Growth rate of Entire functions and Hadamard's factorization theorem

Thm ——— : Weierstrass product thm

$a_n \in \mathbb{C}$, $|a_n| \to \infty$, (can be repeated)

$\Rightarrow$ ① $\exists$ entire f with zeros a_n and nowhere else

② All sol of the form $f e^g$, g entire

$$\prod \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \dots + \frac{1}{k}\left(\frac{z}{a_n}\right)^k}$$

pf: ② easy : $f'/f_2 = h = e^g$

① Canonical factors

$$\log\left(1 - \frac{z}{a_n}\right) + \frac{z}{a_n} + \dots + \frac{1}{k}\left(\frac{z}{a_n}\right)^k$$

$E_0(z) = 1 - z$, $E_k(z) := (1-z) e^{z + \frac{z^2}{2} + \dots + \frac{z^k}{k}}$

$$= -\left(\frac{1}{k+1}\left(\frac{z}{a_n}\right)^{k+1} + \frac{1}{k+2}\left(\frac{z}{a_n}\right)^{k+2} + \dots\right)$$

$f(z) := z^m \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right)$ satisfies the property $*$ $\Leftarrow$ $|*| \le \frac{1}{k+1}\left|\frac{z}{a_n}\right|^{k+1} \frac{1}{1 - \left|\frac{z}{a_n}\right|}$

$k_n = n$ is OK

Remark but no good since $k_n = n$ unbounded !

in practice need only very small k_n

Def'n: order of f: entire $\le \rho$ if

$$|f(z)| \le A e^{B|z|^\rho} \text{ and } \rho_f := \inf \rho$$

eg ① poly $\Rightarrow$ order $= 0$. ① Ch.3 Ex 12.

② $\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

$\Rightarrow |\sin z| \le e^{|z|}$, $\rho \le 1$

actually $=1$ on $z = iy$

Notice that $\sum_{n \ne 0} \frac{1}{(n\pi)^s} = \frac{1}{\pi^s} \sum_{n \ne 0} \frac{1}{|n|^s}$

$< \infty \iff s > 1$ (order)

This is TRUE in general.

require Jensen's formula.

Example

$$\cot z' = \left(\frac{\cos z}{\sin z}\right)' = \frac{-\sin^2 z - \cos^2 z}{\sin^2 z} = \frac{-1}{\sin^2 z}$$

residue $\Rightarrow$

② or Poisson Summation Ch 4. Ex 7

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{-\infty}^{\infty} \frac{1}{(z-n)^2}$$

✓ ③ Liouville thm.

$\int_0^z$

$$\Rightarrow \pi \cot \pi z = \sum_{-\infty}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n}\right)$$

This can be proved directly

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \quad (\text{odd})$$

use this to deduce $\zeta(2k)$

$$\frac{\sin \pi z}{\pi} = z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) e^{g(z)}$$

Hadamard

logarithmic derivative

$\Rightarrow g(z) = 0$

Jensen formula.

① $\log|f(0)| = \underline{\sum_{k=1}^{N} \log\left|\frac{z_k}{R}\right|} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta$

pf $\quad f(z) = (z - z_1)\cdots(z - z_N)\, g(z)$

count with mult $\quad \to$ enough to prove ① $g(z)$ no vanishing ② $(z - z_j)$

①
$$g(z) = e^{h(z)}$$
$$|g(z)| = |e^{\operatorname{Re} h(z) + i \operatorname{Im} h(z)}|$$
$$= e^{\operatorname{Re} h(z)} \quad \Rightarrow \quad \log|g(z)| = \underbrace{\operatorname{Re} h(z)}_{\text{harmonic}}$$

$$\operatorname{Re} h(z) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} h(z)\, d\theta \qquad \text{OK}$$

②
$$\log|\underset{0}{z} - w| \overset{?}{=} \log\left|\frac{w}{R}\right| + \frac{1}{2\pi}\int_0^{2\pi} \log|\underset{Re^{i\theta}}{z} - w|\, d\theta .$$

ie
$$0 \overset{?}{=} \frac{1}{2\pi}\int_0^{2\pi} \log\left|e^{i\theta} - \frac{w}{R}\right| d\theta$$

$\frac{w}{R} = a$ with $|a| < 1$

$$\int_0^{2\pi} \log\underset{1}{|e^{i\theta}|}\,|1 - a e^{-i\theta}|\, d\theta$$
$$= \int_0^{2\pi} \log|1 - a e^{i\theta}|\, d\theta$$
$$= \log|1 - az|$$

the real part of $\log(1 - az)$

- MVT $\Rightarrow = \log 1 = 0$
- or using residue calculus.

Also ② $\underline{\sum_{k=1}^{N} \log\left|\frac{R}{z_k}\right| = \int_0^R n(r)\frac{dr}{r}}$

$$= \int_{|z_k|}^{R} \frac{dr}{r}$$

Application to entire fun of $\rho_f < \infty$

Thm: f order $\le \rho$

(1) $n(r) \le C r^\rho$

(2) $\sum_{k=1}^{\infty} \frac{1}{|z_k|^s} < \infty$ for $s > \rho$

non zero zero

pf: May set $f(0) \neq 0$

$$\int_0^R n(x)\frac{dx}{x} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - \log|f(0)|$$

$$\Rightarrow \int_r^{2r} n(x)\frac{dx}{x} \le \frac{1}{2\pi}\int_0^{2\pi} \log A e^{B(2r)^\rho}\, d\theta + C \sim C' r^\rho$$

$$\vee \!\!\! \vee \qquad n(r)\int_r^{2r}\frac{dx}{x} = n(r)\log 2$$

get (1).

For (2):
$$\sum_{|z_k| \ge 1} \frac{1}{|z_k|^s} = \sum_{j=0}^{\infty} \sum_{z_k \in [2^j, 2^{j+1}]} \frac{1}{|z_k|^s}$$
$$\le \sum_{j=0}^{\infty} \frac{1}{2^{js}}\, n(2^{j+1}) \le c \sum_{j=0}^{\infty} \frac{2^{(j+1)\rho}}{2^{js}} = c\, 2^\rho \sum_{j=0}^{\infty} \frac{1}{2^{j(s-\rho)}} < \infty$$

for $s > \rho$ $\square$

Hadamard factorization : ① E_k is enough $\forall n$, $k \le \rho_f < k+1$
② g is a poly of degree $\le k$. P 3

lem ① $|E_k(z)| \geq e^{-c|z|^{k+1}}$ $|z| < \frac{1}{2}$

$\geq |1-z|\, e^{-c'|z|^k}$ $|z| \geq \frac{1}{2}$

Pf : $|z| \le \frac{1}{2}$ then $E_k(z) = e^{\log(1-z) + \sum_{n=1}^{k} \frac{z^n}{n}} = e^{-\sum_{n=k+1}^{\infty} \frac{z^n}{n}}$

$|E_k(z)| \geq e^{-|\sum_{n=k+1}^{\infty} \frac{z^n}{n}|} \geq e^{-c|z|^{k+1}}$ for some $c > 0$

$|z| \geq \frac{1}{2}$ $\ni$ $|E_k(z)| = |1-z|\,|e^{z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}}| \geq |1-z|\, e^{-|z + \cdots + \frac{z^k}{k}|}$

$\geq |1-z|\, e^{-c'|z|^k}$

e.g. $\frac{|z^i|}{|z|^k} = \frac{1}{|z|^{k-i}} < \frac{1}{2^{k-i}}$

lem ② For any $k \le \rho_0 < s < k+1$ (z, order), $\exists\, c > 0$ indep of z st ☆

$\left| \prod_{n=1}^{\infty} E_k\!\left(\frac{z}{a_n}\right) \right| \geq e^{-c|z|^s}$ for $z \notin B_{a_n}\!\left(\frac{1}{|a_n|^{k+1}}\right)$ $\forall n$.

Pf: for each fixed z, write

$\prod_{n=1}^{\infty} E_k\!\left(\frac{z}{a_n}\right) = \prod_{|\frac{z}{a_n}| \geq \frac{1}{2}} E_k\!\left(\frac{z}{a_n}\right) \times \prod_{|\frac{z}{a_n}| < \frac{1}{2}} E_k\!\left(\frac{z}{a_n}\right) \geq \mathrm{I} \times \mathrm{II}$

finite prod

$|\mathrm{II}| \geq \prod e^{-c|\frac{z}{a_n}|^{k+1}} = e^{-c|z|^{k+1} \sum \frac{1}{|a_n|^{k+1}}}$ $= (k+1-s) + s$

$\geq e^{-c'|z|^s \underbrace{\sum \frac{1}{|a_n|^s}}_{\text{conv.}}}$ OK

$|a_n| \geq 2|z|$

$\frac{-1}{|a_n|} \geq \frac{-1}{2|z|}$

$|\mathrm{I}| \geq \prod_{|\frac{z}{a_n}| \geq \frac{1}{2}} |1 - \frac{z}{a_n}|\, e^{-c'|\frac{z}{a_n}|^k} = \prod |1 - \frac{z}{a_n}| \times e^{-c|z|^k \sum \frac{1}{|a_n|^k}}$ $= (k-s) + s$

$\geq \underbrace{\prod |1 - \frac{z}{a_n}|}_{\shortparallel} e^{-c'|z|^s \underbrace{\sum \frac{1}{|a_n|^s}}_{\text{OK}}}$

$|a_n| \le 2|z|$

$\left| \frac{a_n - z}{a_n} \right| \geq \frac{1}{|a_n|^{k+2}}$

$\frac{1}{|a_n|^{k-s}} \le \frac{1}{(2|z|)^{k-s}}$

$\geq e^{-(k+2) \sum_{|a_n| \le 2|z|} \log |a_n|} \geq e^{-(k+2)\, n(2|z|) \log 2|z|}$

