s$.

(2) Consider $K \subset_{cpt} \mathbb{R}^m$, $f_n \in C^\infty \cap H_s$ with $\mathrm{supp}(f_n) \subset K$ and $|f_n|_s \leq C$.

Let $g \in C_0^\infty(\mathbb{R}^m)$ with $g \equiv 1$ on K. Then $g \cdot f_n = f_n$ and thus $\hat{f}_n = \hat{g} * \hat{f}_n$. We then have

$$\partial_j \hat{f}_n = \partial_j(\hat{g} * \hat{f}_n) = (\partial_j \hat{g}) * \hat{f}_n.$$

$$\begin{aligned}
|\partial_j \hat{f}_n(\xi)| &\leq \int_{\mathbb{R}^m} \left| \frac{\partial \hat{g}}{\partial \xi_j}(\xi - \eta) \hat{f}_n(\eta) \right| d\eta \\
&\leq |f_n|_s \left(\int_{\mathbb{R}^m} \frac{|(\partial_{\xi_j} \hat{g})(\xi - \eta)|^2}{(1 + |\eta|^2)^s} d\eta \right)^{\frac{1}{2}} \quad \text{(Cauchy–Schwarz).}
\end{aligned}$$

We denote $h_j(\xi) := \left(\int_{\mathbb{R}^m} (1 + |\eta|^2)^{-s} |(\partial_{\xi_j} \hat{g})(\xi - \eta)|^2 d\eta \right)^{\frac{1}{2}}$. From theorem 4.25, for any multi-indices α, β,

$$\xi^\beta D_\xi^\alpha \hat{g} = \widehat{D_x^\beta (x^\beta g)},$$

and $D^x \beta(x^\beta g) \in C_0^\infty(\mathbb{R}^m)$ since $g \in C_0^\infty(\mathbb{R}^m)$. We conclude from Plancherel's theorem that $|\xi^\beta D_\xi^\alpha \hat{g}|_0 = |D^x \beta(x^\beta g)|_0 <$

∞ and hence [9] $h_j(\xi) < \infty$, for any $\xi \in \mathbb{R}^m$. Moreover, from continuity of Lebesgue integral with respect to translation, $h_j(\xi)$ is a continuous function in ξ. In conlusion, $|\partial_j \hat{f}_n(\xi)| \leq C_j h_j(\xi)$.

Next, apply the same process to $\hat{f}_n(\xi)$, and we also have $|\hat{f}_n(\xi)| \leq Ch_0(\xi)$. Hence $\hat{f}_n$ is a uniformly bounded and equicontinuous sequence on any compact subset in ξ. By Arzela-Ascoli theorem and a diagonal argument, there exists convergent subsequences of $\hat{f}_n$ (still denoted as $\hat{f}_n$) uniformly on each compact subset of $\xi \in \mathbb{R}^m$.

Now, we show that f_n converges in H_t for $s > t$.

$$|f_j - f_k|_t^2 = \int_{\mathbb{R}^m} |\hat{f}_j - \hat{f}_k|^2 (1 + |\xi|^2)^t \, d\xi$$

We decompose the domain into two parts and have the following estimates.

For $|\xi| \geq r$, $(1 + |\xi|^2)^t \leq (1 + r^2)^{t-s}(1 + |\xi|^2)^s$ and

$$\begin{aligned}
&\int_{|\xi| \geq r} |\hat{f}_j - \hat{f}_k|^2 (1 + |\xi|^2)^t \, d\xi \\
&\leq (1 + r^2)^{t-s} \int_{\mathbb{R}^m} |\hat{f}_j - \hat{f}_k| (1 + |\xi|^2)^s \, d\xi \\
&\leq 2C(1 + r^2)^{t-s}.
\end{aligned}$$

For any $\epsilon > 0$, we can choose r large such that $2C(1 + r^2)^{t-s} < \epsilon/2$.

On the other hand, $\{|\xi| \leq r\}$ is a compact set. We can pick j, k large such that

$$\int_{|\xi| \leq r} |\hat{f}_j - \hat{f}_k|^2 (1 + |\xi|^2)^t \, d\xi < \epsilon/2.$$

Hence, $|f_j - f_k|_t^2 < \epsilon$ for j, k large enough and this implies $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $H_t(\mathbb{R}^m)$.

