Basics in D-modules

Def'n: X sm alg var. $D_X \subset \mathcal{E}nd(\mathcal{O}_X)$ gen by $\mathcal{O}_X$ and Θ_X.

Lemma: $\forall p \in X$, $\exists$ affine nbd U and $x_1, \dots, x_n \in \mathcal{O}_X(U)$, $\partial_1, \dots, \partial_n \in \Theta_X(U)$ st. $[x_i, x_j] = [\partial_i, \partial_j] = 0$ and $\partial_i(x_j) = \delta_{ij}$. Call $\{x_i, \partial_i\}$ local coor.

pf: $\mathcal{O}_{X,p}$ regular local ring $\Rightarrow$ $\exists\, x_i \in m_p$ st. dx_i basis of $\Omega^1_{X,p}$ free $\mathcal{O}_{X,p}$-module $\Rightarrow \exists\, U \ni p$ affine st. $\Omega^1_X(U)$ free with basis dx_i over $\mathcal{O}_X(U)$. Let ∂_i dual base ⌘

Example: $X = \mathbb{C}^n$, $D_n = D_{\mathbb{C}^n} = \mathbb{C}[x_1, \cdots, x_n; \partial_1, \cdots; \partial_n]$ is the Weyl algebra.

Def'n: $F_\ell D_X(V) := \{ P \in D_X(V) \mid P \in \bigoplus_{|\alpha| \le \ell} \mathcal{O}_U \partial^\alpha \ \ \forall U \subset V$ with local coor.$\}$

$$gr(D_X) = \bigoplus_{\ell=0}^{\infty} gr_\ell D_X = \bigoplus_{\ell=0}^{\infty} F_\ell D_X / F_{\ell-1} D_X \simeq \pi_* \mathcal{O}_{T^*X}$$

principal symbol $P \longmapsto \sigma_\ell(P)$ a regular function on T^*X

eg. $gr\, D_n \simeq \mathbb{C}[x_1, \cdots, x_n; \xi_1, \cdots, \xi_n]$ commutative poly ring

Lemma: A (left) D_X-module str on an $\mathcal{O}_X$ module M is equiv. to $\nabla : \Theta \longrightarrow \mathcal{E}nd_{\mathbb{C}}(M)$, $\theta(s) = \nabla_\theta s$ st. ① $\nabla_{f\theta} s = f \nabla_\theta s$, ② $\nabla_\theta(fs) = \theta(f) s + f \nabla_\theta s$ and "integrability condition" ③ $\nabla_{[\theta_1, \theta_2]} s = [\nabla_{\theta_1}, \nabla_{\theta_2}] s$.

pf: These are defining relations of $D_X(U)$: $[\theta, f] = \theta(f)$ and $[\partial_i, \partial_j] = 0$ ⌘

Def'n: If M is locally free of finite rk, call integrable conn, $M \in Conn(X)$.

Thm: A D_X-module M is coherent $/\mathcal{O}_X$ $\Leftrightarrow$ integrable connection.

pf: $\Rightarrow$ Only need to show M is locally free: i.e. M_x is free over $\mathcal{O}_{X,x}$

Let $\{x_i, \partial_i\}$ coor. sys. m_x gen by $x_1, \cdots, x_n$

Nakayama lemma: $\exists\, s_1, \cdots, s_m$ gen M_x st. $\bar{s}_1, \cdots, \bar{s}_m$ basis of $M_x / m_x M_x$

choose $\sum_{i=1}^{m} f_i s_i = 0$ with minimal m_x order of $\{f_i\}$

$\Rightarrow \exists\, j$ st. $\sum_{i=1}^{m} (\partial_j f_i) s_i + f_i(\partial_j s_i) = 0$ a non-trivial rel. order $\downarrow$ $\ast$

Lemma: A right D_X-module str. on an $\mathcal{O}_X$-module M is equiv. to $\nabla' : \Theta \to \mathcal{E}nd_{\mathbb{C}}(M)$ via $s\theta = -\nabla'_\theta s$ st. ①' $\nabla'_{f\theta} s = \nabla'_\theta(fs)$, with the other 2 properties the same ②', ③'.

Example / Ex: $\omega_X = \Omega_X = \wedge^n \Omega^1_X$ is a right D_X-module via Lie derivative:

$$(L_\theta \omega)(\theta_1, \cdots, \theta_n) := \theta(\omega(\theta_1, \cdots, \theta_n)) - \sum_{i=1}^{n} \omega(\theta_1, \cdots, [\theta, \theta_i], \cdots, \theta_n)$$

In local coor. get $(f\, dx_1 \wedge \cdots \wedge dx_n) P(x, \partial) = ({}^t P(x, \partial) f)\, dx_1 \wedge \cdots \wedge dx_n$

$\sum a_\alpha \partial^\alpha \longmapsto \sum (-\partial)^\alpha a_\alpha$ formal adjoint.

Prop: $M, N \in Mod(D_X)$, $M', N' \in Mod(D_X^{op})$ $\Rightarrow$ $M' \otimes_{\mathcal{O}_X} N \in Mod(D_X^{op})$; $(s' \otimes t)\theta := s'\theta \otimes t - s' \otimes \theta t$

$\mathcal{H}om_{\mathcal{O}_X}(M', N') \simeq M'^{\vee} \otimes_{\mathcal{O}_X} N' \in Mod(D_X)$; $(\theta\psi)(s) := -\psi(s)\theta + \psi(s\theta)$

prop (side-changing operation) $\quad \mathrm{Mod}(D_X) \xrightarrow{\sim} \mathrm{Mod}(D_X^{op})$ via $\omega_X \otimes_{O_X} (\cdot)$

with inverse $\omega_X^\vee \otimes_{O_X} (\cdot) = \mathcal{H}om_{O_X}(\omega_X, \cdot)$.