$\frac{-1}{|a_n|^{k-s}} \geq \frac{-1}{(2|z|)^{k-s}}$

$\geq e^{-c|z|^s \log 2|z|} \geq e^{-C'|z|^{s'}}$ for any $s' > s$

By working with a smaller s at the beginning can then choose s' here to be the expected s ※

Rmk: The purpose of ~~lower estimate~~ is to get ~~upper estimate~~ on g in $f = e^g \prod E_k\!\left(\frac{z}{a_n}\right)$ in order to prove g is a polynomial

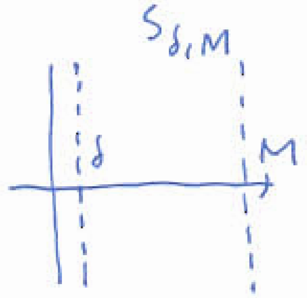
GAMMA FUNCTION

since t^{s-1} integrable

Recall $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt$ originally for $s > 0$

Prop: $\Gamma(s)$ extends to analytic on $\mathrm{Re}(s) > 0$, s+ $\mathbb{C}$

pf:



$$\left| e^{-t} t^{s-1} \right| = e^{-t} t^{\overset{\sigma}{\mathrm{Re}\, s} - 1}$$

~~$t^s = e^{s \log t}$ holo in s~~
~~since $t \in (0,\infty)$~~

$$\Gamma(s) = \lim_{\varepsilon \to 0} F_\varepsilon(s), \quad F_\varepsilon(s) := \int_\varepsilon^{1/\varepsilon} e^{-t} t^{s-1}\, dt$$

claim: $F_\varepsilon \to \Gamma$ unif on $S_{\delta,M}$ — is holo in s, for each $S_{\delta,M}$

every cpt set is contained in such a region

$$\left| \Gamma(s) - F_\varepsilon(s) \right| \le \underbrace{\int_0^\varepsilon e^{-t} t^{\sigma-1}\, dt}_{\le\, \varepsilon\, \varepsilon^{\delta-1} = \varepsilon^{\delta}} + \underbrace{\int_{1/\varepsilon}^\infty e^{-t} t^{\sigma-1}\, dt}_{\le\, e^{-t} t^{M-1} \le c\, e^{-t/2}\ \int \to 0}$$

*

Functional Eq'n I:

lemma: $\mathrm{Re}\, s > 0 \ni \Gamma(s+1) = s\Gamma(s)$

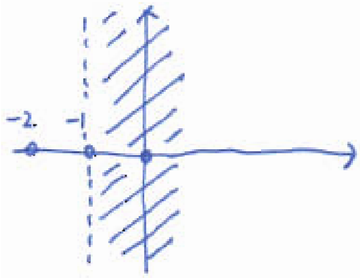
Thus $\Gamma(n+1) = n!$ for $n \in \mathbb{Z}_{\ge 0}$

pf: $\int_0^\infty e^{-t} t^s\, dt = -\int_0^\infty t^s\, d e^{-t} = t^s e^{-t} \Big|_0^\infty + \int_0^\infty e^{-t}\, d t^s$ *

Also $\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$

Thm 1: $\Gamma(s)$ has mero analy conti to $\mathbb{C}$
with (only) simple poles at $s = 0, -1, -2, \dots$
$\mathrm{Res}_{s=-n}\, \Gamma(s) = \dfrac{(-1)^n}{n!}$

pf:



For $\mathrm{Re}\, s > -1$, $F_1(s) := \dfrac{\Gamma(s+1)}{s}$

Then $F_1(s) \equiv \Gamma(s)$ in $\mathrm{Re}\, s > 0$

Similarly for $\mathrm{Re}\, s > -m$,

$$F_m(s) := \frac{\Gamma(s+m)}{s(s+1)\cdots(s+(m-1))}$$

$\equiv \Gamma(s)$ for $\mathrm{Re}\, s > 0$

$$\mathrm{Res}_{s=-n} F_m(s) = \frac{\Gamma(-n+m)}{(-n)(-(n-1))\cdots(-1)\,\hat{}\,(m-n-1)!} = \frac{(-1)^n}{n!}$$ *

Analytic conti in 1-step (reflection)

Thm 2: $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \forall s \in \mathbb{C}$

pf: enough to prove it for $0<s<1$ $\overset{s}{\underset{0 \quad\quad 1}{\circ\!\!\longrightarrow\!\!\circ}}$

$$\Gamma(1-s)\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} \left(\int_0^\infty e^{-u} u^{-s} du \right) dt$$

Trick! let $u = vt$

$$= \int_0^\infty e^{-t} t^{s-1} \left(t \int_0^\infty e^{-vt} (vt)^{-s} dv \right) dt$$

$$= \int_0^\infty \int_0^\infty e^{-(1+v)t} v^{-s} \, \underline{dt} \, dv$$

$$= \int_0^\infty \frac{v^{-s}}{1+v} dv$$

$$= \frac{\pi}{\sin \pi (1-s)} = \frac{\pi}{\sin \pi s} \quad ※$$

• Lemma (Residue): $0<a<1$

$$\int_0^\infty \frac{v^{a-1}}{1+v} dv = \int_{-\infty}^\infty \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin \pi a}$$

In fact, it's more natural to consider $\frac{1}{\Gamma(s)} = \Gamma(1-s)\frac{\sin \pi z}{\pi}$

Thm 3 (i) $\frac{1}{\Gamma(s)}$ is entire with simple zeros at $0, -1, -2, \cdots$

(ii) $\left|\frac{1}{\Gamma(s)}\right| \le c_1 e^{c_2 |s| \log |s|}$, thus of order 1

Key alternative calculation:

$$* \quad \Gamma(s) = \int_0^1 e^{-t} t^{s-1} dt + \int_1^\infty e^{-t} t^{s-1} dt$$

$$\int_0^1 e^{-t} t^{s-1} dt = \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)}$$

this holds $\forall s \in \mathbb{C}$

This also gives a pf to Thm 1

Moreover, we need it for the pf of Thm 3

Pf: (i) $\frac{1}{\Gamma(s)} = \Gamma(1-s)\frac{\sin \pi s}{\pi}$ pole $s = 1, 2, 3,$ cancel out

simple zero $s = 0, -1, -2$ remains

(ii) Apply $*$ to $\Gamma(1-s)$: (next page)

(iii) $$\frac{1}{\Gamma(s)} = e^{\gamma s} s \prod_{n=1}^\infty \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}}$$

where $\gamma := \lim_{N\to\infty} \left(\sum_{n=1}^N \frac{1}{n} - \log N \right)$

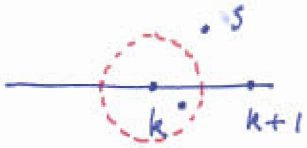
is the Euler constant

Growth ration of $1/\Gamma(s)$: ⊛

$$\frac{1}{\Gamma(s)} = \Gamma(1-s)\frac{\sin\pi s}{\pi} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\boxed{\frac{\sin(\pi s)}{n+1-s}\,\frac{1}{\pi}} + \left(\int_1^{\infty} e^{-t}t^{-s}\,dt\right)\frac{\sin\pi s}{\pi}$$

$\leq$ * below $\qquad\qquad \leq$

$e^{(\sigma+1)\log(\sigma+1)} \qquad e^{\pi|s|}$



- if $|s-k| \geq \frac{1}{2}$ $\forall k \in \mathbb{N}$ then $|⊛| \leq 2e^1 e^{\pi|s|}$ $\qquad \sigma := \mathrm{Re}(s)$ $\qquad$ Notice $\pi > 1$
- if $|s-k| < \frac{1}{2}$ for some $k \in \mathbb{N}$, then k is unique

$$\Rightarrow |⊛| \leq 2e^1 e^{\pi|s|} + \frac{1}{(k-1)!}\left|\frac{\sin\pi s}{\pi(s-k)}\right|$$

↖ bounded

(iii') Hadamard ⇒

$$\frac{1}{\Gamma(s)} = e^{As+B}\, s\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}$$

$\lim_{s\to 0} s\Gamma(s) = 1 \Rightarrow B = 0$

pole at $-n$, res $= \frac{(-1)^n}{n!}$, $n = 0, 1, 2, \cdots$

$$\Gamma(1) = 1 \Rightarrow e^{-A} = e^{\sum_{n=1}^{\infty}\left(\log\left(1+\frac{1}{n}\right)-\frac{1}{n}\right)}$$

$$= e^{\lim_{N\to\infty}\left(\log(N+1)-\sum_{n=1}^{N}\frac{1}{n}\right)} \qquad (1+\tfrac{1}{1})(1+\tfrac{1}{2})\cdots(1+\tfrac{1}{N}) = \tfrac{2}{1}\,\tfrac{3}{2}\cdots\tfrac{N+1}{N} = N+1$$

$= e^{-\gamma}$ by def, i.e. $A = \gamma + 2\pi i k$

Next topic $\qquad$ but $\Gamma(s) \in \mathbb{R}$ for $s \in \mathbb{R}$, $\Rightarrow k = 0$ $\quad A = \gamma$ □

Zeta function: $\mathrm{Re}\, s > 1$

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}} = \prod_p\Big(1+\overset{*}{\underbrace{\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots}}\Big)$$

$\sum_p \frac{1}{p^s}$ conv. absolutely $\Rightarrow \zeta(s) \neq 0$ on $\mathrm{Re}\, s > 1$ as well as *

Will show $\xi(s) := \sqrt{\pi}^{-s}\,\Gamma(\frac{s}{2})\,\zeta(s)$ has $\xi(s) = \xi(1-s)$

Lemma *: very rough estimate

$$\int_1^{\infty} e^{-t}t^a\,dt \leq \int_1^{\infty} e^{-t}t^n\,dt \leq n! < n^n = e^{n\log n} \leq e^{(a+1)\log(a+1)}$$

↗ as $a \in \mathbb{R}^+$, $n = \lceil a \rceil$

Question: Prove (iii) without using Hadamard's thm

Hint: consider $\Gamma_n(s) := \int_0^{\infty}\left(1-\frac{t}{n}\right)^n t^{s-1}\,dt \xrightarrow[n\to\infty]{} \Gamma(s)$

Laplace Transform:

Appendix

Thm: $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ (for $\mathrm{Re}\, s > \sigma$)

$$\Leftrightarrow \quad f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\, ds . \quad (\text{for } t > 0)$$

Example: $\int_0^\infty e^{-st} \log t \; dt$

Q: How to get integral of this form?

