Chapter 3

RIEMANNIAN MANIFOLDS

In his ground-breaking essay in 1855, Riemann investigated the nature of a space M (called n extended quantities by him) through a precise mechanism of measurement of length. Among others, Riemann studied in details the fundamental case when the measurement takes the form

$$ds = \sqrt{\sum g_{ij}\, dx^i dx^j},$$

now called a Riemannian metric g. Through local expansion, Riemann discovered the invariant quantity R_{ijkl} in the second order terms and proved that its vanishing is equivalent to that (M, g) is locally Euclidean.

The Riemann curvature tensor $\sum R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l$, which generalizes the Gaussian curvature K of surfaces, was (and still is) difficult to work with. To understand it, tensor calculus was subsequently developed in the next decades by Levi–Civita, Christoffel, Ricci and others. Notably, for each $X \in T_pM$ the notion of absolute differentiation $\nabla_X T$ was invented. Aided by this language, Hilbert and Einstein wrote down the field equation of General Relativity in 1915, which confirmed the central role of Riemannian geometry in *Geometry* and *Physics* till today.

In this chapter we present the basic elements in Riemannian geometry. We introduce normal coordinates (local) and exponential maps (global) and prove the Hopf–Rinow theorem to characterize complete manifolds. We then discuss geodesics and variations of them. The precise link between the second variation formula and the curvature tensor leads to topological applications like Bonnet and Synge's theorems. Together with Jacobi fields we prove the Cartan–Hadamard theorem which characterizes spaces with constant curvature as quotients of $\mathbb{R}^N$, S^N or $\mathbb{H}^N$ (space forms).

We discuss also the second fundamental form B and the mean curvature vector H of a submanifold $i : M \hookrightarrow N$ and study the variational aspects. Comparing with geodesics, only the simplest properties are addressed here. More advanced aspects will be discussed in later chapters.

1. Riemannian structures and affine structures

Definition 3.1. A *Riemannian manifold* (M, g) is a C^∞ manifold M together with $g \in C^\infty(M, \mathrm{Sym}^2(T^*M))$ such that g_p is positive definite for every $p \in M$.

Such a g is called a *Riemannian metric*, or simply a *metric*. In local chart $(U, \mathbf{x})$,

$$g = \sum_{i,j} g_{ij}\, dx^i \otimes dx^j,$$

where $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ on U. We also denote $\langle X, Y \rangle = g(X, Y)$.

Any differentiable manifold M can be endowed with Riemannian metrics through a P.O.U. $\{U_\alpha, \rho_\alpha\}$: we can assign an arbitrary metric g_α on U_α (e.g. the Euclidean inner product), and then define

$$g = \sum_\alpha \rho_\alpha g_\alpha.$$

It is easily to see that g is a metric. It is also easy to give another construction of metrics on M under a given C^∞ imbedding $M \hookrightarrow \mathbb{R}^N$. We simply consider the inner product on $T_pM \hookrightarrow T_p\mathbb{R}^N \cong \mathbb{R}^N$ induced from the one on $T_p\mathbb{R}^N$, say the Euclidean one $g_{\mathbb{R}^N}$. [1]

With a Riemannian metric g, we can define the concept of measure and integration even for non-orientable M: for $f \in C_c^\infty(M)$,

$$\int_M f := \int_M f\, d\mu_g,$$

where the measure, or the volume form,

$$d\mu_g \equiv dV \equiv d\,\mathrm{vol} := |\omega|$$

[1]A major philosophical question in Riemannian geometry, essentially posed by Riemann, is the *isometric imbedding problem*: for any (M, g), does there exists a C^∞ imbedding $\iota : M \to \mathbb{R}^N$ for some $N \in \mathbb{N}$ such that

$$g = \iota^* g_{\mathbb{R}^N}?$$

The question was finally answered positively by John Nash around 1954–1956 (1954: C^1 isometric imbeddings; 1956: C^∞ isometric embeddings). Nash's proof opened a new era of non-linear analysis and in particular lead to his celebrated inverse function theorem on Fréchet spaces. We will not discuss it in this beginning stage. Will come back to some aspects on it in the later part of this book.

is the absolute value of a top form ω defined at any $p \in M$ by

$$\omega_p := \phi^1 \wedge \ldots \wedge \phi^n,$$

where $\phi^n, \ldots, \phi^n$ of T_p^*M is any orthonormal basis of T_p^*M, say dual to an orthonormal basis $e_1, e_2, \ldots, e_n$ of T_pM.

Exercise 3.1. (2) Given $v_1, \ldots, v_k \in V \cong \mathbb{R}^n$ with inner product $\langle\, , \rangle$, show that the k-dimensional volume of the parallelpipade

$$P(v_1, \ldots, v_k) = \left\{ \sum_{i=1}^k t_i v_i \middle| \, 0 \le t_i \le 1 \right\}$$

is $\sqrt{\det(\langle v_i . v_j \rangle)_{i,j=1}^k}$.

(2) On a local chart $(U, \mathbf{x})$, show that

$$d\mu_g \equiv \omega = \sqrt{\det(g_{ij})}\, dx^1 \wedge \ldots \wedge dx^n.$$

Remark 3.2. In order for the volume form to be defined we need only the manifold and the metric g to be C^1.

With a metric g, we can also measure the size of lower dimensional submanifolds. Given a C^1 immersion $\iota : S \looparrowright M$ of a $k \le n$ dimensional manifold S, the induced tensor $g_S := \iota^* g$ is a Riemannian metric on S and then the k dimensional volume form $d\mu_{g_S} \in A^k(S)$ is defined. In local coordinates $(x^1, \ldots, x^n)$ on M and $(t^1, \ldots, t^k)$ on S, we see from Exercise 3.1 that

$$\begin{aligned} d\mu_{g_S} &= \sqrt{\det\left(g(\iota_* \partial_{t^i}, \iota_* \partial_{t^j}) \right)_{i,j=1}^k}\, dt^1 \wedge \ldots \wedge dt^k, \\ &= \sqrt{\det\left(\sum_{i',j'=1}^n g_{i'j'} \frac{\partial x^{i'}}{\partial t^i} \frac{\partial x^{j'}}{\partial t^j} \right)_{i,j=1}^k}\, dt^1 \wedge \ldots \wedge dt^k, \\ &= \sqrt{\det (D\tilde{\iota})^t G (D\tilde{\iota})}\, dt^1 \wedge \ldots \wedge dt^k, \end{aligned}$$

where $\partial_{t^i} := \frac{\partial}{\partial t^i}$, $\iota_* \partial_{t^i} = \sum_{i'} \frac{\partial x^{i'}}{\partial t^i} \frac{\partial}{\partial x^{i'}}$, and $G = (g_{i'j'})$. The case $k = 1$ goes back to arc length element we start with:

$$ds^2 = \sum g_{ij}\, dx^i dx^j.$$

Next, we study differentiations on (M, g). Let F be a vector-valued function on $\mathbb{R}^n$ and v a unit vector. The directional derivative of F at a point p in the v direction is defined by

$$D_v F := \lim_{t \to 0} \frac{1}{t} \left(F(p + tv) - F(p) \right).$$

To generalize it to a manifold M, we need to compare the difference of vectors (or tensors) at different points. Therefore, we need to "parallel translate" them to the same point p to compare the difference. As a result, to define the differentiation, what we really need is the notion of *parallel translations* or *affine structures*.

We wish to define the directional derivative such that it only depends on $v \in T_pM$ but not on the extension $\tilde{v}$ of it. Also, the resulting differentiation should satisfy linearity and the Leibniz rule.

The "Lie" derivative does not meet the requirement. From

$$L_{fX} Y = [fX, Y] = f[X, Y] - (Yf)X,$$

we see that it is not function-linear in X and the differentiation of Y at p in the direction X_p depends on the extension of X_p to X. [2]

The correct concept, developed by Christoffel, Levi-Civita, Ricci, Koszul and others, start from the following simple definition:

Definition 3.3. A *covariant differentiation* is an operator

$$\nabla_X : C^\infty(TM) \to T_pM$$

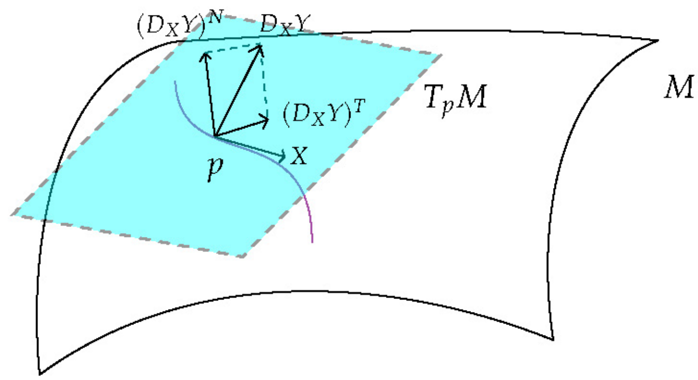
for every $p \in M$ and $X \in T_pM$, such that

(1) $\nabla_X Y$ is $\mathbb{R}$-linear in both X and Y, and
(2) (Leibniz rule) $\nabla_X(fY) = X(f)Y_p + f(p)\nabla_X Y$ for $f \in C_p^\infty$.

In other words, by the Fundamental Theorem of Tensor Calculus (theorem 2.14), the $\mathbb{R}$ bi-linear operator

$$\begin{aligned} \nabla : \qquad C^\infty(TM) \times C^\infty(TM) &\longrightarrow C^\infty(TM) \\ (X, Y) &\longmapsto \nabla_X Y \end{aligned}$$

[2]Different extensions X lead to different integral curves and flows ϕ_t. Even if we take a non-trivial scaling hX of one extension X with $h(p) = 1$ so that we get the same integral curves, they must have different parametrizations and the flows are different. Thus in general we get different values of $L_X Y$ at p.



is function-linear in X. Thus ∇Y is a vector valued 1-form:

$$\nabla : C^\infty(TM) \to A^1(TM) = A^1(M) \otimes C^\infty(TM).$$

More generally, for a vector bundle $E \to M$, an *affine connection* is an operator $D : C^\infty(E) \to A^1(E)$ with Leibniz rule. Thus a covariant differentiation is simply an affine connection on TM.

Example 3.4. (1) $M = \mathbb{R}^n$, $\nabla_X = D_X$, the directional derivative.

(2) Let $M \hookrightarrow \mathbb{R}^n$ be a submanifold and $p \in M$. Let $X \in T_pM$ and Y be a local vector field on M defined near p. Let $\widetilde{Y}$ be any C^∞ extension of Y to $T\mathbb{R}^n$. Then we define the induced connection by

$$\nabla_X Y := (D_X \widetilde{Y})^T,$$

the tangential part of $D_X\widetilde{Y}$ in (1) under the orthogonal projection

$$T_p\mathbb{R}^n = T_pM \overset{\perp}{\oplus} N_p.$$

On a general manifold M, affine connections on a vector bundle $E \to M$ can be constructed by patching the local connections. Let $M = \bigcup_\alpha U_\alpha$ with a partition of unity (U_α, ϕ_α)'s. Given an affine connection ∇^α on U_α for each α, it is natural to ask if

$$\nabla := \sum_\alpha \phi_\alpha \nabla^\alpha$$

is a covariant differentiation on the whole M?

Notice that a simple scaling of a connection is no longer a connection. For $\widetilde{\nabla} = h\nabla$,

$$\widetilde{\nabla}_X(fY) = hX(f)Y + hf\nabla_X Y = hX(f)Y + f\widetilde{\nabla}_X Y.$$

Leibniz rule fails unless $h = 1$. Nevertheless, for a (locally) finite sum $\nabla = \sum h_i \nabla^i$, the same calculation shows that

$$\nabla_X(fY) = \Big(\sum h_i\Big) X(f)Y + f\nabla_X Y.$$

Thus ∇ is a connection if and only if $\sum_i h_i = 1$, i.e. ∇ is a an *affine linear combination* of ∇^i's. In particular, for a P.O.U. $\{\phi_\alpha\}$, $\nabla = \sum_\alpha \phi_\alpha \nabla^\alpha$ is a global covariant differentiation operator.

Exercise 3.2. Show that the set of all affine connections on $E \to M$ is an affine space $\nabla^0 + A^1(\operatorname{End} E)$ modeled on the infinite dimensional vector space $A^1(\operatorname{End} E)$, where ∇^0 is any affine connection.

So far, we have not made use of the Riemannian structure g. When g is present, which connection(s) will be the best choice(s) among the above infinite-dimensionally many ones?

Notice that in the case $M \hookrightarrow \mathbb{R}^n$ as in Example 3.4, two more properties hold for the induced connection ∇:

(1) ∇ is *compatible with metric*, i.e. it satisfies the "Leibniz rule":

$$X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

since $\langle (D_X Y)^N, Z\rangle = 0$ for $Z \in C^\infty(TM)$.

(2) ∇ satisfies the *torsion-free* condition:

$$\nabla_X Y - \nabla_Y X = (XY - YX)^T = [X, Y]^T = [X, Y].$$

Definition 3.5 (Torsion tensor). For any affine connection ∇ on TM, $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ is a tensor, called the torsion of ∇.

Exercise 3.3. Let (N, g) be a Riemannian manifold and $M \hookrightarrow N$. Show that the induced connection $\nabla_X Y := (D_X \widetilde{Y})^T$ is (i) independent of the choices of $\widetilde{Y}$, (ii) compatible with metric and torsion-free.

A fundamental result due to Christoffel (1869) and later used by Levi-Civita (1917) to study parallel transport, asserts that there is an unique connection, the *Levi-Civita connection*, satisfies (1) and (2):

Theorem 3.6 (Fundamental Theorem of Riemannian Geometry)**.**

For any Riemannian manifold (M, g), there is a unique connection $\nabla = \nabla^{LC}$ on TM which is metric compatible and torsion-free.

PROOF OF UNIQUENESS. Given any 3 vector fields X, Y, Z, by compatibility and torsion-free condition, we have

$$\begin{aligned} X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \\ Y\langle X, Z\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle, \\ &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_X Y + [Y, X]\rangle \\ Z\langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \\ &= \langle \nabla_X Z + [Z, X], Y\rangle + \langle X, \nabla_Y Z + [Z, Y]\rangle. \end{aligned}$$

By adding the first two and subtracting the third one, we get

$$\begin{aligned} \langle \nabla_X Y, Z\rangle =& \frac{1}{2}\Big(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &+ \langle Z, [X, Y]\rangle + \langle Y, [Z, X]\rangle - \langle X, [Y, Z]\rangle \Big). \end{aligned}$$

Since Z is arbitrary, this implies that $\nabla_X Y$ is uniquely determined by the differentiable structure and the metric. $\square$

Exercise 3.4. Prove the existence of Levi-Civita connection by checking the above formula for $\nabla_X Y$ defines a covariant differentiation.

In local coordinate $(x^1, \ldots, x^m)$, let $\partial_i := \frac{\partial}{\partial x^i}$. The *Christoffel symbol* Γ_{ij}^k of an affine connection ∇ is defined by

$$\nabla_{\partial_i} \partial_j =: \sum_k \Gamma_{ij}^k \, \partial_k.$$

It measures how the frame moves, hence determines the information of the affine connection: for any vector field $v = \sum v^j \, \partial_j$,

$$\begin{aligned} \nabla_i \, v \equiv \nabla_{\partial_i}(\sum_j v^j \partial_j) &= \sum_j (\partial_i v^j)\partial_j + \sum_{j,k} v^j \, \Gamma_{ij}^k \, \partial_k \\ &= \sum_j (\partial_i v^j + \sum_k v^k \, \Gamma_{ik}^j) \, \partial_j =: \sum_j v^j_{;i} \, \partial_j. \end{aligned}$$

Here $v^j_{;i} = \partial_i v^j + \sum_k v^k \, \Gamma_{ik}^j$. We will generalize this to all tensors later.

