Globalization over curves via D-modules   6/7, 2017

Lemma: let N conn/$\hat K$ pure irregular, i.e. no reg. part.
let $M := \hat{\mathcal D}/\hat{\mathcal D}a$, $a = 1+\sum_{i\geq 1} a_i\partial^i$ cyclic gen. then $v(a_i)\geq i+1$ $(\forall i)$
$M \longrightarrow M[x^{-1}] = N$ is bijective.

~~pf~~: I.e., $x \curvearrowright M$ is bijec or $\mathrm{Ext}^i_{\hat{\mathcal D}}(\hat{\mathcal D}\delta, M) = 0$ $i=0,1$, ($\delta = 1 \bmod \hat{\mathcal D}x$)
by duality, since DM is "M" with $a^* = 1+\sum(-1)^i\partial^i a_i$,
enough to show $\mathrm{Ext}^i_{\hat{\mathcal D}}(M, \hat{\mathcal D}\delta) = 0$. But now $a \curvearrowright \hat{\mathcal D}\delta$ is bijective *

Thm: (i) Any $M\in\mathrm{Mod}_h(\hat{\mathcal D})$ $\exists!$ $M = M_r\oplus M_i$, M_i has the form $\hat{\mathcal D}/\hat{\mathcal D}a$.
(ii) Any M' reg. M'' pure irreg $\Rightarrow$ $\mathrm{Ext}^i_{\hat{\mathcal D}}(M', M'') = \mathrm{Ext}^i_{\hat{\mathcal D}}(M'', M') = 0$, $i\geq 0$.

• Now for $M\in\mathrm{Mod}_h(D)$, $D = D_{0,\epsilon}$, it is determined by a pair $(N, \hat M)$:
N conn/K, $\hat M\in\mathrm{Mod}_h(\hat{\mathcal D})$, st. $\hat N \cong j^*\hat M$, where $j: D^\times \hookrightarrow D$.
In $\hat N = \hat N_r\oplus\hat N_i$, $\hat M = \hat M_r\oplus\hat M_i$, get
$$\hat M_i \xrightarrow{\sim} \hat M_i[x^{-1}] \cong j_*j^*\hat M_i = j_*\hat N_i$$
so we only need to consider the pair $(N, \hat M_r)$ with $\hat N_r \xrightarrow{\sim} j^*\hat M_r$.

•* Monodromy $T \curvearrowright j^*\hat M_r$. For $\hat M_r$, need [$\Psi_0 \underset{V}{\overset{U}{\rightleftarrows}} \Phi_0$ st. $Id + VU = T|_{\Psi_0}$]
f.d.v.s. of nearby cycles   vanishing cycles i.e. perverse sheaves

Rmk: This is initiated in (SGA 7), see Malgrange II. §2;3 for details.
[e.g. $\Psi(M) = \mathrm{Tor}_1^{\mathcal D}(\mathcal O'\otimes_{\mathcal O} M_0, \mathcal O_{\tilde D})$, here $\tilde D^\times$ = univ. cover of $D^\times$]

Hence we get
$$M\in\mathrm{Mod}_h(D) \xleftrightarrow{\ \text{equiv.}\ } \begin{matrix} N \\ \text{conn}/K \end{matrix} \text{ with } \begin{matrix}\Psi_0 \underset{V}{\overset{U}{\rightleftarrows}} \Phi_0 \\ \| \\ (\Psi(\hat N_r), T)\end{matrix} \text{ st } Id + VU = T$$

• Stokes structures: Global case.
X curve, $Z\subset X$ discrete, $\tilde X \xrightarrow{\pi} X$ real blow-up along Z.
$$\begin{tikzcd} \coprod_{z\in Z} S_z = \tilde Z \arrow[r, "\tilde i"] \arrow[d, "\pi"] & \tilde X \arrow[d, "\pi"] & Y \arrow[l, "\tilde j"'] \arrow[d, equal] \\ Z \arrow[r] & X & Y := X\setminus Z \arrow[l, "j"'] \end{tikzcd}$$

$M\in\mathrm{Mod}_h(D_X)$, $\mathrm{Sing}(M)\subset Z$, we have the following data:
i) $V = \tilde j_*\mathcal{H}om_{D_Y}(\mathcal O_Y, M|_Y)$ local system on $\tilde X$ (deRham)
ii) The Stokes structure on $V|_{S_z}$ for each $z\in Z$
iii)* A v.s. Φ_0^z with 2 arrows $\Phi_0^z \underset{U^z}{\overset{V^z}{\rightleftarrows}} \Psi_0^z := \Psi(\mathrm{gr}_0(V|_{S_z}))$.

Theorem (3.1). (Riemann-Hilbert-Birkhoff correspondence)
This is an equiv. of cat. // proved abstractly via Deligne's thm.

