Chapter 2

Tensors and Differential Forms

We have seen that a vector field X on a manifold M can be defined without referring to particular choices of coordinates. Nevertheless, $X = \sum_i a^i(\partial/\partial x^i)$ under local coordinates $(x^1, \ldots, x^n)$ and these local expressions transform properly under the change of coordinates. In this chapter, we investigate natural generalizations of this idea: *tensors, or tensor fields*.

Given a chart $(U, \mathbf{x})$ at $p \in M$, the basis of T_pM and T_p^*M are given by

$$T_pM = \left\langle \frac{\partial}{\partial x_i}\Big|_p \right\rangle; \quad T_p^*M = \mathrm{Hom}(T_pM, \mathbb{R}) = \left\langle dx_i\big|_p \right\rangle.$$

Tensors are "algebraic tensor products" of them, namely sections of $T^{r,s}M = \bigotimes^r TM \otimes \bigotimes^s T^*M$, and thus their local expressions between different coordinates also transform properly. This later property was used to define tensors in old days. We will however follow the modern definitions.

We develop some facts from multilinear algebra and extend these concepts over manifolds to define tensors. Specifically we study Lie derivative of tensors. Alternating tensors, or differential forms, are of special importance. We study Cartan's theory on exterior differentiations as well as proving the de Rham theorem which is fundamental in linking the differentiable structure to the global algebraic topology of a manifold.

1. The tensor algebra

We begin with a quick recap on the tensor product of vector spaces. There are several ways to do it. Here we take the most intuitive way. For given real vector spaces V, W, say

$$V = \langle v_1, v_2, \ldots, v_m \rangle, \qquad W = \langle w_1, w_2, \ldots, w_n \rangle,$$

we define $V \otimes W$ as $\langle v_i \otimes w_j \big| 1 \le i \le m, 1 \le j \le m \rangle$. Here $\otimes$ is a symbol subject to the condition that it is bilinear in $\mathbb{R}$. More explicitly, this means that for $v_1, v_2, v \in V$, $w_1, w_2, w \in W$,

$$v \otimes (w_1 + w_2) = v_1 \otimes w_1 + v \otimes w_2;$$
$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w,$$

and for all $a \in \mathbb{R}$,

$$a(v \otimes w) = (av) \otimes w = v \otimes (aw).$$

It is clear that $V \otimes W$ has dimension mn and the definition is independent of the choices of basis up to the unique isomorphism.

As a consequence, $V \otimes W$ is canonically isomorphic to $W \otimes V$ and $(V \otimes W) \otimes U$ is canonically isomorphic to $V \otimes (W \otimes U)$. Therefore, given finitely many vector spaces $V_1 \ldots, V_n$, we can define unambiguously the tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ up to an obvious isomorphism.

In particular, the tensor product of $V^{\otimes r}$, the r-fold tensor product of V, with $V^{*\otimes s}$, the s-fold tensor product of V^*,

$$\bigotimes^{r,s} V := V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$$

is called the *tensor space of type* (r, s) on V and usually denoted as $T^{r,s}(V)$. The first r components are called the *contravariant* part and the last s ones are called the *covariant* part. The element in $T^{r,s}(V)$ is called a *tensor of type* (r, s) or simply an (r, s)-tensor. The *tensor algebra* $T(V)$ of V is the direct sum of tensor spaces of all types:

$$T(V) := \bigoplus_{r,s \geq 0} T^{r,s}(V).$$

The tensor algebra carries a (non-commutative) multiplicative structure given by tensor product $\otimes$ between tensors.

Notice that we can recognize a vector space V with its double dual $V \cong V^{**}$ canonically, thus we may also identify the (r, s)-tensors with the spaces of multilinear maps in various ways: e.g.

$$T^{r,s}(V) = V^{\otimes r} \otimes V^{*\otimes s} \cong \mathrm{Hom}(V^{\otimes s}; V^{\otimes r}) \cong L(\prod^{r} V^*, \prod^{s} V; \mathbb{R}).$$

Example 2.1. Here are some examples of tensors.

(1) $T \in \bigotimes^{1,2} V$ can be identified with $T(\cdot,\cdot) \in L(V,V;V)$ by

$$\begin{array}{ccc} V \otimes V^* \otimes V^* & & L(V,V;V) \\ T = \sum_{i,j,k} a_{ijk}\, v_i \otimes v_j^* \otimes v_k^* & \longleftrightarrow & \sum_i a_{ijk}\, v_i = T(v_i, v_k). \end{array}$$

(2) $\bigotimes^{0,2}(V) = V^* \otimes V^*$. E.g. the metric tensor on $V = T_pM$:

$$g = \sum_{i,j} g_{ij}\, dx^i \otimes dx^j.$$

(3) $\bigotimes^{1,3}(V) = V \otimes V^* \otimes V^* \otimes V^*$. E.g. the curvature tensor:

$$R = \sum_{i,j,k,l} R^i_{jkl} \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^l.$$

There are two special kinds of tensors which are particularly important in differential geometry: the symmetric tensors and the alternating tensors. A *symmetric k-tensor* $T \in \mathrm{Sym}(V^*) \subset \bigotimes^k(V^*) = T^{0,k}(V)$ is a tensor such that for all $1 \le i,j \le k.$,

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

For example, the metric tensor g is a symmetric 2-tensor. Let $V = \langle e_1, \ldots, e_n \rangle$, $v = \sum_{i=1}^n a^i e_i \in V$, and $g_{ij} := g(e_i.e_j)$, then

$$g(v,v) = \sum_{i,j=1}^n g(e_i, e_j)\, a^i a^j, \qquad g = \sum_{i,j=1}^n g_{ij}\, e_i^* \otimes e_j^*.$$

Note that $q(v) = g(v,v)$ is a quadratic form on V.

Conversely, given a quadratic form q on V, we can recover $g(v,w)$ via the *polarization formula*:

$$g(v,w) = \frac{1}{2}(q(v+w) - q(v) - q(w)).$$

In fact, symmetric tensors on a vector space correspond precisely to polynomial functions. One direction is easy: if $T \in \bigotimes^k(V^*)$ then

$$P(v) := T(v, \ldots, v) = \sum T(v_{i_1}, \ldots, v_{i_k})\, a^{i_1} \cdots a^{i_k}$$

is a degree k polynomial in a^j's. Conversely:

Exercise 2.1. Given a polynomial P of degree k, find the polarization formula of P to get $T \in \mathrm{Sym}^k(V^*)$ for general $k \in \mathbb{N}$.

On the other hand, an *alternating k-tensor* $T \in \mathrm{Alt}^k(V^*) \subset \bigotimes^k(V^*) = T^{0,k}(V)$ is a skew-symmetric k-linear form which satisfies

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

This implies that (in fact equivalent to) $T(\ldots, v, \ldots, v, \ldots) = 0$ for all $v \in V$. Moreover, suppose that $v_i = \sum_j a_i^j e_j$, then

$$\begin{aligned}
T(v_1, \ldots, v_k) &= T(\ldots, \sum_j a_i^j e_j, \ldots) \\
&= \sum_{p \in P(n,k)} a_1^{p(1)} a_2^{p(2)} \cdots a_k^{p(k)} T(e_{p(1)}, \ldots, e_{p(k)}) \\
&= \sum_{c \in C(n,k)} \left(\sum_{\sigma \in S_k(c)} a_1^{\sigma(c(1))} \cdots a_k^{\sigma(c(k))} T(e_{\sigma(c(1))}, \ldots, e_{\sigma(c(k))}) \right) \\
&= \sum_{c \in C(n,k)} T(e_{c(1)}, \ldots, e_{c(k)}) \det V_c.
\end{aligned}$$

Here we assume that $c(1) < c(2) < \cdots < c(k)$ and V_c is the $k \times k$ submatrix of the $n \times k$ matrix

$$V = [v_1\ v_2\ \cdots\ v_k] = \begin{bmatrix} a_1^1 & \cdots & a_k^1 \\ \vdots & \ddots & \vdots \\ a_1^n & \cdots & a_k^n \end{bmatrix}$$

with k rows selected by c. In particular, $\dim \mathrm{Alt}^k(V) = C_k^n$.

A basis of alternating k-tensors $\mathrm{Alt}^k(V^*)$ can be constructed by *anti-symmetrization* of the k-tensors $e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*$: for $\phi_i \in V^*$,

$$\phi_1 \wedge \cdots \wedge \phi_k := \sum_{\sigma \in S_k} (-1)^{\operatorname{sign} \sigma} \phi_{\sigma(1)} \otimes \cdots \otimes \phi_{\sigma(k)}.$$

Then it is clear that $\det V_c = (e_{c(1)}^* \wedge \cdots \wedge e_{c(k)}^*)(v_1, \ldots, v_k)$.

In summary, for $V = \langle e_1, \cdots, e_n \rangle$ and $V^* = \langle e_1^*, \cdots, e_n^* \rangle$, a general tensor $T \in T^{r,s}(V)$ can be decomposed into a linear combination:

$$\begin{aligned}
T &= \sum_{I,J} T^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} (e_{i_1} \otimes \cdots \otimes e_{i_r}) \otimes (e_{j_1}^* \otimes \cdots \otimes e_{j_s}^*) \\
&= \sum_{I,J} T_J^I \, e_I \otimes e^J,
\end{aligned}$$

where $T^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = T(e_{i_1}^*, \ldots, e_{i_r}^*, e_{j_1}, \ldots, e_{j_s})$.

If $T \in \mathrm{Alt}^k(V^*)$ is alternating, then we have

$$T = \sum_{i_1 < \cdots < i_k} T(e_{i_1}, \ldots, e_{i_k})\, e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}.$$

It is customary to use $e^i \equiv e^*_i$ to denote dual basis elements. The art of using upper/lower indices is crucial in working with tensors.

2. Tensor fields on manifolds

We can generalize readily everything above to the case of manifolds with $V = T_pM$, $V^* = T^*_pM$ and with p varies in M. More precisely, the (r,s) tensor bundle on M is simply

$$T^{r,s}(M) := \bigotimes^r TM \otimes \bigotimes^s T^*M \xrightarrow{\pi} M.$$

It has a natural differentiable structure as in the case of tangent bundle. A tensor (field) on M is defined to be a C^∞ section of π.

