Inverse image $Lf^*: D^b_{qc}(\mathcal{D}_Y) \to D^b_{qc}(\mathcal{D}_X)$ for $f: X \to Y$ $\quad f^\dagger := Lf^*[\dim X - \dim Y]$

by def^{n} Lf^*, $f^\dagger$ are functors, but not on D^b_c (e.g. $\mathbb{C}^n \hookrightarrow \mathbb{C}^N$).

Prop: for $f: X \to Y$ smooth morphism, $Lf^* = f^*: \mathrm{Mod}_c(\mathcal{D}_Y) \to \mathrm{Mod}_c(\mathcal{D}_X)$.

Pf: $\mathcal{O}_X$ flat $/ f^{-1}\mathcal{O}_Y$ and $Lf^*M \simeq \mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1}M$ (trivial for $j: U \hookrightarrow X$ open)

as $\mathcal{O}_X$-modules $\Rightarrow H^i(Lf^*M) = 0 \quad \forall i \neq 0$.

For Mod_c, enough to show: $\mathcal{D}_X \to \mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$ is surj. which is obv. $\divideontimes$

Prop: for $i: X^m \hookrightarrow Y^n$, $d := n - m$, (i) $H^j(Li^*M) = 0 \quad \forall j \notin [-d, 0] \quad \forall M \in \mathrm{Mod}(\mathcal{D}_Y)$

(ii) $\forall M^\cdot \in D^+(\mathcal{D}_Y)$: $i^\dagger M^\cdot \equiv Li^*M^\cdot[-d] := \underline{\mathcal{D}_{X\to Y} \otimes^L_{i^{-1}\mathcal{D}_Y} i^{-1}M^\cdot[-d]} \simeq \underline{R\mathcal{H}om_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y\leftarrow X}, i^{-1}M^\cdot)} =: Ri^\natural M^\cdot$

Pf: (i) Take Koszul resolution $0 \to K_d \to K_{d-1} \to \cdots \to K_1 \to K_0 \to \mathcal{O}_X \to 0$ of $i^{-1}\mathcal{O}_Y$-mod.

in l.c. cor $\{y_i, \partial_i\}$, $K_j = \Lambda^j\left(\bigoplus_{k=d+1}^n i^{-1}\mathcal{O}_Y\, dy_k\right) \to K_{j-1}$: $\bigwedge_{i=1}^j dy_{k_i} \mapsto \sum_{p=1}^j (-1)^{p+1} y_{k_p} \bigwedge_{i\neq p} dy_{k_i}$.

$X = (y_{d+1} = \cdots = y_n = 0)$

$\Rightarrow \quad 0 \to K_\cdot \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}\mathcal{D}_Y \to \mathcal{D}_{X\to Y} \to 0$ l.f.r. of right $i^{-1}\mathcal{D}_Y$-mod.

$\Rightarrow Li^*M = (\cdots 0 \to K_\cdot \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}M \to 0 \cdots)$ with $\mathcal{D}_X$-mod str. on $i^{-1}\mathcal{D}_Y$ $\Rightarrow$ (i).

(ii) It suffices to prove the case $M^\cdot = \mathcal{D}_Y$. And by side-changing, equiv to

Q: " $R\mathcal{H}om_{i^{-1}\mathcal{D}_Y^{op}}(\mathcal{D}_{X\to Y}, i^{-1}\mathcal{D}_Y) \overset{?}{\simeq} \mathcal{D}_{Y\leftarrow X}[-d]$ ".

$\simeq R\mathcal{H}om_{i^{-1}\mathcal{D}_Y^{op}}(\mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}\mathcal{D}_Y, i^{-1}\mathcal{D}_Y) \simeq R\mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{O}_X, i^{-1}\mathcal{D}_Y) \simeq i^{-1}\mathcal{D}_Y \otimes_{i^{-1}\mathcal{O}_Y} \underline{R\mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{O}_X, i^{-1}\mathcal{O}_Y)}$.

