Malgrange's book. Ch. III. 5/24 2017

Let $\mathcal{O} = \mathcal{O}_{\mathbb{C},0} = \mathbb{C}\{x\}$, $K = \mathcal{O}[x^{-1}]$, $\hat{K} = \hat{\mathcal{O}}[x^{-1}]$

Prop (1.1) M conn $/\hat{K}$ of rk $m \Rightarrow \exists\, e \in M$ a cyclic generator

i.e. $e, \partial e, \ldots, \partial^{m-1} e$ is a base of $M/\hat{K}$. (HW)

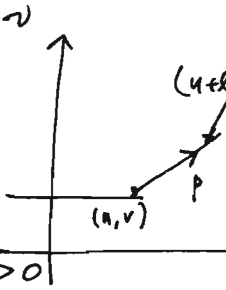
Def$^{\text{in}}$: $0 \neq a \in K\langle\partial\rangle$, $\underline{\text{Newton polygon}}$ $N(a)$ = convex

$\overset{\shortparallel}{\sum a_{k\ell}\, x^\ell \partial^k}$ hull of $\{u \le k,\ v \geq \ell - k \mid a_{k\ell} \neq 0\}$

~~Support function~~, for $t \geq 0$

$w(t, a) = \inf\{v - tu \mid (u,v) \in N(a)\}$

$= \inf\{\ell - (t+1)k \mid a_{k\ell} \neq 0\}$



$N(a) \subset \mathbb{R}^2$

$w'(p^-, a) = -u$

$w'(p^+, a) = -(u+\ell)$

$\cup$ jump $= -\ell$ at slope $t = p$.

as in comm. case: wt of $x^\ell \xi^k$ for $t > 0$

$\Rightarrow t \geq 0$, $ab \neq 0$ then $w(t, ab) = w(t,a) + w(t,b)$

slopes of ab = slopes of a $\cup$ slopes of b

Def$^{\text{in}}$:

Recall the str thm of formal conn:

L rk 1 $\longleftrightarrow$ $\partial + \bar{\alpha}$, $\bar{\alpha} \in \hat{K}$ Then

$L' \simeq L \otimes M$ for some M regular $\Longleftrightarrow$ $\bar{\alpha}' - \bar{\alpha}$ has simple pole

Remark L_ω the conn with $\omega = \alpha\, dx$, $\alpha \in \hat{K}$ mod simple pole,

$I_K := \hat{K}\, dx$ mod simple pole

Thm (1.2) Let M be a conn $/\hat{K}$. After possibly a branching

$\tilde{K} = k[t]$, $t^p = x$ and $\tilde{M} = \tilde{K} \otimes_K M$, we have unique

decomp $\tilde{M} = \bigoplus_{\omega \in I_{\tilde{K}}} L_\omega \otimes M_\omega$ with M_ω regular, ω's distinct.

$\underline{\text{Pf: Step 1}}$: Reduction to the case with only one slope.

Lemma (1.3). Let $a \in \hat{K}\langle\partial\rangle$ with slopes $0 \le p_1 < p_2 < \cdots < p_r$, and pick $s \in \{1, \ldots, r-1\}$

Then $\exists\, b, c \in \hat{K}\langle\partial\rangle$ uniquely st

i) $a = bc$

ii) b has slopes $p_1 \cdots p_s$, c has slopes $p_{s+1} \cdots p_r$

iii) the constant term of $c = 1$.

