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一.二. 晚(7-8)

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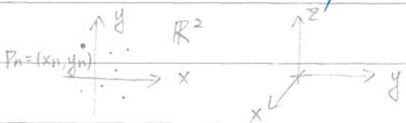
Functions of multivariables and continuity (1.1 − 1.3) :

$f(x,y) = x^2 - y^2$



• Limit of sequence of points

distance $|b-a|$

$P = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m, \quad Q = (x_1', x_2', \cdots, x_m')$

$|P - Q| = \left(\sum_{i=1}^{m} |x_i - x_i'|^2 \right)^{1/2}$ Euclidean distance

Def : " $\lim_{n\to\infty} P_n = Q$ " $\Leftrightarrow \forall \varepsilon > 0, \exists N$ st. $n \ge N \Rightarrow |P_n - Q| < \varepsilon$

$P_n = (x_n, y_n), \quad Q = (a, b)$

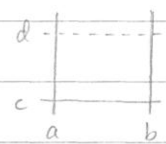
equivalently, $\lim_{n\to\infty} x_n = a, \quad \lim_{n\to\infty} y_n = b$

" $\Rightarrow$ " : $((x_n - a)^2 + (y_n - b)^2)^{1/2} < \varepsilon \Rightarrow |x_n - a| < \varepsilon, \ |y_n - b| < \varepsilon$

" $\Leftarrow$ " : $\forall \varepsilon > 0, \exists N$ st. $n \ge N, \ |x_n - a| < \varepsilon, \ |y_n - b| < \varepsilon \Rightarrow |P_n - Q| < \sqrt{2}\varepsilon$

S : Region (區域) $\subset \mathbb{R}^2$

$\mathbb{R}^2 \quad \longleftrightarrow \quad \mathbb{R}$



$[a,b]$
(a,b)
$[a,b)$
$(a,b]$

$[a,b] \times [c,d) \subset \mathbb{R}^2$

$= \{(x,y) \mid a \le x \le b, \ c \le y < d\}$

• ε - neighborhood : $(a-\varepsilon, a+\varepsilon) \times (b-\varepsilon, b+\varepsilon)$

open box

$B_P(\varepsilon) = \{ Q \in \mathbb{R}^2 \mid |Q - P| < \varepsilon \}$

ε-ball

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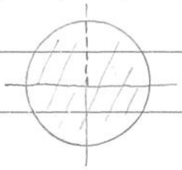


$S = \{(x,y) \mid x^2+y^2 \le 1 \text{ and if } x=0 \text{ then } y<0\}$

- interior points $S^o = \{P \in \mathbb{R}^2 \mid \exists\ \varepsilon\text{-neighborhood of } P \text{ contained in } S\}$
- exterior points $S^e = \{P \in \mathbb{R}^2 \mid \exists\ \varepsilon\text{-neighborhood of } P \text{ which is disjoint from } S\}$

  $\{P \in \mathbb{R}^2 \mid \exists B_P(\varepsilon) \text{ st. } B_P(\varepsilon) \cap S = \phi\}$
- boundary points $\partial S = \{P \in \mathbb{R}^2 \mid \forall B_P(\varepsilon),\ B_P(\varepsilon) \cap S \neq \phi,\ B_P(\varepsilon) \cap S^c \neq \phi\}$

  (S^c = complement of $S := \mathbb{R}^2 \setminus S = \{P \in \mathbb{R}^2 \mid P \notin S\}$)

$\mathbb{R}^2 = S^o \amalg S^e \amalg \partial S$

↑ disjoint union

Key terminology.

- open set : $S = S^o$
- closed set : $S \supset \partial S$

S is open $\Leftrightarrow$ S^c is closed $(\partial S = \partial(S^c))$

domain and range of a function

$f: \mathbb{R}^2 \supset S \longrightarrow \mathbb{R}$ $\quad f(S)$ "image"

eg. $f(x,y) = \log(1-x^2-y^2)$ domain $S = B_0(1)$

eg. $f(x,y) = \tan^{-1}(y/x)$ we need to select a "principal branch" to make the function be single valued

Def : ① $\lim_{P_n \to P} f(P_n) = L \Leftrightarrow \forall \varepsilon>0, \exists N, \text{ st. } n \ge N \Rightarrow |f(P_n) - L| < \varepsilon$

② $f(x,y)$ is conti. at $P=(a,b) \Leftrightarrow \lim_{Q \to P} f(Q) = f(P)$

①' $\lim_{Q \to P} f(Q) = L \Leftrightarrow \forall \varepsilon>0, \exists \delta>0 \text{ st } |Q-P| < \delta \Rightarrow |f(Q)-L| < \varepsilon$

eg 1. $f(x,y) = \dfrac{2xy}{x^2+y^2}$ is conti. outside $(0,0)$

Q: $\lim_{Q \to 0} f(Q)$ exist?

along x-axis: $f(x,0) = 0$; along y-axis: $f(0,y) = 0$

along $y = mx$: $f(x,y) = \dfrac{2mx^2}{x^2+(mx)^2} = \dfrac{2m}{1+m^2}$ varies in m

eg 2. $f(x,y) = \dfrac{xy^2}{x^2+y^2}$

$f(x,mx) = \dfrac{m^2x^3}{x^2(1+m^2)} = \dfrac{m^2}{1+m^2}x$

$\dfrac{xy}{x^2+y^2} \le \dfrac{1}{2}$, $|f(x,y)| \le \dfrac{y}{2}$

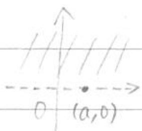
eg 2′ $f(x,y) = \dfrac{xy^2}{x^2+y^4}$

$f(x,mx) = \dfrac{m^2x^3}{x^2(1+m^2x^2)} = \dfrac{m^2x}{1+m^2x^2}$

along $y^2 = mx$

eg 3. Continuous extension of a function to ∂S of its domain S

$f(x,y) = e^{-x^2/y}$ with $S = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ upper half plane



∂S = x-axis

$$\begin{cases} a \ne 0, & \lim_{P \to (a,0)} f(P) = 0 \\ a = 0, & \text{let } y = mx^2, \ f(x,y) = e^{-x^2/mx^2} = e^{-1/m} \end{cases}$$

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Continuity

$f(x,y) = \dfrac{xy^2}{x^2+y^2}$ if $(x,y) \ne (0,0)$

$= 0$ if $(x,y) = (0,0)$

$|f(x,y) - f(0,0)| \le \left|\dfrac{xy}{x^2+y^2}\right| |y|$

$f(x,y)$ is conti. $\Leftrightarrow$ $\lim_{n\to\infty} f(x_n, y_n) = f(\lim_{n\to\infty}(x_n,y_n)) = f(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n)$

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The order of a function "O"; "o" $f(h,k)$, $g(h,k)$

$f = O(g) \Leftrightarrow \left|\frac{f(h,k)}{g(h,k)}\right| \le M$, when $(h,k) \to (0,0)$

$f = o(g) \Leftrightarrow \lim_{(h,k)\to(0,0)} \frac{f(h,k)}{g(h,k)} = 0$

§1.4 Partial derivatives

$z = f(x,y) = x^2 - y^2$ slicing: $x=0,\ z=-y^2$; $x=c,\ z=c^2-y^2$; $y=0,\ z=x^2$; $y=c,\ z=x^2-c^2$

level curves: let $z=$ const. c

☆ $\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$; $\frac{\partial f}{\partial y}(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$

☆ $= D_x f(x_0,y_0) = D_1 f(x_0,y_0) = \partial_1 f(x_0,y_0)$

☆ $= f_x(x_0,y_0)$

Higher Partial derivatives

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = D_x D_x f$; $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx} = D_y D_x f$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = D_x D_y f$; $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = D_y D_y f$

eg1. $f(x,y) = x^2 - y^2$: $f_x = 2x$, $f_{yx} = 0$; $f_y = 2y$, $f_{xy} = 0$

eg2. $f(x,y) = e^{x/y}$: $f_x = e^{x/y}\cdot\frac{1}{y}$, $f_{yx} = e^{x/y}\cdot\frac{-x}{y^2}\cdot\frac{1}{y} + e^{x/y}\cdot\frac{-1}{y^2}$

$f_y = e^{x/y}\cdot\frac{-x}{y^2}$, $f_{xy} = e^{x/y}\cdot\frac{1}{y}\cdot\frac{-x}{y^2} + e^{x/y}\cdot\frac{-1}{y^2}$

eg3. $f(x,y) = \frac{xy}{x^2+y^2}$ $(x,y)\ne(0,0)$ $f_x = \frac{y(x^2+y^2) - xy\cdot 2x}{(x^2+y^2)^2} = \frac{y^3 - x^2 y}{(x^2+y^2)^2}$

$= 0$ $(x,y) = (0,0)$ $f_x(x,mx) = \frac{1}{x}\cdot\frac{m^3-m}{(1+m^2)^2}$

Theorem: If f_x, f_y exists and $|f_x| \le M_1$, $|f_y| \le M_2$ on R, domain of f, then f is continuous

pf. $\Delta = f(a+h,b+k) - f(a,b) = f(a+h,b+k) - f(a+h,b) + f(a+h,b) - f(a,b)$

$= f_y(a+h, b+\theta k)\cdot k + f_x(a+\theta' h, b)\cdot h$ ($\theta \in (0,1)$, $\theta' \in (0,1)$)

$|\Delta| \le M_2\cdot k + M_1\cdot h \xrightarrow[h,k\to 0]{} 0$

harmonic function

eg4. $f(x,y,z) = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}$ $\Delta =$ Laplace operator $= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$f_x = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x$, $f_{xx} = \frac{3}{4}(x^2+y^2+z^2)^{-5/2}\cdot(2x)^2 - (x^2+y^2+z^2)^{-3/2}$

$\Delta f = f_{xx} + f_{yy} + f_{zz} = \frac{3(x^2+y^2+z^2) - 3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$ → Laplace equation eg 5. $f(x,t) = \frac{1}{\sqrt{t}} e^{-(x-a)^2/4t}$ satisfies $\frac{\partial f}{\partial t} = f_{xx} \leftarrow \Delta f$

(annotations: x, y above x; $(x-a)^2+(y-b)^2$ above the exponent)

Newton's heat equation

Theorem: If f_{xy} and f_{yx} are continuous on an open set R, then $f_{xy} = f_{yx}$

pf: let $A(h,k) := f(a+h, b+k) - f(a, b+k) - f(a+h, b) + f(a,b)$

Wrong method still works

$A(h,k) = f_x(a+\theta_1 h, b+k) - f_x(a+\theta_2 h, b)$

let $\phi(x) = f(x, b+k) - f(x,b)$

$A(h,k) = \phi(a+h) - \phi(a) = \phi'(a+\theta_1 h)\cdot h = \big(f_x(a+\theta_1 h, b+k) - f_x(a+\theta_1 h, b)\big)h = f_{yx}(a+\theta_1 h, b+\theta_2 k)\cdot hk$

$\lim_{(h,k)\to(0,0)} \frac{A(h,k)}{hk} = f_{yx}(a,b)$

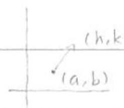
change the role of $x \Leftrightarrow y$, $f_{xy}(a,b) = f_{yx}(a,b)$

3/1 what is the notion of "differentiability" of a multivariable function?

Def. A function $z = f(x,y)$ is differentiable at $(x,y) = (a,b)$ iff

$$f(a+h, b+k) = f(a,b) + Ah + Bk + o(\sqrt{h^2+k^2})$$ linear approximation

$o(\sqrt{h^2+k^2}) = \varepsilon(h,k)\sqrt{h^2+k^2}$ with $\lim_{(h,k)\to(0,0)} \varepsilon = 0$



Corollary: $A = f_x(a,b)$, $B = f_y(a,b)$

set $(h,k) = (h,0)$: $f(a+h, b) = f(a,b) + Ah + o(|h|) \Rightarrow A = \frac{\partial f}{\partial x}(a,b)$

Theorem: if f_x, f_y exist and are conti. in a nbd of (a,b), then f is differentiable at (a,b)

(nbd: neighborhood (ie. open set))

pf. $f(a+h, b+k) - f(a,b)$

$= f(a+h, b+k) - f(a+h, b) + f(a+h, b) - f(a,b)$

$= f_y(a+h, b+\theta_1 k)\cdot k + f_x(a+\theta_2 h, b)\cdot h$

$f(a+h, b+k) - f(a,b) - \big(f_x(a,b)h + f_y(a,b)k\big)$

$= \big(f_x(a+\theta_2 h, b) - f_x(a,b)\big)h + \big(f_y(a+h, b+\theta_1 k) - f_y(a,b)\big)k$

$f(a+h,b) - f(a,b) - f_x(a,b)h = o(|h|)$

$= o(\sqrt{h^2+k^2})$ by continuity of f_x & f_y

Theorem': if $z = f(x_1, \cdots, x_n)$ has at least $(n-1)$ of $\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}$ to be conti., then f is differentiable

Def: $f \in C^k$ iff all partial derivatives of order $\le k$ are continuous

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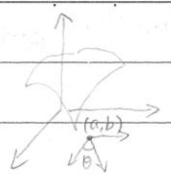
Directional derivatives $\left(\frac{\partial}{\partial r}\right)$ $D_{\vec{v}} f = \nabla f \cdot \vec{v}$



$$D_{(\theta)} f(a,b) := \lim_{r\to 0} \frac{f(a+r\cos\theta,\, b+r\sin\theta) - f(a,b)}{r}$$

Assume that f is differentiable at (a,b)

$$\lim_{r\to 0} \frac{f_x(a,b)\cdot r\cos\theta + f_y(a,b)\cdot r\sin\theta + o(\sqrt{h^2+k^2})}{r} = f_x\cos\theta + f_y\sin\theta$$

chain rule

$u = f(x,y,z)$, $x = x(t)$, $y = y(t)$, $z = z(t)$

$$\Delta f = f_x\cdot\Delta x + f_y\cdot\Delta y + f_z\cdot\Delta z + o(|\Delta\vec{x}|) \quad \sqrt{\Delta x^2+\Delta y^2+\Delta z^2}$$

$$\lim_{\Delta t\to 0}\frac{\Delta f}{\Delta t} = f_x\cdot x' + f_y\cdot y' + f_z\cdot z' \pm \lim_{\Delta t\to 0}\frac{o(|\Delta\vec{x}|)}{|\Delta\vec{x}|}\cdot\left|\frac{\Delta\vec{x}}{\Delta t}\right|$$

$\nabla f\cdot(x', y', z')$

Def: $\nabla f := (f_x, f_y, f_z)$ called the gradient of f
梯度

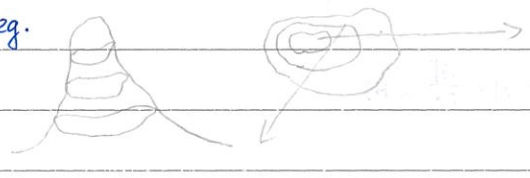
$f(x(t), y(t), z(t)) = c$ constant

$\nabla f\cdot(x', y', z') = 0 \Rightarrow \nabla f$ is the normal vector

eg. $u = f(x,y,z) = 3x^2+2y^2+z^2$ on the level surface $u = 1$

the normal vector is given by $\nabla f = (6x, 4y, 2z)$

eg.



$(x(t), y(t))$

$$\frac{\partial z}{\partial t} = \nabla f\cdot(x'(t), y'(t))$$

$$z = f(x,y) \doteq f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

↖ up to $o(\sqrt{(x-a)^2+(y-b)^2})$

f is diff at (a,b) ⇔ the notion of tangent plane exists

h, k $\Delta x, \Delta y$ dx, dy

$df = f_x\,dx + f_y\,dy$ total differential (全微分)

"d" $= dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}$ operator (算子)

$$d^2 f = f_{xx}dx^2 + f_{yx}dx\,dy + f_{xy}dy\,dx + f_{yy}dy^2 \underset{f\in C^2}{=} f_{xx}dx^2 + 2f_{xy}dx\,dy + f_{yy}dy^2$$

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3/3 1.6 Composite of functions

$u = f(\xi, \eta)$ $\quad (x,y) \in S \xrightarrow{g=(\phi,\psi)} (\xi,\eta) \in R \xrightarrow{f} \mathbb{R}$

$S \subset \mathbb{R}^2$, $R \subset \mathbb{R}^2$, $F = f \circ g$

$\xi = \phi(x,y)$, $\eta = \psi(x,y)$ (phi, psi)

Thm: f, g conti. $\Rightarrow$ $F = f \circ g$ is conti.

pf. given $\varepsilon > 0$, $\exists \delta_1$ st. $|W - g(P)| < \delta_1 \Rightarrow |f(W) - f(g(P))| < \varepsilon$

$\exists \delta$ st. $|Q - P| < \delta \Rightarrow |g(Q) - g(P)| < \delta_1$

Thm: f, g differentiable $\Rightarrow$ $F = f \circ g$ is also differentiable & " formula"

(g differentiable ↳ ϕ, ψ are both differentiable)

$du = f_\xi d\xi + f_\eta d\eta \qquad d\xi = \phi_x dx + \phi_y dy \qquad d\eta = \psi_x dx + \psi_y dy$

$\Rightarrow du = \underbrace{(f_\xi \phi_x + f_\eta \psi_x)}_{\frac{\partial u}{\partial x}} dx + \underbrace{(f_\xi \phi_y + f_\eta \psi_y)}_{\frac{\partial u}{\partial y}} dy$

pf. $\Delta u = f_\xi \Delta\xi + f_\eta \Delta\eta + \varepsilon \cdot \sqrt{\Delta\xi^2 + \Delta\eta^2}$

$\Delta\xi = \phi_x \Delta x + \phi_y \Delta y + \varepsilon_1 \cdot \sqrt{\Delta x^2 + \Delta y^2}$

$\Delta\eta = \psi_x \Delta x + \psi_y \Delta y + \varepsilon_2 \cdot \sqrt{\Delta x^2 + \Delta y^2}$

$(\Delta x, \Delta y) \to (0,0) \Rightarrow (\Delta\xi, \Delta\eta) \to (0,0) \Rightarrow \varepsilon \to 0$

$\Rightarrow \Delta u = (f_\xi \phi_x + f_\eta \psi_x)\Delta x + (f_\xi \phi_y + f_\eta \psi_y)\Delta y + \underline{\varepsilon \sqrt{\Delta\xi^2 + \Delta\eta^2} + (f_\xi \cdot \varepsilon_1 + f_\eta \cdot \varepsilon_2)\sqrt{\Delta x^2 + \Delta y^2}}$

$\sqrt{\Delta\xi^2 + \Delta\eta^2} \le |\Delta\xi| + |\Delta\eta| \le |\phi_x||\Delta x| + |\phi_y||\Delta y| + |\varepsilon_1| \cdot \sqrt{\Delta x^2 + \Delta y^2} + |\psi_x||\Delta x| + |\psi_y||\Delta y| + |\varepsilon_2| \cdot \sqrt{\Delta x^2 + \Delta y^2}$

$\Rightarrow \frac{\sqrt{\Delta\xi^2 + \Delta\eta^2}}{\sqrt{\Delta x^2 + \Delta y^2}} \le$ ______ is bounded

eg. $u = f(x,y) = f(\underset{x(r,\theta)}{r\cos\theta}, \underset{y(r,\theta)}{r\sin\theta})$

$\frac{\partial u}{\partial r} = f_x x_r + f_y y_r = f_x \cos\theta + f_y \sin\theta$

$\frac{\partial u}{\partial \theta} = f_x x_\theta + f_y y_\theta = -f_x r\sin\theta + f_y r\cos\theta$

eg. $u = f(\xi, \eta)$, $\xi = \phi(x,y)$, $\eta = \psi(x,y)$

$u_x = u_\xi \xi_x + u_\eta \eta_x \Rightarrow u_{yx} = (u_\xi \xi_x)_y + (u_\eta \eta_x)_y = (u_{\xi\xi} \xi_y \xi_x + u_{\eta\xi} \eta_y \xi_x + u_\xi \xi_{yx})$

$+ (u_{\xi\eta} \xi_y \eta_x + u_{\eta\eta} \eta_y \eta_x + u_\eta \eta_{yx})$

$\Rightarrow u_{xx} = (u_\xi \xi_x)_x + (u_\eta \eta_x)_x = (u_{\xi\xi} \xi_x^2 + u_{\eta\xi} \eta_x \xi_x + u_\xi \xi_{xx})$

$+ (u_{\xi\eta} \xi_x \eta_x + u_{\eta\eta} \eta_x^2 + u_\eta \eta_{xx})$

$\Rightarrow u_{yy} = (u_\xi \xi_y)_y + (u_\eta \eta_y)_y = (u_{\xi\xi} \xi_y^2 + u_{\eta\xi} \eta_y \xi_y + u_\xi \xi_{yy})$