[9] Alternatively, one can proves this by using the notion of *Schwartz space*. The Schwartz space on $\mathbb{R}^m$, denoted by $\mathcal{S}(\mathbb{R}^m)$, is defined by: $f \in \mathcal{S}(\mathbb{R}^m)$ if $f \in C^\infty(\mathbb{R}^m)$ and for any multi-indices α, β, $\sup_{x \in \mathbb{R}^m} |x^\beta \partial_x^\alpha f(x)| < \infty$. One can show that $C_0^\infty(\mathbb{R}^m) \subset \mathcal{S}(\mathbb{R}^m) \subset L^2(\mathbb{R}^m)$ and Fourier transform is a bijection on $\mathcal{S}(\mathbb{R}^m)$.

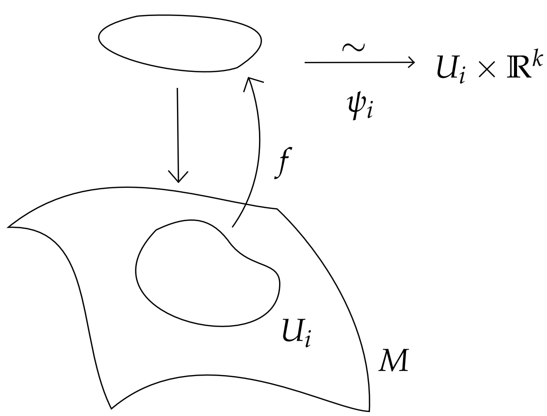
(3) For $s > t > u$, there exists C_ϵ such that
$$(1+|\xi|^2)^t \le \epsilon(1+|\xi|^2)^s + C_\epsilon(1+|\xi|^2)^u.$$
Then we get the interpolation inequality immediately.

□

6. Gårding's Inequality

Now we want to reduce to the Euclidean case and make use of results on Euclidean space in previous sections

Pick a local trivialization (U_i, ϕ_i, ψ_i) where U_i are charts on M, ϕ_i is a P.O.U. and ψ_i is a trivialization of $E\big|_{U_i}$.



For any $f \in C^\infty(M,E)$, $f = \sum_i f^i \sigma_i$, i.e. $\phi_i(f) = (f^1, \cdots, f^k)$. And for $s \in \mathbb{R}$,
$$|f|_s = \sum_i |\psi_i(\phi_i \cdot f)|_s.$$
Then we can define $H_s(M,E)$ to be the completion of $C^\infty(M,E)$ with respect to the norm $|\cdot|_s$.

Denote $H_0(M,E) = L^2(M,E)$.

Remark 4.28.
$$\bigcup_{s\in\mathbb{R}} H_s =: H_{-\infty} \supset \cdots \supset H_s \overset{s<t}{\supset \cdots \supset} H_s \supset \cdots \supset H_\infty := \bigcap_{s\in\mathbb{R}} H_s = C^\infty.$$

Note that $\cup_s H_s = C^\infty$ follows from the Sobolev lemma.

Exercise 4.17.

(1) For any choice of (U_i, ϕ_i, ψ_i) such that ϕ_i, ψ_i (and their inverse) have bounded derivation, show that the Sobolev norms $|\cdot|_s$ are all equivalent.
(2) All the key lemmas still hold.

To ensure the regularity, we need to show that the weak solution belongs to some H_s for higher order s actually and use the key lemmas. This estimate comes from the following theorem.

Theorem 4.29 (Gårding inequality). *P is an elliptic operator of order d on (M, E). There exists $C > 0$ such that*

$$|f|_{s+d} \leq C\left(|Pf|_s + |f|_s\right),$$

for all $f \in H_{s+d}$.

PROOF. Under some local chart (U_i, ϕ_i, ψ_i), we rewrite $P = P_0 + P_1 + P_2$ where

$$P_0 = \sum_{|\alpha|=d} A_\alpha(0) D^\alpha, P_1 = \sum_{|\alpha|<d} A_\alpha(x) D^\alpha, P_2 = \sum_{|\alpha|=d} (A_\alpha(x) - A_\alpha(0)) D^\alpha$$

for some $p \in U_i$, $p \leftrightarrow x = 0$.