~~Pf~~: Let $t \in (p_s, p_{s+1})$ and $a_0 = \alpha_{k\ell}\, x^\ell \partial^k$ the "dominant term"

of $w(t, \cdot)$. Then set $b_0 = \alpha_{k\ell}\, x^\ell \partial^k$ and $c_0 = 1$

to be the expected "dominant term". May assume $\alpha_{k\ell} = 1$

b must have degree k in ∂ by dominance. $\Rightarrow$ deg of $c = \deg a - k$.

may det. b and then c by $\underline{\text{ascending order}}$ wrt. $w(t, \cdot)$ ※

From $a = bc$, we get $M = \hat{K}\langle\partial\rangle/\hat{K}\langle\partial\rangle a$, $N = \hat K\langle\partial\rangle/\hat K\langle\partial\rangle b$, $P = \hat K\langle\partial\rangle/\hat K\langle\partial\rangle c$

$$0 \to N \xrightarrow{c\cdot} M \to P \to 0$$ exact. in fact split uniquely:

Prop (1.4). $\mathrm{Ext}^i_{\hat K\langle\partial\rangle}(N,P) = 0 = \mathrm{Ext}^i_{\hat K\langle\partial\rangle}(P,N)$ $i = 1, 2$.

ie. $b \curvearrowright P$ and $c \curvearrowright N$ are both bijections.

Pf: Do the case $b \curvearrowright P$:

injectivity: if $bb' = c'c$ (ie. 0 in P)

want to show c right-divides b'. Division w.r.t. ∂ $\Rightarrow$ may assume $d^\circ b' < d^\circ c$

if $b' \neq 0$, the jumps of $w'(t, b')$ contain those of c

since $w'(t, b)$ does not jump at $t \in$ slopes of c

but this is incompatible with $d^\circ b' < d^\circ c$ ⁎. (why?)

surjectivity: it suffices to show $bb' + c'c = 1$ (some b', c')

and can assume $d^\circ b' < d^\circ c$, $d^\circ c' < d^\circ b$.

This is done by filtering by $w(t, \cdot)$ and by calculating b', c' step by step as in Lemma (1-3). ⋕

Now M a conn / $\hat K$, e a cyclic vector, let a = min. poly with slopes $0 \le \rho_1 < \rho_2 < \cdots < \rho_r$, we then get

$a = a_1 \cdots a_r$, $M = \bigoplus M_i$, $M_i = \hat K\langle\partial\rangle/\hat K\langle\partial\rangle a_i$, a_i has slope ρ_i.

Thm (1.5). (1) $N(a)$ is indep of choices of e, dep only on M.

(2) We have a unique $M = \oplus M_i$, M_i has one slope, distinct in i.

Cor. M is regular $\Leftrightarrow$ slope $= 0$.

Pf: M_0 is regular (the part with slope $= 0$): let e cyclic vector

get $x^m \partial^m e + \sum a_i x^{m-i} \partial^{m-i} e = 0$ $\quad b_i \in \hat{\mathcal O}$

rewrite $(x\partial)^m e + \sum b_i (x\partial)^{m-i} e = 0$ $\Rightarrow$ matrix with simple pole.

if e regular element but $\notin M_0$, then $\bar e_{\neq 0}$ in M_i for some $i \neq 0$

then $\exists\, f \in \hat K\langle\partial\rangle \bar e \subset M_i$, $\neq 0$ st. $(x\partial - \alpha) f = 0$, $\alpha \in \mathbb C$

by the theory of regular formal connections, but this has slope 0 ⁎

that can be brought into $\frac{1}{x} B$ with B constant matrix. ⋕

Step 2: (Pf of Thm 1.2)

uniqueness: $M = L_w \otimes_{\hat K} P$, $N = L_{w'} \otimes_{\hat K} Q$, P, Q regular, $w \neq w' \Rightarrow \mathrm{Hom}_{\hat K\langle\partial\rangle}(M, N) = 0$

in fact, the space is horizontal sections of

$\mathrm{Hom}_{\hat K}(M, N) = L_{w'-w} \otimes \mathrm{Hom}_{\hat K}(P, Q)$ ← Newton polygon is calculated from $w' - w$

$\Rightarrow$ no non-zero horizontal section. no zero slope

For existence. Assume the ~~Katz inv~~ $\rho > 0$ top (hence the only) (~~ie. the slope~~). ($\rho = 0$ reg. done)

Case 1. $p \in \mathbb{N}$.

