Coherent D-modules and char. variety

Def^{n}: $M \in \mathrm{Mod}_{qc}(D_X)$, a D_X-filtration $F_\cdot$ on M means $F_i M \subset F_{i+1} M$, $i \geq 0$, $\bigcup_{i=0}^{\infty} F_i M = M$ st. $F_i D_X \cdot F_j M \subseteq F_{i+j} M$. $F_\cdot$ is good if $\mathrm{gr}^F M := \bigoplus F_i M / F_{i-1} M$ is coh. over $\mathrm{gr}\, D_X$

Prop: M is coherent $/ D_X$ $\Leftrightarrow$ it has a good $F_\cdot$. (i.e. coh. in the comm. level)

pf: $\Rightarrow$ M is gen over D_X by a $\mathcal{O}_X$-coh submodule M_0 (eg. using affine cover) define $F_i M = F_i D_X \cdot M_0$ which is clearly good.

$\Leftarrow$ if $F_\cdot$ is good but M not coh $/ D_X$, then $\exists$ affine open U st $M|_U$ is not f.g. over $D_X|_U$.

but $\mathrm{gr}^F M|_U$ is f.g. by $m_i \pmod{F_{p_i-1} M|_U}$ with $m_i \in F_{p_i} M|_U$, $i = 1, \cdots, k$

Hence $F_j M|_U = \sum_{j \geq p_i} F_{j-p_i} D_X|_U \cdot m_i$ by induction on $j \geq 0$ $\cancel{\quad}$

Def^{n}: (characteristic variety). for $M \in \mathrm{Mod}_c(D_X)$ with good $F_\cdot$

$\mathrm{ch}(M) :=$ supp of $\widetilde{\mathrm{gr}^F M} := \mathcal{O}_{T^*X} \otimes_{\pi^{-1}(\pi_* \mathcal{O}_{T^*X})} \pi^{-1} \mathrm{gr}^F M$ in T^*X, it $\supset$ Supp M.

it is a cone (inv. under $\mathbb{C}^\times$), and alg. Also has scheme version $CC(M)$.

Fact: $\mathrm{ch}(M)$ is indep of choices of good $F_\cdot$ (pf omitted).

Example I. Fundamental example (PDE) $D_n^h \xrightarrow{P} D_n^k \to M \to 0$. $\sum_{j=1}^{h} P_{ij} f_j = 0$, $i = 1, \ldots, k$

For $k = l = 1$, $P(x, \partial) f = 0$, $M = D/DP$ gen. by $\bar{1}$, hence $M_0 = \mathcal{O} \pmod{DP}$.

$(F_j D + PP)/(F_{j-1} D + DP) \simeq \mathrm{gr}_j D \mod \langle \sigma_\ell(P) \rangle \Rightarrow \mathrm{gr}_\ell M \sim \mathbb{C}[x, \xi]/\langle \sigma_\ell(P) \rangle$. (Ex. 2.2.6)

For general k, l, Need Gröbner basis on Weyl algebra. $\quad \mathrm{ch}(M) \supset X$ unless $\ell = 0$!

Example II. Prop: $M \in \mathrm{Mod}(D_X)$,

$(M \in \mathrm{Mod}(\mathcal{O}_X) \Leftrightarrow$ int. conn$) \Leftrightarrow \mathrm{ch}(M) = X$. ($\Rightarrow$ easy. $\Leftarrow$ Ex.).

Example III: (1) $\mathrm{ch}(\mathcal{O}_X) = X$, i.e. zero section of T^*X. $\mathrm{ch}(D_X) = T^*X$.

(2) $\pi: X \times Y \to Y$, $M \in \mathrm{Mod}_c(D_Y) \Rightarrow \mathrm{ch}(\pi^* M) = \mathrm{ch}(\mathcal{O}_X \boxtimes M) = X \times \mathrm{ch}(M)$; $\mathrm{ch}(M \boxtimes N) = \mathrm{ch}(M) \times \mathrm{ch}(N)$.

Lemma: $i: S \hookrightarrow X$ closed, $M \in \mathrm{Mod}_c(D_S) \Rightarrow \mathrm{ch}(\int_i^{\circ} M) = \omega\, p^{-1} \mathrm{ch}(M)$ where

$$\begin{tikzcd} & T^*X|_S \arrow[dl, "p"'] \arrow[dr, "\omega"] & \\ T^*S & & T^*X \end{tikzcd}$$

Pf: Local + induction, may assume S a hypersurface $x = x_1 = 0$.