Lemma 3.7. *An affine connection ∇ is torsion-free if and only if $\Gamma_{ij}^k = \Gamma_{ji}^k$ in all coordinate systems.*

Indeed, if ∇ is torsion-free, then

$$0 = [\partial_i, \partial_j] = \nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i = (\Gamma^k_{ij} - \Gamma^k_{ji})\partial_k.$$

Conversely, the torsion *tensor* $T(X,Y) = 0$ since $T(\partial_i, \partial_j) = 0$.

We further assume that $\nabla = \nabla^{LC}$ is the Levi-Civita connection. Since $\langle \partial_i, \partial_j \rangle = g_{ij}$, the formula in the proof of Theorem 3.6 says

$$\sum_\ell \Gamma^\ell_{ij}\, g_{\ell k} = \frac{1}{2}\big(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}\big).$$

Denote by $(g^{ij}) = (g_{ij})^{-1}$, i.e. $g^{pq}g_{qr} = \delta^p_r$. Then we arrive at the fundamental formula for the Christoffel symbols:

Lemma 3.8. *The Christoffel symbols for ∇^{LC} are given by*

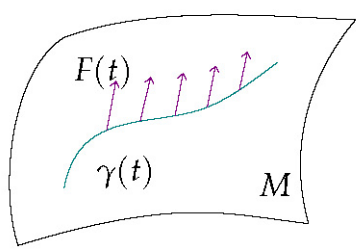
$$\Gamma^k_{ij} = \frac{1}{2}\sum_\ell g^{k\ell}\big(\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij}\big). \tag{3.1}$$

Now we turn to the question of parallel translation for an affine connection ∇. Given a smooth curve $\gamma : [0,1] \to M$. Denote $\dot\gamma(t) = \gamma'(t) = d\gamma(\frac{\partial}{\partial t}) \equiv \frac{\partial}{\partial t} = \partial_t$. Let F be a vector field along γ. Write

$$\frac{DF}{dt} = \nabla_{\partial_t} F = \nabla_t F$$

as the covariant differentiation along γ.

Definition 3.9. We say that F is *parallel* along γ if $\nabla_{\partial_t} F = 0$.



Let $F = \sum f^i \partial_i$, $\gamma(t) = (x^1(t), \dots, x^n(t))$. Then

$$\begin{aligned}\nabla_{\partial_t} F &= \sum \Big(\frac{\partial f^i}{\partial t}\partial_i + f^i \nabla_{\partial_t}\partial_i\Big) = \sum \Big(\frac{\partial f^i}{\partial t}\partial_i + f^i\, \dot x^j\, \Gamma^k_{ij}\,\partial_k\Big)\\ &= \sum \Big(\frac{\partial f^i}{\partial t} + f^k \dot x^j \Gamma^i_{kj}\Big)\partial_i.\end{aligned}$$

The equation $\nabla_t F = 0$ is a first order linear ODE in t. Thus the parallel translation $F(t)$ exists along γ and it is uniquely determined by the initial vector $F(0)$.

Lemma 3.10. *Let (M, g) be a Riemannian manifold. If ∇ is metric compatible then parallel translation preserves inner products.*

PROOF. Let V_1, V_2 be two parallel vector fields along γ. Then

$$\dot\gamma\langle V_1, V_2\rangle = \langle \nabla_{\dot\gamma} V_1, V_2\rangle + \langle V_1, \nabla_{\dot\gamma} V_2\rangle = 0.$$

Thus $\langle V_1, V_2\rangle$ is a constant function in t. □

2. Geodesics, exponential map and Riemann's normal coordinates

From now on we assume that $\nabla = \nabla^{LC}$.

Definition 3.11. A (parametrized) curve γ is a *geodesic* if $\nabla_{\dot\gamma}\dot\gamma = 0$.

Example 3.12. For $M \hookrightarrow \mathbb{R}^n$,

$$\nabla_{\dot\gamma}\dot\gamma = (D_{\dot\gamma}\dot\gamma)^T = (D_{\partial_t}\dot\gamma)^T = \gamma''(t)^T.$$

In particular, geodesics on linear spaces are lines and geodesics on spheres S^m are great circles.

If $\nabla_{\dot\gamma}\dot\gamma = 0$ then

$$\dot\gamma\langle\dot\gamma, \dot\gamma\rangle = 2\langle\nabla_{\dot\gamma}\dot\gamma, \dot\gamma\rangle = 0.$$

So $|\dot\gamma| = c$ is a constant and the arc length $s(t) = \int_0^t |\dot\gamma| = ct$. So t is necessarily proportional to the arc length s.

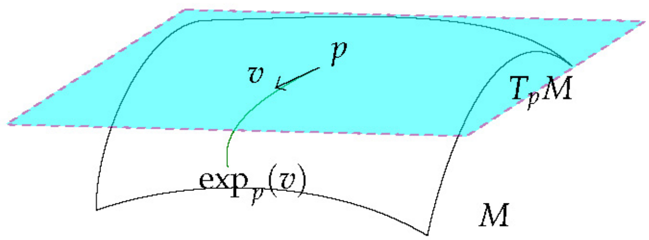
In local coordinates, let $\gamma(t) = (x^1(t), \ldots, x^n(t))$, $\dot\gamma = \sum \dot x^i(t)\partial_i$. The *geodesic equation* is a second order non–linear ODE in t:

$$\nabla_{\dot\gamma}\dot\gamma = \sum_k \Big(\ddot x^k + \sum_{i,j}\Gamma^k_{ij}\dot x^i \dot x^j\Big)\partial_k = 0.$$

Still, by the existence and uniqueness theorem, γ exists and is uniquely determined by $\gamma(t_0) = p$ and $\gamma'(t_0) = v$ for a maximal time interval.

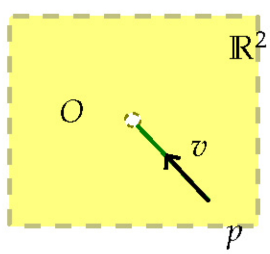
The local existence ensures us to find the geodesics in any direction $v \in T_pM$. This leads to the notion of exponential map:

Definition 3.13 (Exponential map). For $p \in M$, the *exponential map* $\exp_p : U \subset T_pM \to M$ is defined by $\exp_p(tv) = \gamma(t)$, where γ is the geodesic satisfying $\gamma(0) = p, \gamma'(0) = v \in U$. Hence $\exp v = \gamma(1)$.



The exponential map $\exp_p$ is well-defined on some neighborhood U of $0 \in T_pM$. But it may not be extended infinitely:

Example 3.14. Consider $M = \mathbb{R}^2 \setminus \{0\}$. The geodesics are lines on the $\mathbb{R}^2$ plane. It is clear that the geodesic can not across the origin.



Proposition 3.15. $\exp_p$ *is a local diffeomorphism near* $0 \in T_pM$

The smoothness of $\exp_p$ follows from the smooth dependence of ODE on its initial values (cf. Section 8 of Chapter 1). By the inverse function theorem, we only need to compute

$$d(\exp_p)_0 : T_0(T_pM) \cong T_pM \to T_pM.$$

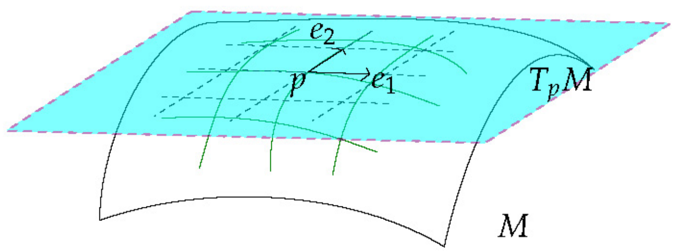
By direct calculation,

$$d(\exp_p)_0(v) = \frac{d}{dt}\Big|_{t=0} \exp_p(tv) = v.$$

Hence $d(\exp_p)_0 = \mathrm{id}$ and the C^∞ local inverse exists.

An easy application of the exponential map is the *Riemann normal coordinates*. Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of T_pM. In the diffeomorphic neighborhood U of the exponential map, we define the Riemannian normal coordinates (RNC) by

$$\exp_p(x^1 e_1 + x^2 e_2 + \ldots + x^m e_m) \mapsto (x^1, x^2, \ldots, x^m) \in U \subset \mathbb{R}^m.$$



Why is the normal coordinates system good? An immediate consequence is the following striking fact:

Lemma 3.16. *Under the normal coordinates, $\Gamma_{ij}^k(p) = 0$ for all i, j, k.*

PROOF. Fix a direction $\mathbf{h} = (h^1, h^2, \ldots, h^m)$ and consider the line $\mathbf{x}(t) = (x^1(t), x^2(t), \ldots, x^m(t)) = (th^1, th^2, \ldots, th^m) = t\mathbf{h}$ with corresponding geodesic $\gamma(t) = \exp_p(t\mathbf{h})$. The geodesic equation is

$$\ddot{x}^k(t) + \sum_{i,j} \Gamma_{ij}^k(\gamma(t))\, \dot{x}^i(t) \dot{x}^j(t) = 0.$$

Since $\mathbf{x}(t)$ is linear in t, we get $\sum_{i,j} \Gamma_{ij}^k(\gamma(t)) h^i h^j = 0$. Let $t = 0$. Then for all k,

$$\sum_{i,j} \Gamma_{ij}^k(p) h^i h^j = 0.$$

Since this is true for arbitrary $\mathbf{h}$ in the existence neighborhood, we conclude that $\Gamma_{ij}^k(p) = 0$. $\square$

As an application of the normal coordinates, we prove

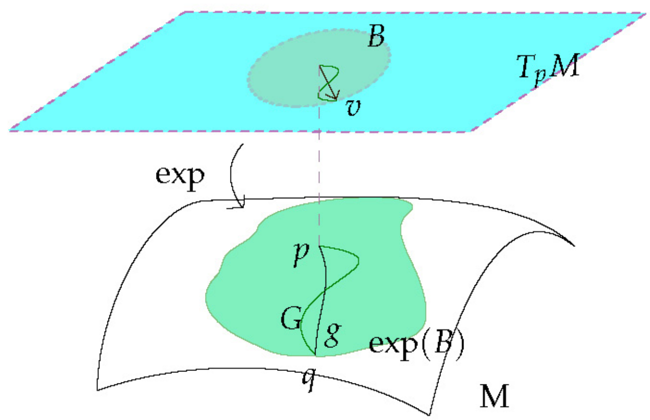
Proposition 3.17. *Geodesics are locally of minimal length.*

More precisely, Take $B = B_\rho(0) \subset T_pM$ such that $\exp_p|_B$ is a diffeomorphism. Any $q \in \exp_p(B)$ is connected to p by a geodesic γ. Then $L(\gamma)$ is minimal among all piecewise smooth curves Γ connecting p and q.

PROOF. Consider the spherical coordinate on T_pM:

$$(r, \theta^1, \ldots, \theta^{m-1}),$$

which defines a "spherical normal coordinates" on $\exp_p(B)$. Denote $g_{rr} := g(\partial_r, \partial_r)$, $g_{r\alpha} := g(\partial_r, \partial_{\theta^\alpha})$, $g_{\alpha\beta} := g(\partial_{\theta^\alpha}, \partial_{\theta^\beta})$.



Lemma 3.18 (Gauss lemma). *$g_{r\alpha} = 0$ for all $\alpha \in [1, m-1]$.*

Assume the Gauss lemma, then

$$ds^2 = dr^2 + \sum_{\alpha,\beta} g_{\alpha\beta}\, d\theta^\alpha \otimes d\theta^\beta.$$

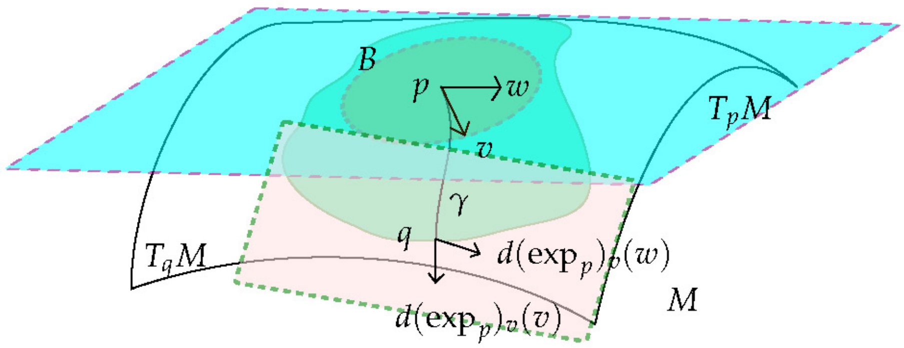
Thus, the arc length of Γ is

$$\begin{aligned} L(\Gamma) = \int_\Gamma ds &= \int \sqrt{\left(\frac{dr}{dt}\right)^2 + \sum_{\alpha,\beta} g_{\alpha\beta} \frac{d\theta^\alpha}{dt}\frac{d\theta^\beta}{dt}}dt \\ &\geq \int \left|\frac{dr}{dt}\right| dt = L(\gamma), \end{aligned}$$

since g is a positive definite symmetric tensor. The equality holds if and only if θ^α's are constants. That is, $\Gamma = \gamma$. □

PROOF OF GAUSS LEMMA. A simple proof using coordinates is left as an exercise (c.f. Exercise 3.5). The proof given below is intended to get us more familiar with the exponential map. Also we prove a stronger form: for $w \in T_v(T_pM) \cong T_pM$,

$$\langle d(\exp_p)_v(v), d(\exp_p)_v(w)\rangle_q = \langle v, w\rangle_p.$$



Let $w = w_1 + w_2$ where $w_1 \perp v, w_2 = \lambda v$. Then for the w_2 part,

$$\langle d(\exp_p)_v v, d(\exp_p)_v w_2 \rangle = \lambda \langle d(\exp_p)_v v, d(\exp_p)_v v \rangle = \lambda \langle v, v \rangle.$$

The last equality follows from the definition

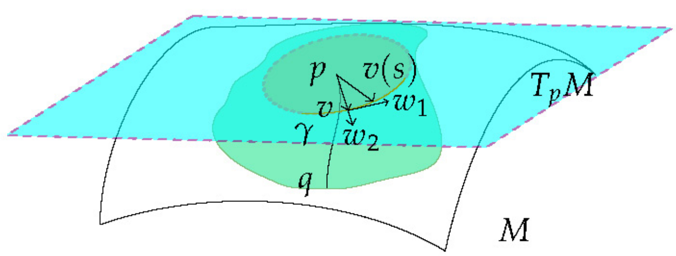
$$d(\exp_p)_v v = \frac{d}{dt}\Big|_{t=0} \exp_p(v + tv) = \frac{d}{dt}\Big|_{t=0} \gamma(t+1) = \gamma'(1)$$

and the fact that $|\gamma'(t)|$ is constant in t. Hence $|\gamma'(1)| = |\gamma'(0)| = |v|$.

For the w_1 part, which is the Gauss lemma, we need to show

$$\langle d(\exp_p)_v v, d(\exp_p)_v w_1 \rangle = 0.$$

Consider a parametrized surface in M, $\mathbb{X}(t,s) := \exp_p(tv(s))$, where $v(0) = v$, $v'(0) = w_1$ and $|v(s)|$ is constant.



Denote $\mathbb{X}_t = \frac{\partial \mathbb{X}}{\partial t} = \mathbb{X}_* \frac{\partial}{\partial t}$, $\mathbb{X}_s = \frac{\partial \mathbb{X}}{\partial s} = \mathbb{X}_* \frac{\partial}{\partial s}$. Then

$$\langle d(\exp_p)_v v, d(\exp_p)_v w_1 \rangle_q = \langle \mathbb{X}_t, \mathbb{X}_s \rangle (1, 0).$$

By definition of geodesics,

$$\begin{aligned}\frac{\partial}{\partial t}\langle \mathbb{X}_t, \mathbb{X}_s\rangle &= \langle \nabla_{\partial_t}\mathbb{X}_t, \mathbb{X}_s\rangle + \langle \mathbb{X}_t, \nabla_{\partial_t}\mathbb{X}_s\rangle \\ &= \langle 0, \mathbb{X}_s\rangle + \langle \mathbb{X}_t, \nabla_{\partial_s}\mathbb{X}_t + [\mathbb{X}_t, \mathbb{X}_s]\rangle \\ &= \langle \mathbb{X}_t, \nabla_{\partial_s}\mathbb{X}_t\rangle = \frac{1}{2}\partial_s\langle \mathbb{X}_t, \mathbb{X}_t\rangle = 0.\end{aligned}$$

since $\langle \mathbb{X}_t, \mathbb{X}_t\rangle = |v(s)|^2$ is constant. □

Exercise 3.5. Prove Gauss lemma by showing that $\Gamma^\alpha_{rr} = 0$ and then $\partial_r g_{r\alpha} = 0$ for all α.