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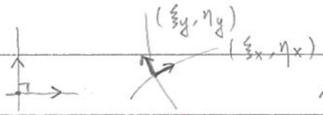
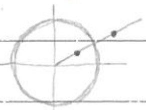
Assume C^2, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = u_\xi(\xi_{xx}+\xi_{yy}) + u_\eta(\eta_{xx}+\eta_{yy}) + (\xi_x^2+\xi_y^2)u_{\xi\xi} + (\eta_x^2+\eta_y^2)u_{\eta\eta}$ $+2u_{\xi\eta}(\xi_x\eta_x+\xi_y\eta_y)$



$\xi_x\xi_y+\eta_x\eta_y=0$

保角 $\xi_x^2+\eta_x^2=\xi_y^2+\eta_y^2$

① $u=f\left(\frac{x}{x^2+y^2},\ \frac{y}{x^2+y^2}\right)$ 反演：圓內翻到圓外 (inversion)



$\xi_x = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$; $\xi_y = \frac{-2xy}{(x^2+y^2)^2}$

$\eta_x = \frac{-2xy}{(x^2+y^2)^2}$; $\eta_y = \frac{x^2-y^2}{(x^2+y^2)^2}$

$\Rightarrow \xi_x\eta_x+\xi_y\eta_y = \frac{-2xy}{(x^2+y^2)^4}\left((y^2-x^2)+(x^2-y^2)\right)=0$

$\Rightarrow \xi_x^2+\xi_y^2=\eta_x^2+\eta_y^2$

$\xi_{xx} = \frac{-2x(x^2+y^2)^2-(y^2-x^2)\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4} = \frac{-2x^3-2xy^2-4y^2x+4x^3}{(x^2+y^2)^3} = \frac{2x^3-6xy^2}{(x^2+y^2)^3}$

$\xi_{yy} = \frac{-2x(x^2+y^2)^2+2xy\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4} = \frac{-2x^3-2xy^2+8xy^2}{(x^2+y^2)^3} = \frac{-2x^3+6xy^2}{(x^2+y^2)^3}$

$u=f(x,y)=f(r\cos\theta, r\sin\theta)$ $u(r,\theta)=f(r)g(\theta)$

$\Delta u = u_{xx}+u_{yy} = u_{rr}+\frac{1}{r^2}u_{\theta\theta}+\frac{1}{r}u_r$

$u_r = u_x x_r + u_y y_r = \cos\theta\, u_x+\sin\theta\, u_y$

$u_\theta = u_x x_\theta + u_y y_\theta = -r\sin\theta\, u_x + r\cos\theta\, u_y$

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Polar coordinates

$u=u(x,y)\in C^2$ $x=r\cos\theta$, $y=r\sin\theta$ $r=\sqrt{x^2+y^2}$, $\theta=\tan^{-1}\frac{y}{x}$

$\Delta u = u_{xx}+u_{yy}$ $r_x=\frac{x}{\sqrt{x^2+y^2}}$, $r_y=\frac{y}{\sqrt{x^2+y^2}}$

$u_x = u_r r_x + u_\theta \theta_x$ $\theta_x = \frac{-y/x^2}{1+(y/x)^2} = \frac{-y}{x^2+y^2}$, $\theta_y = \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2}$

$= u_r\cdot\frac{x}{r} + u_\theta\cdot\frac{-y}{r^2}$ Rmk: $(r_x,\theta_x)\perp(r_y,\theta_y)$, so it is not a conformed change (保角) of coordinate

$u_y = u_r\cdot\frac{y}{r}+u_\theta\cdot\frac{x}{r^2}$

$u_{xx} = \left(u_r\cdot\frac{x}{r}-u_\theta\cdot\frac{y}{r^2}\right)_x$

$= \left(u_{rr}\cdot\frac{x^2}{r^2}+u_{r\theta}\cdot\frac{-xy}{r^3}+u_r\cdot\frac{r-x\cdot\frac{x}{r}}{r^2}\right)-\left(u_{\theta r}\cdot\frac{xy}{r^3}-u_{\theta\theta}\cdot\frac{y^2}{r^4}+u_\theta\cdot\frac{-2y\cdot\frac{x}{r}}{r^3}\right)$ (where $\frac{r-x\cdot\frac{x}{r}}{r^2} = \frac{y^2}{r^3}$)

$= u_{rr}\cdot\frac{x^2}{r^2}-2u_{r\theta}\cdot\frac{xy}{r^3}+u_{\theta\theta}\cdot\frac{y^2}{r^4}+u_r\cdot\frac{y^2}{r^3}+u_\theta\cdot\frac{2xy}{r^4}$

$u_{yy} = u_{rr}\cdot\frac{y^2}{r^2}+2u_{r\theta}\cdot\frac{xy}{r^3}+u_{\theta\theta}\cdot\frac{x^2}{r^4}+u_r\cdot\frac{x^2}{r^3}-u_\theta\cdot\frac{2xy}{r^4}$

$u_{xx}+u_{yy} = u_{rr}+\frac{1}{r^2}u_{\theta\theta}+\frac{1}{r}u_r$

1.7 Mean Value Theorem & Taylor expansion

$$f(Q)-f(P) = f(P+\vec{h}) - f(P) = g(1)-g(0) = g'(\theta)\cdot 1 = (\nabla f)(P+\theta\vec{h})\cdot\vec{h}$$

($\vec{h} = (h,k)$, $\theta \in (0,1)$)

let $g(t) = f(\vec{P}+t\vec{h})$

$g'(t) = \frac{d}{dt} f(\vec{P}+t\vec{h}) = \nabla f\cdot\vec{h}$ $\quad\left(g(t) = f(a+th, b+tk),\ g'(t) = f_x\cdot h + f_y\cdot k = \nabla f\cdot(h,k)\right)$

前提：可全微分

Taylor expansion

$$f(x,y) = f(a,b) + a_{10}(x-a) + a_{01}(y-b) + a_{20}(x-a)^2 + a_{11}(x-a)(y-b) + a_{02}(y-b)^2 + \cdots$$

" $a_{ij}(x-a)^i(y-b)^j$ "

$g(t) = f(P+t\vec{h})$, $g'(t) = \nabla f\cdot\vec{h} = f_x h + f_y k = \left(h\cdot\frac{\partial}{\partial x} + k\cdot\frac{\partial}{\partial y}\right) f = df$

$g''(t) = \left(h\cdot\frac{\partial}{\partial x} + k\cdot\frac{\partial}{\partial y}\right)^2 f$ $\quad$ ↳ differential operator 微分算子

$\Rightarrow$ if $f\in C^k$ then $g^{(k)}(t) = d^k f = \left(h\cdot\frac{\partial}{\partial x} + k\cdot\frac{\partial}{\partial y}\right)^k f$

$$g(1) = g(0) + g'(0)\cdot 1 + \frac{g''(0)}{2!}\cdot 1^2 + \cdots + \frac{g^{(n)}(0)}{n!}\cdot 1^n + R_n,\quad R_n = \frac{g^{(n+1)}(\theta)}{(n+1)!}$$

assume $g\in C^{n+1}$

$$g^{(k)}(0) = h^k\frac{\partial^k f}{\partial x^k}(a,b) + \cdots + C^k_i h^i k^{k-i}\frac{\partial^k f}{\partial x^i\partial y^{k-i}}(a,b) + \cdots$$

$$\Rightarrow f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2!}\Big(f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\Big)$$

$z = f(x,y) = x^2+y^2$

$z = f(x,y) = -x^2-y^2$

$z = f(x,y) = x^2-y^2$

$$(x-a\ \ y-b)\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{pmatrix}\begin{pmatrix} x-a \\ y-b\end{pmatrix}$$

$$(x-a\ \ y-b\ \ z-c)\begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz}\end{pmatrix}\begin{pmatrix} x-a \\ y-b \\ z-c\end{pmatrix}$$

二次型 quadratic form

$\Sigma a_{ij}x_i x_j$, $a_{ij} = a_{ji}$ $\qquad \vec{x}^T A\vec{x}$ $\qquad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n\end{pmatrix}$

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$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$ conformal mapping (保角)

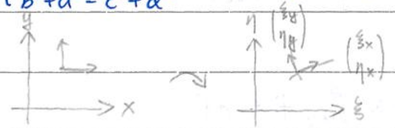
旋轉: $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ · 反射: $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ 只有這兩種

pf. $(1,0) \mapsto (a,c)$, $(0,1) \mapsto (b,d)$ $\Rightarrow$ $ab+cd=0$

$(1,1) \mapsto (a+b, c+d)$, $(-1,1) \mapsto (-a+b, -c+d)$ $\Rightarrow$ $b^2-a^2+d^2-c^2=0$

$(b^2-a^2)^2 = (c^2-d^2)^2 \Rightarrow b^4+a^4 = c^4+d^4 \Rightarrow (b^2+a^2)^2 = (c^2+d^2)^2 \Rightarrow b^2+a^2 = d^2+c^2$

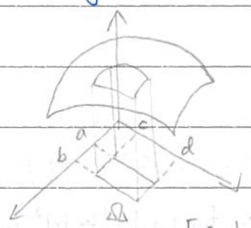
$\begin{cases} b^2-a^2 = c^2-d^2 \\ b^2+a^2 = c^2+d^2 \end{cases}$ $b^2=c^2$, $a^2=d^2$ $b=\pm c$, $a=\mp d$

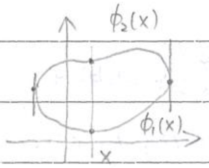


$\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}$ Jacobian 2 possibilities:

① $\begin{cases} \xi_x = \eta_y \\ \xi_y = -\eta_x \end{cases}$ ② $\begin{cases} \xi_x = -\eta_y \\ \xi_y = \eta_x \end{cases}$

1.8 Integrals of functions with parameters



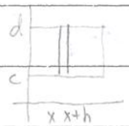


$g(x) = \int_c^d f(x,y)\,dy$ ($d \leftarrow \phi_2(x)$, $c \leftarrow \phi_1(x)$, x: parameter)

Let $f(x,y)$ be continuous

eg. $\int_c^d y^{1/x}\,dy = \frac{1}{\frac{1}{x}+1} y^{1/x+1} \Big|_c^d$

① $g(x)$ is continuous HW: [ ]×[ ] ⇒ uniform conti. Appl.



pf. $|g(x+h)-g(x)| = \left| \int_c^d \big(f(x+h,y) - f(x,y)\big)\,dy \right| \le \varepsilon \cdot (d-c)$

if $|f(x+h,y) - f(x,y)| < \varepsilon \quad \forall y \in [c,d]$ ← uniform continuity for continuous functions with multivariables

② Assume that $f_x(x,y)$ exists and is continuous

Q: $g'(x) \overset{?}{=} \int_c^d f_x(x,y)\,dy$ Ans: Yes

eg. $g(k) = \int_0^1 (x-1)\frac{x^k}{\log x}\,dx$ $k > -1$, $k \in \mathbb{R}$

$\frac{d}{dk} g(k) = \int_0^1 \frac{x-1}{\log x} \cdot x^k \cdot \log x \cdot dx$

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pf. $g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \int_c^d \frac{f(x+h,y)-f(x,y)}{h}\,dy = \lim_{h\to 0}\int_c^d f_x(x+\theta h, y)\,dy$

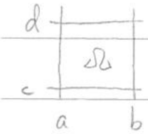
$\left|\int_c^d (f_x(x+\theta h,y) - f_x(x,y))\,dy\right| \le \varepsilon(d-c)$ when $|h|<\delta$ (uniform continuity)

$g(x) = \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy$, f_x conti. ϕ_1, ϕ_2 differentiable

$\Rightarrow g'(x) = f(x,\phi_2(x))\phi_2'(x) - f(x,\phi_1(x))\phi_1'(x) + \int_{\phi_1(x)}^{\phi_2(x)} f_x(x,y)\,dy$

pf. $F(x,u,v) = \int_u^v f(x,y)\,dy$, $g(x) = F(x, \phi_1(x), \phi_2(x))$

$g'(x) = F_x(x)' + F_u\phi_1'(x) + F_v\phi_2'(x) = \int_{\phi_1(x)}^{\phi_2(x)} f_x(x,y)\,dy - f(x,\phi_1(x))\phi_1'(x) + f(x,\phi_2(x))\phi_2'(x)$



$\int_a^b dx \int_c^d f(x,y)\,dy \overset{?}{=} \int_c^d dy \int_a^b f(x,y)\,dx$ (Fubini theorem)

Yes, if $f(x,y)$ is continuous

pf. $v(x,y) = \int_c^y f(x,\eta)\,d\eta$, $u(x,y) = \int_a^x v(\xi,y)\,d\xi = \int_a^x d\xi \int_c^y f(\xi,\eta)\,d\eta$

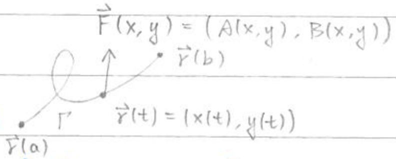
$\rightarrow v_y = f(x,y)$ conti. $\rightarrow u_y(x,y) = \int_a^x v_y(\xi,y)\,d\xi = \int_a^x f(\xi,y)\,d\xi$

$\Rightarrow u(x,y) = u(x,y) - u(x,c) = \int_c^y d\eta \int_a^x f(\xi,\eta)\,d\xi$

(↑ $u(x,c)$ = 0)

3/15 1.9 Differential and Line integrals

Recall arc length, work $\int_a^b \vec{F}\cdot d\vec{r}$ $\quad \vec{r}:[a,b]\to\mathbb{R}^2$



$\int_a^b \vec{F}\cdot d\vec{r} = \int_a^b \left(A(x,y)\cdot\frac{dx}{dt} + B(x,y)\cdot\frac{dy}{dt}\right)dt = \int_\Gamma A\,dx + B\,dy$

↓ $\vec{r}'(t)\,dt = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)dt$

↳ this is independent of the parameter "t" as long as the orientation of Γ is preserved

This is a general form of "1-differential form" (一次微分形式)

eg. total differential (全微分) $df = f_x\,dx + f_y\,dy$

- Usually (in this book) we denote by Γ^* a curve with a fixed orientation.
- Γ with the reverse orientation is denoted by $-\Gamma^*$

$\Rightarrow \int_{-\Gamma^*} = -\int_{\Gamma^*}$

let $L := A\,dx + B\,dy + C\,dz$, A, B, C are functions in x, y, z, C^1 in $\mathbb{R}^3$

if $L = df$ for some f ie. $A = f_x$, $B = f_y$, $C = f_z$ (or equivalently, $(A,B,C) = \nabla f$)

then $\int_\Gamma L = \int_\Gamma df = \int_a^b \frac{df}{dt}\,dt = f(x(t),y(t),z(t))\Big|_a^b = f(Q) - f(P)$

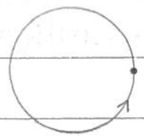
If such an f does exist, we must have $A=f_x$, $B=f_y$, $C=f_z$ ($\Rightarrow f\in C^2$) $\star$: $\left| A_y = B_x, \; B_z = C_y, \; C_x = A_z \right|$ (both $= f_{xy}$)

Q: Does condition $\star$ imply the existence of f?

Examples: ① $L = y\,dx + z\,dy + x\,dz$: $A_y = 1 \neq B_x = 0$

② $L = yz\,dx + zx\,dy + xy\,dz = d(xyz) \Rightarrow f = xyz$

Fake total differential ③ $"d\theta" = d\tan^{-1}\frac{y}{x} = \frac{-y\,dx + x\,dy}{x^2+y^2}$, $A = \frac{-y}{x^2+y^2}$, $B = \frac{x}{x^2+y^2}$



$A_y = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$, $B_x = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

but $\int_{\Gamma^*} \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^{2\pi} -\sin\theta(-\sin\theta)\,d\theta + \cos\theta\cos\theta\,d\theta = \int_0^{2\pi} d\theta = \theta\Big|_0^{2\pi} = 2\pi \neq 0$

Γ^* = unit circle
$(x(t), y(t)) = (\cos\theta, \sin\theta)$

Next time: will show the condition $\star$ is sufficient if the domain $U \subset \mathbb{R}^2$ of $\vec{F} = (A,B)$ is simply connected (單連通) (兩條路徑可以連續變化)

eg. $\vec{F}(x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2+z^2}^{\,3}}, \frac{y}{\sqrt{x^2+y^2+z^2}^{\,3}}, \frac{z}{\sqrt{x^2+y^2+z^2}^{\,3}} \right)$

$= Df$ $\quad f(x,y,z) = \frac{-1}{\sqrt{x^2+y^2+z^2}}$

$\mathbb{R}^3 \setminus (0,0,0)$ is simply connected

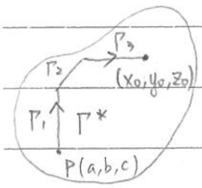
$\mathbb{R}^3 \setminus \mathbb{R}$ is not simply connected 

Fact: The line integral $\int L$ ie. independent of path $\Leftrightarrow$ $L = df$ for some f

pf. "$\Leftarrow$" : finished

"$\Rightarrow$" : we have to "define f" first

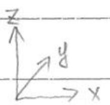
$f(x,y,z) := \int_{P^*} L = \int_{P^*} A\,dx + B\,dy + C\,dz$



fix $P \in \mathbb{R}^3$ ↳ any piecewise C^1 curve connecting P and (x,y,z)

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\int_{\Gamma_1} + \int_{\Gamma_2} + \int_{(a,y_0,z_0)}^{(x,y_0,z_0)} A\,dx + B\,dy + C\,dz \right)$

$= \frac{\partial}{\partial x}\left(\int_{(a,y_0,z_0)}^{(x,y_0,z_0)} A\,dx \right)$



$= A$

similarly, $B = f_y$, $C = f_z$

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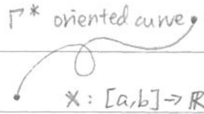
3/17 Line integral



$x : [a,b] \to \mathbb{R}^3$

$x(t) = (x(t), y(t), z(t))$

$$\int_{\Gamma^*} L = \underset{\text{1-form}}{A\,dx + B\,dy + C\,dz} = \int_a^b \left(A \cdot \frac{dx}{dt} + B\frac{dy}{dt} + C\frac{dz}{dt} \right) dt = \int_{\Gamma^*} \vec{F} \cdot d\vec{x}$$

$(\vec{F} = (A, B, C))$

Last time: $\int_{\Gamma^*} L$ is independent of path $\Leftrightarrow$ $L = df$ for some f (potential function) $\;\leftrightarrow \vec{F} = \nabla f$

$\Rightarrow$ curl $\vec{F} = 0$ compatibility coming from $f_{x_i x_j} = f_{x_j x_i}$

$f \quad \nabla f = \text{grad } f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) f$

$\vec{F} = (A, B, C) \quad \text{div. } \vec{F} = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = \nabla \cdot \vec{F}$

(divergence 散度)

$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A & B & C \end{vmatrix} = (C_y - B_z, A_z - C_x, B_x - A_y)$

(旋度)

$(a_1, b_1, c_1) \times (a_2, b_2, c_2) = \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$ why? HW

$L = A\,dx + B\,dy + C\,dz \longleftrightarrow \vec{F} = (A, B, C)$

d : Elie Cartan's d operator

1. $d \to f = df$ 2. $dx\,dy$ $\Delta x \Delta y$ $\curvearrowright$ $dx \wedge dy = -dy \wedge dx$ ($\wedge$: wedge)

$dL = dA \wedge dx + dB \wedge dy + dC \wedge dz$

$= (A_x dx + A_y dy + A_z dz) \wedge dx + (B_x dx + B_y dy + B_z dz) \wedge dy + (C_x dx + C_y dy + C_z dz) \wedge dz$

$= (B_x - A_y)\, dx \wedge dy + (C_y - B_z)\, dy \wedge dz + (A_z - C_x)\, dz \wedge dx$

$L = \sum_{i=1}^{n} A_i\, dx^i \quad (x^1, \cdots, x^n).$

$dL = \sum_{i=1}^{n} dA_i \wedge dx^i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial A_i}{\partial x^j} dx^j \right) \wedge dx^i = \sum_{i<j} \left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \right) dx^i \wedge dx^j$

$= 0$ closed 1-form

Theorem: Let L be a C^1 one form defined on a simply connected open set $U \subseteq \mathbb{R}^3$,

$dL = 0 \Leftrightarrow L = df$ for some f

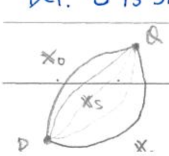
(closed one form) (exact)

