LECTURES ON D-MODULES

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1. Introduction

The theory of *D-Modules* is also known as *Algebraic Analysis*. It studies the *algebraic structures of systems of linear partial differential equations* (PDE). Though this lecture note is mainly on the algebraic side, the analytic side is also reviewed briefly whenever needed. In fact the analytic theory is indispensable when we discuss the Riemann–Hilbert correspondences. In this introduction we present the theory in the complex analytic setting with the hope that some concepts are intuitively clearer even though their precise definitions are not given.

Let $X \subset \mathbb{C}^n$ be an open subset, $\mathcal{O} = \mathcal{O}(X)$ be the space of the holomorphic functions on X, and D be the $\mathbb{C}$-vector space of all linear partial differential operators on X with coefficients in $\mathcal{O}$. Then D is a non-commutative $\mathbb{C}$-algebra generated by $\mathcal{O}$ and $\partial_1, \ldots, \partial_n$, with the Lebnitz relation

$$[\partial_i, f] = \partial_i \circ f - f \circ \partial_i = \partial_i f.$$

The ring $\mathcal{O}$ is naturally a left D-module. For $P \in D$, the study of the PDE:

$$Pu = 0$$

concerns about the solvability/construction of the solution u, in a suitable function space $\mathcal{F} \supset \mathcal{O}$ which admits a natural left D-action extending the one on $\mathcal{O}$, and analyzing the local as well as local structure of it. To put it into an algebraic framework, we consider the left D-module

$$M = D/DP.$$

Then

$$\begin{aligned}\mathrm{Hom}_D(M, \mathcal{F}) &= \mathrm{Hom}_D(D/DP, \mathcal{F})\\ &\cong \{\, \phi \in \mathrm{Hom}_D(D, \mathcal{F}) \mid \phi(P) = 0 \,\}\\ &\cong \{\, u \in \mathcal{F} \mid Pu = 0 \,\},\end{aligned}$$

where we use the identification

$$\mathrm{Hom}_D(D, \mathcal{F}) \cong \mathcal{F}, \qquad \phi \mapsto u := \phi(1),$$

and then $Pu = P\phi(1) = \phi(P \circ 1) = \phi(P) = 0$.

The above consideration extends straightforwardly to a system of linear PDEs $\sum_{j=1}^m P_{ij} u_j = 0$ for $i = 1, \ldots, n$, where $P = (P_{ij}) \in M_{n\times m}(D)$. In this case we consider the left D-module $\mathcal{M}$ defined by the exact sequence

$$D^n \xrightarrow{\times P} D^m \longrightarrow M \longrightarrow 0, \tag{1.1}$$

and view the original PDE system as *one finite presentation* of the module $\mathcal{M}$. The $\mathbb{C}$-vector space of solutions in $\mathcal{F}$ is again given by the functor

$$\begin{aligned} \mathrm{Mod}(D) &\longrightarrow \mathrm{Mod}(\mathbb{C})^{\mathrm{op}}, \\ M &\mapsto \mathrm{Hom}_D(M, \mathcal{F}), \end{aligned} \tag{1.2}$$

from the category of left D-modules with finite presentation to the category of complex vector spaces. The functor is clearly contra-variant.

Simple examples in ordinary differential equations (ODE) show that there might be no global single-valued solutions u to $Pu = 0$. In practice one starts with local solutions and continues them analytically to the global ones by taking into account the monodromy effects.

Thus in order to make the idea of D-modules effective in the study of linear PDEs, it is indispensable to consider *Sheaf Theory and Cohomologies*, and then it is natural to employ *Homological Algebra* in its full strength.

Let $\mathcal{O}_X$ be the sheaf of holomorphic functions and $\mathcal{D}_X$ be the sheaf of linear differential operators generated by $\mathcal{O}_X$ and the tangent sheaf Θ_X. A module which admits a finite presentation locally at each point as in (1.1) is known as a *coherent* module. Thus a (left) coherent $\mathcal{D}_X$-module $\mathcal{M}$ is precisely the global and intrinsic notion of linear PDEs. The analogous "solution functor" counting holomorphic solutions as in (1.2) is

$$\begin{aligned} \mathrm{Mod}_c(\mathcal{D}_X) &\longrightarrow \mathrm{Mod}(\mathbb{C}_X)^{\mathrm{op}}, \\ \mathcal{M} &\mapsto \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X), \end{aligned}$$

which is from the category of (left) coherent $\mathcal{D}_X$-modules to category of sheaves of $\mathbb{C}$-vector spaces. With this setup, then we may allow the space X to be a general complex manifold.