Using Koszul resol. RHS $= K_\cdot^*$ with $K_j^* = \mathcal{H}om_{i^{-1}\mathcal{O}_Y}(K_j, i^{-1}\mathcal{O}_Y)$.

Perfect pairing $K_j \otimes_{i^{-1}\mathcal{O}_Y} K_{d-j} \to K_d$ (l.f. rk $=1$) $\Rightarrow K_j^* \simeq K_{d-j} \otimes_{i^{-1}\mathcal{O}_Y} K_d^*$.

$\Rightarrow [K_0^* \to K_1^* \to \cdots \to K_d^*] \simeq [K_d \to K_{d-1} \to \cdots \to K_1] \otimes_{i^{-1}\mathcal{O}_Y} K_d^* \simeq \mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} K_d^*[-d]$

$\simeq i^{-1}\omega_Y^\vee \otimes_{i^{-1}\mathcal{O}_Y} \omega_X[-d]$. $\Rightarrow \star \cong \mathcal{D}_{Y\leftarrow X}[-d]$ by def of $\mathcal{D}_{X\to Y}$ and side-changing $\divideontimes$

Tensor Products are easier, as in $\mathcal{O}_X$-mod since flat$/\mathcal{D}_X$ is flat$/\mathcal{O}_X$.

- $\boxtimes$: $\mathrm{Mod}(\mathcal{D}_X) \times \mathrm{Mod}(\mathcal{D}_Y) \to \mathrm{Mod}(\mathcal{D}_{X\times Y})$ exact $\leadsto$ on D^b, D^b_{qc}, D^b_c, comm. w. pullback

$\Delta: X \to X \times X$, then $M \otimes_{\mathcal{O}_X} N = \Delta_X^*(M \boxtimes N) \Rightarrow M^\cdot \otimes^L_{\mathcal{O}_X} N^\cdot \simeq L\Delta_X^*(M^\cdot \boxtimes N^\cdot)$ in $D^b(\mathcal{D}_X)$ $\uparrow$ only ϕ !

Hence for $f: X \to Y$, $Lf^*(M^\cdot \otimes^L_{\mathcal{O}_Y} N^\cdot) \simeq Lf^*M^\cdot \otimes^L_{\mathcal{O}_X} Lf^*N^\cdot$ in $D^b(\mathcal{D}_X)$.

Direct Image $f: X \to Y$, $\int_f M^\cdot := Rf_*(\mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_X} M^\cdot)$ and $\int_f^k M^\cdot := H^k(\int_f M^\cdot)$.

Prop (Fubini): $X \xrightarrow{f} Y \xrightarrow{g} Z \Rightarrow \int_{g\circ f} = \int_g \int_f$. (pf: proj. formula + s.s. omit.)

Example: ($i: X \hookrightarrow Y$ closed) $\int_i^k M = 0$ if $k \neq 0$, $\int_i^0 M \cong \mathbb{C}[\partial_{m+1}, \dots, \partial_n] \otimes_{\mathbb{C}} i_*M$.

Prop (Adjoint) $i: X \hookrightarrow Y$ closed, $M^\cdot \in D^-(\mathcal{D}_X)$, $N^\cdot \in D^+(\mathcal{D}_Y) \Rightarrow R\mathcal{H}om_{\mathcal{D}_Y}(\int_i M^\cdot, N^\cdot) \simeq i_* R\mathcal{H}om_{\mathcal{D}_X}(M^\cdot, Ri^\natural N^\cdot)$.

which holds in the Mod level. Hence $i^\natural$ is right adj. to $\int_i^0: \mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_Y)$.

Now for projection $f: X = Y \times Z \to Y$, $g: X \to Z$

Def^{n} (Spencer resolution): $0 \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^n \Theta_X \to \cdots \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^0 \Theta_X \to \mathcal{O}_X \to 0$

and its side-changing $0 \to \Omega_X^{\cdot} \otimes_{\mathcal{O}_X} \mathcal{D}_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^n \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \omega_X \to 0$

idea of pf: use gr to reduce to Koszul on T^*X.