Locally, by uniform ellipticity at p,

$$\begin{aligned}
|P_0 f|_s^2 &= \int (1 + |\xi|^2)^s |\widehat{P_0 f}(\xi)|^2 \, d\xi \\
&= \int (1 + |\xi|^2)^s \left| \sum_{\alpha=d} A_\alpha(0) \xi^\alpha \hat{f}(\xi) \right|^2 \, d\xi \\
&\geq C' \int (1 + |\xi|^2)^s |\xi|^{2d} |\hat{f}(\xi)|^2 \, d\xi.
\end{aligned}$$

$$\begin{aligned}
(|P_0 f|_s + |f|_s)^2 &\geq \int (1 + C''|\xi|^d)^2 (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi \\
&\geq \int C(1 + |\xi|^2)^d (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi \quad \geq C_1^2 |f|_{s+d}^2.
\end{aligned}$$

Under each local chart, we may assume f is compactly supported and we can get easily the following estimates

$$\begin{aligned}
|P_1 f|_s &\leq C_2 |f|_{s+d-1} \\
|P_2 f|_s &\leq \frac{C_1}{2} |f|_{s+d} + C_2 |f|_{s+d-1}
\end{aligned}$$

by choosing U small enough a priori.

Exercise 4.18. Show that in the proof of this inequality 4.29, there exists open neighborhood U of p independent of f such that the above estimate for $P_2 f$ holds. Notice that $P_2 f(0) = 0$.

Hence

$$\begin{aligned} |Pf|_s + |f|_s &\geq |P_0 f|_s - |P_1 f|_s - |P_2 f|_s + |f|_s \\ &\geq \frac{C_1}{2}|f|_{s+d} - 2C_2|f|_{s+d-1}. \\ \Rightarrow |f|_{s+d} &\leq C(|Pf|_s + |f|_s + |f|_{s+d-1}). \end{aligned}$$

By interpolation inequality, ($\epsilon = 1/2C$)

$$|f|_{s+d-1} \leq \frac{1}{2C}|f|_{s+d} + C(\epsilon)|f|_s.$$

Finally we have

$$\frac{1}{2}|f|_{s+d} \leq C|Pf|_s + \tilde{C}|f|_s.$$

□

7. Proof of Compactness and Regularity Theorem

Through out this section, we follow the notations as in the previous section. Particularly, P is an elliptic operator of degree $d \in \mathbb{N}$.

Theorem 4.30 (Compactness theorem)**.** *For a sequence $u_n \in \mathcal{H}$, if $\|u_n\| \leq C$, $\|Pu_n\| \leq C$ are bounded, then $\{u_n\}$ has a Cauchy subsequence (w.r.t. L^2 norm).*

Proof. Since $u_n \in C^\infty$ on compact M, u_n, Pu_n is uniformly bounded in L^2-norm. By Garding inequality,

$$|u_n|_d \leq C(|Pu_n|_0 + |u_n|_0) \leq \tilde{C}.$$

By Rellich lemma, $u_n \in H_d$ and u_n has a Cauchy subsequence in H_0. □

Theorem 4.31 (Regularity theorem)**.** *If $Pu = v$, $v \in H_t$, $u \in H_{-\infty} := \cup_{s\in\mathbb{R}} H_s$, then $u \in H_{t+d}$.*

PROOF. Suppose $u \in H_s$ for some $s \in \mathbb{R}$. Also, it suffices to establish the theorem in local situation. Hence, we may assume $u \in H_s(\mathbb{R}^m)$ and P is an elliptic operator acting on scalar functions. By induction, it suffices to show that

$$Pu \in H_{s-d+1} \Rightarrow u \in H_{s+1}.$$

The key here is to consider that the *difference quotient*

$$u^h(x) := \frac{u(x+h) - u(x)}{|h|},$$

for any sufficiently small $h \neq 0$. Also, for $h \in \mathbb{R}^m$, we denote *translation* $u(x+h)$ by $T_h u = u(x+h)$. In other words, the difference quotient $u^h(x)$ can be written as $\frac{1}{|h|}(T_h u(x) - u(x))$.