e cyclic for M, $\partial^m e + \sum a_i \partial^{m-i} e = 0$ minimal poly. $a(e)=0$

slope p $\Rightarrow$ a_i has pole order at most $(p+1)i$

i.e. $a_i = \dfrac{\alpha_i}{x^{(p+1)i}}$ + higher order, also $\alpha_m \neq 0$

We call $\mu^m + \sum \alpha_i \mu^{m-i} = 0$ the ~~characteristic eq'n~~ or determining eq

let λ be a root of mult q, we set $\omega = \dfrac{\lambda\, dx}{x^{p+1}}$

then $L_\omega \otimes_{\hat{k}} M$ has a cyclic vector "e" with min. poly

$$a' = \left(\partial + \frac{\lambda}{x^{p+1}}\right)^m + \sum a_i \left(\partial + \frac{\lambda}{x^{p+1}}\right)^{m-i} \quad \text{with slope} \leq p$$

if $q=m$ then this reduces to ξ^m.

$w(p,\cdot)$-grading $= \left(\xi + \dfrac{\lambda}{x^{p+1}}\right)^m + \sum \dfrac{\alpha_i}{x^{(p+1)i}}\left(\xi + \dfrac{\lambda}{x^{p+1}}\right)^{m-i}$

$= \xi^m + \sum \dfrac{\beta_i}{x^{(p+1)i}} \xi^{m-i}$

if $q<m$, get $\beta_m = \cdots = \beta_{m-q+1} = 0$, $\beta_{m-q} \neq 0$.

get one or more smaller slopes, by step 1. Can decompose M and decrease m.

Case 2 $p = \frac{q}{r} > 0$, $(q,r)=1$.

Set $t^r = x$, replace M by $\hat{M} = M \otimes_{\hat{k}} \hat{k}[t]$

and $\partial_x = \frac{1}{r} t^{1-r} \partial_t$.

e cyclic in M with min. poly $a(x,\partial_x) = \partial_x^m + \sum a_i \partial_x^{m-i}$

$\Rightarrow$ $\tilde{e}$ = image of e in $\hat{M}$ is cyclic with min. poly $\hat{a} = t^{(r-1)m} a(t^r, \frac{1}{r} t^{1-r} \partial_t)$

After the replacement, the process in case 1 applies. $\divideontimes$

ch IV. **Asymptotic expansions** 5/31, 2017

$D = \{|z|<r\} \subset \mathbb{C}$, $\tilde{D} \xrightarrow{\pi} D$ the real blowup at 0, $S = \pi^{-1}(0)$

$\tilde{D} = [0,r) \times T$ $\quad (\rho,\theta) \stackrel{\pi}{\longmapsto} \rho e^{i\theta}$ $\quad \pi^{-1}(D^\times) \simeq D^\times$.

Sheaf $\mathcal{A} = \mathcal{A}_{\tilde{D}}$: on $D^\times$ it is $\mathcal{O}_{D^\times}$

over S: for $U \subset S$ open, $\tilde{U}$ the sectorial domain $\tilde{U} = (0,r) \times U$

presheaf $\bar{\mathcal{A}}(U)$ = set of germs f at 0, hol in $\tilde{U}$ and with Laurent asymp. expansion at 0.

Then make it a sheaf.

Have $f \mapsto \hat{f} : \bar{\mathcal{A}}(U) \to \hat{k}$ means $\forall p \in \mathbb{Z}$, $\forall x \in \tilde{U}$ near 0

$= \sum_{n \gg -\infty} a_n x^n \qquad \left| f(x) - \sum_{n \leq p} a_n x^n \right| \leq C_p |x|^{p+1}$, $C_p > 0$.

Fact: Surj if $U \neq S$ [Wasow]. In fact, works for "ramified simply conn. sector"

pf: Set $x = y^N$, $N \gg 0$ may assume $U \subset (-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon)$

For $\hat{f} = \sum a_n x^n$, set $f(x) = \sum a_n \big(1 - \underbrace{e^{-\frac{1}{x}\frac{1}{|a_n|}}}_{\to \text{ asymp. to } 0}\big) x^n$ unif. conv. $\divideontimes$