Holonomic D-modules

$M \in \mathrm{Mod}_h(D_X) \Rightarrow \ell(M) < \infty$ since $0 \to L \to M \to N \to 0 \Rightarrow m_c(M) = m_c(L) + m_c(N)$.

Also $\exists$ open dense $U \subset X$ st $M|_U \in \mathrm{Conn}(U)$ since $Ch(M)$ is conic with dim $= d_X$.

Prop: Let $M \in \mathrm{Mod}_{qc}(D_X)$. If $\exists$ holonomic $N \subset M|_U$ then $\exists$ holonomic $\tilde N \subset M$ st. $\tilde N|_U = N$.

pf: $\exists$ D_X-coherent $\tilde N$ which extends N (ok). Replace M by such $\tilde N$.

$L := H^0(\mathbb{D}M) \in \mathrm{Mod}_h(D_X)$ since $\mathrm{codim}\, Ch\, H^i(\mathbb{D}M) \geq d_X + i$ (let $i = 0$)

So $\underline{\tilde N := \mathbb{D}L} \in \mathrm{Mod}_h(D_X)$ too. (same Ch). Claim: $\tilde N \subset M$ and $\tilde N|_U = N$.

From Δ: $\underline{K^\cdot = \tau^{\leq -1}\mathbb{D}M} \to \mathbb{D}M \to \tau^{\geq 0}\mathbb{D}M = H^0\mathbb{D}M = L \xrightarrow{+1}$

Dualize $\Rightarrow$ $\tilde N \to M \to \mathbb{D}K^\cdot \xrightarrow{+1}$ and $\tilde N|_U = \mathbb{D}_U H^0 \mathbb{D}_U M|_U = \mathbb{D}_U^2 N = N$.

To get $\tilde N \hookrightarrow M$, will see $H^{-1}\mathbb{D}K^\cdot = 0$. In fact, $H^i\mathbb{D}\tau^{\geq -k}K^\cdot = 0$ $\underline{\forall i < 0}$, $\underline{k > 0}$. $k \gg 0$ done.

First notice $(k > 0) \Rightarrow H^{-k}(K^\cdot) = H^{-k}(\mathbb{D}M) \Rightarrow \mathrm{codim}\, Ch(\underline{H^{-k}(K^\cdot)}) \geq d_X - k$

Hence for $i < 0$, $H^i\mathbb{D}(\underline{H^{-k}(K^\cdot)}[k]) = H^{i-k}(\mathbb{D}(\underline{H^{-k}(K^\cdot)})) = 0$ since $i - k < -(d - \mathrm{codim}\, Ch)$.

Now apply induction on $k \geq 1$ in $\mathbb{D}\left(H^{-k}(K^\cdot)[k] \to \tau^{\geq -k}K^\cdot \to \tau^{\geq -(k-1)}K^\cdot \xrightarrow{+1}\right)$ ⊠

The Six Operations.

We have seen that $\mathbb{D}_X$ and $\cdot \boxtimes \cdot$ behave well on $\mathrm{Mod}_h(D_X)$, and hence on $D^b_h(D_X)$

Thm: For $f: X \to Y$, both (i) $\int_f$ and (ii) $f^\dagger$ preserve D^b_h.

In particular $\cdot \otimes^L_{\mathcal{O}_X} \cdot$ behaves well too, since $M^\cdot \otimes^L_{\mathcal{O}_X} N^\cdot = L\Delta_X^*(M^\cdot \boxtimes N^\cdot)$.

Lemma: for closed $X \hookrightarrow Y$, $M^\cdot \in D^b_c(D_X)$, then $M^\cdot \in D^b_h(D_X) \iff \int_i M^\cdot \in D^b_h(D_Y)$.

pf: $\int_i$ is exact, may assume $M^\cdot = M \in \mathrm{Mod}_c(D_X)$. But then $\dim Ch \int_i M = \dim Ch(M) + 1$. ⊠

Reduction to the case $\int_p$ for $p: \mathbb{C}^n \to \mathbb{C}^{n-1}$:

Indeed, (i) $\Rightarrow$ (ii): for $f: X = Z \times Y \to Y$, $Ch\, f^*M = Ch(\mathcal{O}_Z \boxtimes M) = T^*_Z Z \times Ch(M)$ done.

for $f = i: X \hookrightarrow Y$, $j: U = Y \setminus X \hookrightarrow Y$, get $\int_i i^\dagger M \to M \to \int_j j^\dagger M \xrightarrow{+1}$

since $j^\dagger M = M|_U \in \mathrm{Mod}_h(D_U)$ for $M \in \mathrm{Mod}_h(D_Y)$, (i) $\Rightarrow \int_j j^\dagger M \in \underline{D^b_h(D_Y)}$ Notice:

the $\Delta \Rightarrow \int_i i^\dagger M \in \mathrm{Mod}_h(D_Y)$. Then Lemma $\Rightarrow i^\dagger M \in \underline{D^b_h(D_X)}$ ⊠ not in Mod_h

Now for (i), f = closed embedding is done by Lemma, so let $f = p: Z \times Y \to Y$.