First, observe that:

$$\begin{aligned}
\widehat{T_h u}(\xi) &= \int e^{-i\xi x} u(x+h)\, dx \\
&= e^{ih\xi} \int e^{-i(x+h)\xi} u(x+h)\, d(x+h) \\
&= e^{ih\xi} \hat{u}(\xi).
\end{aligned}$$

$$\Rightarrow \widehat{u^h}(\xi) = \frac{e^{ih\xi} - 1}{|h|} \hat{u}(\xi) = \left(i\frac{h \cdot \xi}{|h|} + o(|h|) \right) \hat{u}.$$

Therefore,

$$|u^h(x)|_s = \int_{\mathbb{R}^m} (1 + |\xi|^2)^s |\widehat{u^h}(\xi)|^2 d\xi = \int_{\mathbb{R}^m} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 (|\xi|^2 + R(\xi, h)) d\xi,$$

where $R(\xi, h) = o(|h|)$. If for any $h \neq 0$ with $|h|$ small, $|u^h(x)|_s \leq C$, for some constant C independent of h, then

$$\int (1 + |\xi|^2)^s |\xi|^2 |\hat{u}(\xi)|^2 \, d\xi < \infty,$$

since $u \in H_s$. Conversely, if $u \in H_{s+1}$, then clearly $|u^h|_s \leq |u|_{s+1}$, for any small $h \neq 0$. In conclusion, we have proved:

$$u \in H_s, |u^h|_s \leq C \quad \forall \text{ small } h \neq 0 \Longleftrightarrow u \in H_{s+1}.$$

Now, we then turn to show $|u^h|_s$ has a uniform bound C with respect to h. To see this, we need an elementary lemma.

Lemma 4.32.

$$P(u^h) = (Pu)^h - P^h(T_h u)$$

where $P^h = \sum_\alpha A^h_\alpha(x) D^\alpha$.

PROOF OF LEMMA.

$$\begin{aligned} & A_\alpha(x)D^\alpha(u(x+h) - u(x)) \\ = \; & A_\alpha(x+h)D^\alpha u(x+h) - A_\alpha(x)D^\alpha u(x) \\ & - (A_\alpha(x+h) - A_\alpha(x))D^\alpha u(x+h). \end{aligned}$$

□

Back to the proof. From Garding inequality and the above lemma,

$$\begin{aligned} |u^h|_s & \le C(|P(u^h)|_{s-d} + |u^h|_{s-d}) \\ & \le C(|(Pu)^h|_{s-d} + |P^h(T_h u)|_{s-d} + |u^h|_{s-d}) \end{aligned}$$

Notice that $Pu \in H_{s-d+1}$ infers that $|(Pu)^h|_{s-d} \le C_1$, for some constant C_1 uniformly in any small $h \neq 0$. Similarly, $u \in H^s$ implies that $|u^h|_{s-d} \le C_2$, for some uniform constant C_2 in small $h \neq 0$.

Exercise 4.19. Show that the term $|P^h(T_h u)|_{s-d}$ has a uniform bound in h, for any small $h \neq 0$.

As a result, $|u^h|_s \le C$ for any small $h \neq 0$, and hence $u \in H^{s+1}$.

□

Remark 4.33. In the proof of regularity theorem, we a priori assume that $u \in H^s(\mathbb{R}^m)$ for some $s \in \mathbb{R}$. In fact, in the original statement of regularity theorem (cf. theorem 4.10), we assume that u is a weak solution implies that $u \in L^2 = H_0$.

8. Problems

4.1. ([War83] Ch.6 #6) Derive explicit formulas for $d, *, \delta$ and $\triangle$ in Euclidean space. In particular, show that if

$$\alpha = \sum_{i_1 < \cdots < i_p} \alpha_I dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

then

$$\triangle\alpha = (-1) \sum_{i_1 < \cdots < i_p} \left(\sum_{i=1}^{n} \frac{\partial^2 \alpha_I}{\partial x_i^2} \right) dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

4.2. ([War83] Ch.6 #7) Let φ belong to the C^∞ periodic functions $\mathscr{P}$ on the plane. Prove that

$$\left\| \frac{\partial^2 \varphi}{\partial x \partial y} \right\| \leq \frac{1}{2} \|\triangle\varphi\|$$

4.3. ([War83] Ch.6 #8) The Rellich lemma 4.27 says that the natural injection $i : H_t \to H_s$ for $s < t$ is a *compact operator*; that is, it takes bounded sequences into sequences with convergent subsequences. An analogous example of this phenomenon is the following. Let C denote the Banach space of periodic continuous functions on the real line, say with period 2π, and with norm the sup-norm $\|\cdot\|_\infty$. Let C^1 be the subset of C consisting of functions with continuous first derivative. As a norm for C^1 we take

$$\|f\| = \|f\|_\infty + \left\| \frac{df}{dx} \right\|_\infty .$$

Use Arzela-Ascoli theorem to prove that the natural injection $i : C^1 \to C$ is a compact operator.

