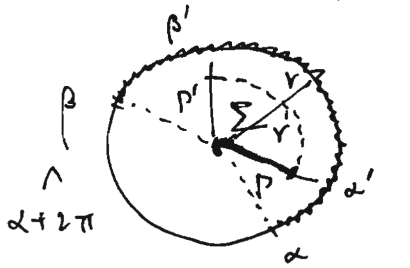
As sheaf, $\mathcal{A}|_S \xrightarrow{\wedge\!\!} \pi^{-1}\hat{K}$, comm with $\frac{d}{dx}$

Let $\mathcal{A}^{<0}$ be the sub sheaf of $\mathcal{A}|_S$ with asymp $= 0$.

we extend it to $\tilde{D}$ by setting $\mathcal{A}^{<0}|_{D^\times} = \mathcal{O}_{D^\times}$.

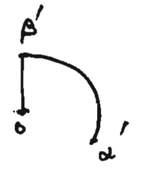
Prop (Shibuya) (1.1). The image $H^1(S, \mathcal{A}^{<0}|_S) \to H^1(S, \mathcal{A}|_S)$ is 0.

~~Pf:~~ It suffices to show that, for f holo in sector $(0,r)\times(\alpha,\beta)$ with $f \sim 0$ at 0, take D', Σ, Σ' as in:



$\exists\, g$ in the ramified sector Σ', $g \sim 0$ at 0 st. the difference of its 2 branches $= f$

Let $\partial\bar{\Sigma} = \Gamma + \Gamma'$ (Γ: angle α', Γ': the rest)



$D' = \{|x| < r'\}$, $r' < r$

$\Sigma := D'^\times \cap \{\arg x \in (\alpha', \beta')\}$

$\Sigma' := D'^\times \cap \{\arg x \in (\alpha', \beta'+2\pi)\}$

Then $g(x) = \frac{1}{2\pi i}\int_\Gamma \frac{f(y)}{y-x}\,dy$; $h(x) = -\frac{1}{2\pi i}\int_{\Gamma'} \frac{f(y)}{y-x}\,dy$

is defined in $D'^\times \setminus \Gamma$; in $D'^\times \setminus \{\text{angle } \beta'\}$

Also, $g - h = f$ in $x \in \Sigma$; $g - h = 0$ in $x \in D'^\times \setminus \bar{\Sigma}$

i.e. h is the analytic continuation of g to the 2nd branch ∗

On the other hand, it is easy to verify that in any closed sector not containing the right-half (angle $x = \alpha'$), generally $\neq 0$

The asymp of g at 0 $= \frac{1}{2\pi i}\sum x^n \int_\Gamma \frac{f(y)}{y^{n+1}}\,dy$, hence $\in \mathcal{A}$.

The same property holds for sector containing (angle $x = \alpha'$) by using $h+f$ to replace g (since same argument applies to h) ∗

Cor. From $0 \to \mathcal{A}^{<0}|_S \to \mathcal{A}|_S \to \pi^{-1}\hat{K} \to 0$, get

$$0 \to \underbrace{H^0(S, \mathcal{A}^{<0}|_S)}_{=0} \to \underbrace{H^0(S, \mathcal{A}|_S)}_{=K} \to \underbrace{H^0(S, \pi^{-1}\hat{K})}_{=\hat{K}} \to H^1(S, \mathcal{A}^{<0}|_S) \to 0$$

$\Rightarrow H^1(S, \mathcal{A}^{<0}|_S) = \hat{K}/K$.

Def$^{\text{n}}$: for $M \in \mathrm{Conn}(D^\times)$, mero at 0, $\mathcal{A}^{<0}(M) := \mathcal{A}^{<0} \otimes_{\pi^{-1}\mathcal{O}_D} \pi^{-1}(M)$ with conn. "∂"

Thm (1.3) $\partial : \mathcal{A}^{<0}(M) \to \mathcal{A}^{<0}(M)$ is surjective.

(~~Fund. Thm. for~~ asymp. expansions, pf omitted).

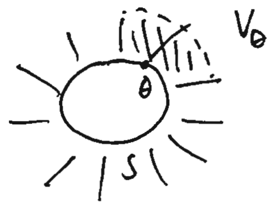
Now from
$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A}^{<0}(M) & \to & \mathcal{A}(M) & \to & \pi^{-1}\hat{M} & \to & 0 \\ & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \end{array}$$
we get

Thm (1.4) [Hukuhara–Turrittin]: $\mathcal{A}(M)_\theta^{\partial} \twoheadrightarrow \hat{M}_\theta^{\partial}$ $\forall\,\theta \in S$ small sector

i.e. formal horizontal section is repr by holo w asymp. exp ...