On a local chart $(U, \mathbf{x})$, the tangent bundle $TU = TM|_U$ has frame $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ and the cotangent bundle $T^*U = T^M|_U$ has a dual-frame $dx^1, \ldots, dx^n$. Then an alternative definition is:

Definition 2.2. An (r,s)-*tensor field* $T \in T^{r,s}(U)$ is C^∞ if all the coefficients $T^I_J(\mathbf{x}) \in C^\infty(U)$, where

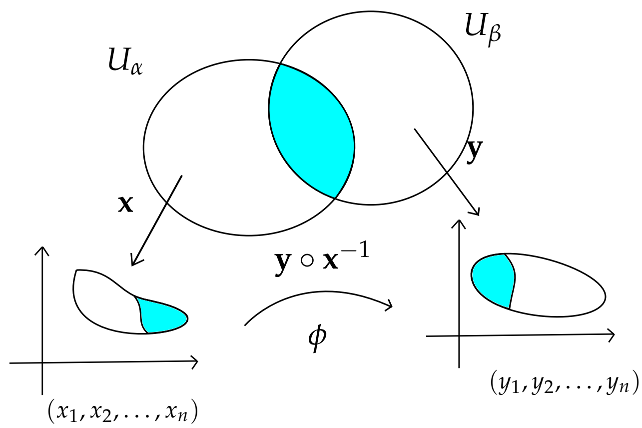
$$T = \sum_{i,j} T^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}(\mathbf{x}) \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}.$$

Exercise 2.2. Determine the C^∞ structure on $T^{(r,s)}(M)$ and show that the above two definitions of C^∞ tensor fields coincide.

Now, we investigate the meaning that "tensors are independent of choices of coordinate systems".

Let $(U_\alpha, \mathbf{x})$, $(U_\beta, \mathbf{y})$ be two charts and $\phi_{\alpha\beta} = \mathbf{y} \circ \mathbf{x}^{-1}$ be the coordinate transformation. Since $y^i = \phi^i_{\alpha\beta}(x^1, x^2, \ldots, x^n)$ has Jacobian matrix $\phi'_{\alpha\beta} = \left[\frac{\partial y^i}{\partial x^j}\right]$, we see that

$$\frac{\partial}{\partial y^i} = \sum_j \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}, \qquad dy^i = \sum_j \frac{\partial y^i}{\partial x^j} dx^j.$$



So the tangent vectors (the contravariant part) vary with the inverse transformation and the dual vectors (the covariant part) vary with the transformation. For an (r,s) tensor field T under the coordinate transition $\phi_{\alpha\beta}$, we have thus

$$\begin{aligned} T &= \sum_{I,J} T^{i_1 \cdots i_r}_{j_1 \cdots j_s}(\mathbf{x}) \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} \\ &= \sum_{I',J'} \widetilde{T}^{I'}_{J'}(\mathbf{y}) \frac{\partial}{\partial y^{i'_1}} \otimes \cdots \otimes \frac{\partial}{\partial y^{i'_r}} \otimes dy^{j'_1} \otimes \cdots \otimes dy^{j'_s} \end{aligned}$$

where

$$\widetilde{T}^{I'}_{J'}(\mathbf{y}) = \sum_{I,J} T^{i_1 \cdots i_r}_{j_1 \cdots j_s}(\mathbf{x}) \frac{\partial y^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{i'_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial y^{j'_1}} \cdots \frac{\partial x^{j_s}}{\partial y^{j'_s}}.$$

Remark 2.3. When the notion of tensors first appeared in the 19th century, it was understood as a collection of local data $(U, \mathbf{x}, \{T^I_J(\mathbf{x})\})$ which obeys the above transform rule.

Example 2.4. Algebraic operations on vector spaces extends over tensor bundles in a straightforward manner. E.g. the tensor bundles of symmetric k forms and alternating k forms are

$$S^k(M) = \mathrm{Sym}^k\, T^*M \subset T^{0,k}(M), \qquad \Lambda^k(M) = \mathrm{Alt}^k\, T^*M \subset T^{0,k}(M).$$

We will also study more general tensor bundles like

$$\mathrm{End}\, TM \otimes \Lambda^2(M) \subset T^{1,3}(M).$$

Given a smooth map $f : M \to N$ between smooth manifolds, in general the tensor fields between them do not transform in a simple manner. One exceptional case with simple transformation rule is as follows. The tangent map $f_* : T_pM \to T_{f(p)}N$ induces a *pull–back map* $f^* : T^*_{f(p)}N \to T^*_pM$ via duality:

$$f^*(w)(v) = w(f_*(v)), \quad \text{for } w \in T^*_{f(p)}N, v \in T_pM.$$

Hence, we can *pullback covariant tensor fields* $T \in T^{0,s}(M)$ by

$$(f^*T)(v_1, \ldots, v_s) = T(f_*v_1, \ldots, f_*v_s).$$

In local coordinates on N,

$$T = \sum_{j_1,\ldots,j_s} h_{j_1\cdots j_s}\, dy^{j_1} \otimes \cdots \otimes dy^{j_s}.$$

The pullback of T via f is the covariant tensor field on M given by

$$f^*T = \sum_{j_1,\ldots,j_s} (h_{j_1\cdots j_s} \circ f)\, d(y^{j_1} \circ f) \otimes \cdots \otimes d(y^{j_s} \circ f).$$

For alternating tensor fields this can be further simplified through determinants. This will be discussed in the next section.

Pullback of general tensors or pushforward of tensors are generally not possible beyond (local) diffeomorphisms.

3. Differential forms and Cartan's d operator

Definition 2.5. A differential p-form w on $U \subset M$ is a an alternating tensor field $w \in A^p(U) := C^\infty(U, \Lambda^p(M))$.

In a chart $(U, \mathbf{x})$, we have a unique presentation

$$\omega = \sum_{i_1 < \ldots < i_p} \omega_{i_1,\ldots,i_p}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}.$$

It is intuitively clear that we may define wedge products of two alternating tensors $\omega \in A^p$ and $\eta \in A^q$ by declaring

$$(e^{i_1} \wedge \cdots \wedge e^{i_p}) \wedge (e^{j_1} \wedge \cdots \wedge e^{j_q}) = e^{i_1} \wedge \cdots \wedge e^{i_p} \wedge e^{j_1} \wedge \cdots \wedge e^{j_q}$$

on the basis elements (which vanishes if there are overlapped indices) and then extend by linearity. In practice, we need to verify that the definition is independent of the choices of basis e_i's.

Exercise 2.3. Show that the above definition of wedge product is independent of the choices of basis.

As a consequence, we see that the wedge product is associative and $\mathbb{Z}_2$-graded in the sense that

$$a \in A^p, \quad b \in A^q \Longrightarrow a \wedge b = (-1)^{pq}\, b \wedge a.$$

Remark 2.6. Following Hermann Grassmann, we give another way to view alternating tensors which gives a natural algebra structure on them, namely the *exterior algebra*.

Consider the two-sided ideal $I(V)$ generated by all elements with repeated entries inside the tensor algebra $T(V)$:

$$I(V) := \langle a \otimes \cdots \otimes a \mid a \in V \rangle \subset \bigotimes V = \bigoplus_{r=0}^{\infty} \left(\bigotimes^r V \right).$$

The exterior algebra is defined to be the quotient algebra

$$\Lambda(V) := \bigotimes V / I(V).$$

The natural homomorphism

$$\begin{array}{ccc} \bigotimes V & \to & \Lambda(V) \\ \alpha & \mapsto & \overline{\alpha} \end{array}$$

induces the multiplication "$\wedge = \overline{\otimes}$" on $\Lambda(V)$ which is associative:

$$(\overline{a} \wedge \overline{b}) \wedge \overline{c} = \overline{a \otimes b} \wedge \overline{c} = \overline{a \otimes (b \otimes c)} = \overline{a} \wedge (\overline{b} \wedge \overline{c}).$$

We call $\wedge$ the *exterior product* or wedge product on $\Lambda(V)$. Often we omit the "bar" notation when no confusion might arise.

We see readily that $\Lambda(V) = \bigoplus_{p=0}^{n} \Lambda^p(V)$ where

$$\Lambda^p(V) = \mathbb{R}\langle e_{i_1} \wedge \cdots \wedge e_{i_p} \mid i_1 < \cdots < i_p \rangle \cong \mathbb{R}^{C_p^n}.$$

Moreover, $\Lambda(V)$ is a $\mathbb{Z}_2$-graded algebra.

Exercise 2.4. Construct a vector space isomorphism $\mathrm{Alt}(V) \to \Lambda(V)$ so that the above two definitions of wedge product agree.

So far our discussion works for a given vector space V and does not require the manifold structure. Over a manifold M, there is surprising d operator acting on differential forms.

For any $f \in C^\infty(M)$, we can view the differential df as a smooth mapping from TM into $\mathbb{R}$. So df can be considered as a 1-form, the *exterior derivative* of a 0-form f.

Another simple example comes from Green's theorem: for (P, Q) a C^1 vector field and $D \subset \mathbb{R}^2$ a domain with smooth boundary ∂D,

$$\int_{\partial D} Pdx + Qdy = \int_D (Q_x - P_y) dxdy.$$

The quantity (2-from) under integration comes from the 1-form $\omega = Pdx + Qdy$ via

$$\begin{aligned} d\omega &= dP \wedge dx + dQ \wedge dy \\ &= (P_x dx + P_y dy) \wedge dx + (Q_x dx + Q_y dy) \wedge dy \\ &= (Q_x - P_y)\, dx \wedge dy, \end{aligned}$$

where the anti-symmetry $dy \wedge dx = -dx \wedge dy$ is used.

We generalize the above two examples to define the *exterior derivative* of any smooth p-form which is characterize by the following theorem due to E. Cartan.

Theorem 2.7 (Cartan's exterior derivative, d)**.** *There exists a unique linear differential operator $d : A^p(M) \to A^{p+1}(M)$ such that*

(1) $df(X) = Xf$, for any $f \in C^\infty(M)$.
(2) $d^2 = 0$.
(3) (Leibniz rule) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$.

Remark 2.8. Actually, Cartan wrote down an explicit formula for d, see Theorem 2.13.

