Meromorphic Connections (Deligne's theory)

Def'n: Mero. conn. (M, ∇) over $B_r^1 = \{|t| < r\}$ as in §5-1. $K = \mathbb{C}\{\{x\}\}[x^{-1}]$, $M \simeq K^{\oplus m}$, with conn. ∇.
(M, ∇) is regular if $\exists$ f.g. $\mathcal{O}$-submodule $L \subset M$ gen M over K and $x\nabla L \subset \Omega^1 \otimes L$ $\Rightarrow$ $L \simeq \mathcal{O}^{\oplus m}$ (lattice)
over alg. curve C, reg. at $p \in C$ is defined by using $K_{C,p}$. Alg. mero. conn is reg at $p \Leftrightarrow (M^{an}, \nabla)$ is
Def'n/Lemma: $M \in \mathrm{Mod}_c(\mathcal{D}_C)$. Then $M \in \mathrm{Mod}_h(\mathcal{D}_C) \Leftrightarrow$ M is generic an int. conn. [$\dim \mathrm{Ch}(M) = 1$].
M is called reg $\triangleq$ the alg. mero. ext. $j_* M$ is reg. at all $p \in \bar{C}$ $\Leftrightarrow$ $\exists$ $C_0 \subset C$, $M|_{C_0}$ a reg. int. conn.
Lemma: Reg. is inv. under $f^\dagger$, $\int_f$ for $f: C \to C'$ dominant.
Example: $j: U = \mathbb{C}^\times \hookrightarrow \mathbb{C}$, $M = \mathcal{D}_U/\mathcal{D}_U \partial$ reg., $N = \mathcal{D}_U/\mathcal{D}_U(x^2\partial - 1)$ irreg.

$M^{an} \simeq N^{an}$ by $p \mapsto p \exp(1/x)$!!

X cpx mfd. $E \subset X$ div. $Y = X \setminus E$.

locally $\mathcal{O}_X[E] = \mathcal{O}_X(*E) = \mathcal{O}_X[h^{-1}] = \mathcal{O}_X[t]/(th-1)$ for "$(h) = E$",

Def'n: Category $\mathrm{Conn}(X; E) \ni (M, \nabla)$ s.t. $M \in \mathrm{Mod}_c(\mathcal{O}_X(*E))$ and $\nabla: M \to \Omega^1_X \otimes_{\mathcal{O}_X} M$. So $M|_Y \in \mathrm{Conn}(Y)$.
clearly $\mathrm{Conn}(X; E) \subset \mathrm{Mod}(\mathcal{D}_X)$ (in fact in $\mathrm{Mod}_h(\mathcal{D}_X)$, will see this in alg case). Ab. cat.
$\varphi: (M, \nabla) \to (N, \nabla)$ is $\mathcal{O}_X(*E)$-linear and $\nabla \circ \varphi = (\mathrm{id} \otimes \varphi) \circ \nabla$, i.e. a $\mathcal{D}_X$-mod hom.
Def'n: $M \otimes_{\mathcal{O}_X(*E)} N$, $\mathcal{H}om_{\mathcal{O}_X(*E)}(M, N)$, $M^* = \mathcal{H}om_{\mathcal{O}_X(*E)}(M, \mathcal{O}_X(*E))$ have obvious def'n of ∇.
Prop: $\varphi|_Y$ isom $\Rightarrow$ φ isom. In fact, $\mathrm{Mod}_c^E(\mathcal{O}_X(*E)) = 0$. In particular, $M^{**} \simeq M$.
pf: $s \in M$, $\mathcal{O}_X s$ supp in E $\Rightarrow$ $h^N s = 0$ $\Rightarrow$ $s = h^{-N} \cdot h^N s = 0$ ※ via Nullstellensatz
Lemma: let $f: Z \to X$ s.t. $f^{-1}E$ is a div. Let $M \in \mathrm{Conn}(X; E)$, Then
$Lf^* M \simeq H^0 Lf^* M = f^* M = \mathcal{O}_Z(*f^{-1}E) \otimes_{f^{-1}\mathcal{O}_X(*E)} f^{-1}M \in \mathrm{Conn}(Z; f^{-1}E)$.
pf: $\mathcal{O}_X(*E)$ is flat over $\mathcal{O}_X$ $\Rightarrow$ $\mathcal{O}_Z \otimes^L_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{O}_X(*E) \simeq \mathcal{O}_Z \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{O}_X(*E) = \mathcal{O}_Z(*f^{-1}E)$.
$\Rightarrow$ $Lf^* M := \mathcal{O}_Z \otimes^L_{f^{-1}\mathcal{O}_X} f^{-1}M \simeq \mathcal{O}_Z(*f^{-1}E) \otimes^L_{f^{-1}\mathcal{O}_X(*E)} f^{-1}M \simeq \mathcal{O}_Z(*f^{-1}E) \otimes_{f^{-1}\mathcal{O}_X(*E)} f^{-1}M$ ※ why?
Def'n (M, ∇) is effective if M is generated by a coherent $\mathcal{O}_X$-submodule over $\mathcal{O}_X(*E)$.
$\mathrm{Conn}^{reg}(X; E)$: (M, ∇) is reg. if $(i^* M)_0$ is regular $\forall$ disk $i: B \to X$ s.t. $i^{-1}(E) = \{0\}$
Lemma: $f: X' \to X$ proper surj, $E' = f^{-1}E$ is a div., $X' \setminus E' \simeq X \setminus E$. Then $\int_f M \simeq \int_f^0 M$, preserving eff/reg.
sketch: $N = \mathcal{O}_{X'}[D'] \otimes_{\mathcal{O}_{X'}} L$ $\Rightarrow$ $\int_f N \simeq \mathcal{O}_X[D] \otimes_{\mathcal{O}_X} Rf_* L$.