Answer:

From $\Gamma(z) = \int_0^\infty e^{-t} t^z \frac{dt}{t}$,

$$\Rightarrow \quad \Gamma'(z) = \int_0^\infty e^{-t} t^z \log t \frac{dt}{t}$$

cf *

$$\underset{-(\gamma)}{\overset{\shortparallel}{}} \; \Gamma'(1) = \int_0^\infty e^{-t} \log t \, dt$$

$$= \int_0^\infty e^{-st} (\log s + \log t)\, s\, dt$$

$$= s \log s \underbrace{\int_0^\infty e^{-st} dt}_{= 1/s} + s \underbrace{\int_0^\infty e^{-st} \log t\, dt}_{F(s)}$$

$$= \log s + s F(s)$$

Laplace transf of $f(t) = \log t$:

$$F(s) = -\frac{\log s}{s} - \frac{(\gamma)}{s} \quad \Leftarrow$$

$$= \boxed{\frac{-1}{s}\left(\log s + \gamma \right)}$$

$$-\frac{\Gamma'(1)}{\Gamma(1)} = \gamma + 1 + \sum \left(\frac{1}{k+1} - \frac{1}{k} \right) = \gamma$$

$$* : \quad \frac{1}{\Gamma(s)} = e^{\gamma s} s \prod_{k=1}^{\infty} \left(1 + \frac{s}{k} \right) e^{-\frac{s}{k}}$$

Riemann's ζ function:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$ for $\overset{\sigma}{\overset{\shortparallel}{\mathrm{Re}\, s}} > 1$

unif bdd by $\sum \frac{1}{n^\sigma} \Rightarrow$ holo.

$$\xi(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$ holo in $\mathrm{Re}\, s > 1$

Thm: ξ has analy conti to $\mathbb{C}$ with only simple pole at $s = 0, 1$. Also $\xi(s) = \xi(1-s)$. $\forall s \in \mathbb{C}$

character

Pf: $\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{s}{2}} \int_0^\infty e^{-t} t^{\frac{s}{2}} \sum_{n=1}^{\infty} \frac{1}{n^s} \frac{dt}{t} = \sum_{n=1}^{\infty} \int_0^\infty e^{-t} \left(\frac{t}{\pi}\right)^{\frac{s}{2}} n^{-s} \frac{dt}{t}$

$\left(\frac{t}{\pi n^2}\right)^{\frac{s}{2}}$, $\frac{t}{\pi n^2} = u$

$$= \sum_{n=1}^{\infty} \int_0^\infty e^{-\pi n^2 u} u^{\frac{s}{2}} \frac{du}{u}$$

why can we do this ! $\;= \int_0^\infty \left(\sum_{n=1}^{\infty} e^{-\pi n^2 u}\right) u^{\frac{s}{2}} \frac{du}{u}$

Need estimate on $\vartheta(u)$ near 0

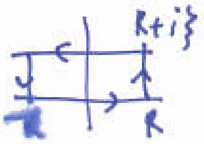
cf. funct. eq'n next page.

$\shortparallel \; \psi(u) = \frac{1}{2}(\vartheta(u) - 1)$

$\shortparallel \; \sum_{n=-\infty}^{\infty} e^{-\pi n^2 u}$ the theta function (general form in ch. 10)

In our 1st example of Cauchy's thm (p. 42, ch 2)

character



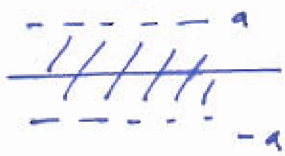
$$e^{-\pi \xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi}\, dx$$

in general (ch. 4) $\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx$ is the Fourier transform

Ch 4. Thm 2.4 (Poisson Summation Formula)

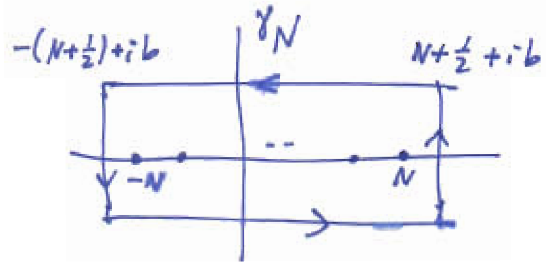
If $f \in \mathfrak{F}_a$ $\left(= \text{set of hol fcn in } |\mathrm{Im}\, z| < a \text{ with } f(x+iy) \le \frac{A}{1+x^2}\right)^*$

Then $\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$.



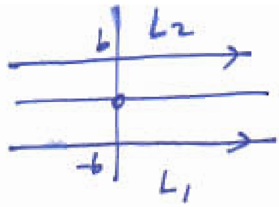
Pf. Res Thm $\Rightarrow$ let $0 < b < a$

$$\sum_{|n| \le N} f(n) = \int_{\gamma_N} \frac{f(z)}{e^{2\pi i z} - 1}\, dz$$



$* \Rightarrow \lim_{N \to \infty}$ exists and both vertical cut $\to 0$.

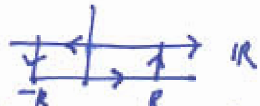
$$\sum_{n\in\mathbb{Z}} f(n) = \int_{L_1} \frac{f(z)\,dz}{e^{2\pi i z}-1} + \left(-\int_{L_2}\frac{f(z)\,dz}{e^{2\pi i z}-1}\right)$$



$$\int_{L_1} \frac{f(z)}{e^{2\pi i z}} \frac{dz}{1-e^{-2\pi i z}} \qquad \int_{L_2}\frac{f(z)\,dz}{1-e^{2\pi i z}} \qquad e^{2\pi i(x+ib)} = e^{2\pi i x}\,\underline{e^{-2\pi b}}$$

$$\sum_{n=0}^{\infty}\int_{L_1} f(z)\,e^{-2\pi i(n+1)z}\,dz \qquad \sum_{n=0}^{\infty}\int_{L_2} f(z)\,e^{2\pi i n z}\,dz$$

→ both L_1, L_2 can be replaced by $\mathbb{R}$ by Cauchy's thm again



L_1 : boundary vertex int $\le b\cdot \frac{A}{1+R^2}\int |e^{-2\pi i(x+iy)}|$ $\quad R\to\infty$

L_2 : similarly and get $e^{-2\pi y}$ ↘ $y>0$. $\quad = e^{2\pi y}$ ↘ $y<0$

$$\Rightarrow \sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty} f(z)\,e^{-2\pi i m z}\,dz = \sum_{m\in\mathbb{Z}}\hat f(m) \quad ※$$

---

apply to $f(x) = e^{-\pi t x^2}$, $\hat f(\xi) = \frac{1}{\sqrt t} e^{-\pi\xi^2/t}$ $\quad x\mapsto\sqrt t\,x$

$(t>0)$

get $\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \frac{1}{\sqrt t}\sum_{n\in\mathbb{Z}} e^{-\pi n^2/t}$ $\quad\left(\text{since } e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\,dx\right)$

i.e. $\vartheta(u) = \frac{1}{\sqrt u}\vartheta\left(\frac1u\right)$. for $u>0$. $\quad\Rightarrow \vartheta(u)\le \frac{C}{\sqrt u}$ as $u\to 0$

- Now back to the pf of $\xi(s) = \xi(1-s)$:

already have $\xi(s) := \pi^{-\frac s2}\Gamma\left(\frac s2\right)\zeta(s) = \int_0^\infty \left(\sum_{n=1}^{\infty} e^{-\pi n^2 u}\right) u^{\frac s2}\frac{du}{u}$

- notice that $\xi(1-s)$ has not yet been defined!