Proof of Cartian's characterization on d. We show the existence and uniqueness of d using local coordinate.

Choose a local chart $(U, \mathbf{x})$ near p. Let

$$\omega = f_I dx^I = f_{i_1, i_2, \ldots, i_p} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}$$

be a typical term of a p-form. By Leibniz rule and $d^2 = 0$, we have $d\omega = df_I \wedge dx^I$, which is uniquely determined. We call it $d_{\mathbf{x}}\,\omega$.

Next we check the consistency in another coordinate chart $(V, \mathbf{y})$ near p. In the $\mathbf{y}$ coordinates, we have

$$\omega = \sum f_{i_1,\ldots,i_p} \frac{\partial x^{i_1}}{\partial y^{\alpha_1}} \cdots \frac{\partial x^{i_p}}{\partial y^{\alpha_p}} \, dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_p} =: \sum_A \tilde{f}_A dy^A.$$

Take the exterior derivative in the $\mathbf{y}$ coordinates we get

$$\begin{aligned} d_{\mathbf{y}}\,\omega = \sum_A d\tilde{f}_A \wedge dy^A &= \sum_A d\left(f_{i_1,\ldots,i_p} \frac{\partial x^{i_1}}{\partial y^{\alpha_1}} \cdots \frac{\partial x^{i_p}}{\partial y^{\alpha_p}} \right) \wedge dy^A \\ &= (df_{i_1,\ldots,i_p}) \sum_A \frac{\partial x^{i_1}}{\partial y^{\alpha_1}} \cdots \frac{\partial x^{i_p}}{\partial y^{\alpha_p}} \wedge dy^A \\ &\qquad + f_{i_1,\ldots,i_p} d\left(\frac{\partial x^{i_1}}{\partial y^{\alpha_1}} \cdots \frac{\partial x^{i_p}}{\partial y^{\alpha_p}} \right) \wedge dy^A \\ &= \; df_I \wedge dx^I + f_I \, d(dx^I) \\ &= df_I \wedge dx^I, \end{aligned}$$

which is precisely $d_{\mathbf{x}}\,\omega$. This proves the theorem. ☐

Recall for any C^∞ map $f : M \to N$, we have the pull-back map $f^* : A^*(N) \longrightarrow A^*(M)$ on tensors induced from the tangent map $df = f_*$: for $\omega \in A^p(N)$, $v_1, \ldots, v_p \in C^\infty(TM)$, we have

$$f^*\omega(v_1, v_2, \ldots, v_p) = \omega(f_* v_1, f_* v_2, \ldots, f_* v_p).$$

Exercise 2.5. Show that f^* is a ring homomorphism with respect to the wedge product: for $\omega, \eta \in A^*(M)$,

$$f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta).$$

The operator d indeed commutes with the pull-back map:

Proposition 2.9. *For any smooth map $f : M \to N$ and $\omega \in A^*(N)$,*

$$df^*\omega = f^* d\omega.$$

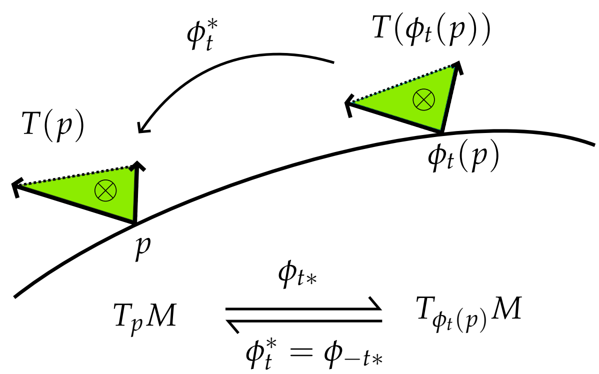
Proof. It suffices to check for a basis element $\omega = g\,dx^I$. Then

$$\begin{aligned} df^*(g\,dx^I) = d((g \circ f) \textstyle\bigwedge_{i \in I} d(x^i \circ f)) &= d(g \circ f) \wedge \textstyle\bigwedge_{i \in I} d(x^i \circ f) \\ &= f^*(dg \wedge dx^I) = f^*\big(d(g\,dx^I)\big), \end{aligned}$$

where the above exercise is used to take out f^*. ☐

4. Lie derivatives on tensors and differential forms

By extending the Leibniz rule, we can apply derivatives on usual tensors. For example, along the flow generated by a vector field, we can take the Lie derivative of a tensor: let $T \in T^{r,s}(M)$ and $X \in C^\infty(TM)$. Let X generate the flow ϕ_t.



$$
\begin{aligned}
L_X T(p) &:= \lim_{t\to 0} \frac{1}{t}\left(\phi_t^* T(\phi_t(p)) - T(p)\right) \\
&= \frac{d}{dt}\Big|_{t=0} \phi_t^* T(\phi_t(p)).
\end{aligned}
$$

(Compare with the formula for Lie derivatives of vector fields in Definition 1.33)

Exercise 2.6. Check the Leibniz rule for Lie derivatives of tensors:

$$L_X(T\otimes S) = L_X T \otimes S + T \otimes L_X S.$$

By Leibniz rule, for any tensor $T \in T^{r,s}$, say

$$
\begin{aligned}
T &= \sum T^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}(\mathbf{x}) \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} \\
&= \sum T^I_J \frac{\partial}{\partial x^I} \otimes dx^J,
\end{aligned}
$$

we can compute the Lie derivative of T along X:

$$L_X T = \sum (L_X T^I_J) \frac{\partial}{\partial x^I} \otimes dx^J + T^I_J (L_X \frac{\partial}{\partial x^I}) \otimes dx^J + T^I_J \frac{\partial}{\partial x^I} \otimes (L_X dx^J).$$

Exercise 2.7. Considering the duality pairing of 1-forms and vectors $(\omega, Y) := \omega(Y)$. Show that $L_X(\omega, Y) = (L_X\omega, Y) + (\omega, L_X Y)$.

There is an intrinsic way to write down the Lie derivatives of differential forms. In stating it, we need some preparation.

Lemma 2.10. *L_X commutes with d: $L_X d = dL_X$.*

PROOF. Let ϕ_t be the flow generated by X. Then

$$L_X d\omega = \frac{d}{dt}\Big|_{t=0} \phi_t^*(d\omega) = d\left(\frac{d}{dt}\Big|_{t=0} \phi_t^*(\omega)\right) = dL_X\omega,$$

where Proposition 2.9 is used. □

Definition 2.11. Given $X \in T_pM$, the *interior product* $\iota_X : \Lambda^p(T^*M) \to \Lambda^{p-1}(T^*M)$ is the linear map defined by

$$(\iota_X\omega)(v_1, \dots, v_{p-1}) = \omega(X, v_1, \dots, v_{p-1}).$$

Exercise 2.8. Show that ι_X satisfies the "Leibniz rule":

$$\iota_X(\omega \wedge \eta) = \iota_X(\omega) \wedge \eta + (-1)^{\deg \omega} \omega \wedge \iota_X(\eta).$$

Theorem 2.12 (Cartan's homotopy formula)**.**

$$L_X = \iota_X d + d\iota_X.$$

PROOF. The key point is that, by the Leibniz rule of L_X, ι and d respectively, it suffices to prove the case $\omega \in A^1(M)$. We may further assume that $\omega = fdh$. With these preparations, the proof of the theorem can be carried out by straightforward calculations.

$$\begin{aligned} L_X(fdh) &= (L_X f)dh + f(L_X dh) \\ &= X(f)dh + fd(L_X h) \\ &= df(X)dh + fd(dh(X)) \\ &= df(X)dh + d(fdh(X)) - dfdh(X). \end{aligned}$$

On the other hand,

$$\begin{aligned} (d\iota_X + \iota_X d)fdh &= d(fdh(X)) + \iota_X df \wedge dh \\ &= d(fdh(X)) + df(X)dh - dfdh(X), \end{aligned}$$

which coincides with $L_X(fdh)$ as expected. □

5. Tensorial criterion and the intrinsic formula of d

Theorem 2.13 (Cartan's intrinsic formula)**.** *Let $\omega \in A^p(M)$, $x \in M$ and $v_0, \dots, v_p \in T_xM$. Then*

$$\begin{aligned} d\omega(v_0, v_1, \dots, v_p) \\ &= \sum_{i=0}^{p} (-1)^i v_i(\omega(\tilde{v}_0, \dots, \widehat{\tilde{v}_i}, \dots, \tilde{v}_p)) \\ &\quad + \sum_{0 \le i < j \le p} (-1)^{i+j} w([\tilde{v}_i, \tilde{v}_j], \tilde{v}_0, \dots, \widehat{\tilde{v}_i}, \dots, \widehat{\tilde{v}_j}, \dots, \tilde{v}_p) \end{aligned}$$

where $\tilde{v}_0, \dots, \tilde{v}_p$ are any vector fields near x extending $v_0, \dots, v_p$.

Notice that $d\omega$ is a tensor which acts on T_xM. However, in order for the RHS to make sense we must extend v_i to $\tilde{v}_i$ locally.

Before proving Cartan's formula, we introduce an important criterion for the tensorial property for a given quantity.

Lemma 2.14 (Fundamental Theorem of Tensor Calculus)**.** *Let*

$$F : C^\infty(TM) \times \cdots \times C^\infty(TM) \to C^\infty(TM)$$

be a $\mathbb{R}$-multilinear map on vector fields. Then F comes from a tensor (i.e. F is pointwisely defined) if and only if F is function-linear in each variable.

Here F is function-linear in the i-th variable means that for any $f \in C^\infty(M)$,

$$F(\dots, f\tilde{v}_i, \dots) = fF(\dots, \tilde{v}_i, \dots).$$

Exercise 2.9. Prove the fundamental theorem of tensor calculus.

Proof of Cartan's intrinsic formula.