4.4. ([War83] Ch.6 #9) We shall consider a number of elliptic equations of the form $Lu = f$ on the real line. In each case, f will be smooth and periodic of period 1, and we look for solution u also periodic of period 1. This restriction to periodic functions makes this in essence a problem on a *compact* space, the unit circle. We let $u' = du/dx$, etc.

(1) $u' = f$. This is the simplest example of an elliptic operator which exhibits all of the essential ingredients of the theory. What is the formal adjoint of this differential operator? Show that there is a solution u (periodic) if and only if f is orthogonal to the kernel of this adjoint.
(2) $u' - u = f$. What is the kernel (in the periodic functions) in the case? What are the necessary and sufficient conditions on f for there to exist a periodic solution?
(3) $u'' = f$. Show that this operator is formally self-adjoint. Show that there is a periodic solution if and only if f is orthogonal to the kernel; and using the fact that

$$\int_0^x \left(\int_0^t f(s)\, ds \right) dt = \int_0^x f(s) \left(\int_0^x dt \right) ds,$$

show that the unique solution orthogonal to the kernel is

$$u(x) = \int_0^x t(x-1)f(t)\,dt + \int_x^1 x(t-1)f(t)\,dt - \frac{1}{2}\int_0^1 t(t-1)f(t)\,dt.$$

This explicitly exhibits the Green's operator for this case.

(4) $u'' + 4\pi^2 u = f$. Show that this operator is formally self-adjoint. What is the kernel? Derive an explicit formula for the solution u, and show that u is periodic if and only if f is orthogonal to the kernel.

4.5. ([War83] Ch.6 #12) Let α and β be n-forms on a compact oriented manifold M^n such that $\int_M \alpha = \int_M \beta$. Prove that α and β differ by an exact form.

4.6. ([War83] Ch.6 #13) Show that the compactness theorem cannot be strengthened to the assertion of the existence of a subsequence which is convergent in $A^p(M)$.

4.7 (The Eigenvalues of the Laplacian)**.** This is an extended exercise in which the fundamental properties of the eigenfunctions and eigenvalues of the Laplacian are developed.

Consider the Hodge Laplacian $\triangle$ acting on the p-forms $A^p(M)$ for some fixed p. A real number λ corresponding to which there exists a not identically zero p-form u such that $\triangle u = \lambda u$ is called an *eigenvalue* of $\triangle$. If λ is an eigenvalue, then any p-form u such that $\triangle u = \lambda u$ is called an *eigenform* of $\triangle$ corresponding to the eigenvalue λ. The eigenforms corresponding to a fixed λ form a subspace $E_\lambda(M)$ of $A^p(M)$ called the *eigenspace* of the eigenvalue λ.

(1) Prove the following properties of eigenvalues of $\triangle$ regardless the existence.
  (a) Prove that the eigenvalues of $\triangle$ are non-negative.
  (b) Prove that the eigenspaces of $\triangle$ are finite dimensional.
  (c) Prove that the eigenvalues have no finite accumulation point.
  (d) Prove that eigenfunctions corresponding to distinct eigenvalues are orthogonal.

(Hint: use compactness theorem for (b) and (c))

(2) (Existence) Now, we shall now establish that $\triangle$ has a positive eigenvalue. Consider $\triangle : (\mathbb{H}^p)^\perp \to (\mathbb{H}^p)^\perp$, and also we have the Green's operator $G : (\mathbb{H}^p)^\perp \to (\mathbb{H}^p)^\perp$ with $\triangle G\alpha = \alpha$ and $G\triangle\alpha = \alpha$ for all $\alpha \in (\mathbb{H}^p)^\perp$.

(a) Show that the eigenvalues of $G\big|_{(\mathbb{H}^p)^\perp}$ are the reciprocals of the eigenvalues of $\triangle\big|_{(\mathbb{H}^p)^\perp}$.

