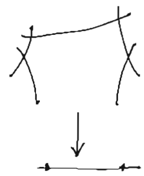
• Regularity of the Gauss-Manin Connection:

Surface → curve, $f(\mathrm{Sing}\,\Delta_S)\subset\Delta_C$: $M\in \mathrm{Conn}^{reg}(S_0) \Rightarrow \int_f i_*M \in D^b_{rh}(D_C)$.



$$\begin{tikzcd} S \arrow[d, "f"] & S_0, \Delta_S \arrow[l, hook', "i"'] \arrow[d, "sm"] & \text{NCD} \\ C & C_0, \Delta_C \arrow[l, hook'] \end{tikzcd}$$

pf: Let $D_{S\to C}\langle\Delta\rangle := \mathcal{O}_S\otimes_{f^{-1}\mathcal{O}_C} f^{-1}D_C\langle\Delta_C\rangle$.

which is a $(D_S\langle\Delta_S\rangle, f^{-1}D_C\langle\Delta_C\rangle)$ bi-module. We prove the right mod. case.

Define $\int_f^{\langle\Delta\rangle} M^\cdot = Rf_*(M^\cdot\otimes^L_{D_S\langle\Delta_S\rangle} D_{S\to C}\langle\Delta\rangle)$: $D^b(\mathrm{Mod}(D_S\langle\Delta_S\rangle^{op}))\to D^b(\mathrm{Mod}(D_C\langle\Delta_C\rangle^{op}))$

It is trivial that $\int_f^{\langle\Delta\rangle} i_*M = \int_f i_*M$ for $M\in\mathrm{Mod}(D_{S_0}^{op})$. ($\Delta,\Delta_S$ disappear on S_0)

Claim: Let $L\in\mathrm{Mod}(D_S\langle\Delta_S\rangle^{op})$, $\mathcal{O}_S$-coh. $\Rightarrow \forall i$, $H^i(\int_f^{\langle\Delta\rangle}L)\in\mathrm{Mod}(D_C\langle\Delta_C\rangle^{op})$ and $\mathcal{O}_C$-coh.

pf: Define $\Theta_{S/C}\langle\Delta\rangle$ by $0\to\Theta_{S/C}\langle\Delta\rangle\to\Theta_S\langle\Delta_S\rangle\to f^*\Theta_C\langle\Delta_C\rangle\to 0$. By $\otimes_{\mathcal{O}_S} D_S\langle\Delta_S\rangle$:

get exact sequence of $D_S\langle\Delta_S\rangle$-modules: $0\to D_S\langle\Delta_S\rangle\otimes_{\mathcal{O}_S}\Theta_{S/C}\langle\Delta\rangle\to D_S\langle\Delta_S\rangle\to D_{S\to C}\langle\Delta\rangle\to 0$?

which is a loc. free resol. of $D_{S\to C}\langle\Delta\rangle$. Thus $\int_f^{\langle\Delta\rangle}L = Rf_*[L\otimes_{\mathcal{O}_S}\Theta_{S/C}\langle\Delta\rangle\to L]$ ※

Now, $i_*M=\underbrace{\bigcup_\alpha L_\alpha}_{\mathcal{O}_S\text{-coh. } D_S\langle\Delta_S\rangle\text{-submodule}} = \varinjlim_\alpha L_\alpha \Rightarrow H^k\int_f i_*M = H^k\int_f^{\langle\Delta\rangle}\varinjlim_\alpha L_\alpha = \varinjlim_\alpha H^k\int_f^{\langle\Delta\rangle} L_\alpha = \bigcup_\alpha$ ($\mathcal{O}_C$-coh. $D_C\langle\Delta_C\rangle$-module) ※

Theorem: (a) $\int_f$ preserves D^b_{rh}.

(b) curve testing criterion: $M^\cdot\in D^b_h(D_X)$ then $M^\cdot\in D^b_{rh}(D_X) \Leftrightarrow i^\dagger M^\cdot\in D^b_{rh}(D_C)$ $\forall\, i: C\to X$.

(enough to consider $i: C\hookrightarrow X$)

pf: Induction on dim Supp$(M^\cdot)$ = max dim Supp $H^j(M^\cdot)$.

I. (a) for affine embedding $i:X\hookrightarrow Y$, $M^\cdot=M\in\mathrm{Conn}^{reg}(X)$:

$\int_{i!}M\to\int_i M\to C_i(M)\xrightarrow{+1}$ in $D^b_h(Y)$ $\Rightarrow$ Any comp. factor of $\int_{i!}M=H^0\int_{i!}M$ or $\int_iM=H^0\int_iM$

is isom. to a comp. factor of $L(X,M)$, which is regular, or of $H^*C_i(M)$.