$\Rightarrow \int_f M = Rf_*(\mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_X} M) =: Rf_*(DR_{X/Y}(M))$. resol. for $\omega_Z \otimes_{\mathcal{D}_X} M$ get rel. deRham cpx $\Rightarrow$ all $\mathcal{O}_X$-q.c.

$\uparrow \mathrm{Mod}_{qc}(\mathcal{D}_X)$ $\qquad \overset{\shortparallel}{\mathcal{D}_Y \boxtimes \omega_Z}$ (why?)

Cor: $\int_f^j M = 0$ for $j \notin [-\dim Z, \dim Z]$, $D^b_{qc}(\mathcal{D}_X) \to D^b_{qc}(\mathcal{D}_Y)$.

More generally, for any $f: X \to Y$, X, Y smooth $\int_f: D^b_{qc}(\mathcal{D}_X) \to D^b_{qc}(\mathcal{D}_Y)$ by factoring $X \xrightarrow{\Gamma_f} X \times Y \to Y$.

Kashiwara's equivalence

Thm: Let $i : X \hookrightarrow Y$ be a closed embedding, then

i) the exact functor $\int_i^0 : Mod_{(q)c}(D_X) \xrightarrow{\sim} Mod^X_{(q)c}(D_Y)$ (i.e. D_Y-mod supp in X) is an equiv of ab. cat. with quasi-inverse $i^\natural := H^0 i^\dagger$.

ii) $N \in Mod^X_{qc}(D_Y) \Rightarrow H^j i^\dagger N = 0 \quad \forall j \neq 0$.

pf: The problem is local, by induction on codim, may assume loc. coor. $\{y_k, \partial_k\}_{k=1}^n$, $X = (y_n = 0)$. let $y = y_n$, $\theta = y\partial_y$. Then for $M \in Mod_{qc}(D_X)$, $N \in Mod^X_{qc}(D_Y)$:

$\int_i^0 M = \mathbb{C}[\partial_y] \otimes_{\mathbb{C}} i_* M$, $\quad 0 \to H^0 i^\dagger N \to i^{-1}N \xrightarrow{y} i^{-1} N \to H^1 i^\dagger N \to 0$

consider $N^j := \{ s \in N \mid \theta s = js \}$ $\quad j \in \mathbb{Z}$ $\qquad H^j i^\dagger N = 0$ for $j \neq 0, 1$

Claim: ① $N = \bigoplus_{i=1}^{\infty} N^{-i}$. $\quad N \in M^X_{qc}(D_Y) \Rightarrow$ any $s \in N$ is annihilated by some y^k

so enough to prove ② $\mathrm{Ker}(y^k : N \to N) \subset \bigoplus_{j=1}^{k} N^{-j}$, $\forall k \geq 1$.

$k=1$ ok: $ys = 0 \Rightarrow \theta s = y\partial s = (\partial y - 1)s = -s$.

In fact, $[\partial, y] = 1 \Rightarrow yN^j \subset N^{j+1}$, $\partial N^j \subset N^{j-1}$, $\partial y = \theta + 1 : N^j \xrightarrow{\sim} N^j$ if $j \neq -1$

in particular $N^j \xrightarrow[\sim]{y} N^{j+1} \xrightarrow[\sim]{\partial} N^j$ for $j < -1$.

Assume ② is proved up to $k-1 > 0$. For $0 = y^k s = y^{k-1}(ys) = 0$

$\Rightarrow ys \in \oplus_{j=1}^{k-1} N^{-j} \Rightarrow \partial y s \in \oplus_{j=2}^{k} N^{-j}$. notice $\partial y s = \underline{\theta s} + s$ ③

Also $0 = \partial(y^k s) = (y^k \partial + k y^{k-1}) s = y^{k-1}(\theta s + ks) \xrightarrow[\text{induction}]{} \underline{\theta s} + ks \in \bigoplus_{j=1}^{k-1} N^{-j}$ ④

④ − ③ $\Rightarrow (k-1)s \in \oplus_{j=1}^{k} N^{-j} \Rightarrow s \in \oplus_{j=1}^{k} N^{-j}$. i.e. ① is proved.