3. Metric space structure and completeness

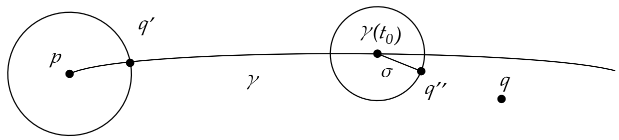
A Riemannian manifold (M, g) can be endowed with a metric space structure (M, d) with $d(p, q) = \inf L(\gamma)$ among all curves γ joining p and q. From the Gauss lemma, we know that $B_p(r) \subset M$, the ball of radius r centered at p, is precisely the diffeomorphic image of $B_0(r) \subset T_pM$ for small r. In particular the manifold topology coincides with the metric space topology.

Definition 3.19. A Riemannian manifold (M, g) is *complete* if (M, d) is a complete metric space.

In the definition of exponential map, $\exp_p$ is only defined on an open set U of $0 \in T_pM$. It turns out that the question whether the map can be defined on whole T_pM for every point $p \in M$ is equivalent to completeness of (M, d). We first need the following

Lemma 3.20. *Let $p \in M$, assume that $\exp_p$ is defined on whole T_pM, then any point $q \in M$ can be joined by a minimal geodesic from p.*

Proof. Let $B_p(r)$ be the ball centered at p in which $\exp_p$ is a diffeomorphism. If $q \in B_p(r)$, then we are done. Now, we consider $q \notin B_p(r)$, we claim that there exists $q' \in \partial B_p(r)$ such that $d(p, q) = r + d(q', q)$. To see this, we easily have $d(p, q) \le r + d(q', q)$ from triangle inequality. For the reverse inequality, let $q' \in \partial B_p(r)$



such that $d(q', q) = \inf_{q'' \in \partial B_p(r)} d(q'', q)$. By definition of our distance d, we get the reverse inequality.

Let $\gamma : [0, \infty) \to M$ be the geodesic enumerating from p and passing through q'. We next prove the statement by continuity method. Let $Z = \{t \in [0, \infty) : d(p, \gamma(t)) + d(\gamma(t), q) = d(p, q)\}$. First, $Z \neq \emptyset$ since $r \in Z$. Obviously, Z is closed. Let $t_0 = \sup Z$, we claim that $\gamma(t_0) = q$. Assume $\gamma(t_0) \neq q$. First, there exists $r_1 > 0$, consider $q'' \in \partial B_{\gamma(t_0)}(r_1)$ such that

$$d(\gamma(t_0), q'') + d(q'', q) = d(\gamma(t_0), q),$$

where $d(\gamma(t_0), q'') = r_1$. Let σ be the unique geodesic joining from $\gamma(t_0)$ to q''. Then from

$$d(p, \gamma(t_0) + d(\gamma(t_0), q) = d(p, q),$$

we have:

$$d(p, \gamma(t_0)) + d(\gamma(t_0), q'') = d(p, q'').$$

However, this shows that $\gamma|_{[0,t_0]} \cup \sigma|_{[0,r_1]}$ is a minimal geodesic, which implies that γ and σ must fit together to form a smooth geodesic $\gamma|_{[0,t_0+r_1]}$. Thus, $t_0 + r_1 \in Z$, a contradiction. $\square$

Theorem 3.21 (Hopf-Rinow-de Rham)**.** *The following are equivalent:*

1. *$\exp_p$ is defined on all T_pM for one p.*
2. *(M, g) is complete.*
3. *$\exp_p$ is defined on all T_pM for all $p \in M$.*

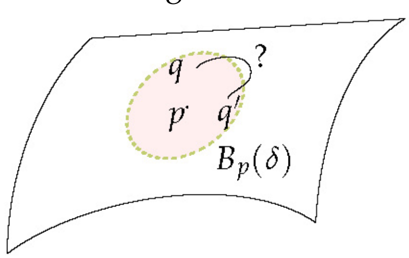
PROOF. (3) $\Rightarrow$ (1) is trivial. For (1) $\Rightarrow$ (2), let $\{q_i\}_{i=1}^\infty$ be a Cauchy sequence of (M, d), and let $\gamma_i : [0, t_i] \to M$ be a sequence of minimal geodesic with $\gamma(t_i) = q_i$. Since γ_i are parametrized by arc–length, $\{t_i\}_{i=1}^\infty$ is also a Cauchy sequence in $[0, \infty)$, whose limit is denoted by t_0. Moreover, by compactness of $B_1(0) \subset T_pM$, we can pass to a subsequence γ_{i_n} such that $\gamma'_{i_n} \to v \in T_pM$ as $n \to \infty$. Then consider the geodesic equation with initial value $\gamma(0) = p$, $\gamma'(0) = v$, the Picard–Lindelöf theorem guarantees the continuity of solution of the ODE on initial data. As a result, we have $q_{i_n} = \gamma_{i_n}(t_{i_n}) \to \gamma(t_0)$, and thus $q_i \to \gamma(t_0)$ (since q_i is Cauchy).

Next, for (2) $\Rightarrow$ (3), if there exists $p \in M$ such that a geodesic γ with $\gamma(0) = p$, $\gamma'(0) = v$ only defines on $[0, t_0)$. Pick any increasing sequence $t_i \nearrow t_0$, $\gamma(t_i)$ is a Cauchy sequence with limit q. Define $\gamma(t_0) = q$, then it is a continuous on $[0, t_0]$. Moreover, for sufficiently large i, $\gamma(t_i)$ lies in a normal coordinate ball of q, and thus we can join $\gamma(t_i)$ from q by a minimal geodesic σ. Thus, the piecewise smooth curve $\gamma \cup \sigma$ must coincide together with a smooth geodesic, and γ extends past t_0, a contradiction. $\square$

Corollary 3.22. *That (M, g) is complete implies it is geodesic convex, which means any two points can be joined by a minimal geodesic.*

One should notice that even if M is complete, two points may not be joined by a unique minimal geodesic, e.g. the north and the south poles of a sphere (cf. Section 6 for more on this).

At the end of this section, we prove Fact 2.37, namely the existence of convex neighborhood.



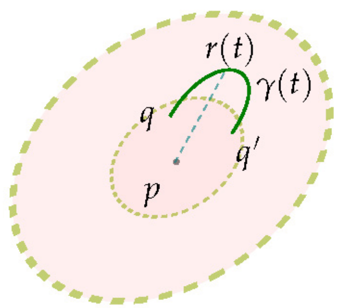
Given $p \in M$. Fix $\delta > 0$ such that $\forall\ q \in B_p(\delta)$, $\exp_q$ is a diffeomorphism on some $B_q(r_q)$. Simply let

$$r_0 = \min_{q \in \overline{B_p(\delta_0)}} r_q > 0.$$

As in the picture, any q, q' may not be jointed by a geodesic inside the neighborhood.

Now, we regard (M, g) as a metric space (M, d). Reset $\delta = r_0/4$. Let $W := B_p(r_0/4)$. Now for all $q \in W$, $\exp_q$ is a diffeomorphism on W since $d(q, q') \le d(q, p) + d(p, q') < r_0/2$ for all $q, q' \in W$. So there at least exists a minimal geodesic γ connecting q and q' with $\gamma \subset B_q(r_0/2) \subset B_p(r_0)$.

We want to prove $\gamma \subset W = B_p(r_0/4)$.



Let $\gamma(t) = (x^1(t), \ldots, x^m(t))$. Recall the geodesic equation:

$$\ddot{x}^k(t) + \Gamma_{ij}^k \dot{x}^i(t)\dot{x}^j(t) = 0.$$

We will apply the *maximum principle* to it. Define the distance function $r(t) := d(p, \gamma(t))$. $r^2 = \sum_{k=1}^m x^k x^k$, $rr' = \sum_{k=1}^m x^k \dot{x}^k$,

$$\begin{aligned}(r')^2 + rr'' &= \sum_k \ddot{x}^k x^k + (\dot{x}^k)^2 \\ &= -\sum_k \Gamma_{ij}^k \dot{x}^i \dot{x}^j x^k + \sum_k (\dot{x}^k)^2.\end{aligned}$$

Suppose r reaches the max value at $t = t_0$. If $\gamma(t_0) = q$ or q' then we are done. Otherwise $r'(t_0) = 0$, $r''(t_0) < 0$, hence LHS ≤ 0. When r_0 is chosen small enough, RHS > 0 which leads to a contradiction.

4. Riemann curvature tensor

Definition 3.23. The *Riemann curvature* detects the non-commutativity of covariant differentiations, that is:

$$\begin{aligned}R(X,Y)Z &:= \nabla_{\widetilde{X}}\nabla_{\widetilde{Y}}\widetilde{Z} - \nabla_{\widetilde{Y}}\nabla_{\widetilde{X}}\widetilde{Z} - \nabla_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z} \\ &\equiv \left([\nabla_{\widetilde{X}}, \nabla_{\widetilde{Y}}] - \nabla_{[\widetilde{X},\widetilde{Y}]}\right)\widetilde{Z}.\end{aligned}$$

for $X, Y, Z \in T_pM$ and $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ any vector fields that extend X, Y, Z.

The last term in definition is a correction to make it function-linear. R is a $(1,3)$ tensor. In local coordinates, we write

$$R(\partial_i, \partial_j)\partial_\ell =: \sum R_{\ell ij}^k \partial_k.$$

For convenience, we drop all summation symbols and use the convention that the appearance of an index j on both upper and lower indices means summing over j. By computations,

$$\begin{aligned} R(\partial_i,\partial_j)\partial_\ell &= \nabla_i\nabla_j\partial_\ell - \nabla_j\nabla_i\partial_\ell - \nabla_{[i,j]}\partial_\ell \\ &= \nabla_i(\Gamma^s_{j\ell}\partial_s) - \nabla_j(\Gamma^t_{i\ell}\partial_t) \\ &= (\partial_i\Gamma^k_{j\ell})\partial_k + \Gamma^s_{j\ell}\Gamma^k_{is}\partial_k - (\partial_j\Gamma^k_{i\ell})\partial_k - \Gamma^t_{i\ell}\Gamma^k_{jt}\partial_k \\ &= (\partial_i\Gamma^k_{j\ell} - \partial_j\Gamma^k_{i\ell} + \Gamma^s_{j\ell}\Gamma^k_{is} - \Gamma^s_{i\ell}\Gamma^k_{js})\partial_k. \end{aligned}$$

Also, the metric g leads to $T_pM \cong T_p^*M$ via $v \mapsto \langle v, \bullet\rangle$. In coordinates, this means we may use g_{ij} to lowering the indices and define

$$R_{k\ell ij} := g_{km}R^m_{\ell ij} = \langle R(\partial_i,\partial_j)\partial_\ell, \partial_k\rangle.$$

Example 3.24. Let $Z = f^p\partial_p$ be a vector field. We denote

$$\nabla_i(f^p\partial_p) =: f^p_{;i}\partial_p, \qquad \nabla_i\nabla_j(f^p\partial_p) =: f^p_{;ji}\partial_p,$$

where indices after ";" means covariant differentiations. So

$$(f^p_{;ji} - f^p_{;ij})\partial_p = R(\partial_i,\partial_j)f^p\partial_p = f^q R^p_{qij}\partial_p.$$

In other words, we can change the order of covariant differentiation via the *commutation formula*

$$f^p_{;ji} = f^p_{;ij} + f^q R^p_{qij}.$$

Exercise 3.6. Extend the covariant derivative to all tensors and prove the commutation formula:

$$\begin{aligned} T^{i_1\cdots i_r}_{j_1\cdots j_s;ji} &= T^{i_1\cdots i_r}_{j_1\cdots j_s;ij} + T^{qi_2\cdots i_r}_{j_1\cdots j_s}R^{i_1}_{qij} + T^{i_1qi_3\cdots i_r}_{j_1\cdots j_s}R^{i_2}_{qji} + \cdots \\ &\quad + T^{i_1\cdots i_r}_{pj_2\cdots j_s}R^p_{j_1ji} + \cdots + T^{i_1\cdots i_r}_{j_1\cdots j_{s-1}p}R^p_{j_sji}. \end{aligned}$$

Proposition 3.25 (Symmetries on R)**.**

(0) $R^k_{\ell ij} = -R^k_{\ell ji}$.

(1) *Torsion free: the first Bianchi identity.*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0,$$

i.e. $R^k_{[\ell ij]} := R^k_{\ell ij} + R^k_{ij\ell} + R^k_{j\ell i} = 0$.

(2) *Metrical: skew symmetry in* $R_{k\ell ij} = -R_{\ell kij}$.

PROOF. (0) follow from $R(X,Y) = -R(Y,X)$. For (1), since these are all tensors, we may check it in any basis. Then

$$R(\partial_i,\partial_j)\partial_k + R(\partial_j,\partial_k)\partial_i + R(\partial_k,\partial_i)\partial_j$$
$$= \nabla_i\nabla_j\partial_k - \nabla_j\nabla_i\partial_k + \nabla_j\nabla_k\partial_i - \nabla_k\nabla_j\partial_i + \nabla_k\nabla_i\partial_j - \nabla_i\nabla_k\partial_j = 0$$

since $\nabla_i\partial_j - \nabla_j\partial_i = [\partial_i,\partial_j] = 0$.

(2) is equivalent to $\langle R(X,Y)Z,W\rangle + \langle Z, R(X,Y)W\rangle = 0$. Let $X = \partial_i$, $Y = \partial_j$. Then

$$\begin{aligned}
&\langle\nabla_i\nabla_j Z,W\rangle - \langle\nabla_j\nabla_i Z,W\rangle\\
&= \partial_i\langle\partial_j Z,W\rangle - \langle\nabla_j Z,\nabla_i W\rangle - \partial_j\langle\nabla_i Z,W\rangle + \langle\nabla_i Z,\nabla_j W\rangle\\
&= \partial_i\partial_j\langle Z,W\rangle - \partial_i\langle Z,\nabla_j W\rangle - \langle\nabla_j Z,\nabla_i W\rangle\\
&\qquad - \partial_j\partial_i\langle Z,W\rangle + \partial_j\langle Z,\nabla_i W\rangle + \langle\nabla_i Z,\nabla_j W\rangle,
\end{aligned}$$

which is anti-symmetric in Z,W. □

Exercise 3.7. Show that

(1) The symmetries in Proposition 3.25 implies $R_{k\ell ij} = R_{ijk\ell}$.
(2) The second Bianchi identity holds:

$$0 = R_{ij[k\ell;m]} := R_{ijk\ell;m} + R_{ij\ell m;k} + R_{ijmk;\ell}.$$

Definition 3.26 (Ricci and scalar curvature). The *Ricci curvature tensor* Ric $:= \sum R_{ij}\,dx^i \otimes dx^j \in C^\infty(\mathrm{Sym}^2(T^*M))$ is the trace of R:

$$R_{ij} = \sum_\ell R^\ell_{i\ell j} = \sum_{k,\ell} g^{k\ell} R_{ki\ell j}.$$

The further trace is a function called the *scalar curvature*:

$$s := \sum_{i,j} g^{ij} R_{ij}.$$

Definition 3.27. A metric g satisfying $\mathrm{Ric} = \lambda g$ for some $\lambda \in C^\infty(M)$ is called an *Einstein metric*.[3]

[3]In Einstein's general theory of relativity in 1915, curvatures of spacetime are related by *Einstein's field equation*:

$$R_{ij} - \frac{s}{2}g_{ij} = T_{ij},$$

Exercise 3.8. Let (M, g) has $\dim M = m \geq 3$, if

$$R_{ij} = \lambda g_{ij},$$

then λ is a constant. (Hint: use the Bianchi identity.)