Inverse image. let $f : X \to Y$ with X, Y smooth, $M \in \mathrm{Mod}(D_Y)$

O module inverse $\quad f^* M = O_X \otimes_{f^{-1}O_Y} f^{-1}M$

D_X-module str : $\quad \underset{\in D_X}{\theta} (\underset{\in M}{\psi} \otimes s) = \theta(\psi) \otimes s + \psi \sum_{i=1}^{n} \theta(y_i \circ f) \otimes \partial_i s$

in local coor $\{y_i, \partial_i\}$ $\qquad$ i.e. chain rule

Def'n (Transfer module I) $\quad D_{X\to Y}$ is the $(D_X, f^{-1}D_Y)$-module $f^* D_Y = O_X \otimes_{f^{-1}O_Y} f^{-1}D_Y$.

$\Rightarrow f^* M = O_X \otimes_{f^{-1}O_Y} (f^{-1}D_Y \otimes_{f^{-1}D_Y} f^{-1}M) \simeq D_{X\to Y} \otimes_{f^{-1}D_Y} f^{-1}M$

$f^* = D_{X\to Y} \otimes_{f^{-1}D_Y} (\cdot) : \mathrm{Mod}(D_Y) \longrightarrow \mathrm{Mod}(D_X)$ is a right exact functor $(fg)^* = g^* f^*$.

Example : $\iota : \mathbb{C}^n \hookrightarrow \mathbb{C}^N$, $\quad D_{n\to N} := D_{\mathbb{C}^n \hookrightarrow \mathbb{C}^N} \cong D_n \otimes_{\mathbb{C}} \mathbb{C}[\partial_{n+1}, \cdots, \partial_N]$

which is locally free D_n-module of ∞ rank.

Direct image. let $M \in \mathrm{Mod}(D_X^{op}) \Rightarrow \underset{\text{sheaf theoretic direct image}}{f_*} (M \otimes_{D_X} D_{X\to Y}) \in \mathrm{Mod}(D_Y^{op})$

Get left module f_* by side-changing :

Def'n (Transfer module II) $\quad D_{Y\leftarrow X} := \omega_X \otimes_{O_X} D_{X\to Y} \otimes_{f^{-1}O_Y} f^{-1}\omega_Y^\vee$: $(f^{-1}D_Y, D_X)$-module

$\Rightarrow \underset{\text{left exact}}{f_*} (\underset{\text{right exact}}{D_{Y\leftarrow X} \otimes_{D_X} (\cdot)}) : \mathrm{Mod}(D_X) \longrightarrow \mathrm{Mod}(D_Y)$ $\qquad$ So No exactness !!

Correct Def'ns : "f_*" : $D^b(D_X) \longrightarrow D^b(D_Y)$ is defined as $\int_f M^\cdot := Rf_*(D_{Y\leftarrow X} \otimes^L_{D_X} M^\cdot)$

and "f^*" : $D^b(D_Y) \longrightarrow D^b(D_X)$ by $Lf^* M^\cdot := D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}M$

Example : $D_{N\leftarrow n} \simeq \mathbb{C}[\partial_{n+1}, \cdots, \partial_N] \otimes_{\mathbb{C}} D_n$. $\quad$ (Ex. check it as in Example 1.3.5)

Some categories of D-modules : $\quad D_X$ loc. free $/O_X \Rightarrow$ q.c. O_X-module

$\mathrm{Mod}_{qc}(D_X) \subset \mathrm{Mod}(D_X)$ is the subcat of O_X-q.c. D_X-module $\Rightarrow$ abelian cat.

Def'n (Coherence) : R sheaf of rings on X (top space)

$M \in \mathrm{Mod}(R)$ is coherent if M is loc. f.g. and $\forall$ open $U \subset X$, any l.f.g. submodule of $M|_U$ is l.f. presented.

Prop : $M \in \mathrm{Mod}(D_X)$ is coherent $/D_X \iff M$ is O_X-q.c. and l.f.g. $/D_X$

pf : $\Rightarrow$ is easy. $\Leftarrow$: let $U \subset X$ affine open, for any

$\mathrm{Ker}(\alpha) \hookrightarrow D_U^p \xrightarrow{\alpha} M|_U$ $\qquad$ since $D_U(U)$ is left Noetherian ($\Leftarrow$ gr $D_U(U)$ is)

q : f.g. over D_U ? $\qquad$ get $D_U(U)^q \to D_U(U)^p \longrightarrow M(U)$ for some q

But $\Gamma(U, \cdot)$ is exact $\Rightarrow$ Q. $\divideontimes$

Cor : D_X is a coherent sheaf of rings. i.e. coh over itself.

Prop : Let X be q.proj. Then any $M \in \mathrm{Mod}_{qc}(D_X)$ is a quotient of loc. free D_X-module

if $M \in \mathrm{Mod}_c(D_X)$, then it can be taken to be of finite rank.