Clearly, $d\omega$ in the LHS is a tensor. Denote the RHS as $R(\tilde{v}_0, \dots, \tilde{v}_p)$. We claim that R is functional-linear. Let $g \in C^\infty(M)$. By relabeling it suffices to check that $R(g\tilde{v}_0, \dots, \tilde{v}_p) = gR(\tilde{v}_0, \dots, \tilde{v}_p)$. We compute

$$\begin{aligned} R(g\tilde{v}_0, \dots, \tilde{v}_p) = gR(\tilde{v}_0, \dots, \tilde{v}_p) + \sum_{i=1}^{p} (-1)^i v_i(g) \omega(\tilde{v}_0, \dots, \widehat{\tilde{v}_i}, \dots, \tilde{v}_n) \\ - \sum_{j=1}^{p} (-1)^j \omega((\tilde{v}_j g)\tilde{v}_0, \dots, \widehat{\tilde{v}_j}, \dots, \tilde{v}_p). \end{aligned}$$

Since $\tilde{v}_j g$ is a scalar function and ω is tensorial, the last two terms cancel out with each other. Hence R is a tensor by Lemma 2.14.

Since both LHS and RHS are tensorial, to evaluate them it does not matter how we choose the vector field $\tilde{v}_i$ to extend v_i. Also we may check the formula in any coordinate system.

Consider a local coordinate $(U, x^1, \ldots, x^n)$ at $x \in M$ such that $v_i = \frac{\partial}{\partial x^{n_i}}|_x$ for $i \in [0, p]$. Then we define

$$\tilde{v}_i = \partial_{n_i} \equiv \frac{\partial}{\partial x^{n_i}}, \qquad 0 \le i \le p.$$

In particular $[\tilde{v}_i, \tilde{v}_j] = 0$ on U for all i, j.

Let $\omega \in A^p(M)$. It suffices to consider the typical case $\omega = f dx^1 \wedge dx^2 \wedge \cdots \wedge dx^p$ (after reordering).

With the above choices, we see readily that both sides are zero if $v_i = v_j$ for some $i \neq j$. Indeed, the only two possibly non-vanishing terms in the first sum differ in sign only.

Now we compare both sides by direct calculations:

$$d\omega(v_0, \ldots, v_p) = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} dx^j \wedge dx^1 \wedge \cdots \wedge dx^p(v_0, \ldots, v_p).$$

If $n_i = j$, we get a a term

$$(-1)^i (v_i f)\, dx^1 \wedge \ldots \wedge dx^p(v_0, \ldots, \widehat{v}_i, \ldots, v_p)$$

which is precisely $(-1)^i v_i(\omega(\tilde{v}_0, \ldots, \widehat{\tilde{v}}_i, \ldots, \tilde{v}_p))$ in the RHS. $\square$

One of the major applications of Cartan's intrinsic formula is the reformulation of Frobenius integrability theorem in terms of the so-called "differential ideals".

Definition 2.15. An ideal $\mathscr{I} \subset A(M)$ under $(+, \wedge)$ is called a *differential ideal* if it is closed under exterior differentiation, i.e. $d\mathscr{I} \subset \mathscr{I}$.

Let E be a p-dimensional distribution. There is a subspace of differential forms consisting of elements vanish on E:

$$E \rightsquigarrow \mathscr{I}_E := \{\omega \in A(M) : \omega\big|_E = 0\}.$$

It is clear that $\mathscr{I}_E$ is an ideal in $A(M)$.

Proposition 2.16 (Frobenius integrability via differential ideals)**.** *A distribution E is involutive, i.e. $[E, E] \subset E$, if and only if $d\mathscr{I}_E \subset \mathscr{I}_E$.*

PROOF. Let $\omega \in \mathscr{I}_E$. If $[E, E] \subset E$, by Cartan's intrinsic formula we get $d\omega|_E = 0$ immediately. Hence $d\mathscr{I}_E \subset \mathscr{I}_E$.

To prove the converse, the key point is to show that $\mathscr{I}_E$ is locally generated by 1-forms: consider on a local chart U such that

$$E\big|_U = \{ f_1 v_1 + \cdots + f_p v_p \mid f_i \in C^\infty(U) \}.$$

That is, $v_1, \ldots, v_p$ are vector fields which are linearly independent at each $q \in U$. We complete it into a basis (frame)

$$\{v_1, \ldots, v_p, v_{p+1}, \ldots, v_n\}$$

of $T(U)$ and consider its the dual basis $\{\theta_1, \ldots, \theta_n\}$. Then

$$\begin{aligned} \mathscr{I}_E\big|_U &= A(U) \wedge \theta_{p+1} + \cdots + A(U) \wedge \theta_n \\ &= \langle \theta_{p+1}, \ldots, \theta_n \rangle. \end{aligned}$$

Now let $X, Y \in E$ and $\omega = \theta_j$ for $j \in [p+1, n]$. Then

$$\begin{aligned} d\omega(X, Y) &= X(\omega(Y)) - Y(\omega(X)) + \omega([X, Y]) \\ &= \omega([X, Y]) \end{aligned}$$

since $\theta_j|_E = 0$. Hence $d\theta_j\big|_E = 0$ for all j implies that $\theta_j([X, Y]) = 0$ for all j and hence $[X, Y] \in E$. The proof is complete. $\square$

Corollary 2.17. *Let $\mathscr{I}$ be a differential ideal $\mathscr{I}$ which is locally generated by 1-forms. Then for any $x \in M$ there exists an integral submanifold M_x containing x, in the sense that $\mathscr{I}|_{M_x} = 0$.*

Exercise 2.10. Let $M = \mathbb{R}^3$ and $\mathscr{I} = \langle \theta \rangle$. Determine if $\mathscr{I}$ is integrable in the following cases. If so, integrate it and find the integrable submanifolds.

(1) $\theta = (xy - zx^2)dx + dy$.
(2) $\theta = yzdx + xzdy - dz$.
(3) $\theta = dy - zdx$.

6. Integration on forms and Stokes's theorem

Let $\omega = f dx^1 \wedge \cdots \wedge dx^n \in A^n_c(\mathbb{R}^n)$ be a top differential form with compact support. We define the integration of the form as the usual Riemann integral:

$$\int_{\mathbb{R}^n} \omega := \int_{\mathbb{R}^n} f\, dx^1 dx^2 \cdots dx^n.$$

Let $\mathbf{y}$ be another coordinate system. Then $\omega = f J\, dy^1 \wedge \cdots \wedge dy^n$ where $J := \det\left(\frac{\partial x^i}{\partial y^j}\right)$ is the Jacobian. In view of the change of variable formula in Calculus:

$$\int_{\mathbb{R}^n} f\, dx^1 dx^2 \cdots dx^n = \int_{\mathbb{R}^n} f|J|\, dy^1 dy^2 \cdots dy^n,$$

the integral is well defined if and only if $J > 0$.

Definition 2.18 (Orientable manifold). A C^k manifold ($k \geq 1$) is *orientable* if it is possible to pick a sub-atlas such that $\det(\phi'_{UV}) > 0$ for any two coordinate system $(U, \mathbf{x})$ and $(V, \mathbf{y})$ with $U \cap V \neq \emptyset$. A choice of such an atlas is called an *orientation*.

Note that an orientable manifold always has two orientations.

Lemma 2.19. *M^n is orientable if and only if there exists $\omega \in A^n(M)$ such that $\omega \neq 0$ everywhere on M.*

PROOF. Let ω be a non-vanishing top form on M. On a local chart $(U, \mathbf{x})$, write $\omega = f dx^1 \wedge \cdots \wedge dx^n$. Since $\omega \neq 0$, $f > 0$ or $f < 0$ everywhere. If $f > 0$, we pick $(U, \mathbf{x})$. Otherwise, pick $(U, (x^2, x^1, x^3, \ldots, x^n))$. Hence by changing coordinate and use $\omega \neq 0$, we see that $\det(\phi'_{UV}) > 0$ for any two coordinate systems.

Conversely, if M is oriented by the atlas $\{(U, \mathbf{x})\}$, $\phi_U : U \to \mathbb{R}^n$. Let $\omega_U = \phi_U^*(dx^1 \wedge \cdots \wedge dx^n)$ on U and (ρ_U, U)'s be a partition of unity. Set

$$\omega = \sum_U \rho_U \omega_U.$$

Then ω is non-vanishing everywhere. $\square$

Such an ω in the lemma is called a *volume form*. Sometimes it is also referred to as a measure or a density.

Assume that M is oriented with an atlas and let $\eta \in A^n_c(M)$. We define the integration $\int_M \eta$ using a partition of unity $\{\rho_U\}$.

$$\begin{aligned}\int_M \eta &\equiv \int_M \sum_U \rho_U \eta \\ &:= \sum_U \int_U \rho_U \eta = \sum_U \int_{\mathbb{R}^n} (\phi_U^{-1})^* (\rho_U \eta).\end{aligned}$$

Since all $\det \phi'_{UV} > 0$, the integration is well-defined. Moreover:

Proposition 2.20. *The integration is independent of the choices of partition-of-unity's.*

PROOF. Given two partition-of-unity's (ρ_i, U_i) and (τ_j, V_j),

$$\sum_i \int_{U_i} \rho_i \eta = \sum_i \int_M \rho_i (\sum_j \tau_j \eta) = \sum_{i,j} \int_M \tau_j (\rho_i \eta) = \sum_j \int_M \tau_j \eta.$$

(Caution: where is the orientation used in the proof?) $\square$

Now we come to the analogous result of the fundamental theorem of Calculus $\int_a^b f'(t)\, dt = f(t)|_a^b$ in the generalized setup of differential forms. We have learned such generalizations in Calculus for lower demensions, namely the Green's theorem in $\mathbb{R}^2$ and Stokes' theorem in $\mathbb{R}^3$. All of them can be viewed as special cases ($n = 1, 2, 3$ respectively) of the following formula: for $\omega \in A^{n-1}(M)$,

$$\int_M d\omega = \int_{\partial M} \omega.$$

But what is ∂M? To make sense of it in the general setup we need the notion of manifolds with boundary.

Definition 2.21 (manifold with boundary). (1) M is a *manifold with boundary* if M is Hausdorff and of second countable, and there exists an open cover of local charts $\{(U, \mathbf{x})\}$ such that

$$\begin{aligned} &\mathbf{x} : U \to \mathbb{R}^n \\ \text{or} \quad &\mathbf{x} : U \to \mathbb{H}^n := \{(x^1, x^2, \ldots, x^n) \mid x^n \geq 0\}\end{aligned}$$

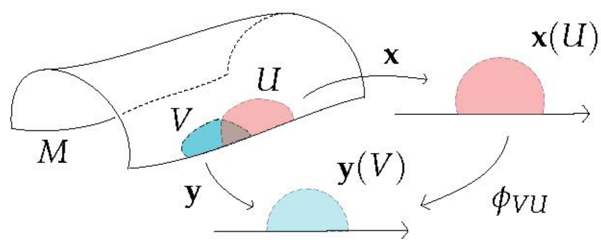
is a homeomorphism onto an open set.

