Constructibility Thm (Kashiwara's PhD Thesis)

For $f: X \to Y$ in analytic spaces, we have $f^{-1}: \mathrm{Mod}(\mathbb{C}_Y) \to \mathrm{Mod}(\mathbb{C}_X)$ exact,
$f_*, f_!$ (coh. w/proper supp) : $\mathrm{Mod}(\mathbb{C}_X) \to \mathrm{Mod}(\mathbb{C}_Y)$ left-exact, $\Rightarrow \exists\ \otimes, R\mathcal{H}om,$
$f^{-1}, Rf_*, Rf_!$ on D^b. Also, $\mathrm{Hom}_{D^b(\mathbb{C}_Y)}(Rf_! K^\cdot, L^\cdot) \simeq \mathrm{Hom}_{D^b(\mathbb{C}_X)}(K^\cdot, f^! L^\cdot)$.

Def'n : For $a_X : X \to pt$, $\omega_X^\cdot := a_X^! \mathbb{C} \in D^b(\mathbb{C}_X)$ is the dualizing cpx.

$\mathbb{D}_X F^\cdot := R\mathcal{H}om_{\mathbb{C}_X}(F^\cdot, \omega_X^\cdot)$ is the Verdier dual. If X is smooth, $\omega_X^\cdot = \omega_X = \mathbb{C}_X[2d_X]$.

Def'n : A locally finite $X = \coprod_{\alpha \in A} X_\alpha$ is a stratification if X_α is loc. closed, smooth,
$\overline{X_\alpha} = \coprod X_\beta$ some β's. A sheaf $F \in \mathrm{Mod}(\mathbb{C}_X)$ is constructible if $\exists$ stratification
st. $F|_{X_\alpha} \in \mathrm{Loc}(X_\alpha)$ $\forall \alpha$. Let $D_c^b(X) \subset D^b(\mathbb{C}_X)$ be cpx with constr. coho.

For X alg. we consider X_α also alg. and $F \in D_c^b(X) \subset D^b(\mathbb{C}_{X^{an}})$, not in $D^b(\mathbb{C}_X)$.

Thm : $\omega_X^\cdot \in D_c^b(X)$, $f^{-1}, f^!, \mathbb{D}_X$ preserve D_c^b, $\mathbb{D}_X^2 = id$, and $f^! = \mathbb{D}_X f^{-1} \mathbb{D}_Y$.
$Rf_*, Rf_!$ also prev D_c^b, if f is proper in the analytic case, and $Rf_! = \mathbb{D}_Y Rf_* \mathbb{D}_X$.
They commute with $\boxtimes$ under the same conditions.

Def'n (Peverse Sheaf) : $F^\cdot \in D_c^b(X)$. Then $F^\cdot \in \mathrm{Perv}(\mathbb{C}_X)$ if

$$\dim \mathrm{Supp}\, H^j(F^\cdot) \leq -j \quad \text{and} \quad \dim \mathrm{Supp}\, H^j(\mathbb{D}_X F^\cdot) \leq -j \quad \forall j \in \mathbb{Z}.$$

GAGA for D-Modules

For X alg. Let $\iota: X^{an} \to X$ and $\iota^{-1}\mathcal{O}_X \to \mathcal{O}_{X^{an}}$, $\iota^{-1} D_X \xrightarrow{f.f.} D_{X^{an}} \simeq \mathcal{O}_{X^{an}} \otimes_{\iota^{-1}\mathcal{O}_X} \iota^{-1} D_X$.
$\Rightarrow (\cdot)^{an} : \mathrm{Mod}(D_X) \longrightarrow \mathrm{Mod}(D_{X^{an}})$ exact $\Rightarrow (\cdot)^{an} : D^b(D_X) \longrightarrow D^b(D_{X^{an}})$, prev. coherence.