The problem is local in Y, so assume Y is affine.

Take affine covering $Z = \bigcup_{i=0}^r Z_i$, $Z \setminus Z_i$ = divisor, and $X = \bigcup_{i=0}^r X_i$, $X_i = Z_i \times Y$ affine cover.

Let $j_{i_0 \cdots i_k}: X_{i_0 \cdots i_k} = \bigcap_{p=0}^k X_{i_p} \hookrightarrow X$ (also affine), then $M \in \mathrm{Mod}_h(D_X)$ is q.-isom to

the Čech complex $0 \to C^0(M) \to C^1(M) \to \cdots \to C^r(M) \to 0$ with

$$C^k(M) = \bigoplus_{i_0 < \cdots < i_k} j_{i_0 \cdots i_k *}(M|_{X_{i_0 \cdots i_k}}) \simeq \bigoplus \int_{j_{i_0 \cdots i_k}} j^*_{i_0 \cdots i_k} M. \quad \text{(by affineness)}$$

So it suffices to show $\int_{p \circ j_{i_0 \cdots i_k}} j^*_{i_0 \cdots i_k} M \in \mathrm{Mod}_h(D_Y)$, ie. may assume X also affine

Say,
$$\begin{tikzcd} X = Z \times Y \arrow[r, hook, "i"] \arrow[d, "p"'] & \mathbb{C}^n \times \mathbb{C}^m \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"] & \mathbb{C}^m \end{tikzcd}$$

If (i) holds for α, then $\int_\beta \int_p M = \int_\alpha \int_i M \in D^b_h(D_{\mathbb{C}^m})$ and Lemma $\Rightarrow \int_p M \in D^b_h(D_Y)$ done.

Also, clearly it suffices to prove the case $p: \mathbb{C}^n \to \mathbb{C}^{n-1}$.

Weyl Calculus and Fourier Transform

Recall $D_n := D_{\mathbb{C}^n}(\mathbb{C}^n) = \bigoplus_{\alpha,\beta} \mathbb{C}\, x^\alpha \partial^\beta$ is the Weyl algebra.

Then $\Gamma(\mathbb{C}^n, \cdot) : \mathrm{Mod}_c(D_{\mathbb{C}^n}) \xrightarrow{\sim} \mathrm{Mod}_f(D_n)$ with inv. $M \mapsto \tilde{M}$. Call M holonomic if $\tilde{M}$ is.

Def^n (Fourier transf). Let $N \in \mathrm{Mod}(D_n)$. Then $\hat{N} = N$ as a group, but with a new D_n-mod str. $x_i \circ s := -\partial_i s$; $\partial_i \circ s := x_i s$. (OK since $[x_i, \partial_j] = -\delta_{ij}$ is preserved.)
$\wedge$ induces equiv. of $\mathrm{Mod}(D_n)$, $\mathrm{Mod}_f(D_n)$, hence $\mathrm{Mod}_{qc}(D_{\mathbb{C}^n})$, $\mathrm{Mod}_c(D_{\mathbb{C}^n})$.

Q: How about $\mathrm{Mod}_h(D_{\mathbb{C}^n})$? Also can we define $\wedge$ for general $\mathrm{Mod}_{qc}(D_X)$??

Prop: Consider

$$\mathbb{C}^{n-1} = \{0\} \times \mathbb{C}^{n-1} \xrightarrow{i} \mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1} \xrightarrow{p} \mathbb{C}^{n-1}.$$

Then $H^k(Li^* \hat{M}) \simeq \widehat{H^k(\int_p M)}$ for $M \in \mathrm{Mod}_{qc}(D_{\mathbb{C}^n})$, $\forall k$.

Pf: Let $N = \Gamma(\mathbb{C}^n, M)$. Then $\int_p M \simeq Rp_*\left(\underline{DR_{\mathbb{C}^n/\mathbb{C}^{n-1}}}(M)\right) \simeq p_*\left(\Omega^0_{\mathbb{C}}(M) \xrightarrow{\partial_1} \Omega^1_{\mathbb{C}}(M)\right)[1] \simeq [p_* M \xrightarrow{\partial_1} p_* M]$

$\Rightarrow \Gamma(\mathbb{C}^{n-1}, H^k \int_p M) = \begin{cases} \ker\, N \xrightarrow{\partial_1} N & k=-1 \\ \mathrm{coker}\, N \xrightarrow{\partial_1} N & k=0 \end{cases} \Rightarrow \Gamma(\mathbb{C}^{n-1}, \widehat{H^k \int_p M}) = \begin{cases} \ker\, \hat{N} \xrightarrow{x_1} \hat{N} & k=-1 \\ \mathrm{coker}\, \hat{N} \xrightarrow{x_1} \hat{N} & k=0 \end{cases}$

which is precisely $\Gamma(\mathbb{C}^{n-1}, H^k Li^* \hat{M})$ via the Koszul resolution for i. ✱

Rmk: Koszul $\longleftrightarrow$ Spencer I $\longleftarrow$(side changing)$\longrightarrow$ Spencer II $\longleftrightarrow$ de Rham.