While both categories in (1.2) are abelian categories, the functor is in general only left exact and not exact. This means that the naive solution functor does not capture all informations contained in the $\mathcal{D}_X$-module. The natural framework to resolve this issue is the natural extension of the problem in (bounded) derived categories and derived functors. Namely,

$$\begin{aligned} \mathbf{D}^b_c(\mathcal{D}_X) &\longrightarrow \mathbf{D}^b(\mathbb{C}_X)^{\mathrm{op}}, \\ \mathcal{M}^\bullet &\mapsto \mathrm{Sol}(\mathcal{M}^\bullet) := R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{O}_X). \end{aligned} \tag{1.3}$$

The (derived) category of sheaves of complex vector spaces is a rather large and loose category, and the formalism is not practically useful if the solution functor Sol does not admit more *rigid structures*. The typical examples of $\mathcal{D}_X$-modules come from integrable (i.e. flat) connections

$$\nabla : \mathcal{M} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M},$$

where $\mathcal{M}$ is a locally free sheaf (over $\mathcal{O}_X$) of finite rank r. The $\mathcal{D}_X$-module structure is given by the left action: for $\theta \in \Theta_X$, $s \in \mathcal{M}$,

$$\theta\, s := \nabla_\theta\, s.$$

In this case the classical Frobenius theorem implies that $\mathrm{Sol}(\mathcal{M})$ is a local system, i.e. a locally constant sheaf of rank r generated by parallel sections. This leads to an equivalence of categories

$$\mathrm{Conn}(X) \cong \mathrm{Loc}(X) \cong \mathrm{Rep}(\pi_1(X)).$$

Simple examples show that such a finiteness property fails for general coherent $\mathcal{D}_X$-modules. Say, for $\mathcal{M} = \mathcal{D}_X$ we get $\mathrm{Sol}(\mathcal{D}_X) = \mathcal{O}_X$. Intuitively we need to insert more equations to cut out the dimension of local solutions. To make the intuition meaningful, we are lead to consider the characteristic variety, or singular support,

$$\mathrm{Ch}(\mathcal{M}) \equiv \mathrm{SS}(\mathcal{M}) \subset T^*X$$

associated to a coherent $\mathcal{D}_X$-module $\mathcal{M}$. It is a *conic subvariety* of the cotangent bundle, locally on $T^*X|_U = T^*U$ defined by the top symbols

$$\sigma_{\mathrm{top}}(P_i)(x,\xi) = 0, \qquad i = 1,\ldots,n,$$

where $\mathcal{M}|_U \cong \mathcal{D}_U / \sum_{i=1}^n \mathcal{D}_U P_i$ and $(x,\xi) \in T^*U$. In the general situation, namely $m > 1$ in the local presentation (1.1), we need to invoke more algebra on graded modules to define this notion.

The celebrated *Bernstein's inequality* says that for $\mathcal{M} \in \mathrm{Mod}_c(\mathcal{D}_X)$,

$$\dim \mathrm{Ch}(\mathcal{M}) \geq \dim X. \tag{1.4}$$

In fact, $\mathrm{Ch}(\mathcal{M})$ is *involutive* with respect to the canonical symplectic structure on T^*X, hence (1.4) holds on each irreducible component of $\mathrm{Ch}(\mathcal{M})$. A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is called *holonomic* if the equality holds in (1.4).

Holonomic modules form an abelian and artinian sub-category

$$\mathrm{Mod}_h(\mathcal{D}_X) \subset \mathrm{Mod}_c(\mathcal{D}_X).$$

It appears to be the fundamental notion in D-Modules theory due to the finiteness property proved in Kashiwara's PhD thesis:

$$\mathcal{M} \in \mathrm{Mod}_h(\mathcal{D}_X) \Longrightarrow \mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \text{ is constructible for all } i \in \mathbb{Z}.$$

This is equivalent to saying that all the stalks of $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O})$ are finite dimensional $\mathbb{C}$-vector spaces. Hence the solution functor is an exact functor

$$\mathbf{D}^b_h(\mathcal{D}_X) \xrightarrow{R\mathcal{H}om_{\mathcal{D}_X}(\bullet, \mathcal{O}_X)} \mathbf{D}^b_c(\mathbb{C}_X)^{\mathrm{op}} \tag{1.5}$$

between the derived category of holonomic modules and the derived category of $\mathbb{C}$-constructible sheaves.