Recall: $\Leftarrow$ trivial

$\Rightarrow$ is reduced to prove that the line integral is independent of path

Simply connectedness (單連通):

Def. U is simply connected if any two curves with the same end points can be deformed to each other continuously



$x_0 : [0,1] \to \mathbb{R}^3 \quad x_0(0) = P,\ x_1(1) = Q$

$x_1 : [0,1] \to \mathbb{R}^3 \quad x_1(0) = P,\ x_1(1) = Q$

ie. $\exists\, x : [0,1] \times [0,1] \to U$ continues $\quad x(t,s)$ st. $x(t,0) = x_0(t)$

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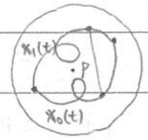

Ball, $B_p(r)$ $\quad X(t,s) = (1-s)X_0(t) + sX_1(t)$

Fact : Any convex set is simply connected

$$\int_{\Gamma_1^*} L - \int_{\Gamma_0^*} L = \int_0^1 \left(A\frac{\partial x}{\partial t} + B\frac{\partial y}{\partial t} + C\frac{\partial z}{\partial t}\right)\Big|_{s=1} dt - \int_0^1 \left(A\frac{\partial x}{\partial t} + B\frac{\partial y}{\partial t} + C\frac{\partial z}{\partial t}\right)\Big|_{s=0} dt$$

$$A\frac{\partial x}{\partial t}\Big|_{s=1} - A\frac{\partial x}{\partial t}\Big|_{s=0} = \int_0^1 \frac{\partial}{\partial s}\left(A\frac{\partial x}{\partial t}\right) ds = \int_0^1 \left((A_x X_s + A_y y_s + A_z z_s) X_t + A X_{st}\right) ds$$

$$= \int_0^1 \left((B_x X_s + B_y y_s + B_z z_s) y_t + B y_{st}\right) ds$$

$$= \int_0^1 \left((C_x X_s + C_y y_s + C_z z_s) z_t + C z_{st}\right) ds$$

$$\int_0^1 (AX_t + By_t + Cz_t)_s \, ds = \int_0^1 (AX_s + By_s + Cz_s)_t \, dt$$

$\because B_x = A_y,\ C_y = B_z,\ A_z = C_x$

(Poincaré lemma)

$$= \int_0^1 ds\,(AX_s + By_s + Cz_s)\Big|_{t=1} - \int_0^1 ds\,(AX_s + By_s + Cz_s)\Big|_{t=0}$$

$= 0$

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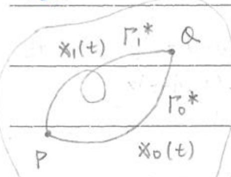


$U \subset \mathbb{R}^n$ open set simply-connected

in $\mathbb{R}^n \ni (x^1, \cdots, x^n)$ $\quad \int_{\Gamma_1^*} L$

$$L = \sum_{i=1}^n A_i\, dx^i$$

$$dL = \sum_{i=1}^n dA_i \wedge dx^i = \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial A_i}{\partial x^j} dx^j\right) \wedge dx^i$$

$$= \sum_{j<i} \left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right) dx^i \wedge dx^j$$

The necessary condition for the line integral to be independent of path is $dL = 0 \equiv \frac{\partial A_j}{\partial x^i} = \frac{\partial A_i}{\partial x^j}$ $\searrow\swarrow$ $\frac{\partial^2 f}{\partial x^i \partial x^j}$

(indep. of path $\equiv \exists f$ st. $L = df$ ie. $A_i = \frac{\partial f}{\partial x^i} = f_i$)

~~homotopy~~

$\exists X : [0,1] \times [0,1] \longrightarrow U$ continuous $\quad X(t,s) = (x^1(t,s), x^2(t,s), \cdots x^n(t,s))$
(with $t \in [0,1]$, $s \in [0,1]$)

st. $X(t,0) = X_0(t)$, $X(t,1) = X_1(t)$

$$(*) = \int_{\Gamma_1^*} L - \int_{\Gamma_0^*} L = \int_0^1 dt \sum_{i=1}^n \left(A_i(X)\frac{\partial x^i}{\partial t}\Big|_{s=1} - A_i(X)\frac{\partial x^i}{\partial t}\Big|_{s=0}\right)$$

(with $\frac{\partial x^i}{\partial t}\big|_{s=1} = \frac{dx_1^i}{dt}$, $\frac{\partial x^i}{\partial t}\big|_{s=0} = \frac{dx_0^i}{dt}$)

$$= \int_0^1 dt \int_0^1 ds \sum_{i=1}^n \frac{\partial}{\partial s}\left(A_i \frac{\partial x^i}{\partial t}\right)$$

$$\sum_{j=1}^n \left(\frac{\partial A_i}{\partial x^j} \cdot \frac{\partial x^j}{\partial s} \cdot \frac{\partial x^i}{\partial t}\right) + A_i \cdot \frac{\partial^2 x^i}{\partial s \partial t}$$

($\frac{\partial A_i}{\partial x^j} \to \frac{\partial A_j}{\partial x^i}$)

$$= \int_0^1 dt \int_0^1 ds \sum_{i=1}^{n} \frac{\partial}{\partial t}\left(A_i \frac{\partial x^i}{\partial s}\right) = \int_0^1 ds \int_0^1 dt \sum_{i=1}^{n} \frac{\partial}{\partial t}\left(A_i \cdot \frac{\partial x^i}{\partial s}\right) = \int_0^1 ds \left(\underbrace{\sum_{i=1}^{n} A_i \frac{\partial x^i}{\partial s}}_{0} \right)\Bigg|_{t=0}^{t=1} = 0$$

Problem: $f(x,y)$ conti. in x, y, we want to approximate f by a C^2 function.

$f_h(x,y) := \frac{1}{4h^2} \int_{x-h}^{x+h} d\xi \int_{y-h}^{y+h} d\eta \, f(\xi,\eta)$ h fixed small number

blur

$= \frac{u(x+h,y+h) - u(x+h,y-h) - u(x-h,y+h) + u(x-h,y-h)}{4h^2}$ $\Rightarrow \frac{\partial}{\partial x} f_h, \frac{\partial}{\partial y} f_h, \frac{\partial^2}{\partial x \partial y} f_h = \frac{\partial^2}{\partial y \partial x} f_h$ conti.

↑ check the textbook

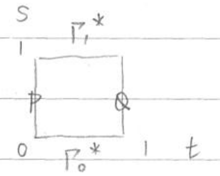
u is a "good" function

$|f_h(x,y) - f(x,y)| = \frac{1}{4h^2} \int_{x-h}^{x+h} \int_{y-h}^{y+h} (f(\xi,\eta) - f(x,y))\, dy \cdot dx < \varepsilon$

↑ if $< \varepsilon$

$x^i(t,s)$ is now approximated by $\overline{x^i}(t,s)$ ← "good" function $i = 1, 2, \cdots, n$

$x(t,s)$ conti. $\bar{x}(t,s)$ good



$$\begin{aligned}\tilde{x}(t,s) = {} & \bar{x}(t,s) - (1-s)(\bar{x}(t,s) - x_0(t)) - s(\bar{x}(t,s) - x_1(t)) \\ & - (1-t)(\bar{x}(t,s) - x_0(0)) - t(\bar{x}(t,s) - x_1(1)) \\ & + (1-t)(1-s)(\bar{x}(0,0) - x_0(0)) + (1-t)s(\bar{x}(0,1) - x_1(0)) \\ & + t(1-s)(\bar{x}(1,0) - x_0(1)) + ts(\bar{x}(1,1) - x_1(1))\end{aligned}$$

四角

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$\mathbb{R}^1$ "$[a,b]$" →

$\mathbb{R}^n$ $[a_1,b_1] \times \cdots \times [a_n,b_n]$ "closed and bounded subset"

- bounded means $S \subset B_0(R)$ for some R
- closed means S contains all its boundary points

Theorem. (Bolzano-Weierstrass) Let $S \subset \mathbb{R}^n$ be a closed and bounded set. Then any sequence $P_n \in S$, $n = 1, 2, \cdots$ has a convergent subsequence in S

pf. divided into 2^n subcubes, $\exists$ a subcube which still contains ∞-many P_i's

Def A set $S \subset \mathbb{R}^n$ is called (sequentially) compact (緊緻) iff it is closed and bounded

Thm: Let $f: \mathbb{R}^n \supset S \to \mathbb{R}$ be conti. with S being compact Then $\exists\, p \in S$ st.

$f(p) = \max_S f$

pf. claim $\exists M$ st. $f \le M$.

If not, $\forall n \in \mathbb{N}$, $\exists P_n \in S$ st. $f(P_n) \ge n$

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let P_{n_i} be a convergent subsequence, $\lim_{i\to\infty} P_{n_i} = q \in S$

$f(P_{n_i}) > n_i$ $\quad \lim_{i\to\infty} f(P_{n_i}) \to \infty$ $\quad \to\leftarrow$

$\qquad\qquad\qquad \| $

$\qquad\qquad f(\lim_{i\to\infty} P_{n_i}) = f(q)$

Chapter 3. Properties of differential Calculus for multivariables

- Implicit functions (隱函數)

Ex 1. $F(x,y) = (x^2+y^2)^2 - 2a^2(x^2-y^2) = 0$ Find the maximal value of y

雙紐線: $r^2 = \cos 2\theta$

Regard y as a function in x. "$y=f(x)$"

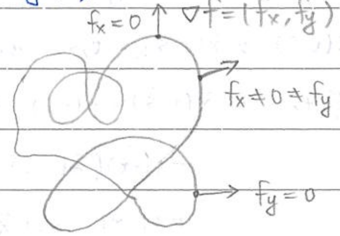
Apply $\frac{d}{dx}$ to it: $2(x^2+y^2)(2x+2yf') - 2a^2(2x-2yf') = 0$

$f' = \cdots$; Here we consider $f'(x) = 0$

$(x^2+y^2)\cdot x - a^2x = 0 \Rightarrow x(x^2+y^2-a^2) = 0$

$x = 0 \Rightarrow y^4 + 2a^2y^2 = 0 \Rightarrow \cdots$

or $x^2+y^2-a^2 = 0 \Rightarrow a^4 - 2a^2(a^2-2y^2) = 0 \Rightarrow \cdots$

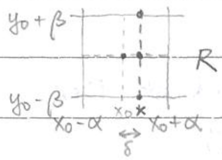


Theorem (Implicit Function Theorem): Let $F(x,y)$ be C^1 and $F(x_0,y_0)=0$, $F_y(x_0,y_0) = m \neq 0$ (>), then $\exists$ nbd of (x_0,y_0) st. $\exists! f$, $y=f(x)$ st. $F(x,f(x))=0$ and $f\in C^1$

in fact, $f'(x) = -\frac{F_x}{F_y}$

$F(x,y) = 0 \qquad \frac{d}{dx}: F_x + F_y\cdot\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$

pf. Step 1. : "$\exists!$" : $\exists R = [x_0-\alpha, x_0+\alpha]\times[y_0-\beta, y_0+\beta]$ st. $F_y > \frac{m}{2}$ on R and let $|F_x| \le M$ on R



Notice that $F(x,y) \nearrow$ in y for any fixed x

$|F(x,y_0) - F(x_0,y_0)| = |F_x(\xi, y_0)|\cdot|x-x_0| \le M\cdot|x-x_0|$

$F(\vec{x}, y_0+\beta) = \big(F(\vec{x}, y_0+\beta) - F(\vec{x}, y_0)\big) + F(\vec{x}, y_0)$

$\qquad = F_y(x,\eta)\cdot\beta$

$\qquad > \frac{m\beta}{2} - M\cdot|\vec{x}-\vec{x}_0|$

$F(\vec{x}, y_0-\beta) = \big(F(\vec{x}, y_0-\beta) - F(\vec{x}, y_0)\big) + F(\vec{x}, y_0)$

$\qquad = F_y(x,\eta')\cdot(-\beta)$

$\qquad < -\frac{m\beta}{2} + M\cdot|\vec{x}-\vec{x}_0|$

we just require that $|\vec{x}-\vec{x}_0| < \frac{m\beta}{2M} =: \delta$

$\Rightarrow \exists!\ y$ st. $F(\vec{x}, y) = 0$

Call this $\vec{x} \longmapsto y$ by $y = f(x)$

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Step 2. "$f \in C^1$"

let x be fixed $f(x+h) - f(x) = k$

$0 = F(\vec{x}+\vec{h}, \underbrace{f(\vec{x}+\vec{h})}_{y+k}) - F(x, \underbrace{f(x)}_{y})$

$= F_x(\vec{x}+\theta\vec{h}, y+\theta k)h + F_y(\vec{x}+\theta\vec{h}, y+\theta k)\cdot k, \quad \theta \in (0,1)$

$| \ | \le M \cdot |h|$; $\ge \frac{m}{2}$

$\Rightarrow |k| \le \frac{2M}{m}|\vec{h}| \Rightarrow f$ is (Lip) conti.

$\frac{f(\vec{x}+h_i) - f(x)}{h_i} = \frac{k}{h} = -\frac{F_{x_i}(x+\theta h, y+\theta k)}{F_y(x+\theta h, y+\theta k)} \to -\frac{F_x(x,y)}{F_y(x,y)}$ as $h \to 0$ ($\Rightarrow k \to 0$)

Multi-variable case: $F(x^1, x^2, \cdots x^n, y)$ st. $F(\vec{x_0}, y_0) = 0$; $F_y(\vec{x_0}, y) = m > 0$

$F(x,y) = 0, \quad f' = -\frac{F_x}{F_y}$

$\Rightarrow f'' = -\frac{(F_{xx} + F_{yx}f')F_y - F_x(F_{xy} + F_{yy}f')}{F_y^2} = -\frac{F_{xx}F_y + F_{yx}\cdot F_y \cdot (-\frac{F_x}{F_y}) - F_x \cdot F_{xy} + F_x \cdot F_{yy} \cdot (-\frac{F_x}{F_y})}{F_y^2}$

$= -\frac{F_y^2 F_{xx} - 2F_x F_y F_{xy} + F_x^2 F_{yy}}{F_y^3}$

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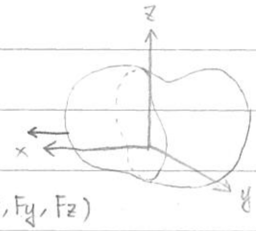
$u = F(x,y,z)$

level set $u =$ constant $= 0$



$\subset \mathbb{R}^3$ "surface"

$N = \nabla F = (F_x, F_y, F_z)$

$F(x_0, y_0, z_0) = 0, \quad F_x(x_0, y_0, z_0) \neq 0$

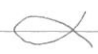
$\Rightarrow \exists!$ implicit function $x = f(y,z)$ near (x_0, y_0, z_0) st. $F(f(y,z), y, z) = 0$

What happens if $\nabla F(x_0, y_0, z_0) = 0$

Consider any curve through (x_0, y_0, z_0) on $F = 0$

$F(x(t), y(t), z(t)) = 0$

$0 = \nabla F \cdot (x'(t), y'(t), z'(t))\big|_{t=0}$ (若切向量張開全空間, $\nabla f = 0$)



$F(x,y) = y^2 - x^2(x+1)$

$\nabla F = (-3x^2 - 2x, 2y)$

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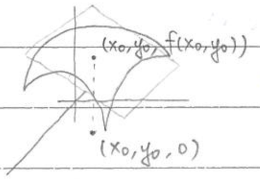
Def. A point $p \in \{\vec{x} \mid F(\vec{x}) = 0\}$ is a singular point (奇異點) if the tangent vectors at p span the whole space ($\Rightarrow \nabla F(p) = 0$) ※ ex. $F(x,y) = y^3 - x^4$

Example 1 $z = f(x,y)$



① $z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

② $(x-x_0, y-y_0, z-z_0)\cdot(-f_x, -f_y, 1) = 0$

$(x,y,f(x,y)) \xrightarrow{\frac{\partial}{\partial x}} (1,0,f_x)$, $(x,y,f(x,y)) \xrightarrow{\frac{\partial}{\partial y}} (0,1,f_y)$; $(1,0,f_x) \times (0,1,f_y) = (-f_x, -f_y, 1) = \vec{N}$

③ $F(x,y,z) := z - f(x,y) = 0$ level set

$\nabla F = (-f_x, -f_y, 1)$

Example 2. $F = 0$, $G = 0$ in $\mathbb{R}^3$

$$\cos\omega = \frac{\nabla F \cdot \nabla G}{|\nabla F|\cdot|\nabla G|} = \frac{F_x G_x + F_y G_y + F_z G_z}{\sqrt{F_x^2+F_y^2+F_z^2}\cdot\sqrt{G_x^2+G_y^2+G_z^2}}$$

$u = a_1x_1 + a_2x_2 + \cdots + a_nx_n + by$ trivial linear model

$u \leftrightarrow y$ ie. $y = -\frac{a_1}{b}x_1 - \frac{a_2}{b}x_2 - \cdots - \frac{a_n}{b}x_n + \frac{u}{b}$

Non-linear version:

※ $u = F(\vec{x}, y)$ for a fixed $(\vec{x_0}, y_0)$, $\frac{\partial u}{\partial y} \equiv F_y \neq 0 \Rightarrow y = G(\vec{x}, u)$ st. $u = F(\vec{x}, G(\vec{x}, u))$

($\downarrow$ $u = u_0$ fixed)

Cor. set $u = 0$, $y = g(\vec{x})$ implicit function

Actually, ※ follows from the $u=0$ case

pf. let $H(\vec{x}, u, y) := u - F(\vec{x}, y)$

$H_y = -F_y \neq 0 \Rightarrow y = G(\vec{x}, u)$ st. $H(\vec{x}, u, G(\vec{x},u)) = 0 = u - F(\vec{x}, G(\vec{x}, u))$

Theorem. General form of IFT (implicit/inverse function theorem)

$u = F(\vec{x}, y, z)$, $v = G(\vec{x}, y, z)$ with $D := \begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix} \neq 0$ at $(\vec{x_0}, y_0, z_0)$

then in a nbd of $(\vec{x_0}, y_0, z_0)$, $\exists\, y = A(\vec{x}, u, v)$, $z = B(\vec{x}, u, v)$

st. $u = F(\vec{x}, A(\vec{x},u,v), B(\vec{x},u,v))$, $v = G(\vec{x}, A(\vec{x},u,v), B(\vec{x},u,v))$

pf. may assume that $F_y \neq 0$ (otherwise change $y \leftrightarrow z$) (first row)

$\frac{\partial u}{\partial y} = F_y \neq 0 \overset{☆}{\Rightarrow} \exists\, y = \Phi(\vec{x}, u, z)$ st. $u = F(\vec{x}, \Phi(\vec{x}, u, z), z)$

Now, the variables are $u, z, \vec{x}$

$V = G(\vec{x}, \Phi(\vec{x}, u, z), z)$ need to compute $\frac{\partial V}{\partial z}$ ($\neq 0$?)

$\frac{\partial V}{\partial z} = G_y \cdot \Phi_z + G_z$

☆ $0 = \frac{\partial u}{\partial z} = F_y \cdot \Phi_z + F_z \Rightarrow \frac{\partial V}{\partial z} \neq 0 \Rightarrow \exists\, z = \Psi(\vec{x}, u, v)$ st. $V = G(\vec{x}, \underbrace{\Phi(\vec{x}, u, \Psi(\vec{x}, u, v))}_{A}, \underbrace{\Psi(\vec{x}, u, v)}_{B})$

(↑ independent variables; ↑ the u equation)

$u = F(\vec{x}, \Phi(\vec{x}, u, \Psi(\vec{x}, u, v)), \Psi(\vec{x}, u, v))$

for any $(\vec{x}, u, v)$ in the nbd

special cases.