Let

$$\eta = \sup_{\|\varphi\|=1, \varphi\in(\mathbb{H}^p)^\perp} \|G\varphi\|.$$

Then $\eta > 0$ and $\|G\varphi\| \leq \eta\|\varphi\|$ for every $\varphi \in (\mathbb{H}^p)^\perp$. We shall prove that $1/\eta$ is an eigenvalue of $\triangle$. Let $\{\varphi_i\} \in (\mathbb{H}^p)^\perp$ be a maximizing sequence for η; that is, $\|\varphi_j\| = 1$ and $\|G\varphi_j\| \to \eta$. First, we observe that $\|G^2\varphi_j - \eta^2\varphi_j\| \to 0$, for

$$\begin{aligned}\|G^2\varphi_j - \eta^2\varphi_j\|^2 =& \|G^2\varphi_j\|^2 - 2\eta^2\langle G^2\varphi_j, \varphi_j\rangle + \eta^4 \\ \leq& \eta^2\|G\varphi_j\|^2 - 2\eta^2\|G\varphi_j\|^2 + \eta^4 \to 0.\end{aligned}$$

Second, let $\psi_j = G\varphi_j - \eta\varphi_j$.

(b) Show that $(\psi_j, G\psi_j) \geq 0$.

From this, we have

$$\begin{aligned}0 \leftarrow \langle\psi_j, G^2\varphi_j - \eta^2\varphi_j\rangle& \\ =&\langle\psi_j, G\psi_j + \eta\psi_j\rangle \\ =&\langle\psi_j, G\psi_j\rangle + \eta\|\psi_j\|^2 \geq \eta\|\psi_j\|^2,\end{aligned}$$

Thus, we conclude that $\|\psi_j\| \to 0$. Now there is a subsequence of the φ_j, call it $\{\varphi_j\}$, such that $\{G\varphi_j\}$ is Cauchy. Define a linear functional ℓ on $A^p(M)$ by setting

$$\ell(\beta) = \lim_{j\to\infty} \eta\langle G\varphi_j, \beta\rangle, \quad \beta \in A^p(M).$$

(c) Show that ℓ is a non-trivial weak solution of

$$(\triangle - 1/\eta)u = 0.$$

(d) Show that $\triangle - 1/\eta$ is elliptic,

From (c) and (d), we conclude that $\lambda = 1/\eta$ is an eigenvalue of $\triangle$.

(3) (Existence of Other Eigenvalues) Suppose that we have eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and corresponding orthonormalized eigenforms $u_1, u_2, \ldots, u_n$ for $\triangle\big|_{(\mathbb{H}^p)^\perp}$. Let R_n be the subspace of

$(\mathbb{H}^p)^\perp$ spanned by $\{u_1, \ldots, u_n\}$. Observe that G and $\triangle$ map $(\mathbb{H}^p \oplus R_n)^\perp$ into itself, then define

$$\eta_{n+1} = \sup_{\|\varphi\|=1,\ \varphi \in (\mathbb{H}^p \oplus R_n)^\perp} \|G\varphi\|$$

and proceed as in part (2) to establish that $\lambda_{n+1} = 1/\eta_{n+1}$ is an eigenvalue of $\triangle$. Clearly $\lambda_{n+1} \geq \lambda_n$.

(4) (L^2 Completeness) Let $\lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of $\triangle$ on $A^p(M)$, where each eigenvalue is included as many times as the dimension of its eigenspace, with a corresponding orthonormalized sequence of eigenfunctions $\{u_i\}$. Let $\alpha \in A^p(M)$. Then

$$\lim_{n\to\infty} \left\| \alpha - \sum_{i=1}^{n} (\alpha, u_i) u_i \right\| = 0.$$

To prove this, let k be the dimension of $\mathbb{H}^p$.

(a) Show that there exists $\beta \in (\mathbb{H}^p)^\perp$ such that $G\beta = \alpha - \sum_{i=1}^{k} (\alpha, u_i) u_i$. It follows that

$$\left\| \alpha - \sum_{i=1}^{n} (\alpha, u_i) u_i \right\| = \left\| G \left(\beta - \sum_{i=k+1}^{n} (\beta, u_i) \beta_i \right) \right\|$$

for $n > k$. But, by the definition of λ_{n+1},

$$\begin{aligned} \left\| G \left(\beta - \sum_{i=k+1}^{n} (\beta, u_i) u_i \right) \right\| &\leq \frac{1}{\lambda_{n+1}} \left\| \beta - \sum_{i=k+1}^{n} (\beta, u_i) u_i \right\| \\ &\leq \frac{1}{\lambda_{n+1}} \|\beta\| \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

(5) (Uniform Completeness) The uniform norm $\|\alpha\|_\infty$ is defined on $A^p(M)$ by

$$\|\alpha\|_\infty = \sup_{m \in M} \left(*(\alpha \wedge *\alpha)(m) \right)^{1/2}.$$