For later case, by Hironaka: $\exists$

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "i"'] & \bar X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

$\bar X\setminus X=E$ is a NCD (special case 1), f proper

Apply $f_!=f_*$ to $\int_{j!}M\to\int_jM\to C_j(M)\xrightarrow{+1}$ $\Rightarrow C_j(M)\in D^b_{rh}(D_{\bar X})$ with smaller supp than M

get $\int_{i!}M\to\int_iM\to\int_f C_j(M)\xrightarrow{+1}$. i.e. $C_i(M)\simeq\int_f C_j(M)\in D^b_{rh}(D_{\bar X})$ ※

II. (b) part $\Rightarrow$: let $i:C\hookrightarrow X$, $M^\cdot\in D^b_{rh}(D_X)$. Induction on cohomology length, may set $M^\cdot=M$,

which is simple in $\mathrm{Mod}_{rh}(D_X)$. i.e. $M=L(Y,N)$, $j:Y\hookrightarrow X$ affine. $N\in\mathrm{Conn}^{reg}(Y)$.

Then $Q:=\int_jN/M$ has smaller support $\Rightarrow Q\in D^b_{rh}(D_X)\Rightarrow i^\dagger Q\in D^b_{rh}(D_C)$.

From $i^\dagger M\to i^\dagger\int_jN\to i^\dagger Q\xrightarrow{+1}$, enough to prove $i^\dagger\int_jN\in D^b_{rh}(D_C)$.

By base change thm on

$$\begin{tikzcd} Y\cap C \arrow[r, hook, "i_0"] \arrow[d, hook, "j_0"'] & Y \arrow[d, hook, "j"] \\ C \arrow[r, hook, "i"] & X \end{tikzcd}$$

, $i^\dagger\int_jN=\int_{j_0}\underbrace{i_0^\dagger N}_{\text{reg. conn up to degree shift}}\Rightarrow$ regular ※

III. (b) part $\Leftarrow$: left as HW reading ※ (Idea: $M^\cdot\in D^b_h(D_X)\Rightarrow H^*(M^\cdot)$ is generically an int. conn.)

IV. (a) for general f: For $f:X\to Y$ closed embedding, may assume $M^\cdot=M=L(X_1,N)$, $i:X_1\hookrightarrow X$ affine, $N\in\mathrm{Conn}^{reg}(X_1)$. Coker of $L(X_1,N)\hookrightarrow\int_iN$ has smaller supp. $\Rightarrow$ follows from I. ※

For projection $f:X=Z\times Y\to Y$. This can be reduced to $\mathbb{A}^1\times Y\to Y$ and to special case 2.

Details are left as HW reading. ※

Cor: $f^\dagger$ preserves D^b_{rh}. (This follows from Thm (b) on curve testing.) ※

Classical Riemann-Hilbert Correspondence (Regular Case)

Theorem : Let $f: X \to Y$. Then under the deRham functors DR_X, DR_Y, we have canonical isomorphisms

① $DR_Y \circ \int_f \simeq Rf_* \circ DR_X$;

also $\int_{f!} \leftrightarrow Rf_! : D^b_{rh}(\mathcal{D}_X) \to D^b_c(\mathbb{C}_Y)$ (since $\mathbb{D}$ comm. with DR.);

② $DR_X \circ f^+ \simeq f^! \circ DR_Y$; also $f^* \leftrightarrow f^{-1} : D^b_{rh}(\mathcal{D}_Y) \to D^b_c(\mathbb{C}_X)$.

Sketch : ① is already known for f proper, in general, Hironaka $\Rightarrow$ $f = p \circ j$,
$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & \bar{X} \arrow[d, "p"] \\ & Y \end{tikzcd}$$
$\bar{X} \setminus X$: NCD, p projective,
so may assume $f = j$. If $M^\cdot = M \in \mathrm{Conn}^{reg}(X)$, this follows from Deligne's theory of mero and reg. conn. For $M \in \mathrm{Mod}_{rh}(\mathcal{D}_X)$, ind. on dim supp M.