This is achieved by:

$$\varphi(u) = \frac12(\vartheta(u)-1) = \frac12\left(\frac{1}{\sqrt u}\vartheta\left(\frac1u\right)-1\right) = \frac{1}{\sqrt u}\varphi\left(\frac1u\right) + \frac12\left(\frac{1}{\sqrt u}-1\right)$$

$$\Rightarrow \xi(s) = \int_0^1 u^{\frac s2}\left[u^{-\frac12}\varphi\left(\frac1u\right) + \frac12\left(\frac{1}{\sqrt u}-1\right)\right]\frac{du}{u} + \int_1^\infty u^{\frac s2}\varphi(u)\frac{du}{u}$$

$$= \int_0^1\left(\frac12 u^{\frac s2-\frac32} - \frac12 u^{\frac s2-1}\right)du + \int_1^\infty\left(u^{-\frac s2+\frac12} + u^{\frac s2}\right)\varphi(u)\frac{du}{u}$$

$= \frac{1}{s-1} - \frac1s$ $\quad$ This $\Rightarrow \xi(1-s) = \xi(s)$ too. ※

$$\xi(s) = \pi^{-\frac{s}{2}}\Gamma\left(\tfrac{s}{2}\right)\zeta(s) \quad \Rightarrow \quad \xi(s)=\xi(1-s) \quad \forall s\in\mathbb{C}$$

Cor: ① $\zeta(s)$ has a unique (simple) pole at $s=1$

② the only zeros in $Re(s)<0$ are $s=-2,-4,-6,$

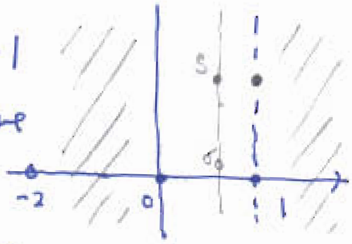
① since $\xi(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \left(u^{-\frac{s}{2}+\frac{1}{2}} + u^{+\frac{s}{2}}\right)\psi(u)\frac{du}{u}$

and the pole at $s=0$ is canceled out by $\Gamma(\frac{s}{2})$ at $s=0$

② $\zeta(\underset{\overset{\shortparallel}{s}}{1-s_1}) = \pi^{-s_1+\frac{1}{2}} \dfrac{\Gamma(\frac{s_1}{2})}{\Gamma(\frac{1-s_1}{2})}\zeta(s_1)$

$Re(1-s_1)<0 \Rightarrow Re\, s_1>1$, only need to see $1-s_1 = 0\ -2, -4$. ※

STUDY $Re(s)=1$: let $\delta_n(s) = \int_n^{n+1}\left(\frac{1}{n^s}-\frac{1}{x^s}\right)dx$



Theorem: $\zeta(s)$ has NO zeros on $Re(s)=1$

lemma: $\zeta(s) - \frac{1}{s-1} = H(s) = \sum_{n=1}^\infty \delta_n(s)$ entire

s.t. $|\delta_n(s)| \le |s|/n^{\sigma+1}$ for $s=\sigma+it$, $\Rightarrow$ holo in $Re(s)>0$

(i) $|\zeta(s)| \le c_\varepsilon |t|^{1-\sigma_0+\varepsilon}$ for each $\overset{\sigma}{\sigma_0}\in[0,1]$, $\varepsilon>0$ $(|t|\ge 1)$

(ii) $|\zeta'(s)| \le c_\varepsilon |t|^\varepsilon$ if $\sigma\ge 1$ $(|t|\ge1)$

Pf: $\sum_{n=1}^{N-1}\delta_n(s) := \sum_{n=1}^{N-1}\frac{1}{n^s} - \int_1^N \frac{dx}{x^s} = \sum_{n=1}^{N-1}\int_n^{n+1}\left(\frac{1}{n^s}-\frac{1}{x^s}\right)dx$ holo in $s\in\mathbb{C}$

the sum conv abs. for $\sigma>0$ $\Leftarrow$ $\le \frac{|s|}{n^{\sigma+1}}$ by MVT

& unif with $\sigma\ge\delta>0$ any δ fixed

If $Re(s)=\sigma>1$ get for $N\to\infty$,

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^\infty \delta_n(s) =: H(s)$$

this extends ζ to $Re(s)>0$ this part ok also for $\sigma>0$

(i) by def'n, $|\delta_n(s)| \le 2/n^\sigma$ $(x\ge n)$, so $\forall\, 1\ge\delta\ge0$

$$|\delta_n(s)| \le \left(\frac{|s|}{n^{\sigma_0+1}}\right)^\delta\left(\frac{2}{n^{\sigma_0}}\right)^{1-\delta} \le \frac{2^{1-\delta}|s|^\delta}{n^{\underbrace{\sigma_0+\delta}_{=1+\varepsilon}}}$$ let $\delta=(1-\sigma_0)+\varepsilon>0$

$$\Rightarrow |\zeta(s)| \le \left|\frac{1}{s-1}\right| + 2^{1-\delta}|s|^{1-\sigma_0+\varepsilon}\sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}}$$

how about $\sigma_0=0$? not allowed

(ii) $\zeta'(s) = \frac{1}{2\pi r}\int_0^{2\pi}\zeta(s+re^{i\theta})e^{i\theta}d\theta$ by Cauchy's int formula,

simply choose $\sigma_0=1$ and $r=\frac{\varepsilon}{2}$ (say). ※

pf of thm: $\operatorname{Re} s > 1 \Rightarrow$

$$\log \zeta(s) = \log \prod_p \frac{1}{1-p^{-s}} = \sum_p \sum_m \frac{p^{-ms}}{m} =: \sum_{n=1}^{\infty} \frac{c_n}{n^s} \quad \text{with } c_n \geq 0$$

let $s = \sigma + it$ $(\sigma > 1)$, then

$$\operatorname{Re} n^{-s} = \operatorname{Re} e^{-(\sigma+it)\log n} = n^{-\sigma} \cos(\underbrace{t \log n}_{\theta_n})$$

so $\log |\zeta^3(\sigma)\, \zeta^4(\sigma+it)\, \zeta(\sigma+2it)|$

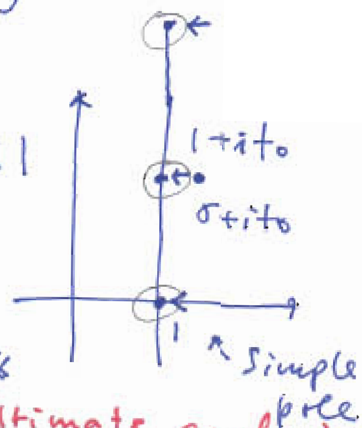
$$= 3 \operatorname{Re} \log \zeta(\sigma) + 4 \operatorname{Re} \log \zeta(\sigma+it) + \operatorname{Re} \zeta(\sigma+2it)$$

$$= \sum_{n=1}^{\infty} c_n \underbrace{(3 + 4\cos\theta_n + \cos 2\theta_n)}_{= 2(1+\cos\theta_n)^2} \geq 0$$

Now Suppose $\zeta(1+it_0) = 0$, then

$$\sigma \to 1 \Rightarrow \underbrace{|\zeta^3(\sigma)|}_{\leq c'(\sigma-1)^{-3}} \underbrace{|\zeta^4(\sigma+it_0)|}_{\leq c(\sigma-1)^4} \underbrace{|\zeta(\sigma+2it_0)|}_{\leq c''} \leq c_1 |\sigma - 1| \to 0$$

thus $\log|\cdots| \to -\infty$ a contradiction ※



In fact, a quantitative version holds. ultimate goal is estimate for $\zeta'(s)/\zeta(s)$

Prop: $\forall \varepsilon > 0$, $\exists C_\varepsilon$ st $\dfrac{1}{|\zeta(s)|} \leq C_\varepsilon |t|^\varepsilon$ for $s = \sigma + it$, $\sigma \geq 1$, $|t| \geq 1$

pf. stating from $|\zeta^3(\sigma)\, \zeta^4(\sigma+it)\, \zeta(\sigma+2it)| \geq 1$ for $\sigma \geq 1$ (since log ≥ 0 and let $\sigma \to 1$)

use $\sigma_0 = 1$ in (i)

$$\Rightarrow |\zeta^4(\sigma+it)| \geq c|\zeta^{-3}(\sigma)|\, |t|^{-\varepsilon} \geq c_1 (\sigma-1)^3 |t|^{-\varepsilon}$$

ie $|\zeta(\sigma+it)| \geq c'(\sigma-1)^{3/4} |t|^{-\varepsilon/4}$ ⊛

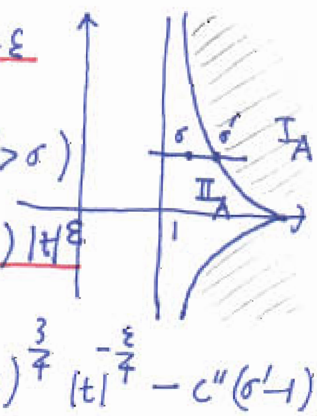
For A to be choosed later,

I_A: $\sigma - 1 \geq A|t|^{-5\varepsilon}$, $\Rightarrow |\zeta(\sigma+it)| \geq Ac' |t|^{-4\varepsilon}$

II_A: $\sigma - 1 < A|t|^{-5\varepsilon}$. let $\sigma' - 1 = A|t|^{-5\varepsilon}$ $(\sigma' > \sigma)$

$$\Rightarrow \underbrace{|\zeta(\sigma'+it) - \zeta(\sigma+it)|}_{\geq |\zeta(\sigma'+it)| - |\zeta(\sigma+it)|} \leq c'' |t|^\varepsilon |\sigma' - \sigma| \leq c''(\sigma'-1)|t|^\varepsilon$$

use ⊛ for σ'



$$\Rightarrow |\zeta(\sigma+it)| \geq c'(\sigma'-1)^{\frac{3}{4}} |t|^{-\frac{\varepsilon}{4}} - c''(\sigma'-1)|t|^\varepsilon$$

$$= (c'A^{\frac{3}{4}} - c''A)|t|^{-4\varepsilon}$$

choose $c'A^{\frac{3}{4}} - c''A = c''A$ ie $A = \left(\frac{c'}{2c''}\right)^4$

and declare that $C_{4\varepsilon} = \min(c', c'')\cdot A$ then done. □

Proof of Prime Number Theorem.

$$\pi(x) \sim \frac{x}{\log x} \overset{(1)}{\Longleftarrow} \psi(x) \sim x \overset{(2)}{\Longleftarrow} \psi_1(x) \sim \frac{x^2}{2} \overset{(3)}{\Longleftarrow} \zeta(1+it) \neq 0$$