1. set $u = V = 0$, get $y(\vec{x}), z(\vec{x})$ st. $F(\vec{x}, y(\vec{x}), z(\vec{x})) = 0$, $G(\vec{x}, y(\vec{x}), z(\vec{x})) = 0$

2. Let $n = 0$, (ie. no $\vec{x}$). get $u = F(\underset{y}{A(u,v)}, \underset{z}{B(u,v)})$, $v = G(\underset{y}{A(u,v)}, \underset{z}{B(u,v)})$ 反函數

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eg. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

$= F(x, y, z)$ $\quad N = \nabla F = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{-2z}{c^2}\right)$

tangent plane of $P(x_0, y_0, z_0)$: $N_p \cdot (x - x_0, y - y_0, z - z_0) = 0$

$\Rightarrow \frac{x_0(x - x_0)}{a^2} + \frac{y_0(y - y_0)}{b^2} + \frac{z_0(z - z_0)}{c^2} = 0$

$\Rightarrow \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - \frac{z_0 z}{c^2} = 1$

$u = F(x, y, z)$, $v = G(x, y, z)$ are both C^1, condition $D = \begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix} \neq 0$ at (x_0, y_0, z_0)

$\Rightarrow y = f(x, u, v)$, $z = g(x, u, v)$

eg. $u = v = 0$, get $y = f(x)$, $z = g(x)$ st. $F(x, f(x), g(x)) = 0$, $G(x, f(x), g(x)) = 0$

ie. F, G 曲面交線參數化 $y = f(x)$, $z = g(x)$

§3.7 Maxima & minima problem

$y = f(x)$, $f'(x) = 0$

eg. $f(x,y) = (ax^2+by^2)\, e^{-(x^2+y^2)}$, $a,b \neq 0$ $\quad f(x,y) \to 0$ as $x^2+y^2 \to \infty$

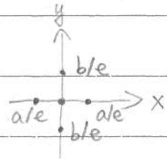
(x_0,y_0) is a extremal point $\Rightarrow f_x(x_0,y_0) = f_y(x_0,y_0) = 0$ $\left(\vec{N} = (-\nabla f, 1),\ \nabla f = 0 \right)$

$f_x = 2ax\, e^{-(x^2+y^2)} + (ax^2+by^2)\, e^{-(x^2+y^2)} \cdot (-2x) = 2x\, e^{-(x^2+y^2)} (a - ax^2 - by^2)$

$f_y = 2y\, e^{-(x^2+y^2)} (b - ax^2 - by^2)$

$\nabla f = 0 \Leftrightarrow x = 0 \begin{cases} y=0 \\ y=\pm 1 \end{cases}$; $x \neq 0,\ a - ax^2 - by^2 = 0 \begin{cases} y=0,\ x=\pm 1 \\ y \neq 0 \begin{cases} \text{if } a \neq b, \text{ no such case} \\ \text{if } a=b, \text{ get } x^2+y^2=1 \end{cases} \end{cases}$

- if $a > b > 0$



$f(x,y) = f(x_0,y_0) + \frac{1}{2}\left(f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2 \right) + \cdots$

$f(x+h, y+k) = f(x_0,y_0) + \frac{1}{2}(h,k) \begin{pmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} + \cdots$ $\quad \vec{x}^t A \vec{x}$ quadratic form

symmetric matrix at (x_0,y_0) $\quad \vec{y}^t \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \vec{y}$

Next, we consider min/max problem with side conditions

Eg. $\frac{x^2+y^2+z^2}{3} \geq \sqrt[3]{x^2y^2z^2}$

Think as the min/max problem for $f(x,y,z) = x^2y^2z^2$ under $x^2+y^2+z^2 = c$

Lagrange multiplier (乘子)

$u = f(x,y,z)$, $v = g(x,y,z) = c$ ⋮ $u = f(\vec{x},y,z)$, $g_1(\vec{x},y,z) = 0$, $g_2(\vec{x},y,z) = 0$

$\nabla f = \lambda \nabla g$ for some λ ⋮ $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$

pf. ① $z = h(x,y) \Rightarrow u = f(x,y,h(x,y))$

$0 = \frac{\partial u}{\partial x} = f_x + f_z \cdot h_x \qquad 0 = \frac{\partial u}{\partial y} = f_y + f_z \cdot h_y$

$g_x + g_z \cdot h_x = 0 \qquad g_y + g_z \cdot h_y = 0$

$\Rightarrow \nabla f = \lambda \nabla g$

② $u = f(\vec{x}, h(\vec{x}), k(\vec{x}))$, $g_1(\vec{x}, h(\vec{x}), k(\vec{x})) = 0$, $g_2(\vec{x}, h(\vec{x}), k(\vec{x})) = 0$

$\frac{\partial u}{\partial x_i} = f_{x_i} + f_y \cdot h_{x_i} + f_z \cdot k_{x_i} = 0 \quad \forall i = 1, \cdots, n$

$\frac{\partial g_1}{\partial x_i} = g_{1x_i} + g_{1y} \cdot h_{x_i} + g_{1z} \cdot k_{x_i} = 0$, $\frac{\partial g_2}{\partial x_i} = g_{2x_i} + g_{2y} \cdot h_{x_i} + g_{2z} \cdot k_{x_i} = 0$

$\begin{pmatrix} 1 & & & h_{x_1} & k_{x_1} \\ & 1 & & h_{x_2} & k_{x_2} \\ & & \ddots & & \\ & & 1 & h_{x_n} & k_{x_n} \end{pmatrix}_{n\times(n+2)}$ ~~$X = 0 \Rightarrow X = \lambda_1 X_1 + \lambda_2 X_2$~~

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Ex 1. $\frac{x_1+\cdots+x_n}{n} \geq \sqrt[n]{x_1\cdots x_n}$ $(x_i \geq 0)$

method 1. Let $x_1+\cdots+x_n = c > 0$, c fixed constant

$f(x_1,\cdots,x_n) = x_1\cdots x_n$

equivalent to $g(x_1,\cdots,x_{n-1}) = x_1\cdots x_{n-1}(c-x_1-\cdots-x_{n-1})$

For simplicity we write out the case $n=3$

$g(x_1,x_2) = x_1x_2(c-x_1-x_2)$ ($f\equiv 0$ (g) on $\partial\Omega$ ($\partial\Omega_1$), the maxima of f (g) exists, and is in the interior)

$\frac{\partial g}{\partial x_1} = x_2(c-x_1-x_2) - x_1x_2 = x_2(c-2x_1-x_2) = 0$

$\frac{\partial g}{\partial x_2} = x_1(c-x_1-2x_2) = 0$

$\Rightarrow x_1, x_2 \neq 0 \Rightarrow 2x_1+x_2 = c,\ x_1+2x_2 = c$

$\Rightarrow x_1 = x_2$ in this case "=" holds

method 2. Let $h(x) = x_1+\cdots+x_n - C$

under $h=0$, solve maxima of f

$\nabla f = \lambda\nabla h \Rightarrow (\frac{f}{x_1},\cdots,\frac{f}{x_n}) = \lambda(1,\cdots,1)$

$\Rightarrow x_1 = \cdots = x_n$

Ex 2. Hölder inequality : $uv \leq \frac{1}{\alpha}u^\alpha + \frac{1}{\beta}v^\beta$, $u,v\geq 0$, $\frac{1}{\alpha}+\frac{1}{\beta} = 1$
(↑ $f(u,v)$)

fix $uv = c \geq 0$ (↑ $h(u,v)$)

$\nabla f = \lambda\nabla h \Rightarrow (u^{\alpha-1}, v^{\beta-1}) = \lambda(v,u) \Rightarrow \frac{u^{\alpha-1}}{v} = \frac{v^{\beta-1}}{u} \Rightarrow u^\alpha = v^\beta$

$u^\alpha = (\lambda v)\cdot u = \lambda c = v^\beta$, $uv = (\lambda c)^{\frac{1}{\alpha}}\cdot(\lambda c)^{\frac{1}{\beta}} = \lambda c = c \Rightarrow \lambda = 1$

$\Rightarrow \frac{1}{\alpha}u^\alpha + \frac{1}{\beta}v^\beta = c$

general form : $\sum_{i=1}^n u_iv_i \leq (\sum u_i^\alpha)^{1/\alpha}\cdot(\sum v_i^\beta)^{1/\beta}$ (A: first factor, B: second factor) $\alpha=\beta=2$ (Cauchy)

pf. let $u = \frac{u_i}{A}$, $v = \frac{v_i}{B}$

$uv \leq \frac{1}{\alpha}\cdot\frac{u_i^\alpha}{\sum_{j=1}^n u_j^\alpha} + \frac{1}{\beta}\cdot\frac{v_i^\beta}{\sum_{j=1}^n v_j^\beta}$

" $\sum_{i=1}^n$ " $\mapsto \frac{\sum_{i=1}^n u_iv_i}{AB} \leq \frac{1}{\alpha}+\frac{1}{\beta} = 1$

Ex 3. More constraints :

$u = f(\vec{x},y,z)$, $g(\vec{x},y,z) = 0$, $h(\vec{x},y,z) = 0$ assume that $\begin{vmatrix} g_y & g_z \\ h_y & h_z \end{vmatrix} \neq 0$ at the extremal point

$\Rightarrow$ get $y = A(\vec{x})$, $z = B(\vec{x})$

$\Rightarrow 0 = \frac{\partial u}{\partial x_i} = f_{x_i} + f_y A_{x_i} + f_z B_{x_i}$ $\forall i = 1,\cdots,n$

$0 = \frac{\partial g}{\partial x_i} = g_{x_i} + g_y A_{x_i} + g_z B_{x_i}$

$\frac{\partial h}{}$ [cut off]

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$$\begin{pmatrix} f_{x_i} & f_y & f_z \\ g_{x_i} & g_y & g_z \\ h_{x_i} & h_y & h_z \end{pmatrix}\begin{pmatrix} 1 \\ A_{x_i} \\ B_{x_i} \end{pmatrix} = 0$$

linearly independent (g_{x_i}, h_{x_i} rows); the boxed submatrix $\begin{pmatrix} g_y & g_z \\ h_y & h_z \end{pmatrix}$

$\Rightarrow$ ↑ has det $= 0$

$$(f_{x_i}, f_y, f_z) = \lambda_1 (g_{x_i}, g_y, g_z) + \lambda_2 (h_{x_i}, h_y, h_z)$$

λ_1, λ_2 are uniquely determined by (y, z) components, independent of $i = 1, \cdots, n$

$\Rightarrow \nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h$

3.3 We have proved the inverse function theorem by a composition primitive mappings.
ie. replace one variable each time

$$\mathbb{R}^2 \xrightarrow[F=\binom{f}{g}]{} \mathbb{R}^2 \qquad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

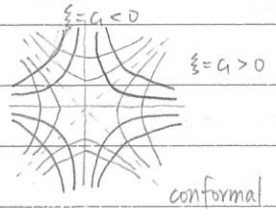
under the assumption that F is C^1 and $D = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} \neq 0$ at (x_0, y_0)

Ex 1. Curvilinear coordinates

$\xi = f(x,y) = x^2 - y^2$, $\eta = g(x,y) = 2xy$

$D \equiv J \equiv \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2) \neq 0$ unless $(x,y) = 0$ ← (x,y), $(-x,-y)$ give the same (ξ, η)

(D: determinant; J: Jacobian)

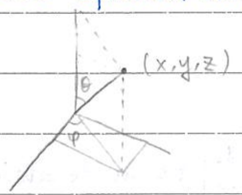


conformal mapping

$$\xi + i\eta = x^2 - y^2 + 2ixy = (x+iy)^2 = z^2$$

$$\underset{(u,v)}{\mathbb{R}^2} \xrightarrow{\mathbb{X}} \underset{(x,y,z)}{\mathbb{R}^3} \qquad \mathbb{X}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$$

Ex 2. Spherical coordinates $\begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix}$



$r = \sqrt{x^2+y^2+z^2}$	$x = r\sin\theta\cos\varphi$
$\theta = ?$	$y = r\sin\theta\sin\varphi$
$\varphi = ?$	$z = r\cos\theta$

$$J = \det[\mathbb{X}_r\ \mathbb{X}_\theta\ \mathbb{X}_\varphi] = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

matrix notations & chain rule :

$dx := x_u du + x_v dv \qquad dy := y_u du + y_v dv$

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix} \Rightarrow d\mathbb{X} = [\mathbb{X}_u \ \mathbb{X}_v]\, d\vec{u} \Leftarrow \mathbb{X}(p+\vec{h}) - \mathbb{X}(p) = A\cdot\vec{h} + o(|\vec{h}|)$$

$[\mathbb{X}_u \ \mathbb{X}_v] \equiv \mathbb{X}' = D\mathbb{X} = \frac{d\vec{\mathbb{X}}}{d\vec{u}}$, $\quad A = \mathbb{X}'(P)$

$$\vec{x}\ \begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow[G=\binom{\varphi}{\psi}]{} \vec{u}\ \begin{pmatrix} u \\ v \end{pmatrix} \xrightarrow[F=\binom{\Phi}{\Psi}]{} \vec{\xi}\ \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$d\vec{u} = G' d\vec{x} \Rightarrow d\vec{\xi} = F' d\vec{u} = F' \circ G' d\vec{x}$

def : $= (F\circ G)' d\vec{x}$

chain rule : $(F\circ G)'(p) = F'(G(p)) \circ G'(p)$

if $F = G^{-1}$ ie. $F\circ G = id \qquad F(G(\vec{x})) = \vec{x}$

$F'(G(\vec{x}))\cdot G'(\vec{x}) = I_n$

Ex. $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \frac{\begin{pmatrix} v_y & -u_y \\ -v_x & u_x \end{pmatrix}}{D}$$

4/19 3.3 (cont.)

Dependent functions

$D = \begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix} = 0$

1, 2 : 不確定反函數是否存在 3 : dependent function

$D \neq 0$ at $(x_0, y_0) \overset{IFT}{\Longrightarrow} \exists$ inverse locally near (x_0, y_0)

eg. $y = f(x) = x^3$

$\frac{dy}{dx} = 3x^2 = 0$ (at $x = 0$) $\Rightarrow x = y^{1/3}$ still exists though it is not differentiable

eg. $u = x^3, \ v = y \qquad D = \begin{vmatrix} 3x^2 & 0 \\ 0 & 1 \end{vmatrix} = 3x^2 \qquad D = 0$ along the y-axis

eg. $\begin{cases} u = x+y+z \\ v = x^2+y^2+z^2 \\ w = xy+yz+zx \end{cases} \Rightarrow v + 2w = u^2$

$$D = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 0$$

(row 2 $\times \frac{1}{2}$ added to row 3)

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Condition: if $u=\phi(x,y)$, $v=\psi(x,y)$ satisfies $\begin{vmatrix}\phi_x & \phi_y\\ \psi_x & \psi_y\end{vmatrix} \equiv 0$

- If $\phi_x \equiv 0 \equiv \phi_y$, then ϕ = constant.
- Otherwise, we may assume that $\phi_x \neq 0$ in $U \ni (x_0, y_0)$

Then we may solve $x = \Phi(u,y)$ st. $u=\phi(\Phi(u,y),y)$, $v=\psi(\Phi(u,y),y) = \chi(u,y)$ for any (u,y)

$\frac{\partial v}{\partial y} = \psi_x \Phi_y + \psi_y = \psi_x\left(-\frac{\phi_y}{\phi_x}\right)+\psi_y = \frac{D}{\phi_x} \equiv 0 \Rightarrow v = \chi(u,y)=\chi(u)$ is independent of y

$0 = \frac{\partial u}{\partial y} = \phi_x \Phi_y + \phi_y$ (↑ (u,y) independent variables)

i.e. $\psi(\Phi(u,y),y) = \chi(u) = \chi(\phi(\Phi(u,y),y))$

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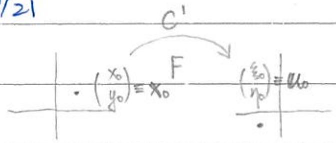

$F' = \begin{pmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{pmatrix}$

if $\frac{d(\xi,\eta)}{d(x,y)} = D(x_0,y_0) = \det F'(x_0,y_0) \neq 0$, $\exists F^{-1}$ locally

Solve the inverse mapping near $\mathbf{u}_0 = F(\mathbf{x}_0)$

Assumption: $F \in C^1$, $F'(\mathbf{x}_0)$ is invertible as a matrix

Given $\mathbf{u}$ near $\mathbf{u}_0$, we want to solve $\mathbf{x} = G(\mathbf{x}) := \mathbf{x} + A(\mathbf{u} - F(\mathbf{x}))$

(↑ expect to hold for some $\mathbf{x}$; ↑ A: any invertible matrix)

$\mathbf{x} \mapsto G(\mathbf{x})$ is a dynamical system

fixed point: $\mathbf{x} = G(\mathbf{x}) \Leftrightarrow \mathbf{u} = F(\mathbf{x})$

$G'(\mathbf{x}) = I - AF'(\mathbf{x})$

pick $A = F'(\mathbf{x}_0)^{-1}$, then $G'(\mathbf{x}_0) = 0$

$|G(\mathbf{y}) - G(\mathbf{x})| \leq \sum_{i=1}^{n} |\nabla g_i(\mathbf{x}+\theta_i(\mathbf{y}-\mathbf{x}))\cdot(\mathbf{y}-\mathbf{x})|$

$\leq \left(\sum_{i=1}^{n} \max_{\mathbf{x}\mathbf{y}} |\nabla g_i|\right)|\mathbf{y}-\mathbf{x}|$

$|G(\mathbf{y})-G(\mathbf{x})| = \sqrt{\sum_{i=1}^{n}(g_i(\mathbf{y})-g_i(\mathbf{x}))^2} = \sqrt{\sum_{i=1}^{n}(\nabla g_i\cdot(\mathbf{y}-\mathbf{x}))^2} \leq \sqrt{\sum_{i=1}^{n} nM^2|\mathbf{y}-\mathbf{x}|^2} = nM|\mathbf{y}-\mathbf{x}|$

Pick $\delta > 0$ small st.

$|\mathbf{x}-\mathbf{x}_0| < \delta$ st. ① $\left|\frac{\partial g_i}{\partial x_j}(\mathbf{x})\right| < \frac{1}{2n^2}$ $\forall i,j$ (since $G'(\mathbf{x}_0)=0$) ② $|G(\mathbf{x}_0)-\mathbf{x}_0| < \frac{\delta}{2}$

i.e. $|\mathbf{u}-F(\mathbf{x}_0)| < \frac{\delta}{2|A|}$

let $\mathbf{x}_{n+1} = G(\mathbf{x}_n)$, $n = 0, 1, \cdots$

$|\mathbf{x}_{n+1}-\mathbf{x}_0| \leq |\mathbf{x}_{n+1}-\mathbf{x}_1| + |\mathbf{x}_1-\mathbf{x}_0| = |G(\mathbf{x}_n)-G(\mathbf{x}_0)| + |\mathbf{x}_1-\mathbf{x}_0| \leq \frac{1}{2}|\mathbf{x}_n-\mathbf{x}_0| + \frac{\delta}{2} < \delta$

(Claim $\mathbf{x}_n \in B_\delta(\mathbf{x}_0)$ $\forall n$, if $n=0$, $|G(\mathbf{x}_0)-\mathbf{x}_0| = |A(\mathbf{u}-F(\mathbf{x}_0))| \leq |A|\cdot|\mathbf{u}-F(\mathbf{x}_0)| < \frac{\delta}{2}$)

$\mathbf{x} = \lim_{n\to\infty} \mathbf{x}_n$ exists by absolute convergence   check $|A\vec{x}| \leq |A||\vec{x}|$

$\mathbf{x}_n = \mathbf{x}_0 + (\mathbf{x}_1-\mathbf{x}_0) + \cdots + (\mathbf{x}_n - \mathbf{x}_{n-1})$ $|x_k - x_{k-1}| \leq \frac{1}{2^{k-1}}|x_1-x_0|$

$\mathbf{x}_{n+1} = G(\mathbf{x}_n) \Rightarrow \mathbf{x} = G(\mathbf{x})$

此證明和維數無關(優點)、無法說明一次只須換一個變數(缺點)

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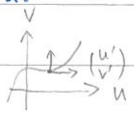
3.4 3.5 微分幾何 intro.