Instead of working on the opposite category, we may use duality on $\mathcal{D}_X$-modules to get an equivalent form of (1.5):

$$\mathbf{D}^b_h(\mathcal{D}_X) \xrightarrow{\mathrm{DR}_X(\bullet)} \mathbf{D}^b_c(\mathbb{C}_X), \tag{1.6}$$

by using the *de Rham functor*

$$\mathcal{M}^\bullet \mapsto \mathrm{DR}_X(\mathcal{M}^\bullet) := \Omega_X^{\dim X} \otimes^L_{\mathcal{D}_X} \mathcal{M}^\bullet. \tag{1.7}$$

In the special case $\mathcal{M}^\bullet = \mathcal{M} \in \mathrm{Mod}_c(\mathcal{D}_X)$, the complex in (1.7) is simply the usual de Rham complex located at indices $-\dim X, \ldots, -1, 0$:

$$\mathrm{DR}_X(\mathcal{M}) = [\,\Omega_X^0 \xrightarrow{d} \ldots \xrightarrow{d} \Omega_X^{\dim X - 1} \xrightarrow{d} \Omega_X^{\dim X}\,] \otimes_{\mathcal{O}_X} \mathcal{M}.$$

In the D-Modules context, the *Riemann–Hilbert problem* becomes a problem in analyzing the properties of the solution functor in (1.5), or equivalently of the de Rhame functor in (1.6). For example, one may ask if they induce equivalence of categories? If not, how should one modify the categories (and/or the functors) to make them equivalent?

The problem was solved in its full generality when one impose regularity on the $\mathcal{D}_X$-modules. Indeed, let E be a divisor in X not necessarily of normal crossing, and $\mathrm{Conn}^{\mathrm{reg}}(X; E)$ be the category of regular meromorphic connections on X with poles along E. Here a connection is regular if its restriction to every curve is a regular ODE. Then Deligne proved the following version of Riemann–Hilbert correspondence in 1970:

$$\mathrm{Conn}^{\mathrm{reg}}(X; E) \cong \mathrm{Conn}(X \setminus E). \tag{1.8}$$

That is, any integrable connection on $X \setminus E$ extends uniquely to a regular meromorphic connection along the divisor E.

The notion on regularity can be defined on holonomic $\mathcal{D}_X$-modules to get a sub-category $\mathrm{Mod}_{rh}(\mathcal{D}_X)$, and hence also on its derived category $\mathbf{D}^b_{rh}(\mathcal{D}_X)$ by requiring each cohomology module being regular holonomic. Based on Deligne's equivalence (1.8) and the theory of D-Modules, Kashiwara in 1984, and Mebkhout in 1984 independently, proved the "by now classical" Riemann–Hilbert correspondence

$$\mathbf{D}^b_{rh}(\mathcal{D}_X) \xrightarrow{\sim} \mathbf{D}^b_c(\mathbb{C}_X) \tag{1.9}$$

under the de Rham functor $\mathrm{DR}_X(\bullet)$.

The theory of D-Modules can also be developed on smooth complex algebraic varieties and most of the discussions above carry over to the algebraic setup. In particular the correspondence (1.9) for smooth complex algebraic varieties were proved by Beilinson and Bernstein.

Remark 1.1. In some steps the algebraic theory even takes simpler structure than the analytic theory. For example, it is well known that Grothendieck's six operations

$$\otimes^L, \quad R\mathcal{H}om, \quad f^*, \quad f_*, \quad f^!, \quad f_!$$

are all defined and well-behaved on $\mathbf{D}^b_c(\mathbb{C}_X)$. This makes the theory of constructible sheaves very useful and the full power of homological methods can be applied. Thus in order for the category of $\mathcal{D}_X$-modules to be useful it is also of fundamental importance to investigate the corresponding six operations. For coherent $\mathcal{D}_X$-modules there are restrictions on the morphisms $f : X \to Y$ in order for the coherence to be preserved under f^+ or f_+ (the analogue functors of f^* and f_* respectively). For derived category of holonomic modules $\mathbf{D}^b_h(\mathcal{D}_X)$, it turns out that in the algebraic category all six operations are well behaved without any additional restriction on the morphism f involved.

It can proved that a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is generically an integrable connection. Thus one may regard the theory of algebraic D-Modules as the "homological closure" of integrable connections. To understand $\mathrm{Conn}(X)$ it is necessary to consider $\mathbf{D}^b_h(\mathcal{D}_X)$ to achieve a complete theory.

However, in the analytic category we need to impose the properness of f to show that the holonomicity is preserved under f_+.