There is another important curvature.

Definition 3.28 (Sectional curvature). For a two-dimensional plane $E \subset T_p M$, the *sectional curvature* is defined by

$$K_E \equiv K(X, Y) := \frac{R(X, Y, X, Y)}{\|X \wedge Y\|^2}$$

for any linearly independent vectors $X, Y \in E$.

In the two-dimensional case (surfaces), the sectional curvature is the only non–vanishing components in the Riemann curvature tensor. Moreover, it coincides with the Gaussian curvature:

Exercise 3.9. Let $X, Y \in T_p M$. Show that $K(X, Y)$ is the Gaussian curvature of the surface $\mathbb{X}(u, v) = \exp_p(uX + vY)$.

Example 3.29. A sphere S^m_r of radius r has constant sectional curvature $K = 1/r^2$. The converse is true and will be proved later.

In fact, the Riemann curvature tensor is completely determined by the sectional curvature ranging over all two-planes.

---

where T_{ij} is the *energy momentum tensor* in spacetime—a pseudo-Riemannian manifold (M, g) where g is non-degenerate but not necessarily positive definite. Most of our discussion on connection and curvature works in the general setup. (The spacetime considered by Einstein is 4-dimensional and is Lorentzian: it has local coordinates $(x^0 = t, x^1, x^2, x^3)$, and g has signature $(1, 3)$.)

The spacetime is vacuum if $T_{ij} \equiv 0$. In this case,

$$R_{ij} - \frac{s}{2} g_{ij} = 0 \quad \Longrightarrow \quad g^{ij}(R_{ij} - \frac{s}{2} g_{ij}) = s - \frac{s}{2} m.$$

If the spacetime dimension $m \neq 2$ then $s = 0$, hence $R_{ij} = 0$ as well.

For the uniform case, $T_{ij} = \mu g_{ij}$ for some function μ, we have $R_{ij} = \lambda g_{ij}$ for $\lambda = \mu + s/2$. This leads to the name of the *Einstein metric*.

Exercise 3.10 (Polarization formula). Let $\overline{K}(X,Y) = R(X,Y,X,Y)$. Then

$$\begin{aligned}\langle R(X,Y)Z,W\rangle = \overline{K}(X+W,Y+Z) - \overline{K}(X+W,Y) - \overline{K}(X+W,Z)\\ - \overline{K}(X,Y+Z) - \overline{K}(W,Y+Z) + \overline{K}(X,Z) + \overline{K}(W,Y)\\ - \overline{K}(Y+W,X+Z) + \overline{K}(Y+W,X) + \overline{K}(Y+W,Z)\\ + \overline{K}(Y,X+Z) + \overline{K}(W,X+Z) + \overline{K}(Y,Z) - \overline{K}(W,X).\end{aligned}$$

Definition 3.30. We say that (M,g) has constant sectional curvature $K(p)$ at p if $K(X,Y) = K(p)$ for all independent vectors $X, Y \in T_pM$.

Exercise 3.11. (M,g) has constant sectional curvature at p if and only if $\langle R(X,Y)W,Z\rangle = K(p)(\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle)$. I.e.,

$$R_{ijk\ell} = K(p)(g_{ik}g_{j\ell} - g_{i\ell}g_{jk}).$$

5. Variations of geodesics

Let $\gamma : [a,b] \to M$ be (piecewise) C^∞ curve on M. We consider the one parameter variation of γ:

$$F : [a,b] \times (-\epsilon,\epsilon) \to M,$$

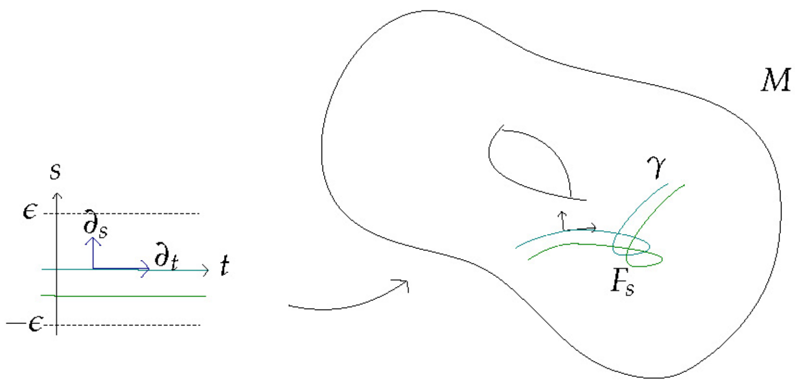
with $F_s(t) := F(s,t)$, $F_0 = \gamma$. Let $T = F_* \frac{\partial}{\partial t}$ be the *tangent vector field* and $V = F_* \frac{\partial}{\partial s}$ be the *variational vector field* for $F_s(t)$. Then

$$[V,T] = [F_*\tfrac{\partial}{\partial s}, F_*\tfrac{\partial}{\partial t}] = F_*[\tfrac{\partial}{\partial s}, \tfrac{\partial}{\partial t}] = 0.$$

Exercise 3.12. Make sense of the above calculations as well as the formula $\nabla_V T = \nabla_T V$. Notice that in this special case we do not need to assume the vector fields are F-related.

The length of F_s is given by the functional

$$L(s) = \int_a^b \langle T,T\rangle^{1/2}(t,s)\, dt.$$



In this section, we will establish the minimality of arclength of geodesics from the variation method. We compute directly that

$$L'(s) = \frac{\partial}{\partial s}\int_a^b \langle T,T\rangle^{1/2}\,dt = \int_a^b V\langle T,T\rangle^{1/2}\,dt \tag{3.2}$$

$$= \int_a^b \frac{2\langle \nabla_V T, T\rangle}{2\langle T,T\rangle^{1/2}}\,dt = \int_a^b \frac{\langle \nabla_T V, T\rangle}{\langle T,T\rangle^{1/2}}\,dt \quad \text{(torsion free)} \tag{3.3}$$

$$= \int_a^b \frac{T\langle V,T\rangle - V\langle \nabla_T T\rangle}{\langle T,T\rangle^{1/2}}\,dt. \quad \text{(metrical)} \tag{3.4}$$

We may assume that γ is parametrized by its arc length, i.e. $\langle T,T\rangle(t,0) = 1$. Then we achieve the

Proposition 3.31 (First variation formula)**.**

$$L'(0) = \langle V,T\rangle\Big|_a^b - \int_a^b \langle V, \nabla_T T\rangle dt. \tag{3.5}$$

Hence, γ is a "critical point" of the arc-length functional L if

$$\frac{d}{ds}L(s)\Big|_{s=0} = 0$$

for any variation F of γ such that $F(a,s) = P$, $F(b,s) = Q$ with $P,Q \in M$ fixed. Note that P,Q are fixed, so $V(a) = V(b) = 0$. From the first variation formula we deduce

Fact 3.32. γ is a critical point of L among all end-points fixed variations if and only if γ is a geodesic, i.e. $\nabla_T T = 0$.

Remark 3.33. Instead of the ends-fixed condition, we may consider $\langle V,T\rangle(t,0) = 0$ for all $t \in [a,b]$, called "normal variations".

Assume γ is parametrized by arc length. Differentiate (3.3) again

$$\begin{aligned} L''(s) &= \frac{\partial}{\partial s}\int_a^b \frac{\langle \nabla_T V, T\rangle}{\langle T, T\rangle^{1/2}}\,dt \\ &= \int_a^b \frac{V\langle \nabla_T V, T\rangle}{\langle T, T\rangle^{1/2}} - \frac{\langle \nabla_T V, T\rangle^2}{\langle T, T\rangle^{1/2}}\,dt. \end{aligned}$$

For the first term, we can change the order of differentiation by the curvature tensor:

$$\begin{aligned} V\langle \nabla_T V, T\rangle &= \langle \nabla_V \nabla_T V, T\rangle + \langle \nabla_T V, \nabla_V T\rangle \\ &= \langle \nabla_T \nabla_V V, T\rangle + \langle R(V,T)V, T\rangle + \|\nabla_T V\|^2 \\ &= T\langle \nabla_V V, T\rangle - \langle \nabla_V V, \nabla_T T\rangle - \langle R(V,T)T, V\rangle + \|\nabla_T V\|^2. \end{aligned}$$

Also, $\langle \nabla_T V, T\rangle = T\langle V, T\rangle - \langle V, \nabla_T T\rangle$. Hence we get the

Proposition 3.34 (Second variation formula-**I**). *Let $\gamma : [a,b] \to M$ be a geodesic parametrized by arc length. Then for any variations,*

$$\begin{aligned} L''(0) &= \langle \nabla_V V, T\rangle\Big|_a^b \\ &\quad + \int_a^b \Big(\|\nabla_T V\|^2 - \langle R(V,T)T, V\rangle - (T\langle V, T\rangle)^2\Big)\,dt. \end{aligned} \tag{3.6}$$

From

$$\|\nabla_T V\|^2 = \langle \nabla_T V, \nabla_T V\rangle = T\langle V, \nabla_T V\rangle - \langle V, \nabla_T \nabla_T V\rangle$$

and

$$\langle V, \nabla_T V\rangle = \langle V, \nabla_V T\rangle = V\langle V, T\rangle - \langle \nabla_V V, T\rangle,$$

we have the second form of this formula:

Proposition 3.35 (Second variation formula-**II**).

$$\begin{aligned} L''(0) &= V\langle V, T\rangle\Big|_a^b \\ &\quad - \int_a^b \Big(\langle \nabla_T^2 V + R(V,T)T, V\rangle + (T\langle V, T\rangle)^2\Big)\,dt. \end{aligned} \tag{3.7}$$

Now we give some applications of these formulas. Recall that in a metric space (M,d), its diameter is defined by

$$\operatorname{diam} M := \sup_{p,q\in M} d(p,q).$$

Theorem 3.36 (Bonnet–Myers 1941)**.** *Let (M^m, g) be complete with* $\mathrm{Ric} \geq (m-1)K > 0$ *(K a constant). Then M is compact with*

$$\operatorname{diam} M \leq \frac{\pi}{\sqrt{K}}.$$

PROOF. If there exist $p, q \in M$ with $d(p,q) > \frac{\pi}{\sqrt{K}}$, let γ be a shortest (minimal) geodesic joining p, q, say $\gamma(0) = p, \gamma(\ell) = q$. Such a γ exists by the Hopf-Rinow-de Rham theorem (Theorem 3.21).

Let $e_1 = T = \gamma'$ and pick $\{e_i(0)\}_{i=2}^m$ so that $\{e_i(0)\}_{i=1}^m$ is an O.N.B. of T_pM. Let $e_i(t)$ be the parallel translation of $e_i(0)$ along γ. Then $e_i(t)$'s are still orthonormal for all t by Lemma 3.10.

For each $i \in [2, m]$, let $V_i(t) = \sin(\pi t/\ell) e_i$ and construct the variation of γ by

$$F_i(t,s) := \exp_{\gamma(t)}(sV_i(t)).$$

Since F_i is an end-points fixed normal variation, the second variation formula in Proposition 3.6 implies that

$$\begin{aligned} 0 \leq \sum_{i=2}^m L_i''(0) &= \sum_{i=2}^m \int_0^\ell \Big(\|\nabla_T V_i\|^2 - \langle R(V_i, T)T, V_i \rangle \Big)\, dt \\ &= \sum_{i=2}^m \Big(\int_0^\ell \| \frac{\pi}{\ell} \cos\left(\frac{\pi t}{\ell}\right) e_i \|^2 - \sin^2\left(\frac{\pi t}{\ell}\right) \langle R(e_i, e_1)e_1, e_i \rangle \Big)\, dt. \end{aligned}$$

Since $\|e_1\| = 1$ and $\mathrm{Ric}(e_1, e_1) = \sum_{m-2}^m \langle R(e_i, e_1)e_1, e_i) \rangle \geq (m-1)K$, a direct integration gives

$$\sum_{i=2}^m L_i''(0) \leq \frac{m-1}{2} \Big(\left(\frac{\pi}{\ell}\right)^2 - K \Big) < 0,$$

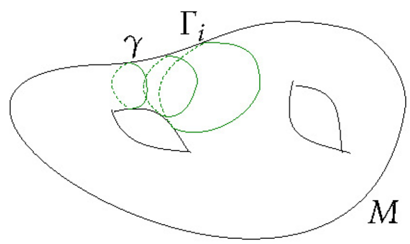
a contradiction! □

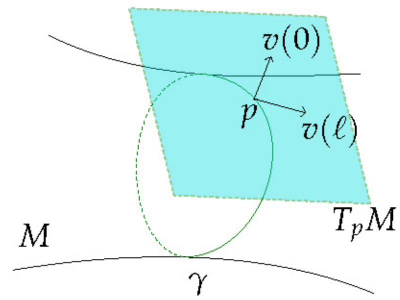
Remark 3.37. This was earlier proved by Bonnet in 1855 under the stronger assumption that the sectional curvature $\geq K > 0$. The constant is sharp as shown by S_r^m. The converse was proved by S.-Y. Cheng in 1975: if there are $p, q \in M$ such that $d(p,q) = \pi/\sqrt{K}$ then M is the round sphere with sectional curvature K.

Theorem 3.38 (Synge 1936)**.** *If (M, g) is compact with sectional curvature > 0, then $\pi_1(M)$ is finite. Moreover,*

(1) *If m is even and M is orientable, then $\pi_1(M) = \{1\}$.*
(2) *If m is odd then M must be orientable.*

SKETCH OF PROOF. (1): if $\pi_1(M) \neq \{1\}$, then any non-trivial $[\gamma_0] \in \pi_1(M)$ can be represented by a C^∞ closed curve γ with shortest length. We leave the existence and smoothness of γ as exercises.





Let $\gamma(0) = p$. For any $v = v(0) \in T_pM$, parallel translations of $v(0)$ along γ gives $v(\ell) \in T_pM$. Parallel translations correspond to solving linear ODE. Hence the map $v(0) \mapsto v(\ell) = Av$ is a linear transformation on T_pM, called the holonomy along γ.

In general we know that $A \in O(m)$ since it preserves inner product. By our construction, $e := \gamma'(0) = \gamma'(\ell)$ is invariant under A, hence we get the induced transformation

$$\tilde{A} : e^\perp \to e^\perp \cong \mathbb{R}^{m-1}.$$

Now $m-1$ is odd and M is orientable, we get $\tilde{A} \in SO(m-1)$ and it must has 1 as its eigenvalue. That is, $\tilde{A}v = v$ for some $v \in e^\perp$.

Let $v(t)$ be the parallel translation of $v = v(0)$ along γ and use $V(t) = v(t)$ as the variation vector field. The second variation formula implies that

$$\begin{aligned} 0 \le L''(0) &= \int_0^\ell \Big(\|\nabla_T V\|^2 - \langle R(V,T)T, V\rangle \Big)\, dt \\ &= -\int_0^\ell K(V,T)\, dt < 0, \end{aligned}$$

which is a contradiction. Thus $\pi_1(M) = \{1\}$.

We leave the proof of (2) as an exercise. $\square$

Exercise 3.13. Let (M, g) be a compact Riemannian manifold. Show that any non-trivial $[\gamma_0] \in \pi_1(M)$ can be represented by a C^∞ closed geodesic $\gamma \in [\gamma_0]$ of minimal length.

Exercise 3.14. Let (M^m, g) be compact with positive sectional curvature. If m is odd show that M is orientable.

6. Jacobi fields

Recall the second variation formula of the form (Proposition 3.7)

$$L''(0) = V\langle V, T\rangle\Big|_a^b - \int_a^b \langle \nabla_T^2 V + R(V, T)T, V\rangle + (T\langle V, T\rangle)^2 \, dt.$$

In case of a normal variation, we found that $\nabla_T^2 V + R(V, T)T$ is the essential quantity to determine this value.