② is known for f smooth (or non-char/M : Cauchy-Kowalevski). In general
$\Rightarrow$ may assume $f = i : X \hookrightarrow Y$ closed embedding. Let $j : Y \setminus X \hookrightarrow Y$. Then
$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & X \times Y : \text{graph} \arrow[d, "p : sm"] \\ & Y \end{tikzcd}$$

$$\begin{tikzcd} DR_Y \int_i i^+ N^\cdot \arrow[r] \arrow[d, "\psi"] & DR_Y N^\cdot \arrow[r] \arrow[d, "id"] & DR_Y \int_j j^+ N^\cdot \arrow[r, "+1"] \arrow[d, "\varphi"] & {} \\ Ri_* i^! DR_Y N^\cdot \arrow[r] & DR_Y N^\cdot \arrow[r] & Rj_* j^! DR_Y N^\cdot \arrow[r, "+1"] & {} \end{tikzcd}$$

φ is an isom by ① and j : sm.

$\Rightarrow \psi$ isom. By ① $\Rightarrow$ $\underline{Ri_*} DR_X i^+ N^\cdot \simeq \underline{Ri_*} i^! DR_Y N^\cdot$ $\quad i^{-1} \circ Ri_* = id \Rightarrow$ result ※

Theorem (R-H, Alg case) : $DR_X : D^b_{rh}(\mathcal{D}_X) \to D^b_c(X)$ is an equiv.

pf : Essentially surj :

For $F^\cdot = Ri_* L$, $i : Z \hookrightarrow X$ loc. closed, Z sm. $L \in \mathrm{Loc}(Z^{an})$ "generators"

Deligne's RH corr. $\Rightarrow \exists!$ $N \in \mathrm{Conn}^{reg}(Z)$ st. $DR_Z N \simeq L[d_Z]$.

Now set $M^\cdot = \int_i N[-d_Z] \in D^b_{rh}(\mathcal{D}_X)$.

$\Rightarrow$ $DR_X(M^\cdot) = DR_X \int_i N[-d_Z] \simeq Ri_* DR_Z N[-d_Z] \simeq Ri_* L = F^\cdot$ ※

Fully faithful (sketch) :

In fact, $R\mathrm{Hom}_{\mathcal{D}_X}(M^\cdot, N^\cdot) \boxed{\simeq} \underline{DR_{pt}} \circ \int_p \mathbb{D}_X M^\cdot \otimes^L_{\mathcal{O}_X} N^\cdot$

$\simeq Rp_* \underline{DR_X} \Delta^+(\mathbb{D}_X M^\cdot \boxtimes N^\cdot) \simeq Rp_* \Delta^! (DR_{X\times X}(\mathbb{D}M^\cdot \boxtimes N^\cdot))$

$$\begin{tikzcd} \Delta : X \arrow[r, hook] & X \times X \arrow[d, "p"] \\ & pt \end{tikzcd}$$
$DR_{pt} = Id$.

$\simeq Rp_* \Delta^! (\underbrace{DR_X \mathbb{D}M^\cdot}_{\simeq\, \mathbb{D}_X DR_X M^\cdot} \boxtimes DR_X N^\cdot)$ ≃ $R\mathrm{Hom}_{\mathbb{C}_X^{an}}(DR_X M^\cdot, DR_X N^\cdot)$ by similar reason as $\boxed{\simeq}$ via Verdier ※

In the complex analytic category, we may prove :

Thm : $DR_X(M) \simeq \mathbb{D}_X \mathrm{Sol}_X(M)[d_X]$ for $M \in \mathrm{Mod}_h(\mathcal{D}_X)$, as perverse sheaves.

pf : Let $Z = \mathrm{supp}\, H^j(\underline{F^\cdot[d_X]})$ it suffices to show $d_Z \leq -j$ ($\forall M$, via duality).

Step 1 ($\exists$ Whitney stratification) in pf of Kashiwara $\Rightarrow \exists Y$ non-char/M. st.

Cauchy-Kowalevski $\Rightarrow$ $\underline{F[d_X]}|_Y \simeq R\mathcal{H}om_{\mathcal{D}_Y}(M_Y, \mathcal{O}_Y)[d_X]$ $\Rightarrow$ $\mathcal{E}xt^{j+d_X}_{\mathcal{D}_Y}(M_Y, \mathcal{O}_Y)_z \neq 0 \Rightarrow j + d_X \leq d_Y$ ※

$Y \pitchfork Z$, $Z \subset X_\alpha$ same dim.

Ex. In alg. cat. we get $DR_X : \mathrm{Mod}_{rh}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Perv}(\mathbb{C}_X)$.