$N \simeq \mathbb{C}[\partial] \otimes_{\mathbb{C}} i_* M$. Let $G_\cdot$ good filtration of M, let

$F_j N = \sum_{\ell=0}^{j} \sum_{k \leq j} \mathbb{C} \partial^k \otimes i_* G_{j-\ell} M$, $F_\cdot$ is good for N

and $\mathrm{gr}_j^F N = \bigoplus_{\ell=0}^{j} \mathbb{C} \partial^\ell \otimes \mathrm{gr}_{j-\ell}^G M \Rightarrow \mathrm{gr}^F N \simeq \mathbb{C}[\xi] \otimes_{\mathbb{C}} \mathrm{gr}^G M \Rightarrow$ Lemma. $\cancel{\quad}$

Thm: (Bernstein inequality) $0 \neq M \in \mathrm{Mod}_c(D_X) \Rightarrow \dim \mathrm{Ch}(M) \geq \dim X$.

pf: May assume Supp $M \subsetneq X$ and Supp $M \subset S$ sm hyp-surface $\overset{i}{\hookrightarrow} X$

Kashiwara equiv. $\Rightarrow M = \int_i^{\circ} L$, $L \in \mathrm{Mod}_c(D_S)$

Lemma + induction $\Rightarrow \dim \mathrm{Ch}(M) = \dim \mathrm{Ch}(L) + 1 \geq \dim S + 1 = \dim X$. $\cancel{\quad}$

Rmk: In fact, by Sato-Kawai-Kashiwara: "$\mathrm{Ch}(M) \supset \mathrm{ch}(M)^{\perp}$" in T^*X (involutive)

Def^{n}: $M \in \mathrm{Mod}_c(D_X)$ is holonomic if $\dim \mathrm{Ch}(M) = \dim X$. (hence Lagrangian in T^*X)

Non-characteristic pullback:

Let $f: X \to Y$, $T^*X \xleftarrow{\rho} f^*(T^*Y) = X\times_Y T^*Y \xrightarrow{\varpi} T^*Y$. $T^*_XY := \rho^{-1}(T^*_XX)$ called the co-normal bundle
$\uparrow$ the zero section $\simeq X$

Def'n: $M\in Mod_c(D_Y)$. f is non-char wrt M if $\varpi^{-1}(Ch(M))\cap T^*_XY \subset X\times_Y T^*_YY \simeq X$.

It is easy to see, smooth morphism is always non-char wrt any M.

Example IV. for $f: X\hookrightarrow Y$ codim 1, $P\in D_Y$, order m, $M = D_Y/D_YP$, $Ch(M)=\{\sigma_m(P)=0\}$

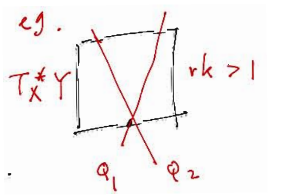
then f is non-char $/M$ $\iff$ $\sigma_m(P)(\xi)\neq 0$ $\forall \xi\in T^*_XY\setminus X$ (zero section)

in local coor $\{z_i,\partial_i\}$, $X=(z_1)$, Let (z_i,ζ_i) be coor of T^*Y, then this means

$\sigma_m(P)(0,z_2,\dots,z_n;1,0\dots,0)\neq 0$ $\forall (z_2,\dots,z_n)$, equiv. $\partial^m_{\zeta_1}\sigma_m(P)(0,z_2,\dots,z_n;0\dots0)\neq 0$.

Lemma: for $f: X\hookrightarrow Y$ codim 1, non-char wrt $M\in Mod_c(D_Y)$. Then $\forall u\in M$, locally $\exists P\in D_Y$ st $Pu=0$ and f non-char wrt D_Y/D_YP.

pf: $Ch(D_Yu)\subset Ch(M)$ $\Rightarrow$ f is non-char wrt any D_Yu. $Ch(D_Yu)$ = zero set of $gr\, I$ where $I = ann(u) = \{Q\in D_Y \mid Qu=0\}\neq 0$. Since T^*_XY has rk $=1$ $\Rightarrow$ P exists. ⋇



Theorem. $f: X\to Y$, $M\in Mod_c(D_Y)$, f non-char $/M$. Then

$Lf^*M\simeq f^*M\in Mod_c(D_X)$ and $Ch(f^*M)\subset \rho_f\varpi_f^{-1}Ch(M)$.

pf (sketch): Write $f: X\xrightarrow{\Gamma_f} X\times Y\xrightarrow{p} Y$ reduce to closed imbed.
induction on codim $\Rightarrow$ reduce to the case of $f: X\hookrightarrow Y$ codim 1 as $z_1=0$.