Definition 3.39. For a geodesic γ, $T := \gamma'$, a vector field V along γ is a *Jacobi field* if it satisfies the **Jacobi equation**:

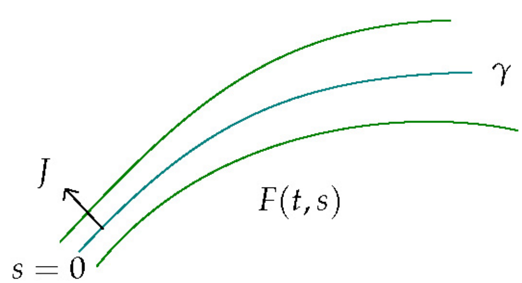
$$\nabla_T^2 V + R(V, T)T = 0.$$

Since the Jacobi equation is a second order linear ODE, the Jacobi field V is uniquely determined by $V(0)$ and $V'(0) = \nabla_T V\big|_{t=0}$. Also the solution space has dimension $2m$.

Theorem 3.40. *A vector field J is Jacobi if and only if J is a variation vector field of geodesics, i.e.*

$$J = F_* \frac{\partial}{\partial s}\Big|_{s=0}$$

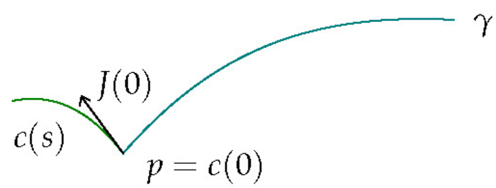
for some $F(t, s)$ which is a geodesic in t for any s.



PROOF. Assume that $F(t,s)$ is a family of geodesics for all s. Let $T = F_* \frac{\partial}{\partial t}$, $V = F_* \frac{\partial}{\partial s}$. Then

$$\begin{aligned}\nabla_T^2 V = \nabla_T \nabla_T V = \nabla_T \nabla_V T\\ = \nabla_V \nabla_T T + R(T,V)T.\end{aligned}$$

So we get the Jacobi equation since $\nabla_T T = 0$ for all s.



Conversely, suppose that J is a Jacobi field along a geodesic γ. Let $c(s)$ be a curve with $c(0) = p$, $c'(0) = J(0)$ along c. Let $\check{T}, \check{J}'$ be the parallel translation of $T(0), J'(0)$. Define

$$F(t,s) := \exp_{c(s)}(t(\check{T} + s\check{J}')), \tag{3.8}$$

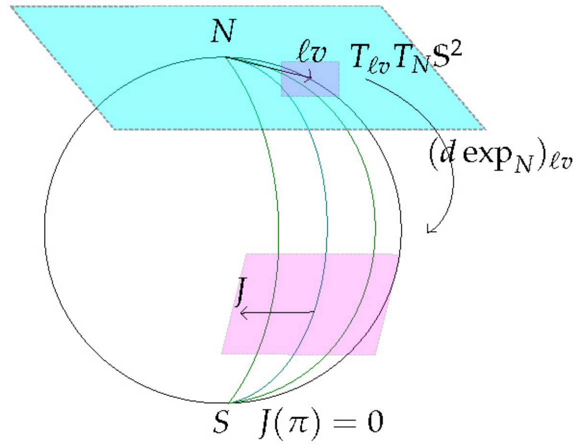
which is clearly a variation of geodesics with $F(0,s) = c(s)$.

It suffice to check $V = F_* \frac{\partial}{\partial s} = J$, i.e. $V(0) = J(0)$, $V'(0) = J'(0)$:

$$\begin{aligned}
V(0) &= \frac{\partial F}{\partial s}\Big|_{(s,t)=(0,0)} \overset{(t=0)}{=} \frac{d}{ds}\exp_{c(s)}(0)\Big|_{s=0} = c'(0) = J(0),\\
V'(0) &= \nabla_T V\,\Big|_{(s,t)=(0,0)} = \nabla_V T\,\Big|_{(s,t)=(0,0)}\\
&= \nabla_V\left[(d\exp_{c(s)})_{t(\check{T}+s\check{J}')}\frac{\partial}{\partial t}(t(\check{T}+s\check{J}'))\right]\Big|_{(s,t)=(0,0)}\\
&\overset{(t=0)}{=} \nabla_{\frac{\partial}{\partial s}}\mathrm{Id}\,(\check{T}+s\check{J}')\Big|_{s=0} = \check{J}'(0) = J'(0).
\end{aligned}$$

The theorem is proved. □

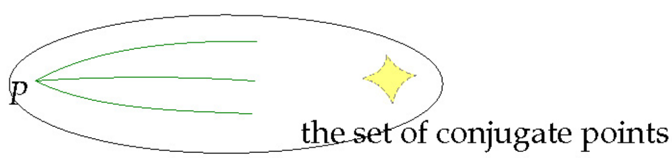
Example 3.41. On $\mathbb{S}^2$, it is clear that $\exp_N$ is well-defined on all $T_N\mathbb{S}^2$. We observe that any Jacobi field J with $J(0) = 0$ must have $J(\pi) = 0$. Indeed, any geodesic passing through the north pole N is a great circle passing through the south pole S as well. Hence the variation constructed above has $F(\pi, s) = S$ for all s.



Definition 3.42. For $P, Q \in \gamma$, we say Q is a *conjugate point* of P if Q is a singular point of $\exp_P$, i.e. if $Q = \exp_P(\ell v)$ then

$$(d\exp_P)_{\ell v} : T_{\ell v}(T_P M) \to T_Q M$$

has a non-trivial kernel.



For any vector w in the kernel, we will construct a Jacobi field with $J(0) = 0$ and with $J(\ell)$ related to w.

Let $F(t, s) := \exp_p(t(v + s\frac{w}{\ell}))$ (c.f. (3.8)). Then

$$\frac{\partial F}{\partial s}\Big|_{s=0,t=\ell} = (d\exp_p)_{\ell v}(w).$$

So if $J = \frac{\partial F}{\partial s}\big|_{s=0}$ is Jacobi with $J(0) = 0$, $J(\ell) = (d\exp_p)_{\ell v}(w)$ for general w. Hence w is in the kernel if and only if $J(\ell) = 0$ and $J'(0) = w/\ell$. And that is to say, $J(\ell)$, $J'(0)$ can determine each other if and only if $(d\exp_p)_{\ell v}$ is invertible. Thus, we have:

Corollary 3.43. *Q conjugates to P if and only if there exists a Jacobi field $J \neq 0$ with $J(P) = J(Q) = 0$. Furthermore, there's a correspondence* *between the space of Jacobi fields and the kernel:*

$$\left\{ \begin{array}{c} J \text{ Jacobi}, \\ J(P) = 0 = J(Q) \end{array} \right\} \leftrightarrow \ker((\exp_p)_*)_{\ell v}$$
$$J \mapsto w := \ell J'(0).$$

Also, Q is conjugate to P if and only if P is conjugate to Q.

Now we give some applications of Jacobi fields.

Theorem 3.44 (Hadamard 1898, Cartan 1928)**.** *If M is complete and with $K \leq 0$, then $\forall p \in M$, $\exp_p : T_pM \to M$ is a covering map. In particular, the universal cover $\tilde{M}$ is diffeomorphic to $\mathbb{R}^m$.*

Lemma 3.45. *If M complete with $K \leq 0$ then $\forall\ p \in M$, any geodesic γ geodesic through p does not have conjugate points of p along γ.*

PROOF. Suppose there exists a conjugate point and let J be a Jacobi field with $J(0) = J(\ell) = 0$. By the Jacobi equation,

$$\begin{aligned} 0 &= -\int_0^\ell \langle \nabla_T^2 J + R(J,T)T, J\rangle\, dt \\ &= -\int_0^\ell \Big(T\langle \nabla_T J, J\rangle - \|\nabla_T J\|^2 + \langle R(J,T)T, J\rangle \Big)\, dt \\ &= \langle \nabla_T J, J\rangle\Big|_0^\ell + \int_0^\ell \Big(\|\nabla_T J\|^2 - \langle R(J,T)T, J\rangle \Big)\, dt. \end{aligned}$$

Since $J(0) = J(\ell) = 0$ and $K \leq 0$, $\nabla_T J \equiv 0$ along γ. But this implies $J \equiv 0$, a contradiction. $\square$

Lemma 3.46. *If X is complete and $f : X \twoheadrightarrow Y$ is a local isometry, then f is a covering map.*

Exercise 3.15. Prove Lemma 3.46 and show by example that it fails if X is not complete.

PROOF OF CARTAN–HADAMARD THEOREM. By Lemma 3.45, $\exp_p$ is a local diffeomorphism on the whole T_pM. Hence

$$(\ ,\ ) := \exp_p^* \langle\ ,\ \rangle$$

is a Riemannian metric on $T_pM \cong \mathbb{R}^m$ which makes $\exp_p$ a local isometry. In particular, $\exp_0 : T_0(T_pM) \to T_pM$ is defined and is an isometry. By Hopf-Rinow theorem (cf. Theorem 3.21), $(T_pM, (\ ,\ ))$ is complete. By Lemma 3.46, $\exp_p$ is a covering map. $\square$

7. Local isometry and space forms

As the second application of Jacobi fields, we investigate how curvatures identifications are related to the constructions of isometries between spaces. In fact this is one of Riemann's original motivations to define his curvature tensor.

Let X and Y be Riemannian manifolds of the same dimension. For any $p \in X, q \in Y$, we may construct an isometry $\Phi : T_pX \to T_qY$ of inner product spaces. Then Φ induces a local diffeomorphism:

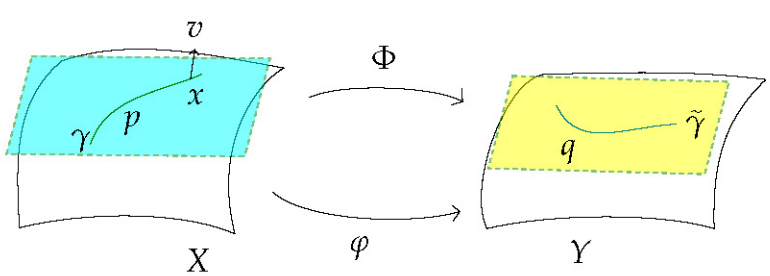
$$y = \varphi(x) := \exp_q \circ \Phi \circ \exp_p^{-1}.$$

That is, we identify points $x \in X$ and $y \in Y$ by identifying the geodesics starting at p and q respectively.

Let γ be a geodesic with $\gamma(0) = p$. Denote by P_γ the parallel translation along γ. Also let $\tilde{\gamma} = \varphi \circ \gamma$ be the corresponding geodesic on Y. For $v \in T_xX$ where x lies on γ, define

$$g_\gamma : T_xX \to T_{\varphi(x)}Y, \qquad v \mapsto g_\gamma(v) := P_{\tilde{\gamma}} \circ \Phi \circ P_\gamma^{-1}(v).$$

Then g_γ is an isometry on inner product spaces.



Theorem 3.47 (Cartan1925, Ambrose 1956)**.** *If*

$$R(g_\gamma(v), g_\gamma(w))g_\gamma(z) = g_\gamma(R(v,w)z)$$

for all $v, w, z \in T_xX$, x in a neighborhood of p, then φ is a local isometry.

PROOF. The key idea is to show that g_γ is indeed the tangent map $(d\varphi)_x$, and then φ is a local isometry.

Since $d\exp_p$ is local diffeomorphism, we may assume that there are no conjugate points in the neighborhood under our consideration. Let J be the Jacobi field along γ with $J(0) = 0, J(\ell) = v$. Then

$$\tilde{J} := g_\gamma(J)$$

is also a Jacobi field on $\tilde{\gamma}$ by the assumption.

Also, $\tilde{J}'(0) = \Phi J'(0)$ by the Chain Rule. Since $J(0) = 0$, we have

$$J(\ell) = (d\exp_p)_{\ell\gamma'(0)}\ell J'(0)$$

as we had seen in the construction Jacobi fields (cf. (3.8)). Then

$$\begin{aligned}
\tilde{J}(\ell) &= (d\exp_q)_{\ell\tilde{\gamma}'(0)}\ell\tilde{J}'(0) \\
&= (d\exp_q)_{\ell\tilde{\gamma}'(0)} \circ \Phi\ell J'(0) \\
&= (d\exp_q)_{\ell\tilde{\gamma}'(0)} \circ \Phi \circ (d\exp_p)^{-1}_{\ell\gamma'(0)} J(\ell) \\
&= d(\exp_q \circ \Phi \circ \exp_p^{-1}) J(\ell) \\
&= (d\varphi)_x J(\ell).
\end{aligned}$$

Here $d\Phi = \Phi$. This applies to all $v \in T_xX$, hence $(d\varphi)_x = g_\gamma$. □

Ambrose indeed proved a global version of the theorem using piecewise geodesics, i.e. a continuous curve $\gamma = \bigcup_i \gamma_i$ which is a finite union of geodesics γ_i (also called broken geodesics in the literature). Although we have not yet a map $\varphi : X \to Y$ as before, applying the exponential maps successively at end points of γ_i's the corresponding piecewise geodesic $\tilde{\gamma}$ in Y is still defined. Then g_γ can be defined similarly.

Theorem 3.48 (Cartan–Ambrose)**.** *Let X, Y be complete Riemannian with an isometry $\Phi : T_pX \to T_qY$. Assume that $\pi_1(M) = \{1\}$. If*

$$R(g_\gamma(v), g_\gamma(w))g_\gamma(z) = g_\gamma(R(v,w)z)$$

for all piecewise geodesics $\gamma : [0,\ell] \to X$ with $\gamma(0) = p$ and $v, w, z \in T_{\gamma(\ell)}X$, then for every other piecewise geodesic γ_1 with $\gamma_1(0) = p$, $\gamma_1(\ell) = \gamma(\ell)$ we also have $\tilde{\gamma}_1(\ell) = \tilde{\gamma}(\ell)$.

This defines $\varphi : X \to Y$ which is a locally isometric covering map.

Exercise 3.16. Prove the global Cartan–Ambrose theorem.

Remark 3.49. Cartan proved his theorem for locally symmetric spaces and used it to classify them. Ambrose proved the general case as well as the global case. In 1959, Hicks generalized the result for general affine connections on the tangent bundle. Thus the theorem is often called the Cartan–Ambrose–Hicks theorem.

As model spaces, we often take manifolds with constant sectional curvature to start with. The more general model spaces including (locally) symmetric spaces require more tools to study and will be discussed in Chapter 5. Here are some simplest examples:

Example 3.50. Complete manifolds with constant sectional curvature.

(1) $\mathbb{S}^n$, $K > 0$.
(2) $\mathbb{R}^n/\Gamma$, $\Gamma \cong \mathbb{Z}^n$. $K \equiv 0$.
(3) $\mathbb{H}^n/\Gamma$, Γ is a discrete group of isometries. $K < 0$. Here

$$\mathbb{H}^n := \{x \in \mathbb{R}^n \mid \|x\| < 1\},\ g = ds^2 = \frac{4\|dx\|^2}{1 - \|x\|^2}$$

is known as the hyperbolic space.

Exercise 3.17. Show that $\mathbb{H}^n$ is complete with $K \equiv -1$.

Definition 3.51 (Space forms). A *space form* is a complete, simply connected Riemannian manifold with constant sectional curvature.