$H^k(\int_f N) = 0$ for $k \neq 0$ since $R^k f_* L$ coh & $= 0$ on $X \setminus E$. Also curve testing lifts to X' (f proper) ※

The case E is NCD and $M = \mathcal{O}_X[E] \otimes_{\mathcal{O}_X} L$ (L v.b.)

Def'n: M has log pole along E wrt L if for a basis e_i of L, $E = (x_1 \cdots x_r)$ locally:
$\nabla e_i = \sum a_{ij}^k \, dx_k \otimes e_j$,
$B^k := x_k A^k$ $k = 1, \dots, r$ and A^k, $k > r$ are holomorphic, $\Rightarrow$ regular.
Prop: Let $\mathrm{Res}^L_{E_k} \nabla := B^k|_{E_k} \in \mathrm{End}_{\mathcal{O}_{E_k}}(L|_{E_k})$. It is indep of choices of e_i's and coor. Also,
(i) $[\mathrm{Res}^L_{E_k} \nabla, \mathrm{Res}^L_{E_\ell} \nabla] = 0$ on $E_k \cap E_\ell$
(ii) $\mathrm{Res}^L_{E_k} \nabla$ is horizontal wrt. induced conn. $\bar{\nabla}$ on $L|_{E_k}$.

pf: $\partial_\ell A^k - \partial_k A^\ell = [A^k, A^\ell]$ $\Rightarrow$ (i) by Laurent expansion. Let $\bar{A}^i = A^i|_{E_k}$, $\bar{B}^k = B^k|_{E_k}$.
$\Rightarrow$ $\partial_j \bar{A}^i - \partial_i \bar{A}^j = [\bar{A}^i, \bar{A}^j]$ for $i, j \neq k$ $\Rightarrow$ get $\bar{\nabla}$ on $L|_{E_k}$.
Also $\partial_i \bar{B}^k - \cancel{x_k \partial_k A^i}|_{E_k} = [\bar{B}^k, \bar{A}^i]$, i.e. $\bar{\nabla}_i \bar{B}^k = 0$ $\Rightarrow$ (ii) ※ (eg. eigenvalues = const.)

Thm : Fix a section $\tau : \mathbb{C}/\mathbb{Z} \hookrightarrow \mathbb{C} \to \mathbb{C}/\mathbb{Z}$. Then $\forall\, M \in \mathrm{Conn}(Y)$, $Y := X \setminus E$, $\exists$ v.b. L_τ on X, $L_\tau|_Y = M$
st. (i) ∇^M extends uniquely to ∇ on $\mathcal{O}_X[E] \otimes_{\mathcal{O}_X} L_\tau$ with log pole along E wrt L_τ.
(ii) for any irred comp $E' \subset E$, the eigenvalues of $\mathrm{Res}^{L_\tau}_{E'} \nabla \subset \tau(\mathbb{C}/\mathbb{Z})$.