Prop A. $M \in \mathrm{Mod}_c(D_{\mathbb{C}^n})$ is holonomic $\Leftrightarrow$ $\hat{M}$ is.

Prop B. Let $j : \mathbb{C}^\times \times \mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$. Then $M \in \mathrm{Mod}_h(D_{\mathbb{C}^n}) \Rightarrow H^0(\int_j j^\dagger M)$ is.

Rmk: j affine $\Rightarrow$ $H^k = 0$ for $k \neq 0$

Pf of Thm via Prop A & B:

For $\mathbb{C}^{n-1} \xrightarrow{i} \mathbb{C}^n \xrightarrow{p} \mathbb{C}^{n-1}$, $M \in \mathrm{Mod}_h(D_{\mathbb{C}^n})$, to show $\int_p M \in \mathrm{Mod}_h$, enough to do $i^\dagger M$. (prop + prop A $\Rightarrow$)

We have $0 \to H^0 \int_i i^\dagger M \to M \to \underline{H^0 \int_j j^\dagger M} \to H^1 \int_i i^\dagger M \longrightarrow 0$. So Prop B $\Rightarrow \int_i i^\dagger M \in D^b_h(\mathbb{C}^n) \Rightarrow i^\dagger M$ is ✱

survives in H^0, H^1.

Let $B_\cdot$ be the Bernstein filt. on D_n using tot. deg. $B_i D_n = \sum_{|\alpha|+|\beta| \le i} \mathbb{C}\, x^\alpha \partial^\beta$.
and for $M \in \mathrm{Mod}_f(D_n)$ we define good filt. $F_\cdot$ wrt. to $B_\cdot$. Then $\dim_{\mathbb{C}} F_i M < \infty$ $\forall i$.
$\Rightarrow \exists$ polynomial $\chi(M, F; T) = \frac{m}{d!} T^d + \cdots$ st. $\chi(M, F; i) = \dim_{\mathbb{C}} F_i M$ for $i \gg 0$.

Lemma: The dim. $d = d_B(M)$ and mult. $m = m_B(M)$ are indep. of F. In fact $\dim \mathrm{Ch}(\tilde{M}) = d_B(M)$
We leave it's pf as Ex. It $\Rightarrow$ Prop A since obviously $d_B(N) = d_B(\hat{N})$ for $N = \Gamma(\mathbb{C}^n, M)$ ✱

Also, $\Gamma(\mathbb{C}^n, H^0 \int_j j^\dagger M) \simeq N_{x_1} = \mathbb{C}[x][x_1^{-1}] \otimes_{\mathbb{C}[x]} N$. $\leftarrow$ Exercise.

Let $F_\cdot$ be good filt. of N wrt. $B_\cdot$. Then $F_i N_{x_1} := \mathrm{image}(F_{2i} N \to N_{x_1})$ $s \mapsto x_1^{-i} s$ is good $/ B$.
$\dim_{\mathbb{C}} F_i N_{x_1} \le \dim_{\mathbb{C}} F_{2i} N = \frac{m_B(M)}{n!}(2i)^n + O(i^{n-1}) = \frac{2^n \cdot m_B(M)}{n!} i^n + \cdots$

But this $\Rightarrow \dim \mathrm{Ch}\, \widetilde{N_{x_1}} \le n$, hence $= n$, ie. holonomic $\Rightarrow$ Prop B ✱

<u>Adjunction Formulas</u>: $\int_{f!} \equiv f_+ \equiv f_! := \mathbb{D}_Y \int_f \mathbb{D}_X$, $f^\star := \mathbb{D}_X f^\dagger \mathbb{D}_Y$ for $f : X \to Y$ on D^b_h.

Thm: $R\mathcal{H}om_{D_Y}(\int_{f!} M^\cdot, N^\cdot) \xrightarrow{\sim} Rf_* R\mathcal{H}om_{D_X}(M^\cdot, f^\dagger N^\cdot)$, and dually " $\int_f \leftrightarrow f_*$; $f^\dagger \leftrightarrow f^!$ "
$Rf_* R\mathcal{H}om_{D_X}(f^\star N^\cdot, M^\cdot) \xrightarrow{\sim} R\mathcal{H}om_{D_Y}(N^\cdot, \int_f M^\cdot)$. Get Hom version by $H^0 R\Gamma(Y, \cdot)$.

Corollary: $\exists$ $f_! \longrightarrow f_* : D^b_h(D_X) \longrightarrow D^b_h(D_Y)$ (which $\exists$ and isom for f proper as seen last time)

Pf: By Hironaka, get $\tilde{f} : \tilde{X} \to \tilde{Y}$ in a compactification. $\Rightarrow$ reduce to open $j : X \hookrightarrow Y$.
Then $\mathrm{Hom}_{D^b_h(D_Y)}(j_! M^\cdot, j_* M^\cdot) \simeq \mathrm{Hom}_{D^b_h(D_X)}(M^\cdot, j^\dagger j_* M^\cdot) \simeq \mathrm{Hom}_{D^b_h(D_X)}(M^\cdot, M^\cdot)$. So $\mathrm{id}_{M^\cdot}$ gives it. ✱