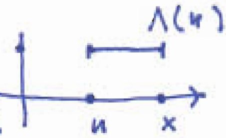
Tchebychev's (over (1), (2))

with estimates (3): Hadamard & de la Vallée Poussin

$$\pi(x) = \sum_{p \le x} 1$$

$$\psi(x) = \sum_{p^m \le x} \log p =: \sum_{n \le x} \Lambda(n) = \sum_p \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p \le \pi(x) \log x$$

$$\psi_1(x) = \int_1^x \psi(u)\,du = \sum_{n \le x} \Lambda(n)(x-n)$$



(1) $\frac{\psi(x)}{x} \le \pi(x) \frac{\log x}{x} \Rightarrow 1 \le \underline{\lim}_{x\to\infty} \pi(x)\frac{\log x}{x}$

fix $\alpha \in (0,1)$, $\psi(x) \ge \sum_{x^\alpha < p \le x} \log p \ge (\pi(x) - \pi(x^\alpha)) \log x^\alpha$

$$\frac{\psi(x)}{x} \ge \alpha\, \pi(x) \frac{\log x}{x} - \alpha \frac{\pi(x^\alpha)}{x^\alpha} \underbrace{\frac{\log x}{x^{1-\alpha}}}_{\to 0} \Rightarrow 1 \ge \alpha\, \overline{\lim_{x\to\infty}}\, \pi(x) \frac{\log x}{x}$$

Now let $\alpha \to 1$ ※

(2) for $\alpha < 1$, $\frac{\psi_1(x) - \psi_1(\alpha x)}{x - \alpha x} \le \psi(x)$ since $\psi \nearrow$

take $\frac{1}{x}\cdot$ and $x \to \infty \Rightarrow \frac{1}{1-\alpha}\left(\frac{1}{2} - \frac{\alpha^2}{2}\right) \le \underline{\lim}_{x\to\infty} \frac{\psi(x)}{x}$

$$\frac{1}{1-\alpha}\left(\frac{1}{2} - \frac{\alpha^2}{2}\right) = \frac{1+\alpha}{2}$$ Now let $\alpha \to 1$

$\overline{\lim_{x\to\infty}} \frac{\psi(x)}{x} \le 1$ similarly ※

To relate ψ or ψ_1 with ζ, consider

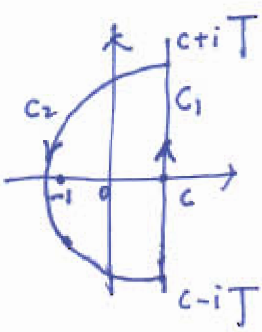
$$\log \zeta(s) = \sum_p \log\left(\frac{1}{1-p^{-s}}\right) = \sum_{m,p} \frac{p^{-ms}}{m} \qquad p^{-ms} = e^{-ms\log p}$$

$$-\frac{\zeta'(s)}{\zeta(s)} = + \sum_{m,p} \log p \; p^{-ms} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

Prop $\forall c > 1$, $\psi_1(x) = \frac{1}{2\pi i} \int_{Re(s)=c} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds$.

Lemma: $\frac{1}{2\pi i} \int_{Re(s)=c} \frac{a^s}{s(s+1)}\, ds = \begin{cases} 1 - \frac{1}{a} & \text{if } a \ge 1 \\ 0 & \text{if } 0 < a \le 1 \end{cases}$

Pf.



if $a \ge 1$,

$$\frac{1}{2\pi i} \int_{C_1} \frac{a^s}{s(s+1)}\, ds + \frac{1}{2\pi i} \int_{C_2} \frac{a^s}{s(s+1)}\, ds = \underset{= 1}{Res_0} + \underset{= -1/a}{Res_1}$$

$$|\cdots| \le \frac{a^c T}{T^2} \longrightarrow 0 \text{ as } T \to \infty$$

if $0 < a \le 1$, do the RHS with no poles ※

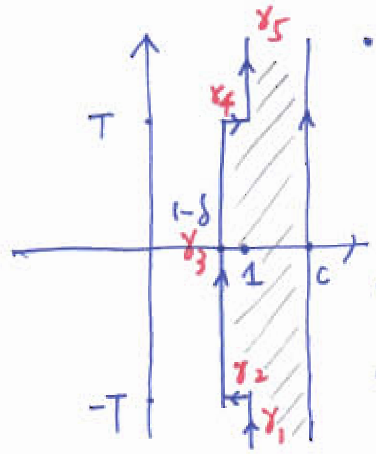
pf of prop:

$$\frac{1}{2\pi i}\int_{Re(s)=c} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds = x\sum_{n=1}^{\infty} \frac{\Lambda(n)}{2\pi i}\int_{Re(s)=c} \frac{\left(\frac{x}{n}\right)^s}{s(s+1)}\, ds$$

$$= x \sum_{n\le x} \Lambda(n)\left(1-\frac{n}{x}\right) = \sum_{n\le x} \Lambda(n)(x-n) = \psi_1(x) \quad \square$$

Now pf of (3): Let $F(s) := \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)$ → simple pole at $s=1$ with coeff $-(-1)=1$

$$Res_{s=1} F(s) = \frac{x^2}{2}$$



- On γ_1 & γ_5, $|x^{1+s}| = x^{1+1} = x^2$

  $|\zeta'(s)/\zeta(s)| \le A|t|^{1/2}$

  $$\left|\int_{\gamma_1} F(s)ds\right| \le c x^2 \int_T^{\infty} \frac{t^{1/2}}{t^2}\, dt \le \epsilon \frac{x^2}{2} \text{ for large } T$$

- Fix T, then $\left|\int_{\gamma_3} F(s)ds\right| \le C_T\, x^{2-\delta}$

- On γ_2 & γ_4,

  $$\left|\int_{\gamma_2} F(s)ds\right| \le C_T' \int_{1-\delta}^{1} x^{1+\sigma}\, d\sigma \le C_T' \frac{x^2}{\log x}$$

Thus $\left|\psi_1(x) - \frac{x^2}{2}\right| \le \epsilon x^2 + C_T x^{2-\delta} + C_T' \frac{x^2}{\log x}$ done. $\square$

Rmk:

We'll need to take δ very small st $\nexists$ zeros of $\zeta(s)$ inside the shadow region, this is OK for T fixed

$\Omega \subset \mathbb{C}$ $\mathcal{F}$ family of (hol) fcn on Ω

Def^{n}: $\mathcal{F}$ normal $\Leftrightarrow$ every sequence in $\mathcal{F}$ has a sub seq conv unif on every cpt subset of Ω

(limit may not in $\mathcal{F}$)

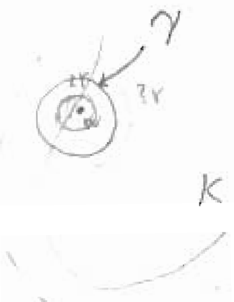
Thm (Arzela-Ascoli)

Any $\mathcal{F}$ $\left(\begin{array}{l}\text{unif bounded} \\ \text{equi conti}\end{array}\right)$ on cpt subset $\Rightarrow$ normal.

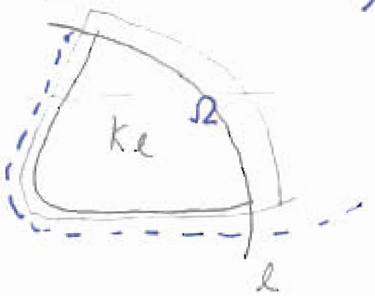
Thm (Montel): $\mathcal{F}$ unif bdd on cpt subset $\Rightarrow$ equi-conti
holo $\Rightarrow$ hence normal
$\Rightarrow \exists$ holo limit

pf: [$\Omega \subset \mathbb{C}$ has an exhaustion: ie
$\{K_\ell\}_{\ell=1}^{\infty}$ "rel cpt" $K_\ell \subset K^{\circ}_{\ell+1}$ & $\Omega = \bigcup_{\ell=1}^{\infty} K_\ell$
(this $\Rightarrow$ every cpt $K \subset \Omega$ is in some K_ℓ)
construction let $K_\ell = \{ z \in \Omega \mid d(z, \partial\Omega) \geq 1/\ell,\ |z| \leq \ell \}$]

This is not needed only need in Arzela-Ascoli



let $K \subset \Omega$ cpt choose $r > 0$ st
$B_z(3r) \subset \Omega$ $\forall z \in K$
let $z, w \in K$ with $|z-w| < r$
and $\gamma := \partial B_w(2r)$



$\Rightarrow$ $f(z) - f(w) = \frac{1}{2\pi i} \int_\gamma f(\zeta)\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right) d\zeta$

$\Rightarrow$ $|f(z) - f(w)| \leq \frac{1}{2\pi} \frac{2\pi \cdot 2r}{r^2} B_K |z-w|$ " $\frac{z-w}{(\zeta-z)(\zeta-w)}$

unif bnd of $\mathcal{F}$ on "$K + 2r$" both have $|\ | > r$

ie. $\mathcal{F}$ is equi-conti on K

i.e. pts $z \in \Omega$, $d(z, K) \leq 2r$ still a cpt set

pf of AA thm: let $f_n \in \mathcal{F}$, $\{w_j\}$ dense in Ω

$\{f_{n,1}\} \subset \{f_n\}$ with $f_{n,1}(w_1)$ conv
$\{f_{n,2}\} \subset \{f_{n,1}\}$ with $f_{n,2}(w_2)$ conv etc
let $g_n := f_{n,n}$ then $g_n(w_j)$ conv $\forall_j$

$\varepsilon > 0$, $\exists \delta$ ~~in equi-conti~~ Also $K \subset \bigcup_{j=1}^{J} B_{w_j}(\delta)$ ~~for some~~ $J < \infty$

let $N \gg 0$ st. $|g_m(w_j) - g_n(w_j)| < \epsilon$ $\forall_j = 1 .. J$ & $n, m > N$

3ϵ proof $\Rightarrow$ unif conv. on K.

Pf of the Riemann Mapping Thm.