3.4 Geometric Application : To describe surfaces in $\mathbb{R}^3$

$$\underset{\text{open} \subset \mathbb{R}^2}{(u,v) \in U} \longmapsto \mathbb{R}^3 \ni (x(u,v), y(u,v), z(u,v))^t = \mathbb{X}(u,v)$$

$\mathbb{X}_u, \mathbb{X}_v$ form a basis of $T_{\mathbb{X}(p)} S$

$$\frac{d}{dt} \mathbb{X}(u(t), v(t)) = \mathbb{X}_u u'(t) + \mathbb{X}_v v'(t) = \mathbb{X}' \begin{pmatrix} u' \\ v' \end{pmatrix} \qquad \mathbb{X}' = (\mathbb{X}_u, \mathbb{X}_v)$$



$\begin{pmatrix} u' \\ v' \end{pmatrix} = u'e_1 + v'e_2$

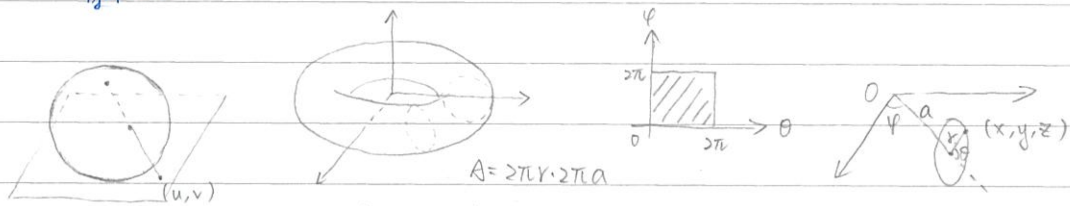
$$\left(\frac{ds}{dt}\right)^2 = \frac{d\mathbb{X}}{dt} \cdot \frac{d\mathbb{X}}{dt} = \left(\mathbb{X}_u \frac{du}{dt} + \mathbb{X}_v \frac{dv}{dt}\right) \cdot \left(\mathbb{X}_u \frac{du}{dt} + \mathbb{X}_v \frac{dv}{dt}\right)$$

$$= \underbrace{(\mathbb{X}_u \cdot \mathbb{X}_u)}_{E} \left(\frac{du}{dt}\right)^2 + 2 \underbrace{(\mathbb{X}_u \cdot \mathbb{X}_v)}_{F} \frac{du}{dt} \cdot \frac{dv}{dt} + \underbrace{(\mathbb{X}_v \cdot \mathbb{X}_v)}_{G} \left(\frac{dv}{dt}\right)^2$$

$ds^2 = E du^2 + 2F du dv + G dv^2$ Gauss First fundamental form

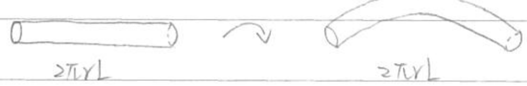
$$= (du \ dv) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}$$

$$= \sum_{i,j=1}^{2} g_{ij} \, du^i du^j$$ 算向量長度



A= 2πr·2πa

$x = (a + r\cos\theta)\cos\varphi$, $y = (a + r\cos\theta)\sin\varphi$, $z = r\sin\theta$



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Multiple integrals : definition of area, volume etc.

$A \subset \mathbb{R}^2$ $|A| \equiv$ area A

- for (closed) rectangle : $R = [a,b] \times [c,d]$, $|R| = (b-a) \times (d-c)$
- if $S_1, S_2, \cdots, S_N$ are disjoint sets and $|S_i|$ is defined, then $|\cup S_i| = \sum_{i=1}^{N} |S_i|$

recall on $\mathbb{R} \supset \mathbb{Q}$ - does not have length

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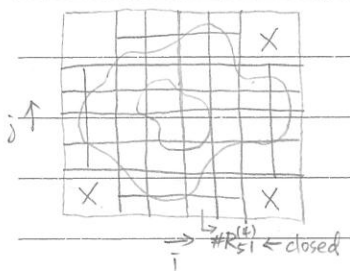
"Jordan measurable set"



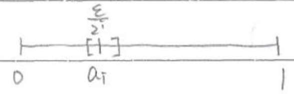
Outer measure $A_n^+(S) \searrow A^+(S)$
↓
the sum of area of those sub rectangles containing points of S

Inner measure $A_n^-(S) \nearrow A^-(S)$

the sum of area of $R_{ij}^{(n)} \subset S$

S is Jordan measurable if $A^+(S) = A^-(S)$

($\mathbb{Q}$ is not Jordan measurable, but it has Lebesgue measure = 0)



Fact: S has a Jordan measure $\Leftrightarrow |\partial S| = 0$ ($|\partial S|$ 可能不存在, $A^-(\partial S) = 0$)

pf. $\Leftarrow$ $A_n^+(S) - A_n^-(S) \le A_n^+(\partial S)$ we may have $R_{ij}^{(n)} \subset S$ but $R_{ij}^{(n)} \cap \partial S \neq \phi$

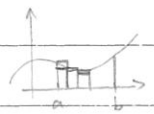
$n \to \infty$ get $A^+(S) = A^-(S)$

$\Rightarrow$ ~~2~~ $2(A_n^+(S) - A_n^-(S)) \ge A_n^+(\partial S)$
$9 = 3^2$

Tarski's Paradox. $\exists$ partition $S^2 = \coprod_{i=1}^{N} A_i$ $\exists T_i \in SO(3) = \{T \mid T^t T = I_3\}$ orthonormal transformation

$\bigcup_{i=1}^{N}$ "$T_i(A_i)$" $= S^2 \amalg S^2 \Rightarrow$ 所有面積定義對 S^2 都不存在

Example. $y = f(x) \in C^0$ $\int_a^b f(x)dx$

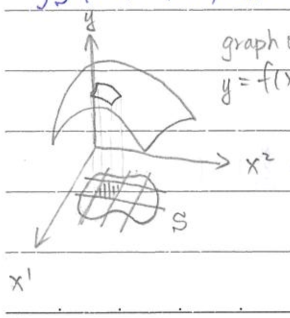


$M_n \ge m_n$

Definition of Riemann integral over a Jordan measurable set

$S \subset \mathbb{R}^n$ st. $|S|$ exists $f: S \to \mathbb{R}$ continuous function

$\int_S f(x^1, \cdots, x^n)\, dx^1 \cdots dx^n$



Theorem (Fubini, the simple form) $R=[a,b]\times[c,d]$ $\int_R f(x,y)\,dx\,dy = \int_c^d dy \underbrace{\int_a^b f(x,y)\,dx}_{\phi(y)}$

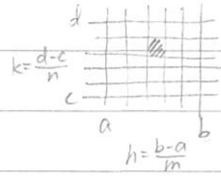
Example $I=\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx$ $a>0,\ b>0$ (上學期習題)

$= \log\frac{b}{a}$

$I=\lim_{T\to\infty}\int_0^T dx\int_a^b e^{-yx}\,dy = \lim_{T\to\infty}\int_a^b dy\int_0^T e^{-xy}\,dx = \lim_{T\to\infty}\int_a^b dy\cdot -\frac{e^{-xy}}{y}\Big|_0^T = \lim_{T\to\infty}\int_a^b \frac{1}{y}-\frac{e^{-Ty}}{y}\,dy$

$= \lim_{T\to\infty} \log y\Big|_a^b - \int_a^b \frac{e^{-Ty}}{y}\,dy \overset{\uparrow\text{check}}{=} \log\frac{b}{a}$

pf. LHS $= \lim_{m,n\to\infty}\sum_{\nu=1}^{n}\sum_{\mu=1}^{m} f(a+\mu h, c+\nu k)\cdot hk$



$\forall\varepsilon>0,\ \exists N$ st. $\forall m,n\geq N \Rightarrow |\_\_ - \int| < \varepsilon$

Denote by $\Phi_\nu = \sum_{\mu=1}^{m} f(a+\mu h, \underbrace{c+\nu k}_{\text{fix}})\cdot h$

$\left|\int_R f(x,y)\,dx\,dy - \sum_{\nu=1}^{n}\Phi_\nu\cdot k\right| < \varepsilon$, true $\forall m,n\geq N$

let $m\to\infty$, $\lim_{m\to\infty}\Phi_\nu = \int_a^b f(x, c+\nu k)\,dx \equiv \phi(c+\nu k)$

$\left|\int_R f(x,y)\,dx\,dy - \sum_{\nu=1}^{n}\phi(c+\nu k)k\right| \leq \varepsilon$

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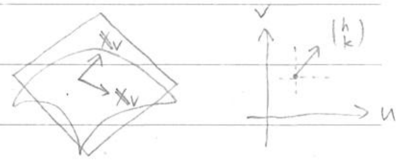
Conformal mapping $\mathbb{X}: U \to \mathbb{R}^3$, $U\subset\mathbb{R}^2$ $(u,v)^t \mapsto (x,y,z)^t$



$\mathbb{X}' = (\mathbb{X}_u\ \mathbb{X}_v)$ $\mathbb{X}'\binom{h}{k} = \mathbb{X}_u\cdot h + \mathbb{X}_v\cdot k \in T_{\mathbb{X}(u,v)}S$

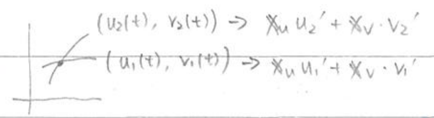
Theorem. $\mathbb{X}$ is conformal $\Leftrightarrow ds^2 = E(du^2+dv^2)$ ie. $E=G$ & $F=0$

pf. $\Rightarrow$ $e_1=\binom{1}{0}$, $e_2=\binom{0}{1}$ $e_1\perp e_2$ $F=\mathbb{X}_u\cdot\mathbb{X}_v=0$

$\downarrow$ $\mathbb{X}_u$ $\downarrow$ $\mathbb{X}_v$

$e_1=\binom{1}{1}$, $e_2=\binom{-1}{1}$ $\Rightarrow (\mathbb{X}_u+\mathbb{X}_v)\cdot(-\mathbb{X}_u+\mathbb{X}_v)=0 \Rightarrow E=G$

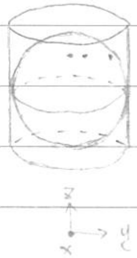
$\Leftarrow$



$\cos\theta = \frac{(\mathbb{X}_u u_1' + \mathbb{X}_v\cdot v_1')\cdot(\mathbb{X}_u u_2' + \mathbb{X}_v v_2')}{|\mathbb{X}_u u_1' + \mathbb{X}_v v_1'|\cdot|\mathbb{X}_u u_2' + \mathbb{X}_v v_2'|}$

$= \frac{E(u_1'u_2' + v_1'v_2')}{\sqrt{E}(u_1'^2+v_1'^2)^{1/2}\cdot\sqrt{E}(u_2'^2+v_2'^2)^{1/2}}$

= cos of the original angle on the u-v plane



check the ratio's cancel out to get 1

$\bigcirc$: $r\sin\theta \mapsto r$

I : $ds \mapsto ds\sin\theta$

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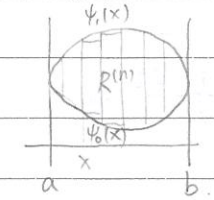
- $R = [a,b]\times[c,d]$ $\quad \int_R f(x,y)\,\overbrace{dx\,dy}^{dA} = \int_c^d dy \int_a^b f(x,y)\,dx$
- R is a convex set $\int_R f\cdot dA = \int_a^b dx \int_{\psi_0(x)}^{\psi_1(x)} f(x,y)\,dy$

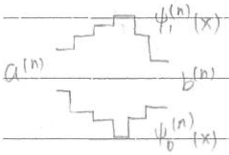


$\int_{R_1\sqcup R_2} f dA = \int_{R_1} f dA + \int_{R_2} f dA$ if $R_1^\circ \cap R_2^\circ = \phi$

→ this holds for $R^{(n)}$

$\int_{R^{(n)}} f dA = \int_{a^{(n)}}^{b^{(n)}} dx \int_{\psi_0^{(n)}(x)}^{\psi_1^{(n)}(x)} f(x,y)\,dy$

now let $n\to\infty$

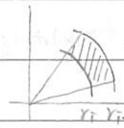


$\to \int_R f dA = \int_a^b dx \int_{\psi_0(x)}^{\psi_1(x)} f(x,y)\,dy$ (ψ_0, ψ_1 conti.)

$\left|\int_{R^{(n)}} f dA - \int_R f dA\right| \le M\, A_n^+(\partial R)$, $|f|\le M$

Mean Value Theorem : $\frac{1}{|R|}\int_R f dA = f(p)$ for some $p\in R$, $f\in C^0$

- Example : $I = \int_{-\infty}^{\infty} e^{-x^2}dx$ Gaussian



$I^2 = \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,\underline{dx\,dy}$ $\quad dA = r\,dr\,d\theta$

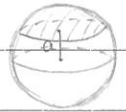
$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,\underline{r}\,dr\,d\theta = 2\pi\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^{\infty} = \pi$

$\Delta A = \frac{1}{2}\left((r+\Delta r)^2 - r^2\right)\Delta\theta$

$D = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$ $\quad = r\Delta r\Delta\theta + \frac{1}{2}\Delta r^2\Delta\theta$

$\Rightarrow I = \sqrt{\pi}$ $\quad \Delta A = \frac{1}{2}(r_{i+1}^2 - r_i^2)\Delta\theta$

$= \frac{r_{i+1}+r_i}{2}\Delta r\Delta\theta$

- 

$\int_R \left(\sqrt{1-x^2-y^2} - a\right) dx\,dy = \int_0^{2\pi}\int_0^{\sqrt{1-a^2}} \left(\sqrt{1-r^2} - a\right) r\,dr\,d\theta$

$R = \{x^2+y^2 \le 1-a^2\}$ $\quad = 2\pi\left(-\frac{1}{3}(1-r^2)^{3/2} - \frac{a}{2}r^2\right)\Big|_0^{\sqrt{1-a^2}} = 2\pi\left(-\frac{a^3}{3}+\frac{1}{3}-\frac{a}{2}+\frac{a^3}{2}\right) = \frac{\pi}{3}(a^3-3a+2)$

$= \frac{\pi}{3}(a-1)^2(a+2)$

- $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$ $\quad (dr)(r\,d\theta)(r\sin\theta\,d\varphi)$

HW. $\int_R \frac{dx\,dy}{(1-x^2-y^2)^2}$

- $v_1, v_2, v_3 \in \mathbb{R}^5$

$|v| = \sqrt{a_1^2+\cdots+a_5^2} = \sqrt{v^t v}$ $\quad v_1\wedge v_2$ $\quad |v_1\wedge v_2| = \sqrt{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3\end{vmatrix}^2 + \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1\end{vmatrix}^2 + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2\end{vmatrix}^2} = \sqrt{\det(V^tV)}$

$v = (a_1,\cdots,a_5)$

↑ the parallelgram spanned by v_1, v_2 $\quad V = (v_1\ v_2)$

i.e. $v_1\wedge v_2 = \{t_1v_1 + t_2v_2 \mid 0\le t_1, t_2\le 1\}$

$V = (v_1,\cdots,v_m)$ $\mathbb{R}^n$ $\quad \sqrt{\det(V^tV)}$

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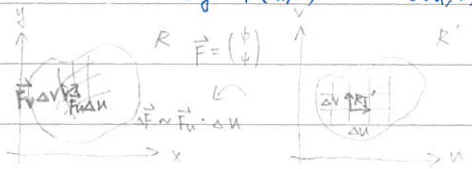
Change of variable formula for multiple integrals

$$\int_R f(x,y)\,dx\,dy = \int_{R'} f(\phi(u,v),\psi(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Assume $\begin{cases} x=\phi(u,v) \\ y=\psi(u,v)\end{cases}$ $D=\dfrac{\partial(x,y)}{\partial(u,v)}$ is a C^1 and 1-1 mapping from R' to R with $D\neq 0$

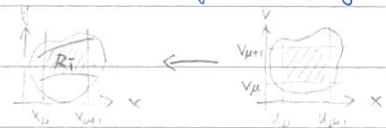


$|\vec{F}_u\Delta u, \vec{F}_v\Delta v| = \dfrac{\partial(x,y)}{\partial(u,v)}\Delta u\Delta v$ at some point in R_i'

primitive mapping eg. $\vec{x}=\vec{u},\ y=\psi(\vec{u},v)$ $\quad \dfrac{\partial(\vec{x},y)}{\partial(\vec{u},v)} = \begin{vmatrix} I & \psi_u \\ 0 & \psi_v\end{vmatrix} = \psi_v$

proof of CVF for primitive mappings:

Assume $x=u,\ y=\psi(u,v)$ gives a 1-1 C^1 mapping between R, R' which has $\psi_v>0$ on R'



$\begin{cases} u_{\nu+1}=u_\nu+h \\ v_{\mu+1}=v_\mu+k\end{cases}$

$\Delta R_i = \int_{x_\nu}^{x_\nu+h}\left(\psi(x,v_\mu+k)-\psi(x,v_\mu)\right)dx = h\cdot\left(\psi(\tilde{x}_\nu,v_\mu+k)-\psi(\tilde{x}_\nu,v_\mu)\right),\quad \tilde{x}_\nu\in[x_\nu,x_\nu+h]$

area of R_i

$= hk\cdot\psi_v(\tilde{x}_\nu,\tilde{v}_\mu),\quad \tilde{v}_\mu\in[v_\mu,v_\mu+k]$ (P_i)

Riemann Sum. $\sum f(\tilde{x}_\nu,\psi(\tilde{x}_\nu,\tilde{v}_\mu))\Delta R_i = \sum_i f(\tilde{x}_\nu,\psi(\tilde{x}_\nu,\tilde{v}_\mu))\,\psi_v(P_i)\,hk$

let $h,k\to 0$. $\int f(x,y)\,dx\,dy = \int f\circ F\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$

Lemma. If CVF holds for $(x,y)=F_1(u,v)$ and $(u,v)=F_2(\xi,\eta)$, then it holds for $(x,y)=(F_1\circ F_2)(\xi,\eta)$

pf. $\int_R f(x,y)\,dx\,dy = \int_{R'} f\circ F_1(u,v)\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|du\,dv = \int_{R''} f\circ F_1\circ F_2(\xi,\eta)\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|_{F_2(\xi,\eta)}\left|\dfrac{\partial(u,v)}{\partial(\xi,\eta)}\right|d\xi\,d\eta$

$\left|\dfrac{\partial(x,y)}{\partial(\xi,\eta)}\right|$

pf for the general case

Assume R is compact (closed and bounded) and Jordan measurable

$\forall p\in R$, $\exists$ cube $C_p(r_p)$ st. F can be decomposed into primitive map

(Problem: r_p could vary when P varies) Then $\{C_p^\circ(r_p)\}$ is an open cover of R

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Heine-Borel Theorem : Any open cover of a compact set admit a finite subcover

$$\bigcup_{i\in\Lambda} U_i \supset R \Rightarrow \exists\, i_1, \cdots, i_N \text{ (finite) st. } U_{i_1} \cup \cdots \cup U_{i_N}$$

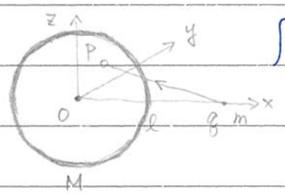
By Heine-Borel, $\exists\, p^{(1)}, \cdots, p^{(N)} \in R$ st. $R \subset C^{\circ}_{p^{(1)}}(r_{p^{(1)}}) \cup \cdots \cup C^{\circ}_{p^{(N)}}(r_{p^{(N)}})$

pick $r = \min_{1\le i\le N}(r_{p^{(i)}})$ and get a finite partition $R = \sqcup R^{(i)}$

Now the "previous step" can be applied to $R^{(i)} \cap C^{\circ}_{p^{(j)}}(r_{p^{(j)}})$

(when F is decomposable into primitive mappings)

Newton



$$\int_R -\frac{G\rho m\, dv}{|p-q|^3}(\vec{p}-\vec{q}) = \frac{GMm}{l^2}$$

(dv = dxdydz)

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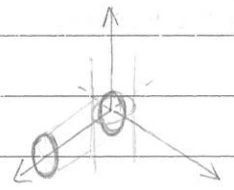
find the volume of the intersection of 2 cylinders in $\mathbb{R}^3$ $\{x^2+z^2\le 1\}$, $\{y^2+z^2\le 1\}$

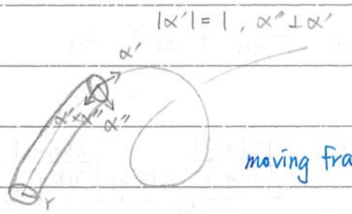


$$8\int_R \sqrt{1-y^2}\, dxdy$$

$R \subset \mathbb{R}^2$, $R = \{x^2+y^2\le 1, x\ge 0, y\ge 0\}$

$$= 8\int_0^1 \sqrt{1-y^2} \int_0^{\sqrt{1-y^2}} 1\, dx\, dy$$

$$= 8\int_0^1 (1-y^2)\, dy = 8\left(y-\frac{y^3}{3}\right)\Big|_0^1 = \frac{16}{3}$$



moving frame $\{\vec{t}, \vec{n}, \vec{b}\}$: $\vec{t} = \alpha'$, $\vec{n} = \frac{\alpha''}{|\alpha''|}$, $\vec{b} = \vec{t}\times\vec{n}$

C: curve: $\alpha(s)$, $s\in[0, l]$

↑ arc length l = length (C)

$$X(s,\theta) = \alpha(s) + r(\cos\theta\,\vec{n} + \sin\theta\,\vec{b})$$

$$\int_0^{2\pi}\int_0^l |X_s \times X_\theta|\, ds\, d\theta$$

$X_s = \vec{t} + r(\cos\theta\,\vec{n}' + \sin\theta\,\vec{b}')$, $X_\theta = r(-\sin\theta\,\vec{n} + \cos\theta\,\vec{b})$

~~$= \vec{t} + r(\cos\theta(-\kappa\vec{t} - \tau\vec{b}) + \sin\theta\,\tau\vec{n})$~~

$(\vec{b}' = \vec{t}'\times\vec{n} + \vec{t}\times\vec{n}' = \vec{t}\times\vec{n}')$

Chuyu culture

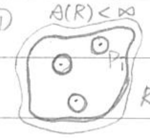
$X_s \times X_\theta = r(-\sin\theta\,\vec{b} + \cos\theta\,\vec{n})$

$\vec{t}' = \kappa\vec{n}$ $\quad \vec{b}' = \alpha\vec{t} + \beta\vec{n}$, $\alpha = \vec{b}'\cdot\vec{t} = (\vec{b}\cdot\vec{t})' - (\vec{b}\cdot\vec{t}')$ ($\alpha = 0$, $\beta = \tau$; $= 0$)

$$\begin{pmatrix}\vec{t}\\ \vec{n}\\ \vec{b}\end{pmatrix}' = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & -\tau\\ 0 & \tau & 0\end{pmatrix}\begin{pmatrix}\vec{t}\\ \vec{n}\\ \vec{b}\end{pmatrix}$$

5/10 4.7 Improper integral

① $A(R) < \infty$



f could be not continuous or even ↗ ∞ at some points $p_i \in R$

② $A(R) = \infty$

$R = \mathbb{R}^2$ $\int_{\mathbb{R}} e^{-x^2-y^2}\,dx\,dy$

Theorem. let R be bounded with area (ie. R is Jordan-measurable)

(i) $R_n \nearrow R$ ie. $R_n \subset R_{n+1}$ & $A(R_n) \to A(R)$ st. f is conti. on R_n $\forall n$
(R_n: closed set)

(ii) $\int_{R_n} |f| \leq \mu$ $\forall n$ (*) not necessary

then $I = \lim_{n\to\infty} \int_{R_n} f$ exists and is independent of choices of $\{R_n\}$ in (i)

Ex 1. $\int_{R=B_0(1)} \frac{dV}{|\vec{r}|^\alpha}$, $dV = dx\,dy\,dz$, $\vec{r} = (x,y,z)$ "=" $\lim_{n\to\infty} \int_{R_n = \{\vec{r} \mid \frac{1}{n} \leq |\vec{r}| \leq 1\}} r^{-\alpha} r^2 \sin\theta \, dr\,d\theta\,d\phi$

$= 2\pi(-\cos\theta)\Big|_0^\pi \cdot \int_{\frac{1}{n}}^1 r^{2-\alpha}\,dr$ (with $(-\cos\theta)\Big|_0^\pi = 2$)

require $2-\alpha > -1$ ie. $\alpha < 3$

Q: on $\mathbb{R}^2$, $1-\alpha > -1$ ie. $\alpha < 2$. How about $\mathbb{R}^n$? $\alpha < n$??