Prop : $(\cdot)^{an}$ commutes with $\mathbb{D}$ on D_c^b, with pull back on D^b, and for $f: X \to Y$,
$\exists\ (\int_f M^\cdot)^{an} \longrightarrow \int_{f^{an}} (M^\cdot)^{an}$ for $M^\cdot \in D^b(D_X)$, which is isom if f is proper, $M^\cdot \in D_c^b(D_X)$.

idea of pf : It reduces to $f: X = Y \times \mathbb{P}^n \to Y$, $M^\cdot = M \in \mathrm{Mod}_c(D_X)$ and

$$\mathcal{O}_{Y^{an}} \otimes_{\iota_Y^{-1}\mathcal{O}_Y} Rf_* DR_{X/Y}(M)^k \simeq Rf_*^{an}(DR_{X^{an}/Y^{an}}(M^{an})^k) \quad \text{by Serre's GAGA} \ \otimes$$

Def'n (DR_X and Sol_X). For X sm alg. $M^\cdot \in D^b(D_X)$,

$$DR_X(M^\cdot) := DR_{X^{an}}((M^\cdot)^{an}) \in D^b(\mathbb{C}_{X^{an}}), \quad Sol_X(M^\cdot) := Sol_{X^{an}}((M^\cdot)^{an}) \in D^b(\mathbb{C}_{X^{an}})^{op}.$$

Rmk : $X = \mathbb{C}$, $P = \partial - \lambda$, $\lambda \in \mathbb{C} \setminus \mathbb{Z}$, $M = D/DP$. $e^{\lambda x} \in \mathcal{O}_{X^{an}}$ but not in $\mathcal{O}_X$!

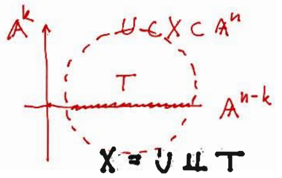
Cor : On D_c^b, DR comm. with non-char pullback and proper pushforward.

Thm : For $M^\cdot \in D_h^b(D_X)$, we have $DR_X(M^\cdot), Sol_X(M^\cdot) \in D_c^b(X)$.

Pf : (Beilinson - Bernstein) Only need to treat $M^\cdot = M \in \mathrm{Mod}_h(D_X)$. It is generically
integrable, $\Rightarrow \exists$ open dense $U \subset X$ st $DR(M|_U)' \in D_c^b(U)$. So it suffices to show
claim: $\exists$ open dense $Y \subset X \setminus U$ st it holds on $U \cup Y$.

For irred. comp. $Z \subset X \setminus U$, $\exists$ étale $V \subset X \to V' \subset \mathbb{A}^n$ st. $V \cap (X \setminus U) \hookrightarrow Z$, $V' \cap \mathbb{A}^{n-k} \hookrightarrow \mathbb{A}^{n-k}$ open dense.
$f_* DR_V(M|_V) = DR_{V'}[\int_f^0 M|_V] \Rightarrow$ reduce to affine case :

$U \subset X \subset \mathbb{A}^n$ open, $T = X \cap \mathbb{A}^{n-k} = X \setminus U$ dense in $\mathbb{A}^{n-k}$. Assume $X \hookrightarrow \mathbb{A}^k \times T$.
Let j_X, j_U, i_S, i_T be embeddings to $\mathbb{P}^k \times T \xrightarrow[p]{} T$. $X \sqcup S = \overset{\cap}{\mathbb{P}^k \times T}$



Let $N^\cdot = \int_{j_X} M$, $K^\cdot = DR_{\mathbb{P}^k\times T}(N^\cdot)$. Apply $Rp_* = Rp_!$ to

$j_{U!} j_U^{-1} K^\cdot \longrightarrow K^\cdot \longrightarrow i_{S!} i_S^{-1} K^\cdot \oplus i_{K!} i_K^{-1} K^\cdot \xrightarrow{+1}$ we get

$R(p\circ j_U)_! \underline{j_U^{-1} K^\cdot} \to \underline{Rp_* K^\cdot} \to R(p\circ i_S)_! i_S^{-1} K^\cdot \oplus R(p\circ i_T)_! i_T^{-1} K^\cdot \xrightarrow{+1}$