(2) Moreover, M is C^k if $\mathbf{y}\circ\mathbf{x}^{-1} : \mathbf{x}(U\cap V)\to\mathbf{y}(U\cap V)$ is C^k for all local charts $(U,\mathbf{x})$ and $(V,\mathbf{y})$. [1]

(3) By restricting the coordinates charts to the boundary $\partial U := \mathbf{x}^{-1}(x^n = 0)$, the restriction of transition functions

$$\mathbf{y}\circ\mathbf{x}^{-1}\big|_{\mathbf{x}(U\cap V)\cap\partial\mathbb{H}^n},$$

gives a C^k diffeomorphism. This atlas defines a C^k manifold, denoted by ∂M, with $\dim\partial M = n-1$.



If M is oriented, we defined the induced orientation on ∂M as follows: let $\mathbf{x},\mathbf{y}$ be two coordinates near a point on the boundary.

$$\mathbf{y}\circ\mathbf{x}^{-1}(x^1,\ldots,x^{n-1},0) = (y^1,\ldots,y^{n-1},0).$$

Since M is oriented,

$$J\Big|_{\partial M} = \det\begin{vmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^{n-1}} & * \\ \vdots & & \vdots & * \\ \frac{\partial y^{n-1}}{\partial x^1} & \cdots & \frac{\partial y^{n-1}}{\partial x^{n-1}} & * \\ 0 & \cdots & 0 & \frac{\partial y^n}{\partial x^n} \end{vmatrix} > 0$$

and

$$\frac{\partial y^n}{\partial x^n}\Big|_{\partial M} > 0 \Longrightarrow \det(\mathbf{y}\circ\mathbf{x}^{-1}|_{\mathbb{H}^n}) > 0.$$

So ∂M is an oriented manifold. However, we add a parity twist to it:

[1] Notice that whenever the chart $\mathbf{x} : U\to\mathbb{H}^n$ meet the boundary, a C^k function ϕ defined on $U' = \mathbf{x}(U)$ means that $\phi = \tilde\phi|_{U'}$ where $\tilde\phi$ is a C^k function defined over some open set $\widetilde{U}\subset\mathbb{R}^n$ containing U'.

Definition 2.22. The induced orientation on ∂M is defined to be the one oriented by

$$(-1)^n dx^1 \wedge \ldots \wedge dx^{n-1}.$$

Now we can state and prove Cartan's version of Stokes' theorem:

Theorem 2.23 (Stokes' theorem)**.** *Let M be an oriented manifold with boundary ∂M, equipped with the induced orientation. Let $\omega \in A_c^{n-1}(M)$. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

PROOF. Let $\{(U_\alpha, \rho_\alpha)\}$ be a C^∞ partition of unity on M. (We leave its existence for manifolds with boundary as an exercise.)

If the theorem holds for $\mathbb{R}^n$ and $\mathbb{H}^n$, then it also holds for M:

$$\begin{aligned}\int_M d\omega &= \sum_\alpha \int_{U_\alpha} \rho_\alpha \, d\omega \\ &= \sum_\alpha \int_{U_\alpha} d(\rho_\alpha \omega) - (d\rho_\alpha)\omega \\ &= \sum_\alpha \int_{\partial U_\alpha} \rho_\alpha \omega - \int_M d(1)\omega = \int_{\partial M} \omega.\end{aligned}$$

To prove the theorem for $\mathbb{R}^n$ and $\mathbb{H}^n$, write

$$\omega = \sum_{i=1}^n f_i(x) dx^1 \wedge \cdots \widehat{dx^i} \cdots \wedge dx^n.$$

On $\mathbb{R}^n$, we apply Fubini's theorem and the case $n = 1$ to get

$$\begin{aligned}\int_{\mathbb{R}^n} d\omega &= \int_{\mathbb{R}^n} \sum_i \frac{\partial f_i}{\partial x^i} (-1)^{i-1} dx^1 \wedge \cdots \wedge dx^n \\ &= \sum_i (-1)^{i-1} \int_{\mathbb{R}^n} \frac{\partial f_i}{\partial x^i} \, dx^1 \cdots dx^n \\ \text{(Fubini)} &= \sum_i (-1)^{i-1} \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f_i}{\partial x^i} dx^i \right) dx^1 \cdots \widehat{dx^i} \cdots dx^n \\ &= \sum_i (-1)^{i-1} \int_{\mathbb{R}^{n-1}} \left(f_i \Big|_{-\infty}^{\infty} \right) dx^1 \cdots \widehat{dx^i} \cdots dx^n = 0\end{aligned}$$

since each f_i is of compact support. On $\mathbb{H}^n$, we have similarly

$$\begin{aligned}\int_{\mathbb{H}^n} d\omega &= \sum_i \int_{\mathbb{H}^n} \frac{\partial f_i}{\partial x^i}(-1)^{i-1} dx^1 \wedge \cdots \wedge dx^n \\ &= (-1)^{n-1} \int_{\mathbb{R}^{n-1}} \left(f_n \Big|_0^\infty \right) dx^1 \cdots dx^{n-1} \\ &= (-1)^n \int_{\mathbb{R}^{n-1}} f_n \, dx^1 \wedge \ldots \wedge dx^{n-1} = \int_{\partial \mathbb{H}^n} \omega,\end{aligned}$$

where the induced orientation on $\partial \mathbb{H}^n$ is used. □

Exercise 2.11. Formulate the C^∞ partition of unity for smooth manifolds with boundary and prove its existence.

For oriented M, our choice of induced orientation on ∂M leads to the sign-free form of the Stokes' theorem as proved.

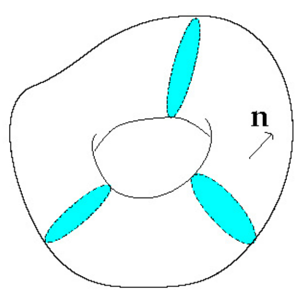
Remark 2.24. In practice, we often apply Stokes' theorem for submanifolds M^m contained in an oriented manifold N^m of the same dimension with more general, say piecewise smooth, boundaries $\partial M \subset N$.

The validity of the Stokes' formula in such cases can usually be derived from Theorem 2.23 through a limiting process.

Example 2.25. The divergence theorem in $\mathbb{R}^3$ states that

$$\int_\Omega \operatorname{div} F \, dV = \int_{\partial\Omega} F \cdot \mathbf{n} \, dA.$$

where $F = (P, Q, R)$ is a vector field in $\mathbb{R}^3$, Ω is a bounded domain with $\partial\Omega$ a smooth surface, and $\mathbf{n}$ is the outer normal along $\partial\Omega$.



It follows from Theorem 2.23: in terms of differential forms, one checks easily that

$$\int_\Omega \operatorname{div} F\, dV = \int_\Omega (P_x + Q_y + R_z)\, dx \wedge dy \wedge dz.$$
$$\int_{\partial\Omega} F \cdot \mathbf{n}\, dA = \int_{\partial\Omega} P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy.$$

Then $\omega = P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy$ does the job.

Example 2.26. The classical Stokes' theorem in $\mathbb{R}^3$ says that

$$\int_\Omega \operatorname{curl} F \cdot \mathbf{n}\, dA = \int_{\partial\Omega} F \cdot d\mathbf{r},$$

where $\Omega \subset \mathbb{R}^3$ is an oriented surface, with boundary $\partial\Omega$ being a finite collection of smooth curves. It also follows from Theorem 2.23.

Via Theorem 2.23, these classical formulae generalize to higher dimensional manifolds once a Riemannian metric is introduced (as we do in the next chapter). Nevertheless we emphasize the intrinsic nature of the Stokes' theorem for forms which does not require any metric structure.

7. De Rham cohomology

Since $d^2 = 0$, we get a complex, called the de Rham complex:

$$\cdots \longrightarrow A^{p-1}(M) \xrightarrow{d_{p-1}} A^p(M) \xrightarrow{d_p} A^{p+1}(M) \longrightarrow \cdots .$$

It is naturally to ask whether the quotient space

$$H^p_{dR}(M) := \frac{\ker d_p}{\operatorname{Im} d_{p-1}}$$

is non-trivial.

Definition 2.27. S form $\alpha \in A^p(M)$ is a *closed p-form* if $d_p\alpha = 0$ (i.e., $\alpha \in \ker d_p$). It is an *exact p-form* if there exists $\beta \in A^{p-1}(M)$ such that $\alpha = d\beta$ (i.e., $\alpha \in \operatorname{Im} d_{p-1}$).

The *p-th de Rham cohomology group (space)* $H^p_{dR}(M)$ of M is the quotient space of the real vector space of closed p-forms modulo the subspace of exact p-forms.

We also define the compactly supported de Rham cohomology by

$$H_c^p(M) := \frac{\ker d_p\big|_{A_c^p(M)}}{\operatorname{Im} d_{p-1}\big|_{A_c^{p-1}(M)}}.$$

These cohomology groups are important invariants for topological and geometric applications. Some of them will be discussed later in this book. Here we study the most basic properties of them.

Let M be connected. For $p = 0$, $f \in A^0(M)$ and $df = 0 \Rightarrow f$ is constant. So $H_{dR}^0(M) \cong \mathbb{R}$; $H_c^0(M) \cong \mathbb{R}$ if M is compact and $H_c^0(M) = 0$ if M is non-compact.

Lemma 2.28 (Poincaré Lemma)**.** *If U is contractible then $H_{dR}^p(U) = 0$ for all $p \geq 1$.*

Poincaré lemma can be regarded as a kind of *homopoty invariance*.

For $f_0, f_1 : M \to N$ being two C^∞ maps, we say that f_0 and f_1 are *smoothly homotopic*, denoted by $f_0 \sim f_1$, if there exists

$$F : M \times [0,1] \to N$$

which is C^∞ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$.