Nevertheless in the discussion of Riemann–Hilbert correspondence one must employ analytic theory in order to define the solution functor and the de Rham functors. As a result one must take care of both the algebraic and analytic versions of Deligne's equivalence in (1.8) and study the GAGA comparison in details.

Finally, the equivalence (1.9) restricts to

$$\mathrm{DR}_X : \mathrm{Mod}_{rh}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Perv}(\mathbb{C}_X), \tag{1.10}$$

where $\mathrm{Perv}(\mathbb{C}_X)$ is the category of perverse sheaves introduced by Beilinson, Bernstein and Deligne in 1980 where they systematically developed the theory in order to apply the arithmetic results on Deligne's solution to the Weil conjectures to much more general contexts.

For example, the BBD theory of perverse sheaves leads to a new proof to the Hard Lefschetz theorem, a proof to the invariant cycle theorem, as well as the so-called *decomposition theorem* of cohomology under a proper morphism $f : Y \to X$ between singular algebraic varieties.

Remark 1.2. A remarkable observation from (1.10) is that the t-structures on both sides of (1.9) looks rather differently at a first sight. On $\mathbf{D}^b_{rh}(\mathcal{D}_X)$ one uses the standard truncation structure $\mathbf{D}^{\geq 0}$ and $\mathbf{D}^{\leq 0}$ on complexes while on $\mathbf{D}^b_c(\mathbb{C}_X)$ one puts middle-perverse t-structure. This is perhaps the first example in the literature so that a non-standard looking structure on one side becomes completely standard on the other (categorically) equivalent side.

To end this introduction we mentioned some recent advances on the D-Modules theory.

I. One of the most significant early applications of D-Modules is in representation theory. It leads to the resolution to the *Kazhdan–Lusztig conjecture* whose statement will not be recalled here. It is remarkable since at the beginning there does not seem to be any relation between representation theory and D-Modules. By now this procedure is more or less standardized under the name *Geometric Representation Theory*.

II. The theory of mixed Hodge modules (MHM) by Saito in 1980's can be regarded as a study of mixed Hodge theory under the framework analogous to that of D-Modules. As a result, the BBD decomposition theorem for perverse sheaves can also be proved in the category of MHM. Recently de Cataldo and Migliorini had found a much more elementary proof to the decomposition theorem and with more detailed description on various special cases of $f : X \to Y$ like the case of semi-small morphisms.

III. Quantum D-Modules? In contrast to **II**, which might be regarded as a theory on the B side form the mirror symmetry point of view, the theory on the A side is less developed. The notion of quantum D-module is basically just an equivalent way to talk about the Dubrovin connection on quantum cohomology. The theory is basically a theory of integrable connections over formal series. Not much is known about the analytic properties, not to say the functorial properties under morphisms. Indeed the main purpose for giving this lecture series is to investigate the possibility to develop a parallel D-Modules theory in the A side.

IV. Holonomic $\mathcal{D}_X$-modules with irregular singularities? In the study of ODE with irregular singularities we need to introduce Stokes structures to capture the "monodromy data" attached to each irregular singular points. Such an idea can be extended to certain higher dimensional cases through works of Sabbah and Mochizuki since early 2000's.

In particular for $E \subset X$ being a smooth divisor, the notions of Stokes-filtered constructible sheaves and Stokes-Peverse sheaves are well behaved and a kind of Riemann–Hilbert correspondence in the algebraic category can be formulated and proved on $\mathrm{Mod}_h(\mathcal{D}_X)$ without any assumption on the regularity.

Very recently, D'Agnolo and Kashiwara (arXiv:1311.2374) announced an analytic version of the Riemann–Hilbert correspondence in its complete and full generality. They defined the so-called category of enhanced Ind-sheaves whose definition roughly takes the form

$$\mathbf{E}^b(\mathbf{I}\mathbb{C}_X) := \mathbf{D}^b(\mathbf{I}_{X\times\mathbb{R}_\infty})/\mathbf{I}\mathcal{C}_{\{t^*\le 0\}}.$$

They also defined the enhanced de Rham functor $\mathrm{DR}_X^{\mathsf{E}}$ and established the following equivalence of categories

$$\mathrm{DR}_X^{\mathsf{E}} : \mathbf{D}_h^b(\mathcal{D}_X) \xrightarrow{\sim} \mathbf{E}^b(\mathbf{I}\mathbb{C}_X).$$

It is important to remark that no Stokes structures are mentioned in the formulation. Instead, the Stokes structure appears only as a topological consequence when one tries to write down the equivalence in local coordinates and frames.