$\underline{\overset{\shortparallel}{DR_U}(M|_U) \in D^b_c(U)}$ $\longrightarrow$ $Rp_* DR_{\mathbb{P}^k\times T}(N^\cdot) \simeq DR_T \underline{\int_p N^\cdot}$ $\Rightarrow \exists\, Y \subset T$ open dense

$\in D^b_c(T)$ $\qquad$ $\in D^b_h(D_T)$ $\qquad$ $Rp_* K^\cdot|_{Y^{an}} \in D^b_c(Y)$.

dist. $\Delta \Rightarrow R(p\circ i_T)_! i_T^{-1} K^\cdot|_{Y^{an}} \in D^b_c(Y)$.

Since $p \circ i_T = id_T$, let $i : T \hookrightarrow X$, get $i_T^{-1} K^\cdot \simeq i^{-1} j_X^{-1} DR_{\mathbb{P}^k\times T} \int_{j_X} M \simeq i^{-1} DR_X(M)$ ※

Similar techniques lead to :

Prop : On D^b_c , get $DR_X(M^\cdot) \boxtimes_{\mathbb{C}} DR_Y(N^\cdot) \longrightarrow DR_{X\times Y}(M^\cdot \boxtimes N^\cdot)$, which is isom if one is D^b_h.

Prop : On D^b_c , have $DR_X(\mathbb{D}M^\cdot) \to \mathbb{D}_X DR_X(M^\cdot)$, $Sol_X \mathbb{D}M^\cdot \to \mathbb{D}_X Sol\, M^\cdot$, isom on D^b_h.

A sketch of Kashiwara's pf for complex Manifolds. Let $M \in Mod_h(D_X)$.

Step 1 : $\exists$ " Whitney stratification " $X = \coprod_{\alpha\in A} X_\alpha$ st. $Ch(M) \subset \coprod_{\alpha\in A} T^*_{X_\alpha} X$.

i.e. (a) if $X_\alpha \ni x_i \longrightarrow y \in X_\beta$ and $T := \lim T_{x_i} X_\alpha$ exists , then $T_y X_\beta \subset T$.

(b) If $X_\alpha \ni x_i \longrightarrow y \longleftarrow x_i' \in X_\beta$, $\alpha \neq \beta$ and $l_i = \overline{x_i x_i'}$, $T_{x_i} X_\alpha \to l, T$ exist , then $l \subset T$.

Rmk : This means the geometrical local structure is locally const along each strata .

Known thm : Any stratification of analytic set can be refined to a Whitney one .

Step 2 : Let $F^\cdot = R\mathcal{H}om_{D_X}(M, \mathcal{O}_X) \in D^b(\mathbb{C}_X)$. Then $H^j(F^\cdot)|_{X_\alpha}$ is locally constant $\forall j, \alpha$.

pf : Fix X_α , may assume $X_\alpha = \mathbb{C}^{n-d} = \{ z_1 = \cdots = z_d = 0 \} \subset X = \mathbb{C}^n$ (locally).

It suffices to show : $\forall z_0 \in X_\alpha$, $\forall j$, $\underline{\exists K := \overline{B_{z_0}(\varepsilon)}}$ in X_α st. $\forall z \in B_{z_0}(\varepsilon)$

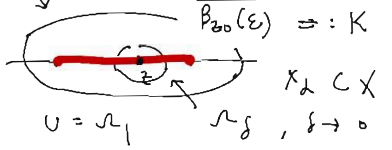
$\Gamma(K, H^j(F^\cdot)) \xrightarrow{\sim} H^j(F^\cdot)_z$. " Non-char deform. lemma " $\Rightarrow \exists$ small open nbd

$U \subset X$ of K st. $R\Gamma(U, F^\cdot) \longrightarrow F^\cdot_z$ quasi-isom.