Now we will show that for each K, say $K \in \{-1, 0, 1\}$, the space forms are isometric:

Theorem 3.52. *Let (M, g) be a space form. Then*

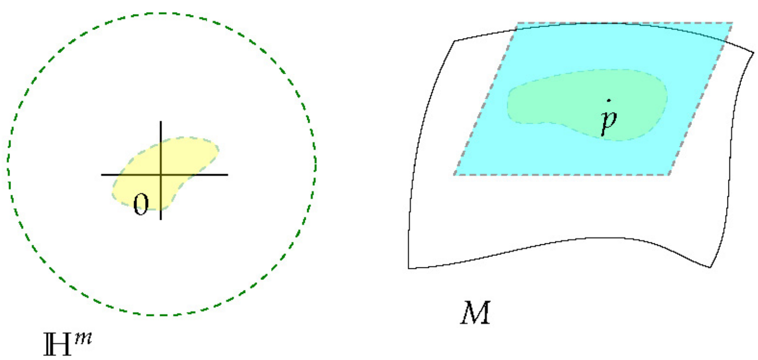
(1) $K \equiv -1 \Longleftrightarrow M \cong \mathbb{H}^m$.
(2) $K \equiv 0 \Longleftrightarrow M \cong \mathbb{R}^m$.
(3) $K \equiv 1 \Longleftrightarrow M \cong \mathbb{S}^m$.

PROOF. For (1) and (2), $K \leq 0$. We compare $\mathbb{H}^m$, $\mathbb{R}^m$ with $M \ni p$.
Step 1: Fix a linear isometry: $L : T_0\mathbb{H}^m \to T_pM$.

Step 2: Matching geodesics as in Cartan–Ambrose's theorem,

$$\varphi = \exp_p \circ L \circ \exp_0^{-1} : U \ni 0 \to M.$$

Since $K \leq 0$, this map is well-defined by Cartan–Hadamard theorem (Theorem 3.44) and we need only geodesics.



In the case $K \equiv -1$, $\exp_0 : T_0\mathbb{H}^m \to \mathbb{H}^m$ is a covering map. So $\varphi : \mathbb{H}^m \to M$ is a diffeomorphism. Cartan–Ambrose's theorem then implies that φ is a local isometry.

The case $K = 0$ is proved by the same argument. We leave the positive case as an exercise. □

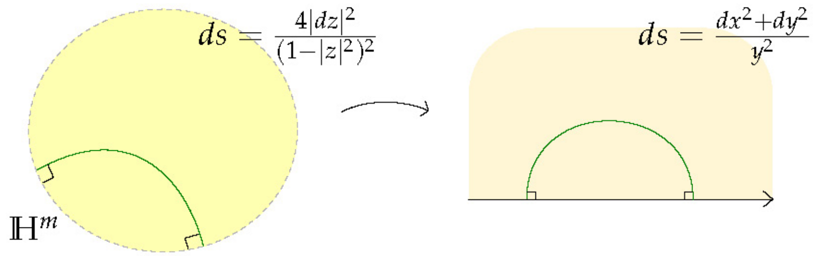
Exercise 3.18. Let M be complete, simply connected with $K \equiv 1$. Show that $M \cong \mathbb{S}^m$.

So if M has constant sectional curvature, $M \cong$ space form$/\Gamma$ where Γ is a discrete group of isometries, $\Gamma \cong \pi_1(M)$.

For example, we have hyperbolic geometry when $K = -1$.

Exercise 3.19. For the Poincaré upper half plane, show that all geodesic are circles that perpendicular to the real line. Use this to prove that the isometry group is precisely $SL(2, \mathbb{R})$ under the action

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(z) = \frac{az+b}{cz+d}.$$

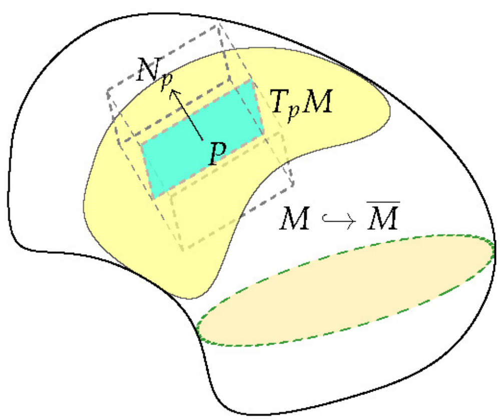


8. Variations of higher dimensional submanifolds

In this final section we consider the variational properties on submanifolds. Let

$$f : M^m \hookrightarrow \overline{M}^{m+k}$$

be a Riemannian imbedding. That is, $(\overline{M}, \bar{g})$ is equipped with the Levi-Civita connection $\overline{\nabla} = \overline{\nabla}^{LC}$, and (M, g) is with the induced metric $g = f^*\bar{g}$ and hence $\nabla := \nabla^{LC} = \overline{\nabla}^T$, where T means the tangential projection under $T_p\overline{M} = T_pM \oplus^\perp N_p$.



In general, for $X, Y, Z, W \in T_pM$,

$$\langle R(X,Y)Z, W\rangle \neq \langle \overline{R}(X,Y)Z, W\rangle.$$

E.g. for $\mathbb{S}^m \hookrightarrow \mathbb{R}^{m+1}$, $K \equiv 1$ on the S^m while $K \equiv 0$ on $\mathbb{R}^m$.

In order to investigate the curvature defect, we start with the defect of covariant differentiations:

Definition 3.53 (Second fundamental form). The *second fundamental form* $B \in C^\infty(M, \mathrm{Sym}^2(N))$ is a symmetric bilinear form with values in the normal bundle: for $X, Y \in T_pM$,

$$\mathbf{II}(X,Y) \equiv B(X,Y) := (\overline{\nabla}_X \tilde{Y})^N.$$

Here $\tilde{Y}$ is any vector field on $\overline{M}$ extending Y, and N is the normal projection to N_p. The trace of B is called the mean curvature (vector):

$$\overrightarrow{H} = H = \mathrm{tr}(B) \in N.$$

Explanations: B is tensorial in X. It is symmetric since

$$B(X,Y) - B(Y,X) = (\overline{\nabla}_X \tilde{Y} - \overline{\nabla}_Y \tilde{X})^N = [\tilde{X}, \tilde{Y}]^N = 0,$$

hence B is also tensorial in Y. The trace of B can be computed by

$$H(p) = \sum_{i=1}^{m} B(e_i, e_j) = \sum_{i,j=1}^{m} g^{ij} B(\partial_i, \partial_j),$$

where $e_1, \ldots, e_m$ is any O.N.B. of T_pM and $\partial_i = \partial/\partial x^i$ are coordinate vector fields in any coordinate system.

The relation between R and $\overline{R}$ is described by

Proposition 3.54 (Gauss equation)**.** *For any $X, Y, Z, W \in T_pM$,*

$$\begin{aligned}\langle R(X,Y)W, Z\rangle &= \langle \overline{R}(X,Y)W, Z\rangle \\ &\qquad + \langle B(X,Z), B(Y,W)\rangle - \langle B(X,W), B(Y,Z)\rangle.\end{aligned} \tag{3.9}$$

PROOF. Denote the extensions of a vector by the same symbol. Direct computation gives

$$\begin{aligned}
&\langle R(X,Y)W, Z\rangle = \langle \nabla_X \nabla_Y W, Z\rangle - \langle \nabla_Y \nabla_X W, Z\rangle - \langle \nabla_{[X,Y]} W, Z\rangle \\
&= \langle \overline{\nabla}_X \nabla_Y W, Z\rangle - \langle \overline{\nabla}_Y \nabla_X W, Z\rangle - \langle \overline{\nabla}_{[X,Y]} W, Z\rangle \\
&= X\langle \overline{\nabla}_Y W, Z\rangle - \langle \nabla_Y W, \overline{\nabla}_X Z\rangle - Y\langle \overline{\nabla}_X W, Z\rangle + \langle \nabla_X W, \overline{\nabla}_Y Z\rangle \\
&\qquad - \langle \overline{\nabla}_{[X,Y]} W, Z\rangle \\
&= X\langle \overline{\nabla}_Y W, Z\rangle - \langle \overline{\nabla}_Y W, \overline{\nabla}_X Z\rangle + \langle B(Y,W), \overline{\nabla}_X Z\rangle \\
&\qquad - Y\langle \overline{\nabla}_X W, Z\rangle + \langle \overline{\nabla}_X W, \overline{\nabla}_Y Z\rangle - \langle B(X,W), \overline{\nabla}_Y Z\rangle \\
&\qquad - \langle \overline{\nabla}_{[X,Y]} W, Z\rangle \\
&= \langle \overline{R}(X,Y)W, Z\rangle + \langle B(X,Z), B(Y,W)\rangle - \langle B(X,W), B(Y,Z)\rangle.
\end{aligned}$$

Notice that $T\overline{M} = TM \oplus^{\perp} N$ is used in most steps. $\square$

Example 3.55. Let $M^m \hookrightarrow \mathbb{R}^{m+1}$ be a hypersurface with prescribed unit normal vector field N. Choose a local coordinate $(x^1, \ldots, x^m)$. Denote $\partial_i = \mathbb{X}_i = \frac{\partial}{\partial x^i}$ and

$$h_{ij}N := B(\partial_i, \partial_j) = (\nabla_{\partial_i}\partial_j)^N = \left(\partial_i \mathbb{X}_j\right)^N = \mathbb{X}_{ji}^N.$$

Then the Gauss equation reads as

$$R_{ijk\ell} = \overline{R}_{ijk\ell} + h_{ik}h_{j\ell} - h_{i\ell}h_{jk}.$$

E.g., for $M = \mathbb{S}^m_r \hookrightarrow \mathbb{R}^{m+1}$ with normal vector $N = \mathbb{X}/r$,

$$h_{ij} = \mathbb{X}_{ij} \cdot N = (\mathbb{X}_i \cdot N)_j - (\mathbb{X}_i \cdot N_j) = -\tfrac{1}{r}\mathbb{X}_i \cdot \mathbb{X}_j = -\tfrac{1}{r}g_{ij}.$$

Hence $R_{ijk\ell} = \frac{1}{r^2}(g_{ik}g_{j\ell} - g_{i\ell}g_{jk})$.

Since an immersion is locally an imbedding, and all of the above discussions are local, hence they works for immersions with small modifications.

Given an immersion $\iota : M^n \looparrowright (N^{n+p}, \bar{g})$ with the induced metric on (M, g), we consider the variation of M in N:

$$F : (-\epsilon, \epsilon) \times M \longrightarrow N,$$

which is C^∞ with $F_0 = \iota$, $F_s = F(s, \bullet) : M \to N$. F_s is still an immersion for s small. Let $\eta := F_* \frac{\partial}{\partial s}$ be the variation vector field.

For simplicity, we assume that M is oriented. Nevertheless we allow M to have non-empty boundary ∂M with outer normal ν.

Recall the n-dimensional area functional:

$$A(s) = \int_{M_s} dA_s = \int_M \sqrt{g(s)}\, dx^1 \wedge \cdots \wedge dx^n$$

where $g_s = F_s^* \overline{g}$ and $g(s) = \det(g_{s,ij})$. Taking derivative in s:

$$A'(s) = \int_M \frac{\partial}{\partial s}\sqrt{g(s)}\, dx^1 \wedge \ldots \wedge dx^n.$$

Let $G_{s,ij}$ be the cofactor of $g_{s,ij}$. To save notations, we omit the subscript s in the presentation when no confusion will arise. Then

$$g'(s) = g'_{ij}G_{ij} = g'_{ij}g^{ij}g.$$

Hence

$$A'(s) = \int_M \frac{g'_{ij} g^{ij} g}{2\sqrt{g}}\, dx^1 \wedge \cdots \wedge dx^n = \frac{1}{2}\int_M g'_{ij} g^{ij}\, dA_s. \tag{3.10}$$

Theorem 3.56 (First variation formula)**.** *For a variation of an immersed submanifold $M \looparrowright N$ with variation field η and outer normal field ν,*

$$A'(0) = \int_{\partial M} \langle \eta^T, \nu\rangle\, d\sigma - \int_M \langle \eta, \overrightarrow{H}\rangle\, dA.$$

PROOF. To calculate $A'(0)$, we will determine the integrand tensor in (3.10) at each point. Let $p \in M$. Pick the Riemann normal coordinate $(x^1, \ldots, x^n)$ of $(M, g = g_0)$ at p. Then

$$g'_{ij}(0) = \frac{\partial}{\partial s}\langle \partial_i, \partial_j\rangle\Big|_{s=0} = \langle \overline{\nabla}_\eta \partial_i, \partial_j\rangle + \langle \partial_i, \overline{\nabla}_\eta \partial_j\rangle.$$

Since $(\nabla_{\partial_i}\partial_j)(p) = \sum_k \Gamma^k_{ij}(p)\partial_k|_p = 0$ by Lemma 3.16, we have

$$\begin{aligned}
\langle \overline{\nabla}_{\partial_i}\eta, \partial_j\rangle &= \partial_i\langle \eta, \partial_j\rangle - \langle \eta, \overline{\nabla}_{\partial_i}\partial_j\rangle \\
&= \partial_i\langle \eta, \partial_j\rangle - \langle \eta, \nabla_{\partial_i}\partial_j + B(\partial_i, \partial_j)\rangle \\
&= \partial_i\langle \eta, \partial_j\rangle - \langle \eta, B(\partial_i, \partial_j)\rangle.
\end{aligned}$$

The other term $\langle \partial_i, \overline{\nabla}_\eta \partial_j\rangle$ is obtained by symmetry, and after contracting with g^{ij} they are equal. Hence

$$\begin{aligned}
A'(0) &= \int_M g^{ij}\big(\partial_i\langle \eta, \partial_j\rangle - \langle \eta, B(\partial_i, \partial_j)\rangle\big)\, dA \\
&= \int_M \operatorname{div} \eta^T\, dA - \int_M \langle \eta, \overrightarrow{H}\rangle\, dA \\
&= \int_{\partial M} \langle \eta^T, \nu\rangle\, d\sigma - \int_M \langle \eta, \overrightarrow{H}\rangle\, dA,
\end{aligned}$$

where the last equality follows from the Stokes theorem. □

It is clear that the critical point condition $A'(0) = 0$ for all η which vanishes on ∂M is equivalent to $\overrightarrow{H} = 0$. Following the tradition, these “critical immersed submanifolds” are called “minimal”:

Definition 3.57. M is a *(immersed) minimal submanifold* in N if $\overrightarrow{H} = 0$.

Remark 3.58. One would ask if a minimal submanifold is indeed "locally minimal" in the usual sense. One would also ask about the existence problem. E.g., given a piecewise smooth curve Γ in $\mathbb{R}^3$, is there an area minimizing surface S with $\partial S = \Gamma$? We will discuss these questions in later chapters.