Before proceeding to homotopy invariance, we need the *pull–back map for de Rham cohomology*. Let $f : M \to N$, we already know that it induces a pull–back map $f^* : A^p(N) \to A^p(M)$. Since pullback map commutes with exterior derivative d (cf. Theorem 2.9) as illustrated in following diagram

$$\begin{tikzcd}
A^{p-1}(N) \arrow[r] \arrow[d, "f^*"] \arrow[dr, phantom, "\circlearrowleft"] & A^p(N) \arrow[r] \arrow[d, "f^*"] \arrow[dr, phantom, "\circlearrowleft"] & A^{p+1}(N) \arrow[d, "f^*"] \\
A^{p-1}(M) \arrow[r] & A^p(M) \arrow[r] & A^{p+1}(M),
\end{tikzcd}$$

we see that f^* preserves both closed p-forms and exact p-forms. Hence, the pullback map f^* descends to the quotient spaces

$$f^* : H_{dR}^p(N) \to H_{dR}^p(M).$$

Theorem 2.29 (Homotopy invariance)**.** *Let $f_0, f_1 \in C^\infty(M,N)$ such that $f_0 \sim f_1$ via $F : M \times [0,1] \to N$, then*

$$f_0^* = f_1^* : H^p_{dR}(N) \to H^p_{dR}(M).$$

PROOF OF POINCARÉ LEMMA. Let $M = N = U$. Let $f_0 = \mathrm{id}$ be the identity map on U. Then U is contractible means that $f_0 \sim f_1$ where f_1 is a constant map, say with value c:

$$M \underset{\text{const}}{\overset{\text{id}}{\rightleftarrows}} M \qquad \begin{array}{l} x \longmapsto x \\ x \longmapsto c \end{array}$$

We have $\mathrm{id}^* = c^* = 0 \implies H^p_{dR}(M) = 0$. □

PROOF OF HOMOTOPY INVARIANCE. Let $\omega \in A(N)$ with $d\omega = 0$. We will show that $f_1^*\omega - f_0^*\omega = d\eta$ for some $\eta \in A(M)$. Recall the homotopy formula: $L_X = \iota_X d + d\iota_X$. Let $X = \partial_t \equiv \frac{\partial}{\partial t}$. Then

$$L_{\partial/\partial t} F^*\omega = \iota_{\partial_t} dF^*\omega + d\iota_{\partial_t} F^*\omega = d\iota_{\partial_t} F^*\omega$$

since $dF^*\omega = F^*d\omega = 0$. Write

$$F^*\omega = \alpha + dt \wedge \beta$$

where α, β involve no dt. Then

$$L_{\partial_t} F^*\omega = d^{M\times I}\iota_{\partial_t}(\alpha + dt \wedge \beta) = d^M\beta + dt \wedge \partial_t\beta.$$

On the other hand, by definition of Lie derivatives we also have

$$L_{\partial_t} F^*\omega = \partial_t\alpha + dt \wedge \partial_t\beta.$$

Therefore $\partial_t\alpha = d^M\beta$.

By the fundamental theorem of Calculus,

$$\begin{aligned} f_1^*\omega - f_0^*\omega &= \alpha(1) - \alpha(0) \\ &= \int_0^1 \partial_t\alpha dt = \int_0^1 d^M\beta dt = d^M\left(\int_0^1 \beta dt\right). \end{aligned}$$

Thus $\eta := \int_0^1 \beta\, dt \in A(M)$ is the form we search for. □

Lemma 2.30 (Mayer-Vietoris sequence)**.** *Let U, V be open sets in M, then there is a short exact sequence of chain complexes:*

$$\begin{tikzcd}
0 \arrow[r] & A^p(U\cup V) \arrow[r, hook, "i"] & A^p(U)\oplus A^p(V) \arrow[r, two heads, "j"] & A^p(U\cap V) \arrow[r] & 0. \\
& \alpha \arrow[r, mapsto] & (\alpha|_U, \alpha|_V) & & \\
& & (\xi,\eta) \arrow[r, mapsto] & \xi|_{U\cap V} - \eta|_{U\cap V} &
\end{tikzcd}$$

PROOF. It is clear that i is injective and $j\circ i = 0$. Also if $j(\xi,\eta) = 0$ then it is clear that ξ and η glue together to a p-form on $U\cup V$.

To show that j is surjective, choose a partition of unity ϕ_U, ϕ_V for the open cover $\{U, V\}$ of the manifold $U\cup V$.

For any $\gamma\in A^p(U\cap V)$, using the partition of unity we are able to extend it to be defined on U or on V respectively: namely

$$(\xi,\eta) := (\phi_V\gamma, -\phi_U\gamma) \in A^p(U)\oplus A^p(V).$$

Then $\phi_V\gamma|_{U\cap V} - (-\phi_U\gamma)|_{U\cap V} = (\phi_V+\phi_U)|_{U\cap V}\,\gamma = \gamma$. This completes the proof that the sequence is exact. $\square$

It is easy to see that both i and j commute with d. With this, then there exists a long exact sequence (which is also usually called the Mayer-Vietoris sequence):

$$\begin{tikzcd}
H^p_{dR}(U\cup V) \arrow[r, "i_*"] & H^p_{dR}(U)\oplus H^p_{dR}(V) \arrow[r, "j_*"] & H^p_{dR}(U\cap V) \arrow[r, "\delta"] & H^{p+1}_{dR}(U\cup V), \\
& & \gamma \arrow[r, mapsto] & d(\phi_V)\wedge\gamma
\end{tikzcd}$$

where the *connecting map* δ is obtained from the diagram chasing:

$$\begin{tikzcd}
& & (\phi_V\gamma, -\phi_U\gamma) \arrow[ddd, bend right, mapsto] & \gamma \arrow[l, mapsto] \\
0 \arrow[r] & A^p(U\cup V) \arrow[r] \arrow[d, "d"] & A^p(U)\oplus A^p(V) \arrow[r] \arrow[d, "d"] & A^p(U\cap V) \arrow[d, "d"] \\
0 \arrow[r] & A^{p+1}(U\cup V) \arrow[r] & A^{p+1}(U)\oplus A^{p+1}(V) \arrow[r] & A^p(U\cap V) \\
& (d\phi_V)\wedge\gamma & (d(\phi_V\gamma), -d(\phi_U\gamma)) \arrow[l, mapsto] &
\end{tikzcd}$$

Exercise 2.12. Show that the (Mayer-Vietoris) long exact sequence arising from the Mayer-Vietoris sequence is an exact sequence.

Exercise 2.13. (1) Let M be a compact oriented manifold (without boundary) of dimension n. Show that $H^n_{dR}(M) \cong \mathbb{R}$.

(2) More generally, prove the Poincaré duality: the pairing

$$H^p_{dR}(M) \times H^{n-p}_{dR}(M) \longrightarrow H^n_{dR}(M) \cong \mathbb{R}$$

defined by $(\omega, \eta) \mapsto \int_M \omega \wedge \eta$ is perfect. (Hint: you may use a "good cover" discussed in the next section and apply Mayer-Vietoris.)

Exercise 2.14. Prove the Brouwer fixed point theorem:

(1) There is no C^∞ map $f : \mathbb{B}^n \to \mathbb{S}^{n-1}$ with $f\big|_{\mathbb{S}^{n-1}} = \mathrm{id}_{\mathbb{S}^{n-1}}$.
(2) For any smooth $g : \mathbb{B}^n \to \mathbb{B}^n$, $\exists\, x \in \mathbb{B}^n$ such that $g(x) = x$.
(3) Prove the original continuous versions of (1) and (2).

8. Singular cohomology and the de Rham theorem

In Physics, we often connect the concepts of particles with the fields they create. In the mathematical perspective, we may associate the topological singular cohomology with the de Rham cohomology via an integration map, known as the *de Rham map*. Let M be a C^∞ manifold. The singular homology can be defined using only C^∞ chains. With this admitted, then the de Rham map is

$$\begin{aligned} H^p_{dR}(M) &\xrightarrow{\int} H^p(M;\mathbb{R}) \equiv H_p(M;\mathbb{R})^* \\ [\omega] &\longmapsto \int \omega : [\sigma] \longmapsto \int_\sigma \omega \end{aligned}$$

where $\omega \in A^p(M)$ and $\sigma \in S^\infty_p(M)$ is a C^∞ p-chain.

The de Rham map $\int$ is well-defined by Stokes' theorem. Furthermore, it is indeed an isomorphism!

Theorem 2.31 (de Rham, 1931)**.** *The de Rham map is a ring isomorphism*

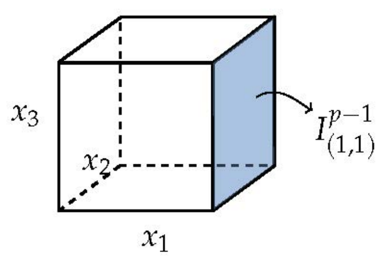
$$\int : H^p_{dR}(M) \xrightarrow{\sim} H^p(M;\mathbb{R}).$$

The remaining of this chapter is devoted to a sketch of proof of de Rham's theorem. To be precise, we recall one of the definitions of *singular homology* which we will use in the proof.

Definition 2.32. For any topological space M, a *singular p-cube* is a continuous map $\sigma : [0,1]^p \to M$. The set of *singular p-chains* $S_p(M)$ is the free abelian group generated by all singular p-cubes, i.e.

$$\sigma = \sum_{i=1}^{N} a_i \sigma_i \in S_p(M), \qquad a_i \in \mathbb{Z},$$

where σ_i is a singular p-cubes. More generally, the coefficients a_i can be taken in a commutative ring R, e.g. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ or even $\mathbb{Z}/n\mathbb{Z}$.