Ex 2.

need only $\alpha < 2$

if $|f(x,y,z)| \leq \frac{M}{\sqrt{y^2+z^2}^{\,\alpha}}$ $R = [a,b] \times$ "$B_0(1)$" ↳ on y-z plane

$\int_R f(x,y,z)\,dx\,dy\,dz$ exists

pf of thm: Step 1. $I_n^+ := \int_{R_n} |f| \nearrow$ bounded by μ

⇒ $I_n^+ \to I^+$ exists (and is a Cauchy sequence)

⇒ $I_n := \int_{R_n} f$ is also a Cauchy sequence:

because $|I_n - I_m| = |\int_{R_n} f - \int_{R_m} f| = |\int_{R_n \setminus R_m} f| \leq \int_{R_n\setminus R_m} |f| = I_n^+ - I_m^+ < \varepsilon$ ($n > m$)
↑ for $n, m > N(\varepsilon)$

⇒ $I = \lim_{n\to\infty} I_n$ exists

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$S \not\subset R_n$ for some n.

Step 2. Now, for any $S \subset R$ closed, J-m, $f|_S$ is conti.

Need to check (*) : " $\int_S |f| \le \mu$ "

$\left| \int_S f - \int_{S \cap R_n} f \right| \le A(S \setminus R_n) \cdot \sup_S |f| \xrightarrow[n\to\infty]{} 0$ (**)

($A(S \setminus R_n) \le A(R \setminus R_n)$)

similarly for $|f|$, get $\int_S |f| = \lim_{n\to\infty} \int_{S\cap R_n} |f| \le \mu$

$\left| \int_S f - \int_{S\cap R_n} f \right| = \lim_{m\to\infty} \left| \int_{S \cap R_m} f - \int_{S\cap R_n} f \right| \le \lim_{m\to\infty} \int_{R_m \setminus R_n} |f| < \varepsilon$ independent of S (for $m, n > N(\varepsilon)$)

($S \cap (R_m \setminus R_n)$)

boot straping

Now, for another sequence $S_m \nearrow R$ satisfying (i),

then (ii) is also satisfied

so, $J = \lim_{m\to\infty} \int_{S_m} f$ also exists

$\left| J - \int_{S_m \cap R_n} f \right| \le \left| J - \int_{S_m} f \right| + \left| \int_{S_m} f - \int_{S_m \cap R_n} f \right| < 2\varepsilon$ as long as $m, n > N(\varepsilon)$

(each term $< \varepsilon$)

similarly, $\left| I - \int_{S_m \cap R_n} f \right| < 2\varepsilon$

$\Rightarrow |I - J| \le 4\varepsilon$ but ε is arbitrary

$\Rightarrow I = J$

The case with unbounded R.

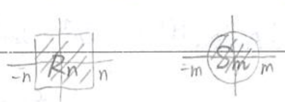
(i)' $R_n \nearrow R$, compact, J-m; now require the "exhausion condition": (every compact $S \subset R$ must be contained in R_n for n large)

(ii)' $\int_{R_n} |f| \le \mu \quad \forall n$

then $I := \lim_{n\to\infty} \int_{R_n} f$ exists and is independent of the choice of $\{R_n\}$

Ex 3. Gauss' integral $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$\lim_{n\to\infty} \int_{R_n} e^{-(x^2+y^2)} dxdy = \lim_{m\to\infty} \int_{S_m}$



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4.8 More Application

Ex 4. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1 \qquad \frac{4}{3}\pi abc$

$$\begin{cases} x = ar\cos\theta \\ y = br\sin\theta \\ z = z \end{cases} \qquad V = 2\int_R c\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}\,dxdy$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = abr \qquad = 2abc\int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\, r\,dr\,d\theta$$

$$= 4\pi abc\left(-\frac{1}{3}(1-r^2)^{3/2}\right)\Big|_0^1 = \frac{4}{3}\pi abc$$

Ex 5. Cylindrical coordinates

$(x,y,z) \longleftrightarrow (r,\theta,z)$

Surface (Solid) of revolution

$$V = \int_R dV = \int_a^b dz\int_0^{2\pi}d\theta\int_0^{\phi(z)} r\,dr = \pi\int_a^b \phi^2(z)\,dz$$

$$S = \int_a^b dz\sqrt{1+\phi'(z)^2}\;2\pi\phi(z)$$

in fact, $\mathbb{X}(z,\theta) = (\phi(z)\cos\theta, \phi(z)\sin\theta, z)$

$\mathbb{X}_z = (\phi'\cos\theta, \phi'\sin\theta, 1)$, $\mathbb{X}_\theta = (-\phi\sin\theta, \phi\cos\theta, 0)$

$F = \mathbb{X}_z\cdot\mathbb{X}_\theta = 0$

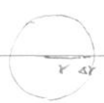
$E = \mathbb{X}_z\cdot\mathbb{X}_z = 1+(\phi')^2$, $G = \mathbb{X}_\theta\cdot\mathbb{X}_\theta = \phi^2$

$dA = \sqrt{EG-F^2}\,d\theta\,dz = \sqrt{1+\phi'^2}\,\phi\,d\theta\,dz$

$\phi(z) = \sqrt{R^2-z^2}$, $\phi'(z) = \frac{-z}{\sqrt{R^2-z^2}}$

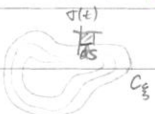
$\int_0^{2\pi}\sqrt{1+\phi'^2}\,\phi\,d\theta = 2\pi\frac{R}{\sqrt{R^2-z^2}}\cdot\sqrt{R^2-z^2} = 2\pi R$

5/12 Multiple Integral in curvilinear coordinates (等高線積分法)



$V(r) = \frac{4}{3}\pi r^3$

$A(r) = 4\pi r^2$



$\phi(x,y) = \xi$ ← const.

$\phi(\sigma(t)) = \xi(t)$

$\sigma(t) = (x(t), y(t))$

$d\xi = \left(\phi_x\frac{dx}{dt}+\phi_y\frac{dy}{dt}\right)dt$

$= \nabla\phi\cdot\sigma'(t)\,dt$

$\Rightarrow |d\vec{\sigma}| = \frac{d\xi}{|\nabla\phi|}\, d\sigma$

$\int f(x,y)\,ds\,\frac{d\xi}{|\nabla\phi|}$, $\frac{d\xi}{|\nabla\phi|} = \frac{d\xi}{\sqrt{\phi_x^2+\phi_y^2}}$

The more rigorous deduction of the "Co-Area formula"

pf. consider $\begin{cases}\xi = \phi(x,y)\\ \eta = y\end{cases}$ $\quad \frac{\partial(\xi,\eta)}{\partial(x,y)} = \phi_x \quad \frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{1}{\phi_x}$

$$\int_R f\,dxdy = \int f\frac{d\xi\,d\eta}{|\phi_x|} = \int f\frac{d\xi}{\sqrt{\phi_x^2+\phi_y^2}}\cdot\underbrace{\left(\frac{\sqrt{\phi_x^2+\phi_y^2}}{|\phi_x|}\,d\eta\right)}_{ds}$$

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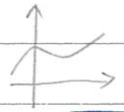
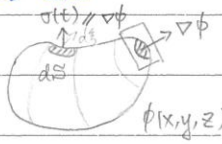


$x = f(y)$ $\quad ds = \sqrt{1+f_y^2}\,dy$

$\phi(x,y) = \xi$ ← fixed (x ↑ $f(y)$) $\qquad \phi_x f_y + \phi_y = 0 \qquad f_y = -\frac{\phi_y}{\phi_x}$

$$ds = \sqrt{1+\left(\frac{\phi_y}{\phi_x}\right)^2}\,dy = \frac{\sqrt{\phi_x^2+\phi_y^2}}{|\phi_x|}\,dy$$

3D case :



$$\int f\,dS\,\frac{d\xi}{|\nabla\phi|} = \int f\,dV$$

$\phi(x,y,z) = \xi$

Example : $f \equiv 1$: calculate the volume

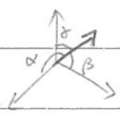
$$\phi(x,y,z) = r = \sqrt{x^2+y^2+z^2} \qquad |\nabla\phi| = |\nabla r| = 1$$

$$dS = \frac{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}}{|\phi_x|}\,dy\,dz \ ?$$ (投影在 yz 平面)

$x = f(y,z) \qquad dS = \sqrt{1+f_y^2+f_z^2}$

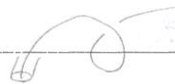
$\phi(f(y,z),y,z) = \xi \qquad \begin{cases} \phi_x f_y + \phi_y = 0 \\ \phi_x f_z + \phi_z = 0 \end{cases}$

$ds = |X_u \wedge X_v|\,du\,dv$, $\quad |X_u \wedge X_v| = \sqrt{EG-F^2}$



$$(\cos\alpha, \cos\beta, \cos\gamma) = \frac{\nabla\phi}{|\nabla\phi|} = \frac{(\phi_x, \phi_y, \phi_z)}{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}}$$

Application.



Area, Volume

(1) $|B_0^n(R)| = ?$ ($V^n(R)$) $\qquad$ (2) $|S_0^{n-1}(R)| = ?$ ($A^{n-1}(R)$)

$\{\vec{x} \in \mathbb{R}^n \mid |\vec{x}| \le R\} \qquad \{\vec{x} \in \mathbb{R}^n \mid |\vec{x}| = R\}$

(1) $x_1^2 + \cdots + x_n^2 \le R^2$

$$V^n(R) = 2\int_0^R V^{n-1}\left(\sqrt{R^2-x_n^2}\right)dx_n = 2V^{n-1}(1)\int_0^R (R^2-x^2)^{n/2}\,dx$$

$V^n(1) = * \; V^{n-1}(1)$?

$\Gamma(n+1) = n! \qquad \Gamma(n+\tfrac{1}{2}) \qquad \Gamma(\tfrac{1}{2}) =$

$\int_0^{\frac{\pi}{2}} \cos^n\theta \, d\theta = \cos^{n-1}\theta\sin\theta \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\cos^{n-2}\theta\,(1-\cos^2\theta)\,d\theta$

$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2}\theta\,d\theta - (n-1)\int_0^{\frac{\pi}{2}}\cos^n\theta\,d\theta$

$\int_0^{\frac{\pi}{2}} \cos^n\theta\,d\theta = \frac{n-1}{n}\int_0^{\frac{\pi}{2}}\cos^{n-2}\theta\,d\theta$

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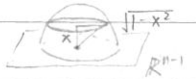
$V^n(R)$ = n-dim volume of $B_0^n(R) = V^n(1)\cdot R^n$

$A^{n-1}(R)$ = (n-1)-dim area of $S_0^{n-1}(R) = A^{n-1}(1)\cdot R^{n-1}$ ← explain it!

$\frac{d}{dR}V^n(R) = A^{n-1}(R) \Rightarrow nV^n(1) = A^{n-1}(1)$

$V^n(1) = 2\int_0^1 V^{n-1}(\sqrt{1-x_n^2})\,dx_n$



$= 2\int_0^1 V^{n-1}(1)(1-x_n^2)^{\frac{n-1}{2}}\,dx_n = 2V^{n-1}(1)\int_0^{\frac{\pi}{2}}\cos^{n-1}\theta\cdot\cos\theta\cdot d\theta = \begin{cases}\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2} & (n \text{ even})\\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3}\cdot 1 & (n \text{ odd})\end{cases}\cdot 2V^{n-1}(1)$

$x_n = \sin\theta$

$n=2k$, $V^{2k}(1) = V^{2k-1}(1)\cdot\frac{2k-1}{2k}\cdots\frac{1}{2}\cdot\pi = V^{2k-2}(1)\cdot\frac{1}{2k}\cdot 2\pi = \cdots = \frac{\pi^k}{k!}$

$n=2k+1$, $V^{2k+1}(1) = V^{2k}(1)\cdot\frac{2k}{2k+1}\cdots\frac{2}{3}\cdot 2 = V^{2k-1}(1)\cdot\frac{2}{2k+1}\cdot\pi = \cdots = V^1(1)\cdot\frac{2^k\pi^k}{(2k+1)\cdots 3}$

$\omega_n := A^{n-1}(1)$ $\qquad = 2\cdot\frac{2^k k!\,\pi^k}{(2k+1)!} = \frac{2^{2k+1}k!\,\pi^k}{(2k+1)!}$

co-area formula: $\int_R f(x_1,\cdots,x_n)\,dx_1\cdots dx_n = \int_R \frac{f}{|\nabla\phi|}\,d\xi\,dA$

$dA = \frac{\sqrt{\phi_{x_1}^2+\cdots+\phi_{x_n}^2}}{|\phi_{x_n}|}\,dx_1\cdots dx_{n-1} \leftarrow \phi(x_1,\cdots,x_n)=\xi$

$dA = \sqrt{1+f_{x_1}^2+\cdots+f_{x_{n-1}}^2}\,dx_1\cdots dx_{n-1} \leftarrow x_n = f(x_1,\cdots,x_{n-1})$

$r = \xi = \phi = \sqrt{x_1^2+\cdots+x_n^2}$ $\quad |\nabla\phi| = 1$

$\int_{\mathbb{R}^n} f\,dV = \int_{\mathbb{R}^n} f\,dr\,dA$

↳ area element on $S_0^{n-1}(r)$

pick $f = e^{-(x_1^2+\cdots+x_n^2)} = e^{-r^2}$

$\pi^{\frac{n}{2}} = \prod_{i=1}^n \int_{-\infty}^{\infty} e^{-x_i^2}\,dx_i = A^{n-1}(1)\int_0^\infty e^{-r^2}r^{n-1}\,dr$

let $s = r^2$ $\quad ds = 2r\,dr$ $\qquad \frac{1}{2}\int_0^\infty e^{-s}s^{\frac{n-1}{2}-\frac{1}{2}}\,ds = \frac{1}{2}\Gamma(\frac{n}{2})$, $\Gamma(s+1) = s\Gamma(s)$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\int_0^\infty e^{-s}s^{k-1}\,ds = \Gamma(k)$ if $k\in\mathbb{N}$, $\Gamma(k) = (k-1)!$

Integration / differentiation of improper integrals with a parameter

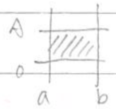
Def: $F(x) = \int_0^\infty f(x,y)\,dy$ (C^0) $x\in[a,b]$ converges uniformly if $\forall\varepsilon>0\ \exists A$ st. $\left|\int_B^\infty f(x,y)\,dy\right| < \varepsilon$ $\forall B\geq A$ ($\forall x\in[a,b]$)

Simple test: "$|f(x,y)| < \frac{M}{y^\alpha}$ for $y\geq y_0$ some $\alpha>1$" ⇒ unif conv.

Thm: Unif convergence ⇒ F(x) is conti. on [a,b]

pf. (Given ε) $|F(x+h)-F(x)| < \left|\int_0^A (f(x+h,y)-f(x,y))\,dy\right| + 2\varepsilon$

↑ choose A



choose h small (depend on A) st. $|f(x+h,y)-f(x,y)| < \frac{\varepsilon}{A}$

$\Rightarrow |F(x+h)-F(x)| < 3\varepsilon$

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Similarly $F(x)=\int_\alpha^\beta f(x,y)\,dy$ but $y\to\alpha$ has ∞-discontinuity unif conv.

$\overset{\text{def.}}{\Leftrightarrow}$ Given $\varepsilon>0$ $\exists k$ st. $\left|\int_\alpha^{\alpha+h} f(x,y)\,dy\right|<\varepsilon$ $\forall h\le k$

Test. $|f(x,y)|<\dfrac{M}{(y-\alpha)^\nu}$ $(\nu<1)$

Integrals. $\int_\alpha^\beta dx\int_0^{\infty\,(A)} f(x,y)\,dy \underset{?}{=} \int_0^{\infty\,(A)} dy\int_\alpha^\beta f(x,y)\,dx$ $\quad A\to\infty$

$+\int_A^\infty$

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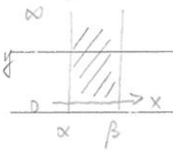

$\int_0^\infty f(x,y)\,dy = F(x)$ uniformly convergent $\Rightarrow$ $F(x)$ conti.

$\quad\parallel$

$\int_0^B f(x,y)\,dy + R_B(x)$ $\quad$ $\forall\varepsilon>0$, $\exists A(\varepsilon)$ st. $\forall B\ge A$, $|R_B(x)|<\varepsilon$

$\int_\alpha^\beta dx\int_0^\infty f(x,y)\,dy \underset{?}{=} \int_0^\infty dy\int_\alpha^\beta f(x,y)\,dx$

pf. $\int_\alpha^\beta dx\int_0^B f(x,y)\,dy + \int_\alpha^\beta R_B(x)\,dx = \int_0^B dy\int_\alpha^\beta f(x,y)\,dx + \int_\alpha^\beta R_B(x)\,dx$

$\left|\int_\alpha^\beta R_B(x)\,dx\right| \le (\beta-\alpha)\varepsilon$

This $\Rightarrow$ "="

For exchange of integral like $\int_0^\infty dy\int_0^\infty f(x,y)\,dx$, so far we only know that it holds if $\int_{\mathbb{R}^2}|f(x,y)|\,dxdy$ exists

Differentiation : Suppose that $f_x(x,y)$ is piece-wise conti. on $x\in[\alpha,\beta]$

and $F(x)=\int_0^\infty f(x,y)\,dy$, $G(x)=\int_0^\infty f_x(x,y)\,dy$ exist uniformly,

then $F'(x)=G$

pf. $\int_\alpha^\xi G(x)\,dx = \int_\alpha^\xi dx\int_0^\infty f_x(x,y)\,dy = \int_0^\infty dy\int_\alpha^\xi f_x(x,y)\,dx = \int_0^\infty \big(f(\xi,y)-f(\alpha,y)\big)\,dy$

$= F(\xi)-F(\alpha)$

$\Rightarrow F'(x)=G(x)$

Fubini Thm. $\Rightarrow$ integral exchange $\Rightarrow$ differentiation exchange is another possible way to prove (review)

$\frac{d}{dx}\int_{\phi(x)}^\infty f(x,y)\,dy$

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Example 1 $\int_0^\infty e^{-xy}\,dy = -\frac{e^{-xy}}{x}\Big|_0^\infty = \frac{1}{x}$ ($x>0$)

↑ unif. conv. $-\frac{e^{-xy}}{x}\Big|_B^\infty$

diff. in x : $\int_0^\infty y e^{-xy}\,dy = \frac{1}{x^2}$

↑ require the unif. conv. of $\int_0^\infty y e^{-xy}\,dy$

$\int_0^\infty y^n e^{-xy}\,dy = \frac{n!}{x^{n+1}}$ $\quad \underbrace{y^n \cdot e^{-\frac{xy}{2}}}_{\le M} \cdot e^{-\frac{xy}{2}} \Rightarrow$ unif conv.