The pf in Stein's book (Uniqueness is easy)

let $\alpha \notin \Omega$

LOGARITHMIC SURFACES AND HYPERBOLICI

$\exists$ $g(z) = \log(z-\alpha)$ is well defined on Ω

ie $e^{g(z)} = z - \alpha$, in particular g is injective

and, fix $w \in \Omega$, then $|g(z) - (g(w) + 2\pi i)| \geq \epsilon > 0$

otherwise $\exists z_n \in \Omega$, $e^{\underbrace{g(z_n) - (g(w) + 2\pi i)}_{\to 0}} = (z_n - \alpha)/(w-\alpha) \to 1$

ie $z_n \to w$ $\;\leftarrow$ to $2\pi i$ gap

$f(z) := \dfrac{1}{g(z) - (g(w) + 2\pi i)} \leq \dfrac{1}{\epsilon}$, injective

by scaling and translation, get $f : \underset{0}{\Omega} \to \underset{0}{D}$ holo. inj, $f(0) = 0$

Now we replace Ω by $f(\Omega) \subset D$ $\quad$ ie $\Omega \xrightarrow{\sim} f(\Omega) \subset D$

Let $\mathcal{F} := \{ f : \Omega \to D$ holo inj, $f(0) = 0 \} \neq \phi$, say id $\Omega \to \Omega$, $z \mapsto z$

this is a unif bounded family

Let $s = \sup_{f \in \mathcal{F}} |f'(0)| < \infty$ (by Cauchy), so $s \geq 1$

choose $f_n \in \mathcal{F}$ st $|f_n'(0)| \to s$

Montel's thm $\Rightarrow$ $\exists$ sub sequence conv. unif on cpt set to a holomorphic function f on Ω

$f \neq$ const ($s \geq 1$), f_n inj $\Rightarrow$ f also inj (why? Use Rouché)

$|f(z)| \leq 1$ hence $|f(z)| < 1$ by maximal principle

Also $f(0) = 0$, hence $f \in \mathcal{F}$ with $|f'(0)| = s$

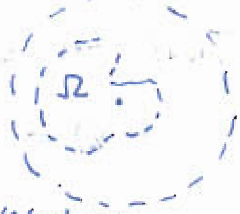
Claim: f is also surjective

if $\alpha \notin f(\Omega)$, $\alpha \in D$, then $U = (\psi_\alpha \circ f)(\Omega) \not\ni 0$

so $\sqrt{\cdot\cdot}$ is defined on U

Define $F = \psi_{\sqrt{\alpha}} \circ \sqrt{\psi_\alpha \circ f} : 0 \mapsto 0$

Then $F \in \mathcal{F}$



From $\underbrace{\psi_\alpha^{-1} \circ h \circ \psi_{\sqrt{\alpha}}^{-1}}_{\varphi} \circ F = f$.

where $h(z) = z^2$ $\quad$ $\varphi : D \to D$, $\varphi(0) = 0$

Schwartz Lemma $\Rightarrow$ $|\varphi'(0)| < 1$

Hence $|f'(0)| = \underset{\uparrow}{|\varphi'(0)|} \cdot |F'(0)| \Rightarrow |F'(0)| > |f'(0)|$ $\;\times$

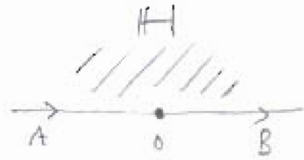
Thus $f : \Omega \xrightarrow{\sim} D$ And $\exists\, \theta$ st. $e^{i\theta} f'(0) > 0$ $\quad$ ※

Uniqueness: $f_1, f_2 : \Omega \xrightarrow{\sim} D \Rightarrow g = f_1 f_2^{-1} : D \to D$, $g(0) = 0$

$\Rightarrow g = e^{i\theta} z = z$ since $g'(0) > 0$ $\quad$ ※

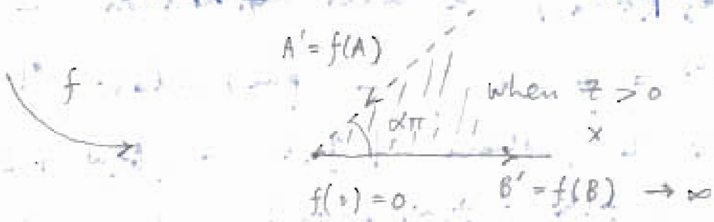
Examples of conformal maps with boundary

①



$$f(z) = z^\alpha = \int_0^z f'(\zeta)\,d\zeta = \alpha \int_0^z \zeta^{-\beta}\,d\zeta$$

$0<\alpha<2$ $\qquad \alpha+\beta=1 \quad (\beta = 1-\alpha < 1)$ integrable



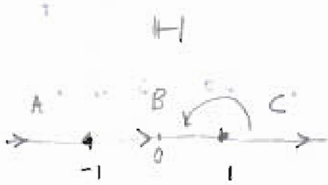
when $\underset{x}{z}<0$, $x^{\alpha-1} = x^{\alpha-1} e^{(\alpha-1)\pi i}$

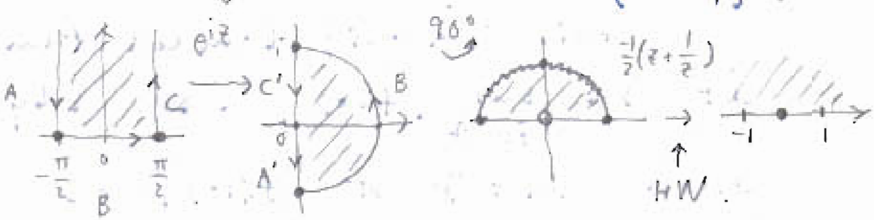
$= -|x|^{\alpha-1} e^{\alpha\pi i}$ $\qquad$ Recall:

② $\quad f(z) = \int_0^z \dfrac{d\zeta}{\sqrt{1-\zeta^2}} = \sin^{-1} z$ $\qquad \sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$

$e^{ix}e^{-y}$ $\qquad = \frac{-1}{2}\left(i\zeta + \frac{1}{i\zeta}\right)$

Direct method





pick a branch of $\sqrt{1-\zeta^2}$ st holo on $\mathbb{H}$ and >0 on $(-1,1)$

for $\zeta>1$: $(1-\zeta^2)^{-1/2} = i(\zeta^2-1)^{-1/2}$ $\quad$ why?

$\quad \| $

$(1+\zeta)^{-1/2}(1-\zeta)^{-1/2}$ $\qquad \zeta = 1 + re^{i\theta}$

$\quad \| $

$(-re^{i\theta})^{-1/2} = r^{-1/2} e^{(i(\theta+\pi)+2k\pi)\times\frac{-1}{2}}$

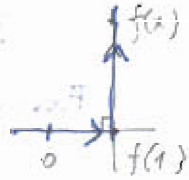
when $\theta=\pi$ should get $1-\zeta>0$

so may pick $k=1$

$\Rightarrow$ when $\theta=0$ get $e^{-\frac{3}{2}\pi i} = e^{\frac{\pi i}{2}} = i$

Thus for $z=x>1$, get

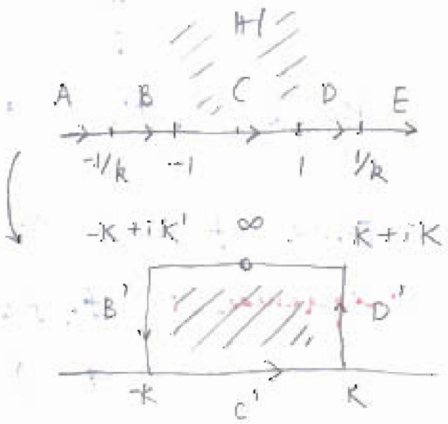
$$f(x) = \underbrace{\int_0^1 \frac{d\zeta}{\sqrt{1-\zeta^2}}}_{f(1)=\frac{\pi}{2}} + i\int_1^x \frac{d\zeta}{\sqrt{|1-\zeta^2|}}$$ turn $90°$ angle



③ Elliptic integral, let $0<k<1$

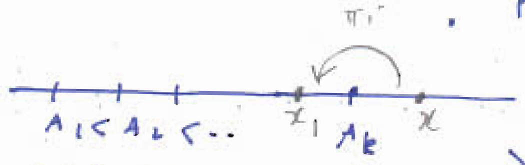
$$f(z) = \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}, \quad z\in\mathbb{H}$$

for $K := f(1)$; $K' := \int_1^{1/k} \dfrac{dx}{\sqrt{(x^2-1)(1-k^2x^2)}}$ $\Rightarrow$



Schwartz - Christoffel integral.

$$S(z) = \int_0^z \frac{d\zeta}{\prod_{i=1}^{n} (\zeta - A_i)^{\beta_i}}$$



$-1 < \beta_i < 1$; $1 < \sum \beta_i < 2$

integrable at A_i — at ∞

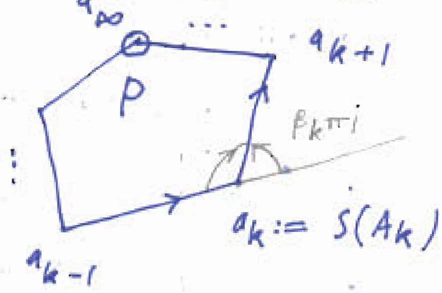
branch: $(z - A_k)^{\beta_k} = \begin{cases} (x - A_k)^{\beta_k} \ ; & x > A_k \\ |x - A_k|^{\beta_k} e^{i\pi\beta_k} \ ; & x < A_k \end{cases}$

for $x \in (A_k, A_{k+1})$,

$$\Rightarrow S(x) = S(A_k) + \int_{A_k}^{x} \frac{dt}{\prod_{j \le k} (t - A_j)^{\beta_j} \prod_{j > k} (t - A_j)^{\beta_j}}$$

S

if $\sum \beta_i \neq 2$



$$\arg(S'(x)) = -\pi \sum_{j > k} \beta_j$$

$a_\infty = S(-\infty) = S(\infty)$ by residue thm on $\mathbb{H}$

$$= \arg(S'(x_1)) + \pi \beta_k \quad ※$$

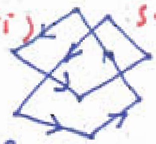
with angle (outer) $\beta_\infty := (2 - \sum \beta_i)\pi$

(A) The side of P depends on positions of $\{A_i\}$ and $\int$

(B) Not sure if conformal $((A) \Rightarrow (B))$ thus not necessary a ~~simple~~ polygon P

(Stein's approach)

Rmk: The pf in Ahlfors uses the Schwartz reflection, which is much better, and rigorous for (ii)



see next page on pf of SC formula

Boundary behavior.