We proceed to study the second variations. Differentiating (3.10) again we get

$$\begin{aligned} A''(s) &= \frac{1}{2}\int_M \frac{\partial}{\partial s}(g'_{ij}g^{ij}\sqrt{g})\, dx^1 \wedge \ldots \wedge dx^n \\ &= \int_M \Big(\underbrace{\frac{1}{2}g''_{ij}g^{ij}\sqrt{g}}_{(1)} + \underbrace{\frac{1}{2}g'_{ij}(g^{ij})'\sqrt{g}}_{(2)} + \underbrace{\frac{1}{2}g'_{ij}g^{ij}(\sqrt{g})'}_{(3)} \Big)\, dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

At $p \in M$, $s = 0$, we pick the Riemann normal coordinate so that $g_{ij}(p) = \delta_{ij}$, $(\nabla_{\partial_i}\partial_j)(p) = 0$. Then

$$\begin{aligned} (1) &= \eta\langle \overline{\nabla}_{\partial_i}\partial_i, \partial_i\rangle = \langle \overline{\nabla}_\eta\overline{\nabla}_\eta\partial_i, \partial_i\rangle + \langle \overline{\nabla}_\eta\partial_i, \overline{\nabla}_\eta\partial_i\rangle \\ &= \langle \overline{\nabla}_\eta\overline{\nabla}_{\partial_i}\eta, \partial_i\rangle + \|\overline{\nabla}_{\partial_i}\eta\|^2 \\ &= \langle \overline{R}(\eta, \partial_i)\eta, \partial_i\rangle + \langle \overline{\nabla}_{\partial_i}\overline{\nabla}_\eta\eta, \partial_i\rangle + \|\overline{\nabla}_{\partial_i}\eta^T\|^2 + \|\overline{\nabla}_{\partial_i}\eta^N\|^2 \\ &= -\overline{R}(\eta, \partial_i, \eta, \partial_i) + \partial_i\langle \overline{\nabla}_\eta\eta, \partial_i\rangle - \langle \overline{\nabla}_\eta\eta, \overline{\nabla}_{\partial_i}\partial_i\rangle \\ &\quad + \|\overline{\nabla}_{\partial_i}\eta^T\|^2 + \|\overline{\nabla}_{\partial_i}\eta^N\|^2. \end{aligned}$$

For (2), recall that $(G^{-1})' = -G^{-1}G'G^{-1}$. Hence

$$\begin{aligned} (2) &= -\frac{1}{2}\sum_{i,j} g'_{ij}(g^{ik}g'_{k\ell}g^{\ell j}) = -\frac{1}{2}\sum_{i,j}^{n}(g'_{ij})^2 \\ &= -\frac{1}{2}\sum_{i,j}\left(\langle \overline{\nabla}_\eta\partial_i, \partial_j\rangle + \langle \partial_i, \overline{\nabla}_\eta\partial_j\rangle\right)^2 \end{aligned}$$

For (3), we have seen that $\sqrt{g}' = \frac{1}{2}\sum g'_{ij}g^{ij}$, hence

$$(3) = \frac{1}{4}\left(\sum_{i,j} g'_{ij}g^{ij}\right)^2 = \left(\sum \langle \overline{\nabla}_\eta\partial_i, \partial_i\rangle\right)^2.$$

By putting together (1), (2) and (3) we get

Theorem 3.59 (Second variation formula)**.** *Under the Riemann normal coordinates,*

$$\begin{aligned} A''(0) = \int_{\partial M} \langle \overline{\nabla}_\eta \eta, \nu\rangle - \int_M \Big(\langle \overline{\nabla}_\eta \eta, \overrightarrow{H}\rangle + \sum_i \langle \overline{R}(\partial_i, \eta)\eta, \partial_i\rangle \Big) \\ + \int_M \sum_{i,j} \langle \overline{\nabla}_{\partial_i}\eta, \partial_j\rangle^2 + \sum_i \|\overline{\nabla}_{\partial_i}\eta^N\|^2 \\ - \frac{1}{2}\int_M \sum_{i,j=1}^n \left(\langle \overline{\nabla}_\eta \partial_i, \partial_j\rangle + \langle \partial_i, \overline{\nabla}_\eta \partial_j\rangle \right)^2 \\ + \int_M \Big(\sum_i \langle \overline{\nabla}_\eta \partial_i, \partial_i\rangle \Big)^2. \end{aligned}$$

The expression is complicate. Nevertheless, in many specific applications it can be simplified considerably.

For $i : M \hookrightarrow N$ (minimal), F normal variation ($\eta \perp TM$), let $\eta = f\mu$ with μ being an unit normal vector field. In this case,

$$\begin{aligned} \langle \overline{\nabla}_\eta \partial_i, \partial_j\rangle = \langle \overline{\nabla}_{\partial_i}\eta, \partial_j\rangle = \partial_i\langle \eta, \partial_j\rangle - \langle \eta, \overline{\nabla}_{\partial_i}\partial_j\rangle \\ = -\langle \eta, B(\partial_i, \partial_j)\rangle, \end{aligned}$$

and the calculation for $\langle \partial_i, \overline{\nabla}_\eta \partial_j\rangle$ is the same. So

$$-\frac{1}{2}\int_M \sum_{i,j=1}^n \left(\langle \overline{\nabla}_\eta \partial_i, \partial_j\rangle + \langle \partial_i, \overline{\nabla}_\eta \partial_j\rangle \right)^2 = -2\int_M \sum_{i,j=1}^n |f|^2 \langle \mu, B(\partial_i, \partial_j)\rangle^2.$$

Combining with the term $\sum \langle \overline{\nabla}_{\partial_i}\eta, \partial_j\rangle^2$, we get

$$-\int_M |f|^2 \|\langle B, \mu\rangle\|^2.$$

Also,

$$\sum_i \langle \overline{\nabla}_\eta \partial_i, \partial_i\rangle = \sum_i \langle \overline{\nabla}_{\partial_i}\eta, \partial_i\rangle = -f\sum_i \langle \overline{\nabla}_{\partial_i}\partial_i, \mu\rangle = -f\langle \overrightarrow{H}, \mu\rangle.$$

For $\sum_i \|\overline{\nabla}^N_{\partial_i}\eta\|^2$, if M is a hypersurface we may write $\eta = fe_{n+1}$ ($\nu = e_{n+1}$), and then

$$\overline{\nabla}_{\partial_i}\eta = \overline{\nabla}_{\partial_i}(fe_{n+1}) = (\partial_i f)e_{n+1} + f\overline{\nabla}_{\partial_i}e_{n+1}.$$

Since $\langle \overline{\nabla}_{\partial_i}e_{n+1}, e_{n+1}\rangle = \frac{1}{2}\partial_i\langle e_{n+1}, e_{n+1}\rangle = 0$, $f\overline{\nabla}_{\partial_i}e_{n+1} \in TM$. So

$$\overline{\nabla}^N_{\partial_i}\eta = (\partial_i f)e_{n+1}, \qquad \sum_i \|\overline{\nabla}^N_{\partial_i}\eta\|^2 = \|df\|^2 = \|\nabla f\|^2.$$

We have thus a corollary:

Corollary 3.60. *For ∂M-fixed normal variations of a minimal immersion $M \looparrowright N$,*

$$A''(0) = \int_M \left(\|\nabla f\|^2 - \left(\overline{\mathrm{Ric}}(\overrightarrow{n}, \overrightarrow{n}) + \|B\|^2 \right) f^2 \right),$$

where $\overrightarrow{n} = e_{n+1}$.

As an application, we prove a theorem due to Schoen and Yau which was used in their 1979 proof of the "positive mass theorem" in general relativity.

Definition 3.61. A minimal immersion $M \looparrowright N$ is stable if $A''(0) \geq 0$ for all variations which fixed ∂M.

Theorem 3.62 (Schoen–Yau)**.** *Let $\overline{M}$ is a compact oriented 3-dimensional manifold with non-negative scalar curvature and $\overline{s}(p) > 0$ for some $p \in M$. Then there is no stable minimal immersed surface $M \looparrowright N$ which is compact orientable, $\partial M = \emptyset$, and $g(M) \geq 1$.*

PROOF. If M is stable, i.e. $A''(0) \geq 0$, then for all $f \in C^\infty(M)$:

$$\int_M (\overline{\mathrm{Ric}}(\overrightarrow{n}, \overrightarrow{n}) + \|B\|^2) f^2 \leq \int_M |\nabla f|^2.$$

Let $f \equiv 1$ and apply the follwoing lemma:

Lemma 3.63. *Let $M^2 \looparrowright \overline{M}^3$ be a minimal immersion. Then*

$$\overline{\mathrm{Ric}}(\overrightarrow{n}, \overrightarrow{n}) = \frac{1}{2}\overline{s} - K - \frac{1}{2}\|B\|^2$$

where K is the Gauss curvature of M.

With this lemma, we then have

$$\int_M \left(\frac{1}{2}\overline{s} + \frac{1}{2}\|B\|^2 \right) - \int_M K \leq 0$$

By the Gauss–Bonnet theorem, we have

$$\int_M K = 2\pi\chi(M) = 4\pi(1 - g(M)) \leq 0$$

if $g(M) \geq 1$. This contradicts to the assumption on $\overline{s}$. $\square$

PROOF OF LEMMA. Pick Riemann normal coordinate at p. Then

$$\begin{aligned}\overline{R} &= \overline{g}^{ij}\overline{g}^{k\ell}\overline{R}_{ikj\ell} = 2(\overline{R}_{1212} + \overline{R}_{1313} + \overline{R}_{2323}) \\ &= 2(\overline{\mathrm{Ric}}(e_3, e_3) + \overline{R}_{1212}).\end{aligned}$$

By Gauss' equation $R_{ijk\ell} = \overline{R}_{ijk\ell} + h_{ik}h_{j\ell} - h_{i\ell}h_{jk}$, we get

$$K = R_{1212} = \overline{R}_{1212} + h_{11}h_{22} - h_{12}h_{21}.$$

Since M is minimal, we have

$$B = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}, \qquad \mathrm{tr}(B) = h_{11} + h_{22} = 0.$$

Hence $\|B\|^2 = h_{11}^2 + h_{12}^2 + h_{21}^2 + h_{22}^2 = 2h_{11}^2 + 2h_{12}^2$ and $h_{11}h_{22} - h_{12}h_{21} = -(h_{11}^2 + h_{12}^2) = \frac{1}{2}\|B\|^2$. The formula follows. $\square$

9. Problems

1. Determine all the geodesics on the sphere $\mathbb{S}^n$.
2. ([Car92] Ch.2 #8) Consider the upper half-plane

$$\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

with the metric given by $g_{11} = g_{22} = \frac{1}{y^2}$, $g_{12} = 0$ (metric of Lobatchevski's non-euclidean geometry).

   (a) Show that the Christoffel symbols of the Riemannian connection are: $\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = 0$, $\Gamma^2_{11} = \frac{1}{y}$, $\Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}$.

   (b) Let $v_0 = (0, 1)$ be a tangent vector at point $(0, 1)$ of $\mathbb{R}^2_+$ (v_0 is a unit vector on the y-axis with origin at $(0, 1)$). Let $v(t)$ be the parallel transport of v_0 along the curve $x = t$, $y = 1$. Show that $v(t)$ makes an angle t with the direction of the y-axis, measured in the clockwise sense.

3. ([Car92] Ch.3 #5) Let M be a Riemannian manifold and $X \in \mathcal{X}(M)$ (the set of all vector field of class C^∞ on M). Let $p \in M$ and let $U \subset M$ be a neighborhood of p. Let $\varphi : (-\varepsilon, \varepsilon) \times U \to M$ be a differentiable mapping such that for any $q \in U$ the curve $t \to \varphi(t, q)$ is a trajectory of X passing through q at $t = 0$ (U and φ are given by the fundamental theorem for ordinary differential equations). X is called a *Killing field* (or an *infinitesimal isometry*) if, for each $t_0 \in (-\varepsilon, \varepsilon)$, the mapping $\varphi(t_0, \cdot) : U \subset M \to M$ is an isometry. Prove that

(a) A vector field v on $\mathbb{R}^n$ may be seen as a map $v : \mathbb{R}^n \to \mathbb{R}^n$; we say that the field is linear if v is a linear map. A linear field on $\mathbb{R}^n$, defined by a matrix A, is a Killing field if and only if A is anti-symmetric.
(b) Let X be a Killing field on M, $p \in M$, and let U be a normal neighborhood of p on M. Assume that p is a unique point of U that satisfies $X(p) = 0$. Then, in U, X is tangent to the geodesic spheres centered at p.
(c) Let X be a differentiable vector field on M and let $f : M \to N$ be an isometry. Let Y be a vector field on N defined by $Y(f(p)) = df_p(X(p))$, $p \in M$. Then Y is a Killing field if and only if X is also a Killing vector field.
(d) X is Killing $\Leftrightarrow \langle \nabla_Y X, Z \rangle + \langle \nabla_Z X, Y \rangle = 0$ for all $Y, Z \in \mathcal{X}(M)$ (the equation above is called the *Killing equation*).
(e) Let X be a Killing field on M with $X(q) \neq 0$, $q \in M$. Then there exists a system of coordinates $(x_1, \dots, x_n)$ in a neighborhood of q, so that the coefficients g_{ij} of the metric in this system coordinates do not depend on x_n.

4. ([Car92] Ch.4 #4) Let M be a Riemannian manifold with the following property: given any two points $p, q \in M$, the parallel transport from p to q does not depend on the curve that joins p to q. Prove that the curvature of M is identically zero, that is, for all $X, Y, Z \in \mathcal{X}(M)$, $R(X,Y)Z = 0$.
5. Define covariant derivatives on differential forms.
6. ([Car92] Ch.4 #8)(Schur's theorem) Let M^n be a connected Riemannian manifold with $n \geq 3$. Suppose that M is *isotropic*, that is, for each $p \in M$, the sectional curvature $K(p, \sigma)$ does not depend on $\sigma \subset T_pM$. Prove that M has constant sectional curvature, that is, $K(p, \sigma)$ also does not depend on p.
7. ([Car92] Ch.4 #10) (Einstein manifolds) A Riemannian manifold M^n is called an *Einstein manifold* if, for all $X, Y \in \mathcal{X}(M)$, $\mathrm{Ric}(X,Y) = \lambda \langle X, Y \rangle$, where $\lambda : M \to \mathbb{R}$ is a real valued funciton. Prove that:
   (a) If M^n connected and Einstein , with $n \geq 3$, then λ is constant on M.
   (b) If M^3 is a connected Einstein manifold then M^3 has constant sectional curvature.
8. Show that the sectional curvature is the Gauss curvature.
9. ([Car92] Ch.4 #6) Let M be a Riemannian manifold. M is a *locally symmetric space* if $\nabla R = 0$, where R is the curvature tensor of M.

(a) Let M be a locally symmetric space and let $\gamma : [0, \ell) \to M$ be a geodesic of M. Let X, Y, Z be parallel vector fields along γ. Prove that $R(X,Y)Z$ is a parallel field along γ.

(b) Prove that if M is locally symmetric, connected, and has dimension two, then M has constant sectional curvature.

(c) Prove that if M has constant (sectional) curvature, then M is a locally symmetric space.

10. ([Car92] Ch.5 #5) (Jacobi fields and conjugate points on locally symmetric spaces) Let $\gamma : [0, \infty) \to M$ be a geodesic in a locally symmetric space M and let $v = \gamma'(0)$ be its velocity at $p = \gamma(0)$. Define a linear transformation $K_v : T_pM \to T_pM$ by

$$K_v(x) = R(v, x)v, \ x \in T_pM.$$

(a) Prove that K_v is self-adjoint.

(b) Choose an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM that diagonalizes K_v, that is,

$$K_v(e_i) = \lambda_i e_i, \quad i = 1, \dots, n.$$

Extend the e_i to fields along γ by parallel transport. Show that, for all t,

$$K_{\gamma'(t)}(e_i(t)) = \lambda_i e_i(t).$$

where λ_i does not depend on t.

(c) Let $J(t) = \sum_i x_i(t) e_i(t)$ be a Jacobi field along γ. Show that the Jacobi equation is equivalent to the system

$$\frac{d^2 x_i}{dt^2} + \lambda_i x_i = 0, \quad i = 1, \dots, n.$$

(d) Show that the conjugate points of p along γ are given by $\gamma(\pi k/\sqrt{\lambda_i})$, where k is a positive integer and λ_i is a positive eigenvalue of K_v.

11. ([Car92] Ch.7 #1) If M, N are Riemannian manifolds such that the inclusion $i : M \hookrightarrow N$ is an isometric immersion, show by an example that the strict inequality of metrics $d_M > d_N$ can occur.

12. ([Car92] Ch.7 #4) Consider the universal covering

$$\pi : M \to \mathbb{R}^2 \setminus \{(0,0)\}$$

of the Euclidean plane minus the origin. Introduce the covering metric on M. (Note: For $\tilde{M}$ a covering space of M, it is possible to give the covering space a Riemannian structure such that the covering map $\pi : \tilde{M} \to M$ is a local isometry.) Show that M is not complete and not extendible, and that the Hopf-Rinow theorem is not true for M (this shows that the definition of non-extendibility, though natural, is not a satisfactory one).