(a) Show that there exists a large enough integer k and a constant $c > 0$ such that

$$\|\alpha\|_\infty \leq c \|(1 + \triangle)^k \alpha\|$$

for every $\alpha \in A^p(M)$. (Hint: Use Sobolev lemma)

Let $\alpha \in A^p(M)$, and let $P_n(\alpha) = \sum_{i=1}^{n} \langle \alpha, u_i \rangle u_i$, where we are continuing with the notation of (4). Now $\triangle P_n = P_n \triangle$, so that

$$\begin{aligned} \|\alpha - P_n(\alpha)\|_\infty &\leq c \|(1 + \triangle)^k [\alpha - P_n(\alpha)]\| \\ &= \|\varphi - P_n \varphi\| \to 0, \end{aligned}$$

where $\varphi = (1+\triangle)^k \alpha$.

4.8. ([War83] Ch.6 #17) We define the operator $\triangle^2 : A^p(M) \to A^p(M)$ by $\triangle^2\alpha = \triangle(\triangle\alpha)$. Discuss the solvability of $\triangle^2\alpha = \beta$.

4.9. ([War83] Ch.6 #18) Consider the operator

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c$$

acting on $C^2(\mathbb{R}^n)$. Show that there is no loss of generality in assuming that $a_{ij} = a_{ji}$, and prove that L is elliptic at a point x if and only if the matrix $(a_{ij}(x))$ is positive (or negative) definite.

In particular, show that the *wave equation*

$$\Box u = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f$$

is not elliptic, and give an example where

$$\Box u = f \in C^\infty, \text{ but } u \notin C^\infty.$$

4.10. ([War83] Ch.6 #19) Consider $\triangle : A^p(M) \to A^p(M)$. Prove that if λ is the minimum eigenvalue of $\triangle$, and if $c > -\lambda$, then $(\triangle + c)\alpha = \beta$ can be solved for every $\beta \in A^p(M)$.

4.11. ([War83] Ch.6 #22) Let φ_n, for $n = 1, 2, \dots$ be a periodic C^∞ function on the plane which agrees with $\log\log(\frac{1}{r+1/n})$ for $0 \le r \le \frac{1}{2}$, where $r = \sqrt{x^2+y^2}$. Show that there is no constant $c > 0$ such that

$$\|\varphi_n\|_\infty \le c\|\varphi_n\|_1, \text{ for all } n.$$

This shows, in the case $n = 2$, that the restriction $t \ge [n/2]+1$ ($s > k + m/2$ in this lecture note) in the Sobolev lemma 4.27 is essential.

4.12 ([Car92] Ch.6 #11). Let $f : \overline{M}^{n+1} \to \mathbb{R}$ be a differentiable function. Define the *Hessian*, Hess$f$ of f at $p \in \overline{M}$ as the linear operator

$$\text{Hess} f : T_p\overline{M} \to T_p\overline{M}, (\text{Hess} f)Y = \overline{\nabla}_Y \operatorname{grad} f, Y \in T_p\overline{M},$$

where $\overline{\nabla}$ is the Riemannian connection of $\overline{M}$. Let a be a regular value of f and let $M^n \subset \overline{M}^{n+1}$ be the hypersuperface in $\overline{M}$ defined by $M = \{p \in \overline{M} \mid f(p) = a\}$. Prove that:

(1) The Laplace–Beltrami operator $\overline{\triangle}_{LB} f$ is given by

$$\overline{\triangle}_{LB} f = \mathrm{tr}(\mathrm{Hess} f).$$

(2) If $X, Y \in \mathcal{X}(M)$, then

$$\langle (\mathrm{Hess} f) Y, X \rangle = \langle Y, (\mathrm{Hess} f) X \rangle.$$

Conclude that Hess$f$ is self-adjoint, hence determines a symmetric bilinear form on $T_p\overline{M}$, $p \in \overline{M}$, given by $(\mathrm{Hess} f)(X, Y) = \langle (\mathrm{Hess} f) X, Y \rangle$, $X, Y \in T_p\overline{M}$.

(3) The mean curvature H of $M \subset \overline{M}$ is given by

$$nH = -\mathrm{div}\left(\frac{\mathrm{grad} f}{|\mathrm{grad} f|}\right).$$

(4) Observe that every embedded hypersurface $M^n \subset \overline{M}^{n+1}$ is locally the inverse image of a regular value. Conclude from (c) that the mean curvature H of such a hypersuperface is given by

$$H = -\frac{1}{n}\mathrm{div} H,$$

where N is an appropriate local extension of the unit normal vector field on $M^n \subset \overline{M}^{n+1}$.