Rmk: A more precise description needs Shibuya's prop 1.1, cf. [M] IV. Thm 3.2 (Next page)

<Malgrange's fund. thm>

Thm (3.2) M, N conn on Y, mero over Z with local system V, W over $\tilde{X} \xrightarrow{\pi} X \overset{j}{\supset} Y$. Then $\exists$ canonical Isom

(i) $R\mathcal{H}om_{D_X}(j_*M, j_*N) \simeq R\pi_* \mathcal{H}om(V, W)^{\circ}$. (This $= \bigoplus_\alpha \mathcal{H}om(gr_\alpha V, gr_\alpha W)$ in graded level)

(ii) In particular, $DR(M) \simeq R\pi_* V^{\circ}[1]$ (i.e. q-iso).

Over $\tilde{D} \xrightarrow{\pi} D$, $S = \pi^{-1}(0)$, $\mathcal{O}^{Nils}$ = the local system of $\sum f_{\alpha p}(x)\, x^{\alpha} (\log x)^p$, $\alpha \in \mathbb{C}$, $p \in \mathbb{N}$

$\mathcal{A}^{\leq 0}$ = sheaf over S with sections has ~~asymp belonging to $\mathcal{O}^{Nils}$~~ $\hat{\mathcal{O}} = \mathbb{C}\{x\}$

over extend to $\hat{D}$ by holo fun on $D^{\times}$

$0 \to \mathcal{A}^{<0} \to \mathcal{A}^{\leq 0} \to \hat{\mathcal{O}}^{Nils} \to 0 \quad \Rightarrow \quad \pi_* \mathcal{A}^{\leq 0} = \mathcal{O}_D[x^{-1}],\ \pi_* \hat{\mathcal{O}}^{Nils} = \hat{K}$.

Lemma (3.4). $R^i \pi_* \mathcal{A}^{\leq 0} = 0 = R^i \pi_* \hat{\mathcal{O}}^{Nils}$ for $i \geq 1$.

~~pf~~: It's enough to show $H^1(S, -) = 0$. $H^1(S, \hat{\mathcal{O}}^{Nils}) = 0$ is clear.

Then $H^1(S, \mathcal{A}^{<0}) \twoheadrightarrow H^1(S, \mathcal{A}^{\leq 0}) \to 0$ the map not really defined, but

$\searrow H^1(S, \mathcal{A})$ the "image" is 0 (Shibuya) as shown in the pf before. $*$

Cor. $R\pi_* \mathcal{A}^{<0} = [\pi_* \mathcal{A}^{\leq 0} \to \pi_* \hat{\mathcal{O}}^{Nils}] = [\mathcal{O}_D[x^{-1}] \to \hat{K}] = [\mathcal{O}_D \to \hat{\mathcal{O}}]$

This refines $H^1(S, \mathcal{A}^{<0}) = \hat{K}/K$

For M conn on $D^{\times}$, mero at 0, $\underline{\mathcal{A}^{\leq 0}(M) := \mathcal{A}^{\leq 0} \otimes_{\pi^{-1}\mathcal{O}} \pi^{-1} M}$

Lemma (3.8): $\partial : \mathcal{A}^{\leq 0}(M) \to \mathcal{A}^{\leq 0}(M)$ is surjective (~~as conn~~).

~~pf~~: This holds for $\mathcal{A}^{<0}$, hence it suffices to prove for $\hat{\mathcal{O}}^{Nils} \otimes_{\pi^{-1}\mathcal{O}} \pi^{-1} M = \hat{\mathcal{O}}^{Nils} \otimes_{\pi^{-1}\mathcal{O}} \pi^{-1} \hat{M}$; and any $\theta \in S$, enough to show $\underline{\mathrm{Ext}^1_{\hat{D}}(D\hat{M}, \hat{\mathcal{O}}^{Nils}_{\theta}) = 0}$ (Ex. why?)

Write $D\hat{M} = N_r \oplus N_i$, for N_i it is done since $\hat{\mathcal{O}}^{Nils}_{\theta}$ is regular

for N_r, may assume it is rank 1 of the form $\hat{K}\langle\partial\rangle/(\partial - \alpha/x)$.

In this case the original surjectivity follows by direct comp. $*$

~~Pf of Thm (3.2)~~: Enough to prove (ii). for (i) set $M := \mathcal{H}om_{\mathcal{O}_X}(M, N)$.

Lemma 3.8 $\Rightarrow$ $0 \to V^{\circ} \to \mathcal{A}^{\leq 0}(M) \xrightarrow{d} \mathcal{A}^{\leq 0}(M) \otimes_{\pi^{-1}\mathcal{O}} \pi^{-1}\Omega_X \to 0$

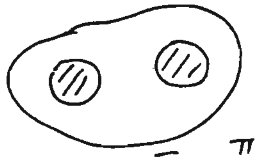
which is a π_* acyclic resolution by Lemma (3.4).

i.e. $R\pi_* V^{\circ} = [M \xrightarrow{d} M \otimes_{\mathcal{O}_X} \Omega_X] = DR(M)[-1]$ $*$

Remark: If $(X, (M, \nabla))$, mero conn, is defined over $k \subset \mathbb{C}$, then the de Rham coh is a k-v.s. Moreover, by Malgrange's fund thm (3.2), if the local system V has a reduction of str to a subfield $K \subset \mathbb{C}$ compatible with Stokes str. on Z, then $H^*_{dR}(X, (M, \nabla))$ also has a K-str. This is useful in "~~Periods~~" [BBDE, 05]