Example 2. $\int_0^\infty \frac{\sin y}{y}\,dy = \frac{\pi}{2}$

交錯數列 ↘ ⇒ 收斂

Let $F(x) = \int_0^\infty e^{-xy}\frac{\sin y}{y}\,dy$ $\quad \int|\ldots| \le \int e^{-xy} = \frac{1}{x} \to 0$

$x \ge 0$ unif conv. e or 交錯級數

$F'(x) = -\int_0^\infty e^{-xy}\sin y\,dy = \frac{-1}{1+x^2}$

$x>0$ unif conv. ↑ check

$F(x) = c - \tan^{-1}x$

$x>0$ but $F(x)$ conti.

let $x\to\infty$ $\quad 0 = c - \frac{\pi}{2}$

$F(0) = c = \frac{\pi}{2}$

Example 3. Fresnel's integral $F_1 = \int_{-\infty}^\infty \sin(x^2)\,dx = 2\int_0^\infty \sin(x^2)\,dx$

$t = x^2 \quad dt = 2x\,dx, \quad F_1 = \int_0^\infty \frac{\sin t}{\sqrt{t}}\,dt$

$f(x) \longmapsto \hat{f}(y) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)\,e^{-ixy}\,dx \qquad f(x) \mapsto \hat{f}(y) \mapsto f(-x)$

oscilation mode (phase)

$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \hat{f}(y)\,e^{iyx}\,dy$ 將微分方程轉化為代數方程

$v_1, \cdots, v_m \in \mathbb{R}^n$

$v_1 \wedge \cdots \wedge v_m = \{\sum_{i=1}^m t_i v_i \mid 0 \le t_i \le 1\}$ $\quad$ m-dim area $= \sqrt{\det(V^tV)}$

$V = (v_1 \cdots v_m)_{n\times m}$

$V^tV = \begin{pmatrix} v_1^t \\ \vdots \\ v_m^t \end{pmatrix}(v_1 \cdots v_m) = (v_i^t \cdot v_j)_{m\times m}$

$T \curvearrowright v_1 \cdots v_m \quad \mathbb{R}^m \hookrightarrow \mathbb{R}^n$

$TV = \begin{pmatrix} \square \\ O \end{pmatrix}_{m\times m} \qquad (TV)^t TV = V^tT^tTV = V^tV$

$(\square^t O)\begin{pmatrix} \square \\ O \end{pmatrix} = \square^t\square \qquad \det V^tV = \det \square^t\square = (\det \square)^2$

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$u_1, \cdots, u_m$

$\mathbb{R}^m \xrightarrow{\ X\ } \mathbb{R}^n$

$X(u_1, \cdots, u_m)$

$DX = \left(\frac{\partial X}{\partial u_1} \cdots \frac{\partial X}{\partial u_m} \right)$

$\int f \sqrt{\det((DX)^t DX)}\, du_1 \cdots du_m$

$\begin{vmatrix} X_1 \cdot X_1 & X_1 \cdot X_2 \\ X_2 \cdot X_1 & X_2 \cdot X_2 \end{vmatrix}^{1/2} = \sqrt{EG - F^2}$

5/24 Vector Calculus

$\mathbb{R}^3$  $B_0(r)$

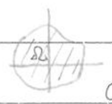
Green's Theorem

Gauss' Theorem (divergence theorem) } Integration by parts

Stokes' Theorem

$\int_a^b f dg = fg\Big|_a^b - \int_a^b g df \Leftarrow \int_a^b df = f\Big|_a^b$ ($df = f'(x)dx$; let $f = gh$)

Green's theorem



$$\int_C P dx + Q dy = \int_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$C = \partial\Omega$

$\omega = P dx + Q dy$ (1-form), $C = \partial\Omega$ on the plane $\mathbb{R}^2$ "form" dx $dx \wedge dy = -dy \wedge dx$ ($\Rightarrow dx \wedge dx = 0$)

- $\int_{\partial\Omega} \omega = \int_\Omega d\omega$ $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$ (total diff)

$d\omega = dP \wedge dx + dQ \wedge dy$

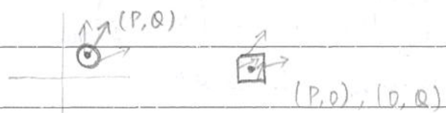
$= (P_x dx + P_y dy) \wedge dx + (Q_x dx + Q_y dy) \wedge dy$

$= (Q_x - P_y)\, dx \wedge dy$

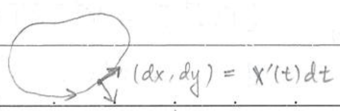
$\Omega = [a, b]$, $\omega = f$ (0-form)

$f\Big|_a^b = \int_a^b f' dx$

An equivalent form : 2-dim divergence thm.



$\vec{F} = (P, Q)$ $\operatorname{div} \vec{F} := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$
(散度)

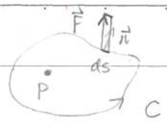


$\int_\Omega (P_x + Q_y) dx dy = \int_{\partial\Omega} (-Q dx + P dy)$

$\int_\Omega \operatorname{div} \vec{F}\, dx dy$ $\int_{\partial\Omega} \vec{F} \cdot (dy, -dx) = \int_{\partial\Omega} \vec{F} \cdot \vec{n}\, ds$

$$\Rightarrow \lim_{\Omega \to P} \frac{1}{|\Omega|} \int_{\partial\Omega} \vec{F}\cdot\vec{n}\, ds = \lim_{\Omega\to P} \frac{1}{|\Omega|}\int_{\Omega} \operatorname{div}\vec{F}\, dA = \operatorname{div}\vec{F} \text{ at } P$$

通量 flux



$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$$

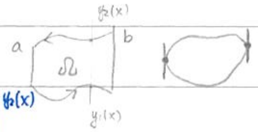
$$* \int_{\partial\Omega} P dx + Q dy = \int_{\Omega} (Q_x - P_y)\, dx dy$$

- for any 2 C^1 functions P & Q in Ω
- $\partial\Omega = \bigcup c_i$, c_i piecewise C^1 curve

It is enough to prove $*$ for P & Q separately.

pf of $*$ (Green's thm) for P:

Step 1. Ω is a region of the form:



$$RHS = \int_{\Omega} -P_y\, dx dy = -\int_a^b dx\, P(x,y)\Big|_{y_1(x)}^{y_2(x)}$$

$$= \int_a^b P(x, y_1(x))\, dx - \int_a^b P(x, y_2(x))\, dx = \int_{\partial\Omega} P dx = LHS$$

Step 2. Divide Ω into subregions Ω_i, $i=1,\cdots,N$ st. each Ω_i is of the form in Step 1.

" $\partial\Omega$ " $= C_1 - C_2 - C_3$ Hint: how to define $\int$?

Then $RHS = \int_{\Omega} -P_y\, dx = \sum_{i=1}^{N} \int_{\Omega_i} -P_y\, dx = \sum_{i=1}^{N} \int_{\partial\Omega_i} P dx = \int_{\partial\Omega} P dx$

Similar pf works for Q, but with step 1. modified to be 

① $\int_{\Omega} \operatorname{div}\vec{F}\, dA = \int_{\partial\Omega} \vec{F}\cdot\vec{n}\, ds$ (outer)

② $\int_{\Omega} \underbrace{\operatorname{curl}\vec{F}\cdot e_3}_{Q_x - P_y} \cdot dA = \int_{\partial\Omega} \vec{F}\cdot dX$

$\vec{F} = \nabla f$ (grad f) $\quad \underset{\nabla\cdot}{\underset{|||}{\operatorname{div}}}\, \nabla f = (f_x)_x + (f_y)_y = f_{xx} + f_{yy} = \Delta f$

①: $\int_{\Omega} \Delta f\, dA = \int_{\partial\Omega} \underbrace{\nabla f\cdot\vec{n}}_{\text{"}\frac{\partial f}{\partial n}\text{"}}\, ds$

$\frac{\partial f}{\partial n}$ special notation for normal derivatives

$\vec{F} = g\nabla f \quad \operatorname{div} F = \nabla\cdot(g\nabla f) = \nabla g\cdot\nabla f + g\Delta f$

$$\int_{\Omega} (\nabla g\cdot\nabla f + g\Delta f) = \int_{\partial\Omega} g \frac{\partial f}{\partial n}\, ds \qquad \int_{\Omega}(\nabla f\cdot\nabla g + f\Delta g) = \int_{\partial\Omega} f\frac{\partial g}{\partial n}\, ds$$

$$\int_{\Omega} (f\Delta g - g\Delta f) = \int_{\partial\Omega} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) ds$$

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$\int_{C=\partial\Omega} \omega \overset{\text{Green's thm.}}{=} \int_{\Omega} d\omega = \int_{\Omega} (Q_x - P_y)\, dxdy$ (1-form: ω)

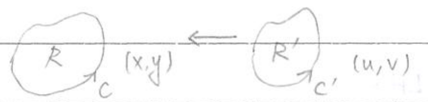
$\int_C Pdx + Qdy$ (C: 封閉曲線), $P(x,y), Q(x,y) \in C^1$

$\longrightarrow \int_C \vec{F} \cdot d\mathbf{x}$ (circulation) $\quad \mathbf{x}(t) = (x(t), y(t)) \quad \vec{F} = (P, Q)$

$\searrow \int \vec{F} \cdot \vec{n}\, ds$ (flux integral) $\quad \vec{F} = (Q, -P)$

1. Change of variable formula

$I = \int_R f\, dxdy$



$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$

Step 1. Let $f = Q_x$ for some Q

Step 2.

$I = \int_R f dxdy = \int_R Q_x\, dxdy = \int_C Q dy = \int_{C'} Q(x(u,v), y(u,v)) (y_u du + y_v dv) = \int_{C'} (Q \cdot y_u) du + (Q \cdot y_v) dv$

$= \int_{R'} ((Qy_v)_u - (Qy_u)_v)\, dudv = \int_{R'} (Q_x x_u \cdot y_v + Q_y y_u \cdot y_v + Qy_{uv} - Q_x x_v \cdot y_u - Q_y y_v \cdot y_u - Qy_{vu})\, dudv$

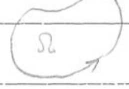
$= \int_{R'} Q_x \cdot \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} dudv = \int_{R'} f \cdot \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} dudv$

$y: C^2 \Rightarrow$ 先前的證明條件較少

the area enclosed by a curve C.

$A = -\int_C y dx = \int_C x dy = \frac{1}{2}\int_C x dy - y dx$

$P = -y, Q = 0 \quad P = 0, Q = x$



isoperimetric inequality. Fix the length of a curve $(= \ell)$, find the maximal area enclosed by such curves

$4\pi A \le \ell^2$, "=" iff C is a circle

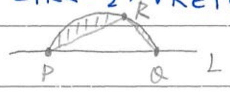
if C bounds Ω with the maximal area

step 1. convex

step 2. for any $|\widehat{PQ}| = \frac{\ell}{2}$, $\overline{PQ}$ divide Ω into two pieces into equal area

step 3. Only need to consider $P, Q \in L$ with $|\widehat{PQ}| = \frac{\ell}{2}$ fixed

$\angle PRQ = \frac{\pi}{2}, \forall R \in \widehat{PQ}$



fix the shadow part, with P, Q allowed to move

$\Rightarrow \widehat{PQ}$ is a half circle

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Poincaré

Suppose that $\int_0^{2\pi} f\,dx = 0$, then $\int_0^{2\pi} f^2 \le \int_0^{2\pi} f'^2$ "=" iff $f = a\cos t + b\sin t$

$f - \frac{1}{\ell}\int_0^{\ell} f\,dx$

pf. $f = (a_1\cos t + b_1\sin t) + (a_2\cos 2t + b_2\sin 2t) + \cdots$

$f' = (-a_1\sin t + b_1\cos t) + 2(-a_2\sin t + b_1\cos t) + \cdots$

$\frac{1}{\pi}\int_0^{2\pi} f^2 = (a_1^2 + b_1^2) + (a_2^2 + b_2^2) + \cdots$

$\frac{1}{\pi}\int_0^{2\pi} f'^2 = (a_1^2 + b_1^2) + 2^2(a_2^2 + b_2^2) + \cdots$

pf of isoperimetric inequality "$\ell^2 \ge 4\pi A$".

for simplicity, assume $\ell = 2\pi$

$A = \int_C xy'\,ds$ ☆ $2\pi = \int_0^{2\pi} (x'^2 + y'^2)\,ds$

we use arc length as parameter of C.

$2(\pi - A) = \int_0^{2\pi} (x'^2 + y'^2) - 2xy'\,ds = \int_0^{2\pi} (x'^2 - x^2)\,ds + \int_0^{2\pi} (x - y')^2\,ds$

≥ 0 (Poincaré (Fourier)) $\quad \ge 0$

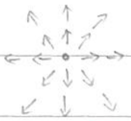
"=" iff $x(s) = a\cos s + b\sin s$, $y(s) = a\sin s - b\cos s + c$

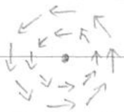
$= \sqrt{a^2+b^2}\cos(s+\theta_0)$ $\quad = \sqrt{a^2+b^2}\sin(s+\theta_0) + c$

$\int_{\partial\Omega} \vec{F}\cdot\vec{n}\,ds = \int_\Omega \operatorname{div}\vec{F}\,dA$ 散度 divergence

$\int_{\partial\Omega} \vec{F}\cdot\vec{t}\,ds = \int_\Omega (\operatorname{curl}\vec{F})_z\,dA$ 旋度 vorticity, vortex

$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$





$\operatorname{div}\vec{F} = 0$ $\quad \Delta f = 0$ harmonic function

$\vec{F} = \nabla f$, $\operatorname{div}\vec{F} = \Delta f$

$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial f}{\partial \theta^2}$

If $f = f(r)$: $rf'' + f' = 0$ $\quad (rf')' = 0$

$rf' = \text{const.}$ $\quad f' = \frac{c}{r}$ $\quad f = c\log r + c_1 = \frac{c}{2}\log(x^2+y^2) + c_1$

$\vec{F} = \nabla f = \left(\frac{c}{2}\cdot\frac{2x}{x^2+y^2}, \frac{c}{2}\cdot\frac{2y}{x^2+y^2}\right) = d\tan\frac{y}{x}$ (corrections: $2x \to -y$, $2y \to x$)

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$\int_{C=\partial\Omega} P\,dx + Q\,dy$ $\quad \int_C \vec{F}\cdot d\mathbf{x} = \int_\Omega (\text{curl}\,\vec{F})_z\, dA$ Stoke's thm (for dim = 2)

Green's theorem $\quad \int_C \vec{F}\cdot\vec{n}\,ds = \int_\Omega \text{div}\,\vec{F}\, dA$ Gauss' thm (divergence thm)

if $\text{curl}\,\vec{F} = Q_x - P_y = 0$, then "locally" $\vec{F} = \nabla f$

$\Rightarrow \text{div}\,\vec{F} = \Delta f$ "$\Delta f = 0$" Laplace equation (f is a harmonic function)

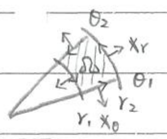
heat equation $\frac{\partial}{\partial t} f(x,t) = \Delta f$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial}{\partial \theta^2}$$

Green's thm $\Rightarrow$

$$\int_\Omega \Delta f\, dA = \int_C \nabla f\cdot\vec{n}\, ds = \int_C \frac{\partial f}{\partial n}\, ds$$

$$\int_\Omega \text{div}(g\nabla f)\, dA = \int_\Omega (\nabla g\cdot\nabla f + g\Delta f)\, dA = \int_C g\frac{\partial f}{\partial n}\, ds$$



$$\int_\Omega \Delta f\, dA = \int_{\partial\Omega} \frac{\partial f}{\partial n}\, ds = \int_{\theta_1}^{\theta_2} \frac{\partial f}{\partial r}\cdot r \Big|_{r_1}^{r_2} d\theta + \int_{r_1}^{r_2} \frac{\partial f}{\partial \theta}\cdot\frac{1}{r}\Big|_{\theta_1}^{\theta_2} dr$$

$$= \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\cdot r\right) dr\, d\theta + \int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial \theta}\cdot\frac{1}{r}\right) d\theta\, dr$$

$\mathbf{x}(r,\theta) = (x,y) = (r\cos\theta, r\sin\theta)$	$= \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} \left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}\right) dA$
$\mathbf{x}_r = (\cos\theta, \sin\theta)$	Apply MVT and let $\Omega \to$ a point
$\mathbf{x}_\theta = (-r\sin\theta, r\cos\theta)$	$\Rightarrow \Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$

$f(\mathbf{x}(r,\theta))$ $\quad \frac{\partial f}{\partial \theta} = \nabla f\cdot\mathbf{x}_\theta$

Q: what kind of f do we have?

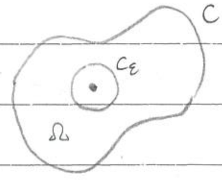
case 1. if $f = f(r)$

$$f'' + \frac{1}{r}f' = 0,\ (rf')' = 0,\ rf' = c,\ f' = \frac{c}{r},\ f = c\log r + C_1$$

$$\vec{F} = \nabla f = c\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right),\ r = (x^2+y^2)^{1/2}$$

$$\int_{C_\varepsilon} \vec{F}\cdot\vec{n}\, ds = \int_0^{2\pi} \frac{\sqrt{x^2+y^2}}{x^2+y^2}\,\varepsilon\, d\theta = \int_0^{2\pi} d\theta = 2\pi \neq 0$$ since (0,0) is singular $\Rightarrow$ cannot apply Green's thm

↳ this is independent of ε!