Stokes Structure ~~(at a point~~)

$S \xrightarrow{\tilde{i}} \tilde{D} \xleftarrow{\tilde{J}} D^\times$ M conn on $D^\times$ mero at 0



V = loc system of horizontal sections over $\tilde{D}$ (via $\tilde{J}_*$)

$\hat{\lambda} : \hat{M} \otimes_K \hat{K} \xrightarrow{\sim} \hat{M}'$, $M' = \bigoplus L_\alpha \otimes M_\alpha$ (cf. *) over $\hat{K}$ ~~via reg. thm.~~

$\rightsquigarrow \lambda_\theta$ horizontal sect. of $\mathcal{A}_\theta \otimes_{\hat{K}} \mathrm{Hom}_{\hat{K}}(M \otimes_K \hat{K}, M')$, isom

The loc. system V' asso to M' has ~~explicit forms~~

horizontal sect : $\sum e^{\int \alpha} f_\alpha$ f_α sol of a reg. eq.

$\to$ explicit asymp expansion of M in sector near θ

Def'n ① local system Ω on S : for a small sector, we take all $\alpha = \sum_{-h}^{\infty} a_k x^{k/p} dx$ (p any $\mathbb{N}$), modulo pole order ≤ 1.

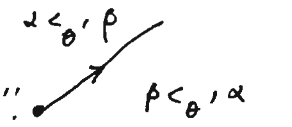
$\alpha \mapsto$ "$e^{2\pi i}\alpha$" $= e^{2\pi i k/p}\alpha$

② Partial order on Ω : for $\theta \in S$,

$\alpha \le_\theta \beta$ if $e^{\int(\alpha - \beta)} \in O(|x|^{-N})$ ie moderate growth in a small sector around θ. (compare ~~top sing part~~)

for given $\alpha \ne \beta$, $\exists$ finite many θ's of (a finite cover of) S st. α, β are incomparable at θ

$\longleftrightarrow$ Stokes lines rel. to (α, β).



③ A Stokes str. on V is a family of ~~subsheaf~~ "~~locally~~": V^α, $\alpha \in \Omega$ st. $\forall \theta \in S$, $\exists V_\theta = \bigoplus V_{\alpha,\theta}$ with

$V^{T(\alpha)} = T(V^\alpha)$

$$V_{\theta'}^{\alpha} = \bigoplus_{\beta \le_{\theta'} \alpha} V_{\beta,\theta} \quad \text{for } \theta' \text{ near } \theta.$$

We get :

(Ω-filtered local system) $\xleftarrow[\text{functor } \Sigma]{}$ conn. on K.

④ It also leads to a Ω-graded local system $\mathrm{gr}\, V$ by

$$(\mathrm{gr}_\alpha V)_\theta := V_\theta^\alpha / \sum_{\beta <_\theta \alpha} V_\theta^\beta \quad ※$$

ie. $W = \bigoplus_{\alpha \in \Omega} W_\alpha$

* Given $\hat{M}$ conn $/\hat{K}$, $\exists$ conn M/K st. completion of $M \simeq \hat{M}$. if M_1/K is another choice, then $\hat{M}_1 \xrightarrow{\hat{\lambda}} \hat{M}$. $\forall \theta \in S$, for a unique $\lambda \in \mathcal{A}_\theta \otimes \mathcal{H}om(M_1, M)$ modulo $\mathcal{A}_\theta^{<0} \otimes \mathcal{H}om(M_1, M)$

$\Rightarrow (\mathrm{gr}\, V_1)_\theta \longrightarrow (\mathrm{gr}\, V)_\theta$ depends only on $\hat{\lambda}$. (~~ie $e^{\int\alpha}$ with $\alpha <_\theta 0$~~)

Thm (Deligne).

$$\begin{array}{ccc}
(\mathrm{Conn}/K) & \xrightarrow[\text{equiv.}]{\Sigma} & (\Omega\text{-filt local system}) \\
\downarrow \hat{} & & \downarrow \mathrm{gr} \\
(\mathrm{Conn}/\hat{K}) & \xrightarrow[\text{equiv.}]{\hat{\Sigma}} & (\Omega\text{-graded local system})
\end{array}$$

e.g. in the str thm given above

Q : How to prove essential surjectivity ?