The case of holonomic $\mathcal{D}_X$-modules M with $\operatorname{Sing} M \subset Z$

Let $\bar{X} = \tilde{X} \cup_{\partial D_z = S_z} D_z$ and associate



i) local system V on $\tilde{X}$

ii) stokes str on S_z $\forall z \in Z$

iii) perverse sheaf F_z over D_z corr to $(\hat{M}_z)_0$ ← i.e. reg. part

$\bar{X} \xrightarrow{\pi} X$

Let ~~the complex F~~ with

$F|_{\overset{\circ}{D}{}_z^{\times}} = F_z[-1]$, $F|_{\tilde{X}} = V^0$,

$F|_{S_z}$ by $V^0 \xrightarrow{\sim} \operatorname{gr}_0 V$ (gluing)

$F_z \simeq \operatorname{gr}_0 V[1] = V^0/V^{<0}[1]$

Then we can prove $\underline{DR(M) = R\pi_* F[1]}$. (Thm 3.10)

Moreover, wrt (V, V_z^{α}, F_z) and (W, W_z^{α}, G_z) corr. to $M, N \in \mathrm{Mod}_h(\mathcal{D}_X)$ with $\operatorname{sing} \subset Z$

define complex H on $\bar{X}$ by

Over $\tilde{X}$, $H = \mathcal{H}om(V, W)^0$

$\overset{\circ}{D}{}_z^{\times}$, $H = R\mathcal{H}om(F_z, G_z)$, with gluing over S_z by

$\mathcal{H}om(V,W)^0 \longrightarrow \operatorname{gr}_0(V,W) = \bigoplus_{\alpha}(\operatorname{gr}_\alpha V, \operatorname{gr}_\alpha W) \longrightarrow \operatorname{Hom}(\operatorname{gr}_0 V, \operatorname{gr}_0 W)$

Theorem (3.13). $\underline{R\mathcal{H}om_{\mathcal{D}_X}(M, N) = R\pi_* H}$ <All pfs are omitted>

Conclusion: Reformulation (Sabbah ch. 5) for generalizations to HD:

- Ω^{et} = etale space of Ω (Sabbah uses $\mathcal{I}^{et}$ and $\mathcal{I}$ instead)

$$0 \to \mathcal{A}^{rd\,Z}_{\Omega^{et}} \to \mathcal{A}^{mod\,Z}_{\Omega^{et}} \to \mathcal{A}^{gr\,Z}_{\Omega^{et}} \to 0$$

~~rapid decay~~ : ~~moderate growth~~ ~~completion of~~ ~~Nilsson class~~

i.e. $\forall K$ cpt, $\exists C_{K,N}$ st. $|f| \le C_{K,N} |g_K|^N$ on K, $N \gg 0$ ← defining eq'n of $Z \subset X$

- ~~de Rham functor with twisted coefficients~~ : $\varphi \in \Omega$:

$\hat{D} \xrightarrow{\pi} D$, $S \mapsto 0$, 1 pt case

$DR_{\le\varphi} M := DR(e^{\varphi} \mathcal{A}^{mod\,0}(M))$, M mero conn.

$DR_{<\varphi} M := DR(e^{\varphi} \mathcal{A}^{rd\,0}(M))$,

$DR_{\operatorname{gr}_\varphi} M := DR([e^{\varphi}\mathcal{A}^{mod\,0}/e^{\varphi}\mathcal{A}^{rd\,0}] \otimes M)$. Then

get corr. local systems ~~$L_{\le\varphi}$, $L_{<\varphi}$, $\operatorname{gr}_\varphi L$ by taking $\mathcal{H}^0$~~, i.e. D-flat sect.

Thm: The RH functor $M \mapsto \underline{(\mathcal{H}^0 \widetilde{DR}(M), \mathcal{H}^0 DR_{\le} M)}$ ($\widetilde{DR}$ = multi-valued sections)

is an equiv. of cat. to "stokes filtered local systems".

- RH for general case on $\mathcal{D}_X$-modules

${}^p DR : D^b(\mathcal{D}_X) \to D^b(\mathbb{C}_X)$: $M^{\cdot} \mapsto \omega_X \otimes^L_{\mathcal{D}_X} M^{\cdot}$, get also

${}^p DR^{mod\,Z}_{\Omega^{et}}$, ${}^p DR^{rd\,Z}_{\Omega^{et}}$, ${}^p DR^{gr\,Z}_{\Omega^{et}}$ for $M^{\cdot} \in \mathcal{D}_X(*Z)$, holonomic

Thm: $\underline{{}^p DR_{\Omega^{et}}} := \underline{\{{}^p DR^{rd\,Z}_{\Omega^{et}} \to {}^p DR^{mod\,Z}_{\Omega^{et}} \to {}^p DR^{gr\,Z}_{\Omega^{et}} \xrightarrow{+1}\}}$ to $\underline{St(\mathbb{C}_{\Omega^{et}, \le})}$

is an equivalence of categories. i.e. sly in (i)(ii)(iii)