As in de Rham cohomology, we have a boundary map for singular p-chains $\partial_p : S_p(M) \longrightarrow S_{p-1}(M)$. We first define the boundary maps $\partial_p : I^p \longmapsto \partial I^p$ for the standard cube $I^p = \mathrm{id}_{[0,1]^p}$ by

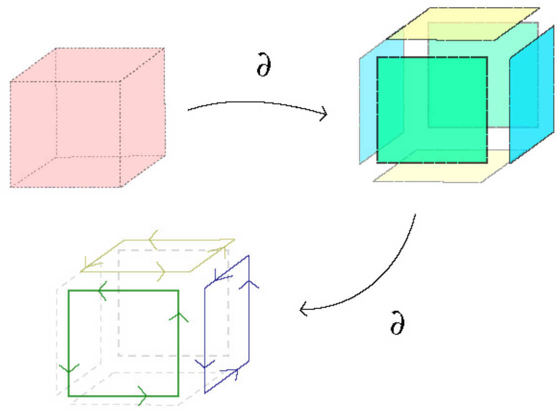
$$\partial I^p = \sum_{i=1}^{p} \sum_{\alpha=0}^{1} (-1)^{\alpha+i} I^{p-1}_{(i,\alpha)},$$

where $I^{p-1}_{(i,\alpha)}$ is one of the two faces ($\alpha = 0, 1$) perpendicular to the i-th axis. As a singular $p-1$ cube in $[0,1]^p$, it is defined as

$$I^{p-1}_{(i,\alpha)} : \begin{array}{ccc} [0,1]^{p-1} & \longrightarrow & [0,1]^p \\ (t_1, t_2, \ldots, t_{p-1}) & \longmapsto & (t_1, \ldots, t_{i-1}, \alpha, t_i, t_{p-1}) \end{array}.$$

Similar to $d^2 = 0$, the boundary map has the same property:

Fact 2.33. $\partial \circ \partial = 0$.



It is easy to see that this holds on the standard cube I^p. The proof is mainly concerned with the signs involved and is left as an exercise.

Now for a singular p-cube $\sigma \in S_p(M)$, we define

$$\begin{aligned}\partial\sigma &= \sum_{i=1}^{p}\sum_{\alpha=0}^{1}(-1)^{i+\alpha}\sigma_{(i,\alpha)}\\ &= \sum_{i=1}^{p}\sum_{\alpha=0}^{1}(-1)^{i+\alpha}\sigma\circ I^{p-1}_{(i,\alpha)} = \sigma\circ\partial I^p.\end{aligned}$$

Here we extend the definition of $\circ$ (composition of functions) over chains by linearity. So $\partial^2\sigma = 0$ as well.

Exercise 2.15. Show that the sign of boundary agrees with the induced orientation on boundary. Also show that $\partial^2 = 0$.

Hence, we arrive at a complex, the singular chain complex,

$$\cdots \longrightarrow S_{p+1}(M) \xrightarrow{\partial_{p+1}} S_p(M) \xrightarrow{\partial_p} S_{p-1}(M) \longrightarrow \cdots.$$

The p-th singular homology group is defined as the quotient

$$H_p(M;\mathbb{Z}) = \frac{\ker\partial_p}{\operatorname{Im}\partial_{p+1}}.$$

The *singular cohomology* is defined by the dual complex:

$$S^*_{p+1}(M) \xleftarrow{\partial^*} S^*_p(M) \xleftarrow{\partial^*} S^*_{p-1}(M)$$

where ∂^* is the dual (co-boundary) map. Namely,

$$(\partial^*\phi)(\sigma) := \phi(\partial\sigma).$$

Hence $\partial^* \circ \partial^* = 0$ and

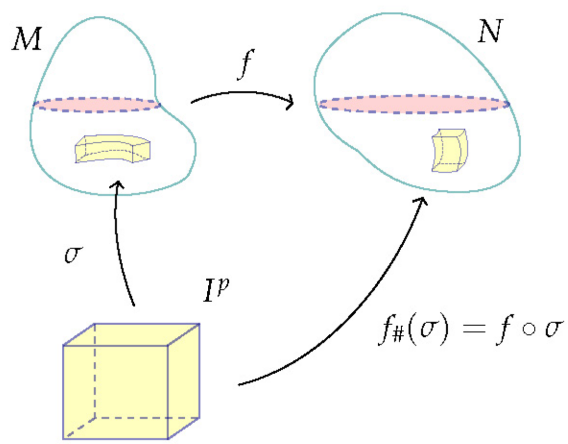
$$H^p(M;\mathbb{Z}) = \frac{\ker \partial^*_{p+1}}{\operatorname{Im} \partial^*_p}.$$

If the coefficient ring $\mathbb{Z}$ is replaced by a field F, then elementary linear algebra shows that $H^p(M;F) \cong H_p(M;F)^*$.

Let $f : M \to N$ be a continuous map. It induces a chain map between chain groups via compositions

$$f_\# : S_p(M) \longrightarrow S_p(N) : \qquad \sigma \longmapsto f_\#(\sigma) := f \circ \sigma.$$

By definition, we have $\partial f_\# = f_\# \partial$: for a singular cube σ, we have



$$\partial f_\# \sigma = f_\# \sigma \circ \partial I^p = f \circ \sigma \circ \partial I^p = f \circ \partial\sigma = f_\# \partial\sigma.$$

Hence it induces $f_* : H_p(M) \to H_p(N)$.

Proposition 2.34 (Homotopy formula)**.** *Let $f_0, f_1 : M \to$ be two continuous maps. If f_0, f_1 are homotopic then there exists $T : S_p(M) \to S_{p+1}(N)$ such that*

$$f_{1\#} - f_{2\#} = \pm\partial T \pm T\partial.$$

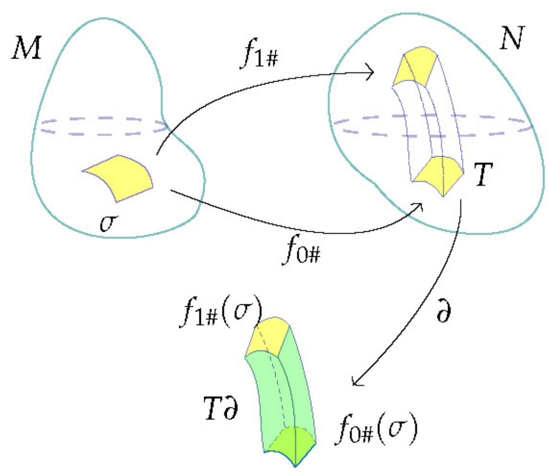
SKETCH OF PROOF. Construction of T: let $F : M \times [0,1] \to N$ be the homotopy such that $F(x,0) = f_0(x)$, $F(x,1) = f_1(x)$. Define

$$T(\sigma)(x) = F(\sigma(x), t).$$

The various maps in the following diagram

$$\begin{tikzcd}
S_{p+1}(M) \arrow[r, "\partial"] \arrow[d, "f_{0\#}"', "f_{1\#}"] & S_p(M) \arrow[r, "\partial"] \arrow[d, "f_{0\#}"', "f_{1\#}"] \arrow[dl, "T"'] & S_{p-1}(M) \arrow[d, "f_{0\#}"', "f_{1\#}"] \arrow[dl, "T"'] \\
S_{p+1}(N) \arrow[r, "\partial"] & S_p(N) \arrow[r, "\partial"] & S_{p-1}(N)
\end{tikzcd}.$$

are visualized in the following graph. Then, up to signs, we obtain



the homotopy formula. $\square$

Exercise 2.16. Determine the sign in the homotopy formula.

As in the de Rham cohomology, a simple consequence is

Corollary 2.35. *If U is contractable, then $H_p(U, \mathbb{Z}) = 0$ for all $p \geq 1$.*

We return to the de Rham map. First we need the definition of integration over C^∞ chains: for $\sigma : [0,1]^p \to M$ being a C^∞ singular p-cube, we define

$$\int_\sigma \omega := \int_{[0,1]^p} \sigma^* \omega$$

as a Riemann integral on $[0,1]^p \subset \mathbb{R}^p$. Then we extend the integration linearly to all combinations $\sigma = \sum_{i=1}^N a_i \sigma_i \in S_p^\infty(M)$.

Lemma 2.36. *The de Rham map is well-defined, i.e. $\int_\sigma \omega$ is independent of the choices of ω in $[\omega] \in H^p_{dR}(M)$ and the choices of σ in $[\sigma] \in H_p(M;\mathbb{R})$.*

PROOF. For any other representative $\omega + d\eta$ of $[\omega] \in H^p_{dR}(M)$, from Stokes' theorem we get

$$\int_\sigma d\eta = \int_{\partial\sigma} \eta = 0.$$

Similarly for any other representative $\sigma + \partial\tau$ of $[\sigma] \in H_p(M;\mathbb{R})$, we have $\int_{\partial\tau} \omega = \int_\tau d\omega = 0$. □

The main idea of the proof of de Rham's theorem to be presented below is a local to global principle. We first prove the theorem for local charts and then show the validity on their union by a glueing argument. This is usually known as the Mayer-Vietoris argument.

Exercise 2.17. Show that the Mayer-Vietoris long exact sequence in de Rham cohomology is compactible with the analogous one in singular cohomology under the de Rham map. Namely all diagrams below commute:

$$\begin{tikzcd}
H^{p-1}_{dR}(U\cap V) \arrow[r] \arrow[d, "\int"] & H^p_{dR}(U\cup V) \arrow[r] \arrow[d, "\int"] & H^p_{dR}(U)\oplus H^p(V) \arrow[r] \arrow[d, "\int"] & H^p_{dR}(U\cap V) \arrow[r] \arrow[d, "\int"] & H^{p+1}_{dR}(U\cup V) \arrow[d, "\int"] \\
H^{p-1}(U\cap V) \arrow[r, "\delta"] & H^p(U\cup V) \arrow[r, "i_*"] & H^p(U)\oplus H^p_{dR}(V) \arrow[r, "j_*"] & H^p(U\cap V) \arrow[r, "\delta"] & H^{p+1}(U\cup V).
\end{tikzcd}$$

SKETCH OF PROOF OF THE DE RHAM THEOREM. One checks easily that the theorem is true for a single ball U.

We need the notion of a good (open) cover $\{U_\alpha\}_{\alpha\in A}$ of M which is characterized by the property that any finite intersection $\bigcap U_\alpha$ is contractible. One way to construct a good cover is to use convex open sets. For this we need the notion of a Riemannina metric and a basic fact that will be proved in the next chapter.

Fact 2.37. On a Riemannian manifold (M, g), there exists convex neighborhoods of any $p \in M$.