pf : M is det. by $\rho : \pi_1(Y) \longrightarrow GL_m(\mathbb{C})$. Locally at $p \in E$, $Y \simeq (B^\times)^r \times B^{n-r}$ and $\pi_1(Y) \simeq \mathbb{Z}^r = \bigoplus_{i=1}^{r} \mathbb{Z}\gamma_i$.
Let $C_i = \rho(\gamma_i)$, then $\exists!\ \Gamma^i$, $i = 1, \cdots, r$ with $\exp(2\pi\sqrt{-1}\,\Gamma^i) = C_i$, eigenvalues $(\Gamma^i) \subset \tau(\mathbb{C}/\mathbb{Z})$,
and $[\Gamma^i, \Gamma^j] = 0$ (Ex.). Now define the extended ∇ by $\nabla e_p = -\sum \frac{\Gamma^i_{pq}}{x_i} dx^i \otimes e_q$
where e_p's are a frame for $L_\tau \simeq \mathcal{O}_X^{\oplus m}$ (locally at p).

Q : How to make this global ?

Key idea : (local) uniqueness up to "unique isomorphism".
Given two extension (L, ∇) and (L', ∇') with conn. 1-form ω, ω' wrt basis e_i's ; e_j''s.
$L|_Y \simeq L'|_Y \Rightarrow \exists!\, S \in GL_m(\mathcal{O}_Y)$ st. $dS = S\omega - \omega' S$ on Y. Will show both $S, S^{-1} \in M_m(\mathcal{O}_X)$.
Hartog's thm $\Rightarrow$ enough to extend over divisors $E_k^\circ = E_k \setminus \bigcup_{j \neq k} E_j$.

$\Rightarrow \quad x_k \partial_k S = S B^k - B'^k S$

$\Rightarrow \quad |x_k| \cdot \|\partial_k S\| \leq (\|B^k\| + \|B'^k\|)\, \|S\| \quad \overset{\star}{\Rightarrow} \quad S$ ext. meromorphically over E_k° (why ?)

Ex. Prove Gronwall's inequality and prove $\star$.

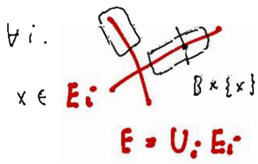
By Laurent expansion $S = \sum_{j=p}^{\infty} S_j \cdot x_i^j$ with $S_p \neq 0$.

Then $p S_p = S_p (\mathrm{Res}^{L}_{E_k} \nabla) - (\mathrm{Res}^{L'}_{E_k} \nabla') S_p$,

i.e. $(p I_m + \mathrm{Res}^{L}_{E_k} \nabla') S_p = S_p (\mathrm{Res}^{L}_{E_k} \nabla)$.

If $p \neq 0$, then by (ii) the eigenvalues of are shifted by p and no overlapped with RHS.
This's contradicts to the commutativity with $S_p \neq 0$. The pf applies to S^{-1} too. ※

For $E \subset X$ a general div, Condition (R) :



Regularity on 1-dim'l slices in $\tilde{U}$: $\emptyset \neq U = \tilde{U} \cap E_i \subset E_{reg}$

Lemma : $(N, \nabla) \in \mathrm{Conn}(X; E)$ satisfying (R) $\Rightarrow$ $\Gamma(X, N^\nabla) \xrightarrow{\sim} \Gamma(Y, N^\nabla)$. (Ex.)

Cor : $\mathrm{Hom}_{\mathrm{Conn}(X;E)}((N_1, \nabla_1), (N_2, \nabla_2)) \simeq \Gamma(X, \mathcal{H}om_{\mathcal{O}_X[E]}(N_1, N_2)^\nabla)$ is determined on Y.

Thm (Deligne) : $\mathrm{Conn}^{reg}(X; E) \simeq \mathrm{Conn}(Y)$.

pf : Reg. $\Rightarrow$ (R) $\Rightarrow$ fully-faithful by Cor.

Essential Surj : Hironaka $\Rightarrow \quad \exists f : (X', E') \longrightarrow (X, E)$ good resol.

We extend $f|^{*}_{X' \setminus E'} M$ to a log conn. $N \in \mathrm{Conn}(X'; E')$
wrt. a lattice L (exists). Then $H^0 \int_f N$ is the extension. ※

Cor : Condition (R) $\Leftrightarrow$ regularity. Also, $N \in \mathrm{Conn}^{reg}(X; E) \Rightarrow N$ is effective.