13. ([Car92] Ch.7 #5) A *divergent curve* in a Riemannian manifold M is a differentiable mapping $\alpha : [0, \infty) \to M$ such that for any compact set $K \subset M$ there exists $t_0 \in (0, \infty)$ with $\alpha(t) \notin K$ for all $t > t_0$ (that is, α "escapes" every compact set in M). Define the length of a divergent curve by
$$\lim_{t\to\infty} \int_0^t |\alpha'(t)|\, dt.$$
Prove that M is complete if and only if the length of any divergent curve is unbounded.
14. ([Car92] Ch.7 #6) A geodesic $\gamma : [0, \infty) \to M$ in a Riemannian manifold M is called a *ray starting from* $\gamma(0)$ if it minimizes the distance between $\gamma(0)$ and $\gamma(s)$, for any $s \in (0, \infty)$. Assume that M is complete, non-compact, and let $p \in M$. Show that M contains a ray starting form p.
15. ([Car92] Ch.7 #12) A Riemannian manifold is said to be *homogeneous* if given $p, q \in M$ there exists an isometry of M which takes p into q. Prove that any homogeneous manifold is complete.
16. ([Car92] Ch.8 #1) Consider, on a neighborhood in $\mathbb{R}^2$, $n > 2$ the metric
$$g_{ij} = \frac{\delta_{ij}}{F^2}$$
where $F \neq 0$ is a function of $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Denote by $F_i = \frac{\partial F}{\partial x_i}$, $F_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j}$, etc.
    (a) Show that a necessary and sufficient condition for the metric to have constant curvature K is
$$\begin{cases} F_{ij} = 0, \quad i \neq j \\ F(F_{jj} + F_{ii}) = K + \sum_{i=1}^n (F_i)^2. \end{cases}$$
    (b) Use (a) to prove that the metric g_{ij} has constant curvature K if and only if
$$F = G_1(x_1) + G_2(x_2) + \cdots + G_n(x_n),$$
where
$$G_i(x_i) = ax_i^2 + b_i x_i + c_i$$
and
$$\sum_{i=1}^n (4c_i a - b_i^2) = K.$$

(c) Put $a = K/4$, $b_i = 0$, $c_i = 1/n$ and obtain the formula of Riemann

$$g_{ij} = \frac{\delta_{ij}}{\left(1 + \frac{K}{4}\sum x_i^2\right)^2}$$

for a metric g_{ij} of constant curvature K. If $K < 0$, the metric g_{ij} is defined in a ball of radius of $\sqrt{\frac{4}{-K}}$.

(d) If $K > 0$, the metric is defined on all of $\mathbb{R}^n$. Show that such a metric on $\mathbb{R}^n$ is not complete.

17. ([Car92] Ch.6 #1) Let M_1 and M_2 be Riemannian manifolds, and consider the product $M_1 \times M_2$, with the product metric. Let ∇^1 be the Riemannian connection of M_1 and let ∇^2 be the Riemannian connection of M_2.
    (a) Show that the Riemannian connection ∇ of $M_1 \times M_2$ is given by $\nabla_{Y_1+Y_2}(X_1 + X_2) = \nabla^1_{Y_1} X_1 + \nabla^2_{Y_2} X_2$, for $X_1, Y_1 \in \mathcal{X}(M_1)$, $X_2, Y_2 \in \mathcal{X}(M_2)$.
    (b) For every $p \in M_1$, the set $(M_2)_p = \{(p,q) \in M_1 \times M_2; q \in M_2\}$ is a submanifold of $M_1 \times M_2$, naturally diffeomorphic to M_2. Prove that $(M_2)_p$ is a totally geodesic submanifold of $M_1 \times M_2$.
    (c) Let $\sigma(x,y) \subset T_{(p,q)}(M_1 \times M_2)$ be a plane such that $x \in T_p M_1$ and $y \in T_q M_2$. Show that $K(\sigma) = 0$.
18. ([Car92] Ch.6 #2) Show that $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$\mathbf{x}(\theta, \varphi) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi), \ (\theta, \varphi) \in \mathbb{R}^2$$

is an immersion of $\mathbb{R}^2$ into the unit sphere $\mathbb{S}^3(1) \subset \mathbb{R}^4$, whose image $\mathbf{x}(\mathbb{R}^2)$ is a torus $\mathbb{T}^2$ with sectional curvature zero in the induced metric.
19. ([Car92] Ch.6 #4) Let $N_1 \subset M_1$, $N_2 \subset M_2$ be totally geodesic submanifolds of the Riemannian manifolds M_1 and M_2 respectively. Prove that $N_1 \times N_2$ is a totally geodesic submanifold of the product $M_1 \times M_2$ with the product metric.
20. ([Car92] Ch.6 #5) Prove that the sectional curvature of the Riemannian manifold $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric, where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$, is non-negative. Find a totally geodesic, flat torus, $\mathbb{T}^2$, embedded in $\mathbb{S}^2 \times \mathbb{S}^2$.
21. ([Car92] Ch.6 #8) (*The Clifford torus*) Consider the immersion $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^4$ given in the above exercise 18.
    (a) Show that the vectors

$$e_1 = (-\sin\theta, \cos\theta, 0, 0), e_2 = (0, 0, -\sin\varphi, \cos\varphi)$$

form an orthonormal basis of the tangent space, and that the vectors

$$n_1 = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi),$$
$$n_2 = \frac{1}{\sqrt{2}}(-\cos\theta, -\sin\theta, \cos\varphi, \sin\varphi)$$

form an orthonormal basis of the normal space.

(b) Use the fact that

$$\langle S_{n_k}(e_i), e_j\rangle = -\langle \overline{\nabla}_{e_i} n_k, e_j\rangle = \langle \overline{\nabla}_{e_i} e_j, n_k\rangle,$$

where $\overline{\nabla}$ is the covariant derivative (that is, the usual derivative) of $\mathbb{R}^4$, and $i, j, k = 1, 2$, to establish that the matrices of S_{n_1} and S_{n_2} with respect to the basis $\{e_1, e_2\}$ are

$$S_{n_1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad S_{n_2} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(c) From the above exercise 18, $\mathbf{x}$ is an immersion of the torus $\mathbb{T}^2$ into $\mathbb{S}^3(1)$ (the Clifford torus). Show that $\mathbf{x}$ is a minimal immersion.

22. ([Car92] Ch.6 #11) Let $f : \overline{M}^{n+1} \to \mathbb{R}$ be a differentiable function. Define the *Hessian*, $\mathrm{Hess} f$ of f at $p \in \overline{M}$ as the linear operator

$$\mathrm{Hess} f : T_p\overline{M} \to T_p\overline{M}, (\mathrm{Hess} f)Y = \overline{\nabla}_Y \operatorname{grad} f, Y \in T_p\overline{M},$$

where $\overline{\nabla}$ is the Riemannian connection of $\overline{M}$. Let a be a regular value of f and let $M^n \subset \overline{M}^{n+1}$ be the hypersuperface in $\overline{M}$ defined by $M = \{p \in \overline{M} \mid f(p) = a\}$. Prove that:

(a) The Laplacian $\overline{\Delta} f$ is given by

$$\overline{\Delta} f = \operatorname{trace}(\mathrm{Hess} f).$$

(b) If $X, Y \in \mathcal{X}(M)$, then

$$\langle (\mathrm{Hess} f)Y, X\rangle = \langle Y, (\mathrm{Hess} f)X\rangle.$$

Conclude that $\mathrm{Hess} f$ is self-adjoint, hence determines a symmetric bilinear form on $T_p\overline{M}$, $p \in \overline{M}$, given by $(\mathrm{Hess} f)(X, Y) = \langle (\mathrm{Hess} f)X, Y\rangle$, $X, Y \in T_p\overline{M}$.

(c) The mean curvature H of $M \subset \overline{M}$ is given by

$$nH = -\operatorname{div}\left(\frac{\operatorname{grad} f}{|\operatorname{grad} f|}\right).$$

(d) Observe that every embedded hypersurface $M^n \subset \overline{M}^{n+1}$ is locally the inverse image of a regular value. Conclude from (c) that the mean curvature H of such a hypersuperface is given by

$$H = -\frac{1}{n}\operatorname{div} H,$$

where N is an appropriate local extension of the unit normal vector field on $M^n \subset \overline{M}^{n+1}$.

23. ([Car92] Ch.8 #8) (Riemannian submersions) A differentiable mapping $f : \overline{M}^{n+k} \to M^n$ is called a *submersion* if f is surjective, and for all $\bar{p} \in \overline{M}$, $df_{\bar{p}} : T_{\bar{p}}\overline{M} \to T_{f(\bar{p})}M$ has rank n. In this case, for all $p \in M$, the *fiber* $f^{-1}(p) = F_p$ is a submanifold of $\overline{M}$ and a tangent vector of $\overline{M}$, tangent to some F_p, $p \in M$, is called a *vertical vector* of the submersion. If, in addition, $\overline{M}$ and M have Riemannian metrics, the submersion f is said to be *Riemannian* if, for all $p \in \overline{M}$, $df_p : T_p\overline{M} \to T_{f(p)}M$ preserves lengths of vectors orthogonal to F_p. Show that:
   (a) If $M_1 \times M_2$ is the Riemannian product, then the natural projections $\pi_i : M_1 \times M_2 \to M_i$, $i = 1, 2$ are Riemannian submersions.
   (b) Let the tangent bundle TM be given the Riemannian metric as:

$$\langle V, W\rangle_{(p,v)} = \langle d\pi(V), d\pi(W)\rangle_p + \langle \frac{Dv}{dt}(0), \frac{Dw}{ds}(0)\rangle_p$$

   for $(p, v) \in TM$, V, W tangent vectors at (p, v) in TM where $V = \alpha'(0)$, $W = \beta'(0)$ for curves α, β chosen such that $\alpha(t) = (p(t), v(t))$, $\beta(t) = (q(s), w(s))$, $p(0) = q(0) = 0$, $v(0) = w(0) = v$ (cf. [Car92] Ch.3 #2). Show that the projection $\pi : TM \to M$ is a Riemannian submersion.

24. ([Car92] Ch.8 #9) (Conneciton of a Riemannian submersion) Let $f : \overline{M} \to M$ be a Riemannian submersion. A vector $\bar{x} \in T_{\bar{p}}\overline{M}$ is *horizontal* if it is orthogonal to the fiber. The tangent space $T_{\bar{p}}\overline{M}$ then admits a decomposition $T_{\bar{p}}\overline{M} = (T_{\bar{p}}\overline{M})^h \oplus (T_{\bar{p}}\overline{M})^v$, where $(T_{\bar{p}}\overline{M})^h$ and $(T_{\bar{p}}\overline{M})^v$ denote the subspaces of horizontal and vertical vectors, respectively. If $X \in \mathcal{X}(M)$, the *horizontal lift* $\overline{X}$ of X is the horizontal field defined by $df_{\bar{p}}(\overline{X}(\bar{p})) = X(f(p))$.
   (a) Show that $\overline{X}$ is differentiable.
   (b) Let ∇ and $\overline{\nabla}$ be the Riemannian connections of M and $\overline{M}$ respectively. Show that

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla_X Y} + \frac{1}{2}[\overline{X}, \overline{Y}]^v, \quad X, Y \in \mathcal{X}(M),$$

where Z^v is the vertical component of Z.

(c) $[\overline{X}, \overline{Y}]^v(\bar{p})$ depends only on $\overline{X}(\bar{p})$ and $\overline{Y}(\bar{p})$.

25. ([Car92] Ch.8 #10) (Curvature of a Riemannian submersion) Let $f : \overline{M} \to M$ be a Riemannian submersion. Let $X, Y, Z, W \in \mathcal{X}(M)$, $\overline{X}, \overline{Y}, \overline{Z}, \overline{W}$ be their horizontal lifts, and let R and $\overline{R}$ be the curvature tensors of M and $\overline{M}$ respectively. Prove that:

    (a)
$$\left\langle \overline{R}(\overline{X}, \overline{Y})\overline{Z}, \overline{W} \right\rangle = \langle R(X,Y)Z, W \rangle - \frac{1}{4}\left\langle [\overline{X}, \overline{Z}]^v, [\overline{Y}, \overline{W}]^v \right\rangle + \frac{1}{4}\left\langle [\overline{Y}, \overline{Z}]^v, [\overline{X}, \overline{W}]^v \right\rangle - \frac{1}{2}\left\langle [\overline{Z}, \overline{W}]^v, [\overline{X}, \overline{Y}]^v \right\rangle$$

    (b) $K(\sigma) = \overline{K}(\overline{\sigma}) + \frac{3}{4}\left|[\overline{X}, \overline{Y}]^v\right|^2 \geq \overline{K}(\overline{\sigma})$, where σ is the plane generated by the orthonormal vectors $X, Y \in \mathcal{X}(M)$ and $\overline{\sigma}$ is the plane generated by $\overline{X}, \overline{Y}$.

26. ([Car92] Ch.8 #11) (The complex projective space) Let
$$\mathbb{C}^{n+1} \setminus \{0\} = \{(z_0, \ldots, z_n) = Z \neq 0 \mid z_j = x_j + iy_j, j = 0, \ldots, n\}$$
be the set of all non-zero $(n+1)$-tuples of complex numbers z_j. Define equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$: $(z_0, \ldots, z_n) \sim W = (w_0, \ldots, w_n)$ if $z_j = \lambda w_j$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$. The equivalence class of Z will be denoted by $[Z]$ (the complex line passing through the origin and through Z). The set of such classes is called, by analogy with the real case, *the complex projective space* $\mathbb{P}^n(\mathbb{C})$ of complex dimension n.

    (a) Show that $\mathbb{P}^n(\mathbb{C})$ has a differentiable structure of a manifold of real dimension $2n$ and that $\mathbb{P}^1(\mathbb{C})$ is diffeomorphic to $\mathbb{S}^2$.

    (b) Let $(Z, W) = z_0\overline{w_0} + \cdots + z_n\overline{w_n}$ be the hermitian product on $\mathbb{C}^{n+1}$, where the bar denotes complex conjugation. Identify $\mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2}$ by putting $z_j = x_j + iy_j = (x_j, y_j)$. Show that
$$\mathbb{S}^{2n+1} = \{N \in \mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2} \mid (N, N) = 1\}$$
is the unit sphere in $\mathbb{R}^{2n+2}$.

    (c) Show that the equivalence relation $\sim$ induces on $\mathbb{S}^{2n+1}$ the following equivalence relation: $Z \sim W$ if $e^{i\theta}Z = W$. Establish that there exists a differentiable map (the Hopf fibering) $f : \mathbb{S}^{2n+1} \to \mathbb{P}^n(\mathbb{C})$

such that

$$\begin{aligned} & f^{-1}([Z]) \\ & \qquad =\{e^{i\theta}N \in \mathbb{S}^{2n+1} \mid N \in [Z] \cap \mathbb{S}^{2n+1}, 0 \le \theta \le 2\pi\} \\ & \qquad =[Z] \cap \mathbb{S}^{2n+1}. \end{aligned}$$

(d) Show that f is a submersion.

27. ([Car92] Ch.8 #12) (Curvature of the complex projective space) Define a Riemannian metric on $\mathbb{C}^{n+1} \setminus \{0\}$ in the following way: If $Z \in \mathbb{C}^{n+1} \setminus \{0\}$ and $V, W \in T_Z(\mathbb{C}^{n+1} \setminus \{0\})$,

$$\langle V, W \rangle_Z = \frac{\mathrm{Real}(V, W)}{(Z, Z)}.$$

Observe that the metric $\langle\ ,\ \rangle$ restricted to $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1} \setminus \{0\}$ coincides with the metric induced from $\mathbb{R}^{2n+2}$.

(a) Show that, for all $0 \le \theta \le 2\pi$, $e^{i\theta} : \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}$ is an isometry, and that, therefore, it is possible to define a Riemannian metric on $\mathbb{P}^n(\mathbb{C})$ in such a way that the submersion f is Riemannian.

(b) Show that, in this metric, the sectional curvature of $\mathbb{P}^n(\mathbb{C})$ is given by

$$K(\sigma) = 1 + 3\cos^2\varphi,$$

where σ is generated by the orthonormal pair X, Y, $\cos\varphi = \langle \overline{X}, i\overline{Y} \rangle$, and $\overline{X}, \overline{Y}$ are the horizontal lifts of X and Y, respectively. In particular, $1 \le K(\sigma) \le 4$.