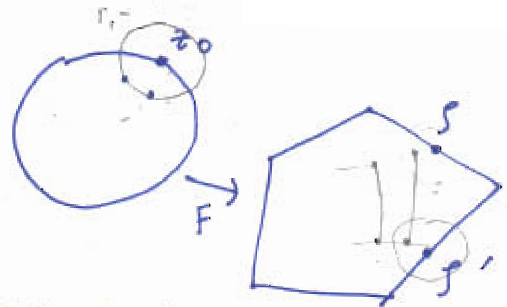
Thm: $F : \mathbb{D} \longrightarrow P$ conformal

then F extends to $\bar{\mathbb{D}} \xrightarrow{\bar{F}} \bar{P}$ conti, bijective

Pf: (i) existence of limit:

if $\exists\ z_i \longrightarrow z_0$ but $F(z_i) \to \zeta$
$\quad\ \ z_i' \nearrow$ $\qquad\qquad\quad F(z_i') \to \zeta'$

say in $\partial B_{z_0}(r_i)$



replace pts by z_r, z_r' $\forall r \in (0, r_0)$

$\rho(r) := |f(z_r) - f(z_r')|$, may assume $\rho(r) \geq c > 0$

$$\leq \int_{\theta_1(r)}^{\theta_2(r)} |f'(z)|\, r\, d\theta \quad \text{via } z = z_0 + re^{i\theta}$$

$$\leq \left(\int_{\theta_1(r)}^{\theta_2(r)} |f'(z)|^2 r\, d\theta \right)^{1/2} \underbrace{\left(\int_{\theta_1(r)}^{\theta_2(r)} r\, d\theta \right)^{1/2}}_{\le 2\pi r}$$

$$\infty \leftarrow\ \Rightarrow \int_0^{r_0} \frac{\rho^2(r)}{r} dr \leq 2\pi \int_0^{r_0} \int_{\theta_1(r)}^{\theta_2(r)} \underbrace{|f'(r)|^2}_{= \text{Jacobian}} r\, d\theta\, dr \leq 2\pi \text{ Area } P \quad \times$$

(ii) Ex. $\bar{F}$ is homeo.

P. 5/5

Schwartz - Christoffel formula:

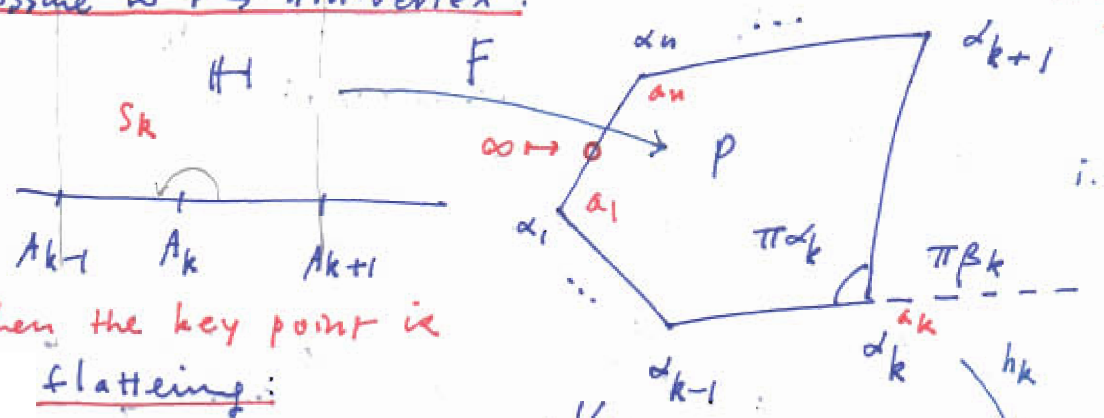
Given F conformal $\mathbb{H} \xrightarrow{\sim} P$, Then $F = c_1 S + c_2$ for some $A_1 < \cdots < A_n$ and c_1, c_2.

i.e. $0 < \alpha_i < 1$

$\sum \beta_i = 2$

$a_i =: F(A_i)$

Pf: Assume $\infty \mapsto$ non-vertex:



Then the key point is flattening:

$$h_k(z) := \left(F(z) - a_k \right)^{1/\alpha_k}$$

a line weight with some angle

- Schwartz reflection $\Rightarrow$ $h_k(z)$ holo on $S_k = \{A_{k-1} < \mathrm{Re}(z) < A_{k+1}\}$
- $\log' \Rightarrow \dfrac{h_k'(z)}{h_k(z)} = \dfrac{1}{\alpha_k} \dfrac{F'(z)}{F(z) - a_k}$ $\Rightarrow h_k'(z) \neq 0$ in S_k

  $F - a_k = h_k^{\alpha_k}$ $\quad F' = \alpha_k h_k^{\alpha_k - 1} h_k'$ $\quad$ if $z \in \mathbb{R}$ use local inj $\Leftrightarrow h_k' \neq 0$

h_k holo, $h_k(A_k) = 0$

$h_k' \neq 0 \Rightarrow \dfrac{F''(z)}{F'(z)} = \dfrac{\alpha_k - 1}{z - A_k} + E_k(z)$ holo in S_k. by gluing over overlap region $S_k \cap S_{k+1}$

$\Rightarrow \dfrac{F''(z)}{F'(z)} + \sum_{k=1}^{n} \dfrac{\beta_k}{z - A_k}$ is entire.

- in fact F is holo at ∞ and thus $\dfrac{F''(z)}{F'(z)} \sim \dfrac{c}{z} \to 0$ as $z \to \infty$ bounded.

Liouville $\Rightarrow$ $\dfrac{F''(z)}{F'(z)} = -\sum_{k=1}^{n} \dfrac{\beta_k}{z - A_k}$

i.e. $F'(z) = c_1 / \prod_{k=1}^{n} (z - A_k)^{\beta_k}$ i.e. $F(z) = c_1 \int_0^z \dfrac{d\zeta}{\prod (\zeta - A_k)^{\beta_k}} + c_2$.

if $\infty \mapsto$ vertex then apply some CVF. $\square$

the details is omitted (cf Stein Thm 4.7 in ch.8)

Rmk. "Same formula" holds for $\mathbb{H}$ being replaced by D

Weierstrass' Elliptic functions

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^\times} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

$$\frac{1}{(z-\omega)^2} = \frac{1}{\omega^2} \frac{1}{(1-\frac{z}{\omega})^2}$$

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots$$

$$\frac{1}{(1-t)^2} = 1 + 2t + 3t^2 + \cdots$$

$$= \frac{1}{z^2} + \sum_{\omega \in \Lambda^\times} \sum_{\ell=1}^{\infty} (\ell+1) \frac{z^\ell}{\omega^{\ell+2}}$$

$$= \frac{1}{z^2} + \sum_{\ell=1}^{\infty} (\ell+1) E_{\ell+2}\, z^\ell \qquad \ell = 2k \quad (E_{odd} = 0)$$

Laurent series

$$= \frac{1}{z^2} + 3E_4 z^2 + 5E_6 z^4 + \cdots$$

E_{2k} modular of wt $2k$ $(k \geq 2)$

Eisenstein series via Poisson summation

$$\pi \cot \pi\tau = \sum_{n=-\infty}^{\infty} \frac{1}{n+\tau}$$

or simply $\rightarrow$

$$\frac{\pi^2}{\sin^2(\pi\tau)} = \sum_{n=-\infty}^{\infty} \frac{1}{(n+\tau)^2}$$

(*)

$$\Rightarrow \sum_{-\infty}^{\infty} \frac{1}{(n+\tau)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{\ell=1}^{\infty} \ell^{k-1} e^{2\pi i \tau \ell}$$

(*)

$$\frac{\pi^2}{\sin^2 \pi\tau} = \left(\frac{2i\pi}{e^{\pi i \tau} - e^{-\pi i \tau}} \right)^2$$

$$= \frac{-4\pi^2 e^{2\pi i \tau}}{(1 - \underbrace{e^{2\pi i \tau}}_{\omega})^2} = \frac{(2\pi i)^2 \omega}{(1-\omega)^2}$$

$$= (2\pi i)^2 \sum_{\ell=1}^{\infty} \ell \omega^\ell$$

For k even,

Thm: $$E_k(\tau) = 2\zeta(k) + \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!} \sum_{r=1}^{\infty} \sigma_{k-1}(r)\, e^{2\pi i \tau r}$$

$$\sigma_{k-1}(r) = \sum_{d \mid r} d^{k-1} \text{ (divisor function)}$$

"pf": $$E_k(\tau) = 2\zeta(k) + 2 \sum_{m>0} \sum_{n=-\infty}^{\infty} \frac{1}{(n+m\tau)^k}$$

$$= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{m>0} \sum_{\ell=1}^{\infty} \ell^{k-1} e^{2\pi i \underline{m\tau\ell}}$$

sum over $r = m\ell$

$$\sum_{m>0} \sum_{\ell=1}^{\infty} \ell^{k-1} e^{2\pi i m\tau\ell} = \sum_{r=1}^{\infty} \sigma_{k-1}(r)\, e^{2\pi i r \tau}$$

~~The case $k=2$:~~

$$E_2(\tau) := \sum_m \left(\sum_n \frac{1}{(n+m\tau)^2} \right) \quad \text{for } (m,n) \neq (0,0)$$

The above pf shows that this indeed converges!

$$\Rightarrow E_2(\tau) = 2\zeta(2) - 8\pi^2 \sum_{r=1}^{\infty} \sigma(r)\, e^{2\pi i r \tau}$$

but $E_2(\tau)$ is NOT modular!

$$F(-1/\tau) \neq \tau^2 F(\tau)$$

it is $\tau^2 \widehat{F}(\tau)$, the reverse sum $\sum_n \sum_m$

$E_2(\tau)$ is crucial in theta functions and sum of 4 squares

$$\wp'(z) = -2\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^3} = \frac{-2}{z^3} + 6E_4 z + 20 E_6 z^3 + \cdots$$

Direct computations $\Rightarrow$ for $g_2 = 60E_4$, $g_3 = 140 E_6$,

Theorem: $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ (cubic curve)

set $\omega_3 = \omega_1+\omega_2$, $e_i := \wp\left(\frac{\omega_i}{2}\right)$, $i=1,2,3$ since $\wp'$ is odd, $\Rightarrow \wp'\left(\frac{\omega_i}{2}\right)=0$

Corollary: $\wp'(z)^2 = 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)$ no other zeros

moreover, $e_1+e_2+e_3=0$

Q Can one see this directly?