$\Rightarrow$ (ii) $H^1 i^\dagger N = 0$ and $i^\natural N = H^0 i^\dagger N = i^{-1} N^{-1}$.

Moreover, ① $\Rightarrow N \simeq \mathbb{C}[\partial] \otimes_{\mathbb{C}} N^{-1} \Rightarrow \int_i^0 i^\natural N = \mathbb{C}[\partial_y] \otimes_{\mathbb{C}} i_* i^{-1} N^{-1} \simeq N$. $\Rightarrow$ i) for qc.

The Mod_c case (f.g. case) is also obvious.

Corollary: $\int_i : D^b_{(q)c}(D_X) \underset{\sim}{\overset{\sim}{\rightleftarrows}} D^{b,X}_{(q)c}(D_Y) : R i^\natural$

pf: Induction on cohomology length $\ell(M^\cdot)$ on $\tau^{\leq k} M^\cdot \to M^\cdot \to \tau^{>k} M^\cdot \xrightarrow{+1}$ ※

Example: δ function. $X^m \hookrightarrow Y^n$ in local coor $\{y_j, \partial_j\}$ $\qquad$ "DE" $y_i f = 0 \ \forall i$

$\Rightarrow \int_i^0 \mathcal{O}_X = D_Y \Big/ \left(\sum_{i=1}^{m} D_Y \partial_i + \sum_{i=m+1}^{n} D_Y y_i \right)$. for $m=0$ $1 \in D_Y / D_Y m_0$ gives δ_0.

A Base Change Theorem. Assume $Y_Z := Y \times_X Z$ is also smooth in

$$\begin{tikzcd} Y_Z \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{f}"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "g"] & X \end{tikzcd}$$

Then $g^\dagger \int_f \simeq \int_{\tilde f} \tilde{g}^\dagger : D^b_{qc}(D_Y) \longrightarrow D^b_{qc}(D_Z)$.

pf: Enough to prove the cases of projection and closed embeddings. proj. case is obvious by $\mathcal{O}_T \boxtimes$.

for $g : Z \hookrightarrow X$ closed,

$$\begin{tikzcd} Y_Z \arrow[r, "\tilde{i}"] \arrow[d, "\tilde{f}"'] & Y \arrow[d, "f"] & f^{-1}(U) \arrow[l, "\tilde{j}"'] \arrow[d, "h"] \\ Z \arrow[r, "i"] & X & U = X \setminus Z \arrow[l, "j"'] \end{tikzcd}$$

$\int_{\tilde f} \tilde{i}^\dagger \simeq \underbrace{i^\dagger \int_i}_{\simeq \text{ id by Kashiwara equiv.}} \int_{\tilde f} \tilde{i}^\dagger \simeq i^\dagger \int_f \underbrace{\int_{\tilde i} \tilde{i}^\dagger}_{\to \text{ id}} \longrightarrow i^\dagger \int_f$

Apply $i^\dagger \int_f$ to ★, since $i^\dagger \int_f \int_{\tilde j} \tilde{j}^\dagger M^\cdot = i^\dagger \int_j \underbrace{\int_h \tilde{j}^\dagger M^\cdot}_{\text{in } D^b_{qc}(D_U)} = 0$. done ※

in ★ $\int_{\tilde i} \tilde{i}^\dagger M^\cdot \to M^\cdot \to \int_{\tilde j} \tilde{j}^\dagger M^\cdot \xrightarrow{+1}$ distinguished Δ, $M^\cdot \in D_{qc}(D_Y)$

Cor (Projection formula). $f : X \to Y$, $\int_f M^\cdot \otimes^L_{\mathcal{O}_X} Lf^* N^\cdot \simeq \int_f M^\cdot \otimes^L_{\mathcal{O}_Y} N^\cdot$ ※