Now $Lf^*M\in D^b(D_X)$ is repr. by $f^{-1}M\xrightarrow{z_1} f^{-1}M$ in degree -1 & 0.

claim: $z_1\cdot$ is injective. Let $u\in M$, $z_1f^{-1}u=0$. By Lemma, $\exists P$, $Pu=0$, order m.

set $ad_{z_1}P := [z_1,P] = z_1P - Pz_1\in D_Y$. Then $ad^m_{z_1}P\in D_Y$ is a mult. by inv. fcn. cf. the Example IV.

But then $(ad^m_z P)u = 0$ $\Rightarrow$ $u=0$. (Ex. read the remaining pf.) ⋇

Proper pushforward: Recall, all var. are assumed sm. quasi-proj.

Theorem. If $f: X\to Y$ proper (projective), then $\int_f: D^b_c(D_X)\longrightarrow D^b_c(D_Y)$.

pf: Write $X\hookrightarrow Y\times\mathbb{P}^n\longrightarrow Y$. reduce to the projection case $X = Y\times\mathbb{P}^n$. Let $M^\cdot\in D^b_c(D_X)$

Thm. below $\Rightarrow$ X is "D-affine", $M^\cdot\simeq N^\cdot$, $N^i\simeq D_X^{n_i}$. (with Y affine, since thm is local)

Then $\int_p D_X = Rp_*(D_{Y\leftarrow X}\otimes^L_{D_X} D_X)\simeq Rp_*(D_Y\boxtimes\omega_{\mathbb{P}^n})\simeq D_Y\otimes_{\mathbb{C}} R\Gamma(\mathbb{P}^n,\omega_{\mathbb{P}^n})\simeq D_Y[-n]$ ⋇

Theorem: $X = Y\times\mathbb{P}^n$, with Y affine, is D-affine, ie. $\Gamma(X,\cdot)$ on $Mod_{qc}(D_X)$ is exact and inj.

pf (sketch): Let $\tilde X = Y\times(\mathbb{C}^{n+1}\setminus 0)\xrightarrow{\pi} X$, then $\mathbb{C}^\times$ acts on $\Gamma(\tilde X,\pi^*M)$.

$\Gamma(\tilde X,\pi^*M) = \bigoplus_{\lambda\in\mathbb{Z}}\gamma(M)^{(\lambda)}$ as z^λ-eigenspace. ie. $\theta u=\lambda u$ for $\theta=\sum_{i=0}^n x_i\partial_i$ on $\mathbb{C}^{n+1}$

$0\to M_1\to M_2\to M_3\to 0$ in $Mod_{qc}(D_X)$ $\Rightarrow$ $0\to j_*\pi^*M_1\to j_*\pi^*M_2\to j_*\pi^*M_3\to R^1j_*\pi^*M_1$

$j: \tilde X\hookrightarrow Y\times\mathbb{C}^{n+1}$, $\uparrow k$, $Z = Y\times\{0\}$

Kashiwara $\Rightarrow$ $R^1j_*\pi^*M\simeq\int^0_k N\simeq\mathbb{C}[\partial_0,\dots,\partial_n]\otimes_{\mathbb{C}} N$, $N\in Mod_{qc}(D_Z)$

θ acts on it as $\theta(\partial^\alpha\otimes u) = \underbrace{-(|\alpha|+(n+1))}_{<0}\partial^\alpha\otimes u$ (Exercise, cf. Ex 1.3.5)

$\Gamma(Y\times\mathbb{C}^{n+1},\cdot)$ exact, $\Gamma(Y\times\mathbb{C}^{n+1}, j_*\pi^*M_i)\simeq\Gamma(\tilde X,\pi^*M_i)$, (0) part $=\Gamma(X,M_i)$
$\uparrow$ affine

$\Gamma(Y\times\mathbb{C}^{n+1}, R^1j_*\pi^*M_1)^{(0)} = 0$ $\Rightarrow$ $\Gamma(X,\cdot)$ exact ⋇