Corollary: Any elliptic function is a rational expression of $\wp(z)$ and $\wp'(z)$

pf f odd $\Rightarrow$ $f/\wp'$ is even $\Rightarrow$ May assume f even

then zeros, poles appear in pairs $\pm a_i$, $\pm b_i$ mod Λ

$\Rightarrow f = c\prod_{i=1}^{k}\frac{\wp(z)-\wp(a_i)}{\wp(z)-\wp(b_i)}$ by Liouville $\square$

check the case if some $a_i, b_i = \omega_j/2$

Q Not explicit enough!

Can we have a factorization like rational functions?

A: Yes, but need to be in terms of $\sigma(z)$

More about Weierstrass functions (usual notation is $\zeta(z)$)

$$\eta(z) := -\int^z \wp(w)\,dw = \frac{1}{z} + \sum_{\omega\in\Lambda^\times}\left(\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2}\right)$$

not elliptic, odd function

$$\eta_i := \eta(z+\omega_i) - \eta(z) = 2\eta\left(\frac{\omega_i}{2}\right)$$

$$\sigma(z) := e^{\int^z \eta(w)\,dw} = z\prod_{\omega\in\Lambda^\times}E_2\left(\frac{z}{\omega}\right)$$ still odd

Weierstrass product

$$\sigma(z+\omega_i) = e^{\int_z^{z+\omega_i}\eta(w)\,dw}\,\sigma(z)$$

$\frac{d}{dz} = \eta(z+\omega_i) - \eta(z) = \eta_i$

hence $= \eta_i z + c_i$

P period parallelogram

$$1 = \frac{1}{2\pi i}\int_{\partial P}\frac{\eta'(z)}{\eta(z)}\,dz$$

$$= A_i e^{\eta_i z}\sigma(z)$$

$\Downarrow$

$z = -\frac{\omega_i}{2} \Rightarrow -1 = A_i e^{-\eta_i\frac{\omega_i}{2}}$ ie $A_i = -e^{\eta_i\omega_i/2}$

Legendre relation

$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$

ie $\sigma(z+\omega_i) = -e^{\eta_i\left(z+\frac{\omega_i}{2}\right)}\sigma(z)$

Thm: Explicit determination of elliptic functions in terms of σ

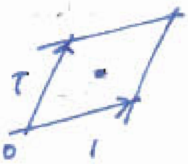
Theta fns: $\tau \in \mathbb{H}$ (brief introduction)

$$\Theta(z;\tau) := \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}$$ entire in $z \in \mathbb{C}$, holo in $\tau \in \mathbb{H}$

① : $\Theta(z+1;\tau) = \Theta(z;\tau)$

② : $\Theta(z+\tau;\tau) = e^{-\pi i \tau - 2\pi i z}\,\Theta(z;\tau)$

③ : $\Theta(z;\tau) = 0$ if $z \equiv \frac{1}{2}\omega_3 \mod \Lambda$



heat eq'n: $\left(\frac{\partial}{\partial \tau} - \frac{1}{4\pi i}\frac{\partial^2}{\partial z^2}\right)\Theta = 0.$

Thm 1. $\Theta = \Pi(z;\tau) := \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2\pi i z})(1+q^{2n-1}e^{-2\pi i z})$

where $q := e^{\pi i \tau}$. (notice $\leftrightarrow e^{2\pi i \tau}$) ← this is more common $\pi i n^2$ $4\pi^2 i^2 n^2$

idea of pf: Π satisfies ①, ② & ③$^+$. $z \equiv \frac{\omega_3}{2}$ are all the zeros (which are also simple). $\Rightarrow \Theta/\Pi$ entire & elliptic ※

Thm 2 (Jacobi's modularity)

(i) $\Theta(z;\tau+2) = \Theta(z;\tau)$ (not $\tau+1$)

(ii) $\Theta(z;\frac{-1}{\tau}) = \sqrt{\frac{\tau}{i}}\,e^{\pi i \tau z^2}\,\Theta(z\tau;\tau)$

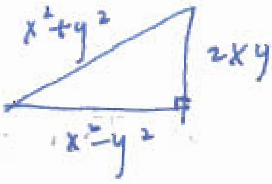
idea of pf: (i) is trivial. For (ii), let $z = x \in \mathbb{R}$, $\tau = it$ then it follows from Poisson Sum formula as $\vartheta(t)$ ※

Special case: $\theta(\tau) := \Theta(0;\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} = \sum q^{n^2}$

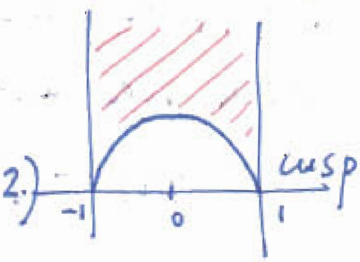
Problem on $n = x^2+y^2$, $r_2(n) :=$ # of ways $= ??$

not all n can do eg $n \neq 4k+3$

$$\theta^2(\tau) = \sum_{n_1,n_2} q^{n_1^2+n_2^2} = \sum_{n=0}^{\infty} r_2(n)\, q^n.$$



$$* \begin{cases} \theta^2(\tau+2) = \theta^2(\tau) \\ \theta^2(-1/\tau) = \tau/i\ \theta^2(\tau) \\ \theta(\tau) \neq 0 \text{ in } \mathbb{H}. \ (\Leftarrow \text{Thm 1.}) \\ \theta^2(1-\frac{1}{\tau}) \sim \frac{4\tau}{i} e^{\pi i \tau/2} \text{ as } \operatorname{Im}\tau \to \infty \ (\Leftarrow \text{Thm 2.}) \end{cases}$$



divisor fcn. $d_1(n)$, $d_3(n)$ div of $4k+1$, $4k+3$

Thm 3: $r_2(n) = 4\left(d_1(n) - d_3(n)\right)$

observation: let $c(\tau) := 2\sum_{n=-\infty}^{\infty} \frac{1}{q^n + q^{-n}} = \sum_{n=-\infty}^{\infty} \frac{1}{\cos 4\pi\tau}$

Thm 3 is equiv to $\theta^2(\tau) = c(\tau)$.

pf: $c(\tau) = 1 + 4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cdot \frac{1-q^{2n}}{1-q^{2n}} \leftarrow \frac{q^n - q^{3n}}{1-q^{4n}}$

$= 1 + 4\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left(q^{n(4m+1)} - q^{n(4m+3)}\right)$ *

idea of pf of Thm 3:

Verify $*_2$ using Poisson Sum formula, then $\Rightarrow$ $*_4$ as in θ^2

Then $f := c(\tau)/\theta^2(\tau)$ is modular, bounded $\Rightarrow$ $f =$ constant

in fact this require $f = 1$ by $*4$ ※ non-trivial

Sum of 4 squares: let $\sigma_1^*(n) := \sum_{4 \nmid d | n} d$.

Thm 4 Every $n \geq 1$ = sum of 4 squares & $r_4(n) = 8\sigma_1^*(n)$.

Observation: $\theta(\tau)^4 = \sum_{n=0}^{\infty} r_4(n) q^n$

want to construct modular function related to it

let $E_2^*(\tau) = \sum_m \sum_n \frac{1}{(\frac{m}{2}\tau + n)^2} - \sum_m \sum_n \frac{1}{(m\tau + \frac{n}{2})^2}$

$\equiv E_2(\frac{\tau}{2}) - 4E_2(2\tau)$

$= -\pi^2 - 8\pi^2 \sum_{k=1}^{\infty} \sigma_1^*(k) q^k$

where $E_2 = F$ is the Forbidden Eisenstein Series

$= \frac{\pi^2}{3} - 8\pi^2 \sum_{k=1}^{\infty} \sigma_1(k) e^{2\pi i k\tau}$

So Thm 4 $\Longleftrightarrow$ $\theta^4 = \frac{-1}{\pi^2} E_2^*$

$\sigma_1^*(n) = \begin{cases} \sigma_1(n) & 4 \nmid n \\ \sigma_1(n) - 4\sigma_1(\frac{n}{4}) & 4 | n \end{cases}$

The pf is more delicate

which uses also Dedekind's η function.

$\eta(\tau) := q^{1/12}\prod_{n=1}^{\infty}(1-q^{2n})$ to prove "almost modularity" of $F(\tau)$ (eg. Lemma 3.9) ※