4.13 (Witten Deformation)**.** In [Wit82], Edward Witten used the idea of supersymmetry, a conjectured (till today) symmetry of spacetime developed in qunatum field theory during 70s, to give a fresh new point of view on Morse theory.

In this series of problems, we will exploit the most elementary aspect of his idea from mathematical standpoint. Let M^m be a closed C^∞ manifold. Given any $f \in C^\infty(M)$, the starting point of Witten's idea is to consider the deformation of Cartan's exterior derivative d:

$$d_t := e^{-tf} d e^{tf}, \quad t \geq 0.$$

(1) Check that $d_t^2 = 0$.

From (1), we obtain the *deformed de Rham complex* $(A^\bullet(M), d_t)$. We denote $H^q_{t,dR}(M, \mathbb{R})$ by the corresponding q-th cohomology of the complex.

(2) Show that the chain map $(A^\bullet(M), d) \to A^\bullet(M), d_t)$ given by $\alpha \mapsto e^{-tf}\alpha$ gives rise to an isomorphism $H^q_{\mathrm{dR}}(M, \mathbb{R}) \cong H^q_{t,\mathrm{dR}}(M, \mathbb{R})$.

Choosing any Riemannian metric g on M, we denote d_t^* by the formal adjoint of d_t with respect to the L^2-inner product $(\alpha,\beta) = \int_M \langle \alpha,\beta\rangle dV_g$ on $A^p(M)$, where dV_g is the Riemannian volume form of (M,g).

(3) Show that $d_t^* = e^{tf} d^* e^{-tf}$ and $d_t^* = d_t + t\iota_{\mathrm{grad} f}$, where $\mathrm{grad} f := \widetilde{df}$ is the vector field dual fo df via g.

We define Witten's Laplacian $\triangle_t^{(q)} := d_t^* d_t + d_t d_t^*$ on $A^q(M)$. Notice that when $t=0$, $\triangle_t^{(q)}$ is just Hodge Laplacian $\triangle$ acting on q-forms.

(4) Show that $\triangle_t^{(q)}$ is self-adjoint and has the same principal symbol as Hodge Laplacian $\triangle$. Conclude that $\triangle_t^{(q)}$ is also an elliptic operator.

The key observation of Witten is the following Bochner type formula.

(5) At any point $p \in M$, let $\mathbf{x} = (x^1,\dots,x^m)$ be normal coordinate around p, show that

$$\triangle_t^{(q)} = \triangle + t^2|df|^2 + t\sum_{k,l=1}^m \mathrm{Hess}(f)(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l})[dx^k\wedge\cdot,\iota_{\partial/\partial x^l}],$$

where $|df|^2 = \langle df, df\rangle$, $\mathrm{Hess}(f) = \nabla(df)$, and $[\cdot,\cdot]$ is a commutator regarded as in $\mathrm{End}(A^q(M))$.

(6) Show that if p is a critical point of f, then for any $X,Y\in T_pM$, $\mathrm{Hess}(f)(X,Y) = d^2f(X,Y)$, where $d^2f(X,Y)$ is the other Hessian defined in problem 1.12.

For surfficiently large $t \gg 0$, we see from Bochner type formula that the term $t^2|df|^2$ becomes the dominant part for Witten Laplacian if $p\notin \mathrm{Crit}(f)$. On the other hand, since $\triangle_t^{(q)}$ is elliptic for any $t>0$, we know that the eigenforms of $\triangle_t^{(q)}$ must be smooth q-form. Combining these two observations together, we then see that as $t\to\infty$, the eigenforms of $\triangle_t^{(q)}$ concentrates near the neighborhoods of $\mathrm{Crit}(f)$.

Remark 4.34. As a first application, when f is a Morse function (cf. definition 1.46), Witten gave a new proof of Morse inequalities by investigating the spectrum of Witten Laplacian as t sufficiently large. The details can be found for instance in [Zha01]. Moreover, incorporating with ideas from Thom and Smale, instead of just estimating Betti number, we can in fact define a complex, known as *Witten-Morse complex* nowadays, whose cohomology is isomorphic to the cellular cohomology. Later, Floer generalized Witten's idea and applied to problems in symplectic topology.