Convex sets are contractible and intersection of convex sets is still convex. Thus convex open covers are good covers.

Now we prove the theorem by induction on $n = |A|$, the number of open sets in the convex open cover. Suppose that the theorem is true for U, V. Using the Mayer-Vietoris sequence, to show that it is true for $U \cup V$ we need it to be true true for $U \cap V$. We observe that for $U_1, \ldots, U_n, V$ being convex, then

$$(U_1 \cup \cdots \cup U_n) \cap V = (U_1 \cap V) \cup \cdots \cup (U_n \cap V).$$

If $U = \bigcup_{i=1}^{n} U_i$, then the theorem holds for U, V as well as for $U \cap V$. Hence it also holds for $U \cup V$. $\square$

For those who knows the Alexander Whitney diagonal approximation to define cup product in singular cohomology:

Exercise 2.18. Show that the de Rham map is a ring isomorphism.

Remark 2.38. The formal procedure in the above proof is known as the Mayer-Vietoris argument. A generalization of Lemma 2.30 to an open cover is known as the Cech complex which gives Cech cohomology. This was used by A. Weil in 1945 to give a new proof of de Rham's theorem. It is one of the main ingredients for the birth of modern theory of sheaf cohomology and derived functor cohomology. Nevertheless, owing to its simplicity, Mayer-Vietoris argument remains useful in cohomology calculations. For more informations see [BT82], [War83].

9. Problems

1. ([War83] Ch.2 #2)
   (a) Show that homogeneous tensor are generally not decomposable.
   (b) Show that if $\dim V \leq 3$, then every homogeneous element in $\Lambda(V)$ is decomposable.
   (c) Let $\dim V > 3$. Give an example of an indecomposable homogeneous element of $\Lambda(V)$.
   (d) Let α be a differential form. Is $\alpha \wedge \alpha \equiv 0$?

2. ([War83] Ch.2 #4) Derive the following three formulae.

$$
\begin{aligned}
& f \wedge_\alpha g(v_1, \ldots, v_{p+q}) \\
= & \sum_{p,q \text{ shuffles}} (\operatorname{sgn} \pi) f(v_{\pi(1)}, \ldots, v_{\pi(p)}) g(v_{\pi(p+1)}, \ldots, v_{\pi(p+q)}), \\
& f \wedge_\beta g(v_1, \ldots, v_{p+q}) \\
= & \frac{1}{(p+q)!} \sum_{\pi \in S_{p+q}} (\operatorname{sgn} \pi) f(v_{\pi(1)}, \ldots, v_{\pi(p)}) g(v_{\pi(p+1)}, \ldots, v_{\pi(p+q)}),
\end{aligned}
$$

$$
f \wedge_\alpha g = \frac{(p+q)!}{p!q!} f \wedge_\beta g.
$$

Here a permutation $\pi \in S_{p+q}$ is called a "p, q shuffle" if $\pi(1) < \pi(2) < \cdots < \pi(p)$ and $\pi(p+1) < \cdots < \pi(p+q)$.

3. ([War83] Ch.2 #13) Let V be an n-dimensional real inner product space. We extend the inner product from V to all of $\Lambda(V)$ by setting the inner product of elements which are homogeneous of different degrees equal to zero, and by setting

$$
\langle w_1 \wedge \cdots \wedge w_p, v_1 \wedge \cdots \wedge v_q \rangle = \det \langle w_i, v_j \rangle
$$

and then extending bilinearly to all of $\Lambda_p(V)$. Prove that if $e_1, \ldots, e_n$ is an orthonormal basis of V, then the corresponding basis

$$
e_\Phi = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}, \quad i_1 < i_2 < \cdots < i_r
$$

where $\Phi = \{i_1, \cdots, i_r\}$ runs over all subsets of $\{1, \ldots, n\}$, is an orthonormal basis for $\Lambda(V)$.

Since $\Lambda_n(V)$ is one-dimensional, $\Lambda_n(V) \setminus \{0\}$ has two components. An *orientation* on V is a choice of a component of $\Lambda_n(V) \setminus \{0\}$. If V is an oriented inner product space, then there is a linear transformation

$$
* : \Lambda(V) \to \Lambda(V),
$$

called *star*, which is well-defined by the requirement that for any orthonormal basis $\{e_1, \ldots, e_n\}$ of V (in particular, for any re-ordering of a given basis),

$$
\begin{aligned}
&*(1) = \pm e_1 \wedge \cdots \wedge e_n, \quad *(e_1 \wedge \cdots \wedge e_n) = \pm 1, \\
&*(e_1 \wedge \cdots \wedge e_p) = \pm e_{p+1} \wedge \cdots \wedge e_n,
\end{aligned}
$$

where one takes "$+$" if $e_1 \wedge \cdots \wedge e_n$ lies in the component of $\Lambda_n(V) \setminus \{0\}$ determined by the orientation and "$-$" otherwise. Observe that

$$
* : \Lambda_p(V) \to \Lambda_{n-p}(V).
$$

Prove that on $\Lambda_p(V)$,

$$** = (-1)^{p(n-p)}.$$

Also prove that for arbitrary $v, w \in \Lambda_p(V)$, their inner product is given by

$$\langle v, w\rangle = *(w \wedge *v) = *(v \wedge *w).$$

4. ([War83] Ch.2 #7) Let $w \in A_p(M)$ and let $X, Y_0, \ldots, Y_p$ be C^∞ vector fields on M. Show that

$$\begin{aligned} L_{Y_0}\big(w(Y_1,\ldots,Y_p)\big) =& (L_{Y_0}w)(Y_1,\ldots,Y_p) \\ &+ \sum_{i=1}^{p} w(Y_1,\ldots Y_{i-1}, L_{Y_0}Y_i, Y_{i+1},\ldots,Y_p) \end{aligned}$$

and use the case $p = 1$ to derive

$$L_X = \iota(X)\circ d + d\circ\iota(X)$$

on $A^*(M)$ for $p = 1$, and then for all p.

5. ([War83] Ch.2 #11) Let $\mathscr{I}$ be an ideal of forms on M locally generated by r independent 1-forms. Say $\mathscr{I}$ is generated by $w_1, \ldots, w_r$ on U. Then the condition that $\mathscr{I}$ be a differential ideal is equivalent to each of
   (a) $dw_i = \sum_j w_{ij} \wedge w_j$ for some 1-forms w_{ij} (for each such $(U, w_1, \ldots, w_r)$).
   (b) If $w = w_1 \wedge \cdots \wedge w_r$, then $dw = \alpha \wedge w$ for some 1-forms α (for each such $(U, w_1, \ldots, w_r)$).
6. ([War83] Ch.4 #1) A d-dimensional manifold X for which there exists an immersion $f : X \to \mathbb{R}^{d+1}$ is orientable if and only if there is a smooth nowhere-vanishing normal vector field along (X, f).
7. ([War83] Ch.4 #2) Prove that the real projective space $\mathbb{P}^n$ is orientable if and only if n is odd.
   (Hint: Observe that the antipodal map on the n-sphere $\mathbb{S}^n$ is orientation-preserving if and only if n is odd.)
8. ([War83] Ch.4 #3) Carry out in detail the proof of the existence of local orthonormal frame fields on a Riemannian manifold (cf. Exercise 1.7 and the next chapter).
9. ([War83] Ch.4 #6) Let w be the volume form of an oriented Riemannian manifold of dimension n. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be vector fields on M. Prove that

$$w(X_1,\ldots,X_n)\cdot w(Y_1,\ldots,Y_n) = \det\{\langle X_i, Y_j\rangle\}.$$

Prove also that

$$w(X_1, \dots, X_n)w = \tilde{X}_1 \wedge \cdots \wedge \tilde{X}_n,$$

where $\tilde{X}_i$ is the 1-form dual (via the Riemannian structure) to the vector field X_i.

10. ([War83] Ch.4 #17) Using de Rham cohomology, prove that the torus $\mathbb{T}^2$ is not diffeomorphic with the 2-sphere $\mathbb{S}^2$.
11. ([War83] Ch.4 #18)
    (a) Prove that every closed 1-form in the open shell $1 < \left(\sum_{i=1}^{3} r_i^2\right)^{1/2} < 2$ in $\mathbb{R}^3$ is exact.
    (b) Find a 2-form in the above shell that is closed but not exact.
    (c) Prove that the above shell is not diffeomorphic with the open unit ball in $\mathbb{R}^3$.
12. ([War83] ch.4 #4) Prove the divergence theorem. First assume M is oriented and use Stokes' theorem together with the identity

$$\int_{\partial D} *\tilde{V} = \int_{\partial D} \langle V, \vec{n} \rangle.$$

The easiest way to see the identity is to choose a local oriented orthonormal frame field $e_1, \dots, e_n$ on a neighborhood of a point of ∂D, such that at points of ∂D, e_1 is the outer unit normal vector and $e_2, \dots, e_n$ form an oriented basis of the tangent space to ∂D. Then express $*\tilde{V}$ and $\langle V, \vec{n} \rangle$ in terms of this local frame field and its dual coframe field $w_1, \dots, w_n$. Finally, show that the theorem holds for a regular domain D in a Riemannian manifold M which is not necessarily orientable.

13. ([War83] ch.4 #16)
    (a) Prove that every closed 1-form on $\mathbb{S}^2$ is exact.
    (b) Let

$$\sigma = \frac{r_1 dr_2 \wedge dr_3 - r_2 dr_1 \wedge dr_3 + r_3 dr_1 \wedge dr_2}{(r_1^2 + r_2^2 + r_3^2)^{3/2}}$$

in $\mathbb{R}^3 \setminus \{0\}$. Prove that σ is closed.

    (c) Evaluated $\int_{\mathbb{S}^2} \sigma$. How does this show that σ is not exact?
    (d) Let

$$\alpha = \frac{r_1 dr_1 + r_2 dr_2 + \cdots + r_n dr_n}{(r_1^2 + r_2^2 + \cdots + r_n^2)^{n/2}}$$

in $\mathbb{R}^n \setminus \{0\}$. Find $*\alpha$ and prove that $*\alpha$ is closed.

    (e) Evaluate $\int_{\mathbb{S}^{n-1}} *\alpha$. Is $*\alpha$ exact?