$H^j(F^\cdot) = 0$ for $j < 0 \Rightarrow \underset{\shortparallel}{\Gamma(U, H^0(F^\cdot))} \xrightarrow{\sim} H^0(F^\cdot)_z$.

taking limt $U \to K$ we get $\Gamma(K, H^0(F^\cdot))$.

see ↓



$\partial\Omega_t$ is C^∞ real hypersurface

For $H^1(F^\cdot)|_{X_\alpha}$, similarly get $R\Gamma(K, F^\cdot) \xrightarrow{\sim} F^\cdot_z$ q.i.

$$\begin{tikzcd}
R\Gamma(K, H^0(F^\cdot)) \arrow[r] \arrow[d, "*"] & R\Gamma(K, F^\cdot) \arrow[r] \arrow[d, "S"] & R\Gamma(K, \tau^{\geq 1} F^\cdot) \arrow[r, "+1"] \arrow[d, "\star"] & {} \\
H^0(F^\cdot)_z \arrow[r] & F^\cdot_z \arrow[r] & \tau^{\geq 1} F^\cdot_z \arrow[r, "+1"] & {}
\end{tikzcd}$$

Step 1 $\Rightarrow T^*_{\partial\Omega_t}(X) \cap Ch(M) \subset T^*_X X$.

Lemma : coh. is stable when δ varies

Since $H^0(F^\cdot)$ is locally const . and K is contractible $\Rightarrow *$ is q.i. $\Rightarrow \star$ is q.i . (dist. Δ)

Take H^1 on $\star$ get $\Gamma(K, H^1(F^\cdot)) \xrightarrow{\sim} H^1(F^\cdot)_z$, $\forall z \in B_{z_0}(\varepsilon)$, $\Rightarrow$ loc. const . Now by induction ※

Step 3 : All the stalks $H^j(F^\cdot)_z$ are finite dimensional , $\forall j \in \mathbb{Z}$, $\forall z \in X$. Say $z \in X_\alpha$:

The " Non-char deform lemma " gives also $R\Gamma(B_z(\varepsilon_1), F^\cdot) \xrightarrow{\sim} R\Gamma(B_z(\varepsilon_2), F^\cdot)$ q.i , $\varepsilon_1 > \varepsilon_2$

Since the open balls $B_z(\varepsilon_i)$ are Stein , the q.i. is represented by some

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(B_z(\varepsilon_1))^{N_0} \arrow[r, "P_1"] \arrow[d] & \mathcal{O}_X(B_z(\varepsilon_1))^{N_1} \arrow[r, "P_2"] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{O}_X(B_z(\varepsilon_2))^{N_0} \arrow[r, "P_1"] & \mathcal{O}_X(B_z(\varepsilon_2))^{N_1} \arrow[r, "P_2"] & \cdots
\end{tikzcd}$$

where P_i is an $N_i \times N_{i-1}$ PD op the vertical maps are restrictions $\Rightarrow$ " COMPACT " embedding ! $\Rightarrow$ finite dimensional cohomology (Ex.) ※

Thm : $DR_X(M) \simeq \mathbb{D}_X \underline{Sol_X(M)[d_X]}$ for $M \in Mod_h(D_X)$, as perverse sheaves .

pf : Let $Z = supp\, H^j(\underline{F^\cdot[d_X]})$ it suffices to show $d_Z \leq -j$ ($\forall M$, via duality). Step 1 $\Rightarrow \exists\, Y$ non-char/M. $Y \pitchfork Z$ $Z \subset X_\alpha$ same dim.

Cauchy-Kowalevski $\Rightarrow \underline{F^\cdot[d_X]}|_Y \simeq R\mathcal{H}om_{D_Y}(M_Y, \mathcal{O}_Y)[d_X] \Rightarrow \mathcal{E}xt^{j+d_X}_{D_Y}(M_Y, \mathcal{O}_Y)_z \neq 0 \Rightarrow j + d_X \leq d_Y$ ※