$\int_C \vec{F}\cdot\vec{n}\, ds$ $\quad \partial\Omega = C - C_\varepsilon$

$$\int_{C-C_\varepsilon} \vec{F}\cdot\vec{n}\, ds = \int_\Omega \text{div}\,\vec{F}\, dA = 0$$

↑ apply Green's thm

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case 2. if $f = f(\theta) \Rightarrow f(\theta) = a\theta + b = a\tan^{-1}\frac{y}{x} + b$ is not a well-defined function of (x,y)

$\nabla f = a\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$

$\int_C \vec{F}\cdot\vec{n}\, ds = 0$

$\int_C \nabla\vec{F}\cdot dx = f\Big|_{\theta=0}^{\theta=2\pi} = (a\theta+b)\Big|_{\theta=0}^{\theta=2\pi} = a\cdot 2\pi$

3D ① $\int_{\partial\Omega} P\,dx + Q\,dy = \int_\Omega (Q_x - P_y)\,dx\,dy$; ② $\int_{\partial\Omega} -Q\,dx + P\,dy = \int_\Omega \underbrace{(P_x+Q_y)}_{\text{div}\,\vec{F}}\,dx\,dy$

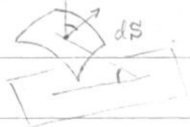
② $\vec{F} = (a,b,c)$ $\text{div}\,\vec{F} = a_x + b_y + c_z$

$\int_\Omega \text{div}\,\vec{F}\, \underset{dx\,dy\,dz}{dV} = \int_{\partial\Omega} \vec{F}\cdot\vec{n}\, dS$ area of $\partial\Omega = S$

$\partial\Omega$: boundary surface

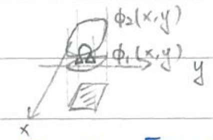
$\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$ $\vec{F}\cdot\vec{n}\,dS = (a\cos\alpha, b\cos\beta, c\cos\gamma)\,dS$

$= a\,dy\,dz + b\,dz\,dx + c\,dx\,dy$



pf of divergence theorem. (We prove the theorem for "c")

Step 1. Ω is bounded by 2 graphs of functions $\phi_1(x,y)$, $\phi_2(x,y)$ over some region in (x,y) plane



Step 2. For general Ω, find a partition $\Omega = \sqcup\, \Omega_i$

Möbius band 不可定向曲面 non-orientable

Theorem. (Jordan curve theorem): Any "closed" surface in $\mathbb{R}^3$ is orientable and $S = \partial\Omega$ for a bounded region Ω

Prove closed curve on $\mathbb{R}^2$ divides $\mathbb{R}^2$ into two parts

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Divergence thm in $\mathbb{R}^3$ 若先給S，可能不確定Ω是否存在，避免解釋何謂 closed surface

(先前假設) which can be partitioned into a finite union of simple regions Ω_i (ie. x,y,z direction

Let Ω be a bounded open set in $\mathbb{R}^3$ with $\partial\Omega$ be a surface S (which is C^1), and let $\vec{F}$ be a C^1 vector field, then $\int_\Omega \text{div}\,\vec{F}\,dV = \int_{\partial\Omega} \vec{F}\cdot\vec{n}\,dS$ ($\partial\Omega$ = S; $\vec{n}$ → outer normal)

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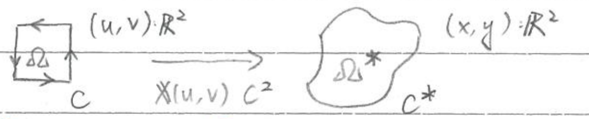
In order to talk about the "boundary of a surface", we need to consider the "induced topology" from $\mathbb{R}^3$ to S $\quad S \hookrightarrow \mathbb{R}^3$ $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$ topology

Definition. A set $u \subset S$ is open iff $u = V \cap S$ for some open set $V \subset \mathbb{R}^3$

- a point $p \in S$ is an interior if $\exists$ open set $u \ni p$ st. u looks like a disk
- a point $p \in S$ is a boundary point of S if $\nexists$ $u \ni p$ st. u looks like a disk

2-dim Green's thm



$$\int_{C^*} P\,dx + Q\,dy = \int_C P(x_u du + x_v dv) + Q(y_u du + y_v dv) = \int_C (Px_u + Qy_u)\,du + (Px_v + Qy_v)\,dv$$

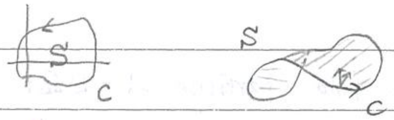
$$= \int_\Omega \left((Px_v + Qy_v)_u - (Px_u + Qy_u)_v\right) du\,dv = \int_\Omega (P_u x_v + Px_{uv} + Q_u y_v + Qy_{uv} - P_v x_u - Px_{vu} - Q_v y_u - Qy_{vu})\,du\,dv$$

$$= \int_\Omega \left((P_x x_u x_v + P_y y_u x_v + Q_x x_u y_v + Q_y y_u y_v) - (P_x x_v x_u + P_y y_v x_u + Q_x x_v y_u + Q_y y_v y_u)\right) du\,dv$$

$$= \int_\Omega (-P_y + Q_x) \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du\,dv = \int_{\Omega^*} (-P_y + Q_x)\,dx\,dy$$

finite polyhedron

$\int_C \vec{F} \cdot d\mathbf{x} = \int_S (\operatorname{curl} \vec{F})_z\, dA$
$C = \partial S$



$S \subset \mathbb{R}^3$ be an oriented surface (ie. with a given continuously defined normal vectors)

"$C = \partial S$" has the induced (positive) orientation

Stoke's thm in $\mathbb{R}^3$ $\quad \int_C \vec{F} \cdot d\mathbf{x} = \int_S (\operatorname{curl} \vec{F}) \cdot \vec{n}\, dA$

Its primitive form is $\int_C \underbrace{P\,dx + Q\,dy + R\,dz}_{\omega} = \int_S \left((R_y - Q_z)\cos\alpha + (P_z - R_x)\cos\beta + (Q_x - P_y)\cos\gamma\right) dS$

$$= \int_S \underbrace{(R_y - Q_z)\,dy\,dz + (P_z - R_x)\,dz\,dx + (Q_x - P_y)\,dx\,dy}_{d\omega}$$

parameterize S by $\mathbf{x}(u,v)$ $\quad \underset{\mathbb{R}^2}{\Omega} \xrightarrow{\mathbf{x}\ C^2} \mathbb{R}^3$ $\quad \mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$, $\partial\Omega = C^*$, $\partial S = C$

pf. $\int_C P\,dx + Q\,dy + R\,dz = \int_{C^*} (Px_u + Qy_u + Rz_u)\,du + (Px_v + Qy_v + Rz_v)\,dv$

Green's thm on $\mathbb{R}^2$

$$= \int_\Omega \left((Px_v + Qy_v + Rz_v)_u - (Px_u + Qy_u + Rz_u)_v\right) du\,dv = \int_\Omega (P_u x_v + Q_u y_v + R_u z_v) - (P_v x_u + Q_v y_u + R_v z_u)\,du\,dv$$

$$= \int_\Omega \Big((P_y y_u x_v + P_z z_u x_v - P_y y_v x_u - P_z z_v x_u) + (Q_z z_u y_v + Q_x x_u y_v - Q_z z_v y_u - Q_x x_v y_u)$$

$$+ (R_x x_u z_v + R_y y_u z_v - R_x x_v z_u - R_y y_v z_u)\Big)\,du\,dv$$

$$= \int_{\Omega}\left((R_y-Q_z)\begin{vmatrix} y_u & y_v \\ z_u & z_v\end{vmatrix} + (P_z-R_x)\begin{vmatrix} z_u & z_v \\ x_u & x_v\end{vmatrix} + (Q_x-P_y)\begin{vmatrix} x_u & x_v \\ y_u & y_v\end{vmatrix}\right) du\,dv$$

$$= \int_{\Omega} (R_y-Q_z)\,dy\,dz + (P_z-R_x)\,dz\,dx + (Q_x-P_y)\,dx\,dy$$

$dS = |X_u \times X_v|\,du\,dv \Rightarrow \begin{vmatrix} y_u & y_v \\ z_u & z_v\end{vmatrix} du\,dv = \cos\alpha\, dS$

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$R \xrightarrow{g} R'$, (x,y) → (u,v)

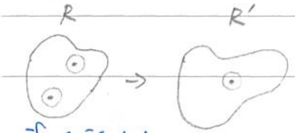
$$\int_{R'} f(u,v)\,du\,dv = \int_R f\frac{\partial(u,v)}{\partial(x,y)}\,dx\,dy$$

↑ if g is 1-1 and $J(g) = \frac{\partial(u,v)}{\partial(x,y)} > 0$

$$= -\int_R f\frac{\partial(u,v)}{\partial(x,y)}\,dx\,dy$$

↑ if g is 1-1 and $J(g) < 0$

recall single-variable
two proofs (one requires 1-1, ~~the other not~~)



~~if g is 1-1,~~
Define $\varepsilon_R(u,v) = \begin{cases} 0, & (u,v) \notin \operatorname{Im} g \\ \operatorname{sgn}\left(\frac{\partial(u,v)}{\partial(x,y)}\right), & (u,v) = g(x,y)\end{cases}$ $u,v \in \mathbb{R}^2$

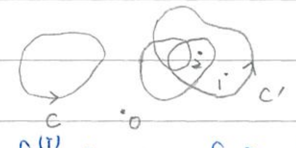
Then CVF: $\int_{\mathbb{R}^2} f\,\varepsilon_R\,du\,dv = \int_R f\frac{\partial(u,v)}{\partial(x,y)}\,dx\,dy$

General case: $R \xrightarrow{g} R'$ not 1-1, but (Assume) $R = \bigcup_{i=1}^m R_i$ st. g is 1-1 on R_i (let $C_i = \partial R_i$)

$$\int_R f\frac{\partial(u,v)}{\partial(x,y)}\,dx\,dy = \sum_{i=1}^m \int_{R_i} f\frac{\partial(u,v)}{\partial(x,y)}\,dx\,dy = \sum_{i=1}^m \int_{\mathbb{R}^2} f\,\varepsilon_{R_i}\,du\,dv = \int_{\mathbb{R}^2 (R')} f\cdot\underbrace{\sum_{i=1}^m \varepsilon_{R_i}}_{\chi_R(u,v)}\,du\,dv$$

$\chi_R(u,v) =$ " degree of the mapping g at (u,v) "

Identity: (*) $\chi_R(u,v) = \mu_C(u,v)$ winding number of C' = image of C at the point (u,v)



pf. (i) (*) is true if $R \to R'$ is 1-1

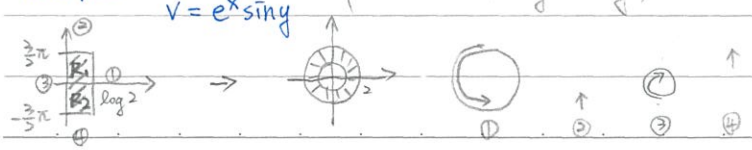
$J(g) > 0$, $\mu_C = 1$
$J(g) < 0$, $\mu_C = -1$

(ii) (*) is additional $R = \cup A_i$

$R = \cup\, \partial A_i$

$$\chi_R(u,v) \overset{\downarrow}{=} \sum_i \chi_{A_i}(u,v) = \sum_i \mu_{C_i}(u,v) \overset{\downarrow}{=} \mu_C(u,v)$$

Example. $u = e^x\cos y$, $v = e^x \sin y$ $\left(u+iv = e^x(\cos y + i\sin y) = e^{x+iy}\right)$



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harmonic functions

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \theta^2}$$

$\Delta f = 0$ for $f = f(r)$ or $f = f(\theta)$ singularity
($\log r$, $\tan^{-1}\frac{y}{x}$)

$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ conformal $\Rightarrow$ satisfy Cauchy-Riemann equation $\begin{cases} u_x = v_y \\ v_x = -u_y \end{cases}$

$\Rightarrow \Delta u = u_{xx} + u_{yy} = v_{yx} + (-v_{xy}) = 0, \quad \Delta v = 0$

$w = z^3 = (x+iy)^3 \quad ; \quad w = e^z$

in general, we need "f(x) to be analytic" ie. $f(x)$ = Taylor series $\Rightarrow f(z)$ well defined

$w = x^3 + 3x^2 y i - 3xy^2 - y^3 i = \underbrace{(x^3 - 3xy^2)}_{u(x,y)} + i\underbrace{(3x^2 y - y^3)}_{v(x,y)}$

$\Rightarrow u, v$ satisfy Cauchy-Riemann equation $\Rightarrow u, v$ harmonic

$\log z = \log(re^{i\theta}) = \log r + (i\theta + 2\pi k i)$ $\quad$ $\log x$ has Taylor series at $x \neq 0$

$\log x = \log(a + (x-a)) = \log a + \log(1 + \frac{x-a}{a})$

$f(z) = f(x+iy)$

$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) f = f'' + f''\cdot i^2 = 0$

Δ in spherical coordinates :

$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$ $\quad$ ✗

$\int_\Omega \Delta f \, dV = \int_{\partial\Omega} \frac{\partial f}{\partial n} dS$ ($\frac{\partial f}{\partial n} = \nabla f \cdot \vec{n}$)

$\partial\Omega$:

r: $\int_{r=r_2} r^2 \sin\theta \frac{\partial f}{\partial r} d\theta d\phi - \int_{r=r_1} \cdots$

$= \int_\Omega \frac{\partial}{\partial r}(r^2\sin\theta \frac{\partial f}{\partial r}) dr d\theta d\phi$

θ: $\int_\Omega \frac{\partial}{\partial \theta}(\frac{1}{r}\frac{\partial f}{\partial \theta} r\sin\theta) dr d\theta d\phi$

ϕ: $\int_\Omega \frac{\partial}{\partial \phi}(\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\cdot r) dr d\theta d\phi$

$$\Rightarrow \Delta f = \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}(r^2\sin\theta\frac{\partial f}{\partial r}) + \frac{\partial}{\partial\theta}(\frac{\partial f}{\partial\theta}\sin\theta) + \frac{\partial}{\partial\phi}(\frac{\partial f}{\partial\phi}\cdot\frac{1}{\sin\theta})\right)$$

n-dim, Δ in spherical coordinates

 $x_n = r\cos\theta_1, \; x_{n-1} = r\sin\theta_1\cos\theta_2, \; x_{n-2} = r\sin\theta_1\sin\theta_2\cos\theta_3, \cdots$

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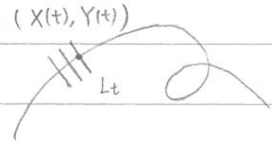
$X(t) = \frac{1}{|L_t|}\int_{L_t} x\, ds$

$(X(t), Y(t))$



4.10 b area swept out by a moving line in $\mathbb{R}^2$

Co-area formula (等高線積分法)

Equation of a line $\xi(t)x + \eta(t)y = p(t)$ at time t.

$(\xi, \eta) \parallel \vec{n}$ $\quad \vec{v} = (x'(t), y'(t))$

$\xi' x + \eta' y + \xi x' + \eta y' = p'$

velocity vector of the centroid

$\vec{n}\cdot\vec{v} = \xi \frac{dx}{dt} + \eta \frac{dy}{dt} = \frac{dp}{dt} - \frac{d\xi}{dt}x - \frac{d\eta}{dt}y$ $\quad \vec{n}\cdot\vec{w} = \xi\frac{dX}{dt} + \eta\frac{dY}{dt} = \frac{dp}{dt} - \frac{d\xi}{dt}X - \frac{d\eta}{dt}Y$

$A = \int dt \int_{L_t} \frac{ds}{|\nabla\phi|}$ $\quad \frac{1}{|\nabla\phi|} = \pm \vec{v}\cdot\vec{n}$ $\quad \left(\phi(x(t), y(t)) = t, \quad \underbrace{\nabla\phi}_{\pm|\nabla\phi|\vec{n}} \cdot \underbrace{(x'(t), y'(t))}_{\vec{v}} = 1 \right)$

$= \pm\int dt \int_{L_t} \vec{v}\cdot\vec{n}\, ds$

$= \pm\int dt \left(\frac{dp}{dt}\cdot|L_t| - \frac{d\xi}{dt}X(t)\cdot|L_t| - \frac{d\eta}{dt}Y(t)\cdot|L_t| \right)$

$= \pm\int_a^b |L_t|\, \vec{w}\cdot\vec{n}\, dt$

special case: if $L_t \perp \vec{n}$ $\forall t$, $A = \int |L_t| \frac{d\sigma}{dt} dt = \int |L|\, d\sigma$ $\quad \sigma$, arc length of $(X(t), Y(t))$

application:

$V(r) = (\pi r^2)\ell$ true for r small

$S(r) = V'(r) = 2\pi r \ell$

不考慮

Stoke's thm in higher dimension

Green - divergence - Gauss $(n=N)$

step 1. $B = [0,1]^n \subset \mathbb{R}^n = \{ (x_i)_{i=1}^n \mid 0 \le x_i \le 1 \}$

$\int_B d\omega = \int_{\partial B}\omega$ for any $(n-1)$ form ω

step 2. $f: \underset{\substack{\cap\\ \mathbb{R}^n}}{B^{(n)}} \xrightarrow{C^1} \underset{\substack{\cap\\ \mathbb{R}^N}}{\Omega^{(N)}}$ $\quad \int_\Omega d\omega = \int_{\partial\Omega}\omega$

step 1 for $n=1$, FTC $\int_{[0,1]} df = \int_{\partial[0,1]} f$ $\quad$ 0 B 1 $\quad \partial[0,1] = "1" - "0"$

$n=2$, Green $\int_B (Q_x - P_y)\, dx\wedge dy = \int_{\partial B} P\,dx + Q\,dy$ $\quad$ c_1, c_2, c_3, c_4 $\quad \partial B = c_1 + c_2 + c_3 + c_4$

for step 2. $\underset{u_1, u_2}{B} \xrightarrow{g: C^2} \underset{x_1, \dots, x_N}{\mathbb{R}^N}$ $\quad g(B) = \Omega$

ω: 1-form on $\mathbb{R}^N$, $\omega = \sum f_i\, dx_i$ $\quad \int_\Omega d\omega = \int_{\partial\Omega}\omega$

$\int_{\partial\Omega}\omega = \int_{\partial\Omega} \sum_i f_i\, dx_i = \int_{\partial B} \sum_{i,j} f_i \frac{\partial x_i}{\partial u_j} du_j = \int_B \sum_{i,j,k} \underbrace{\frac{\partial}{\partial u_k}\left(f_i \frac{\partial x_i}{\partial u_j}\right)}_{\frac{\partial f_i}{\partial u_k}\cdot\frac{\partial x_i}{\partial u_j} + \cancel{f_i \frac{\partial^2 x_i}{\partial u_k \partial u_j}}} du_k \wedge du_j$

$= \int_\Omega \sum_i df_i \wedge dx_i = \int_\Omega d\omega$

CH3. iteration inverse function

primitive map → CVF

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CH4,5 Fubini thm (domain) Co-area formula

improper 微積分次序

for step 2 (n=3)

$$\int_{\partial B} a\,dy\wedge dz + b\,dz\wedge dx + c\,dx\wedge dy = \int_B (a_x + b_y + c_z)\,dx\,dy\,dz$$

$B \to \mathbb{R}^N$

$u_1, u_2, u_3 \quad x_1, \cdots, x_N$

ω : 2-form on $\mathbb{R}^N$, $\omega = \sum f_{i_1 i_2}\, dx_{i_1} \wedge dx_{i_2}$

$$\int_{\partial\Omega} \omega = \int_{\partial\Omega} \sum_i f_{i_1 i_2}\, dx_{i_1} dx_{i_2} = \int_{\partial B} \sum_{i,j} f_{i_1 i_2} \frac{\partial x_{i_1}}{\partial u_{j_1}} \frac{\partial x_{i_2}}{\partial u_{j_2}}\, du_{j_1} \wedge du_{j_2}$$

$$= \int_B \sum_{i,j,k} \frac{\partial}{\partial u_k}\left(f_{i_1 i_2} \frac{\partial x_{i_1}}{\partial u_{j_1}} \frac{\partial x_{i_2}}{\partial u_{j_2}}\right) du_k \wedge du_{j_1} \wedge du_{j_2}$$

$$\frac{\partial f_{i_1 i_2}}{\partial u_k} \cdot \frac{\partial x_{i_1}}{\partial u_{j_1}} \cdot \frac{\partial x_{i_2}}{\partial u_{j_2}} + f_{i_1 i_2} \frac{\partial^2 x_{i_1}}{\partial u_k \partial u_{j_1}} \frac{\partial x_{i_2}}{\partial u_{j_2}} + f_{i_1 i_2} \frac{\partial x_{i_1}}{\partial u_{j_1}} \cdot \frac{\partial x_{i_2}}{\partial u_k \partial u_{j_2}}$$

$$= \int_\Omega \sum_i df_i \wedge dx_{i_1} \wedge dx_{i_2}$$

$$= \int_\Omega d\omega$$

$d : \Omega^p \to \Omega^{p+1}$

$\omega \to d\omega$

$\omega = \sum f_I\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$

$d\omega = \sum_I df_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}$

$$d^2\omega = d(d\omega) = \sum_{I,i,j} d\left(\frac{\partial f}{\partial x_i}\right) \wedge dx_i \wedge \cdots$$

$$\frac{\partial^2 f}{\partial x_j \partial x_i}\, dx_j \wedge dx_i \wedge \cdots$$

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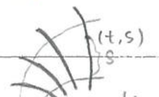


$\phi(x,y) = t$ height $\qquad \{(x,y) \mid \phi(x,y) = t\} =: L_t \qquad s$: arc length on L_t

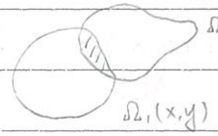
$\xi(t)\, x + \eta(t)\, y = p(t) \qquad \xi' x + \eta' y + \xi \frac{\partial x}{\partial t} + \eta \frac{\partial y}{\partial t} = p'$ integrate on s variable

$x(t,s) \quad y(t,s)$

$$\int_{L_t} x\, ds = |L_t| \cdot \bar{X}(t)$$

Orientation (方向)

Def



$\subset \mathbb{R}^2$, $\Omega_1 \cap \Omega_2 \neq \phi$ has the same orientation if $\frac{\partial(x,y)}{\partial(u,v)} > 0$ on $\Omega_1 \cap \Omega_2$

$\left(\Leftrightarrow \frac{\partial(u,v)}{\partial(x,y)} > 0\right)$

電磁場 Maxwell equation

$d{*}F = 0$ abelian gauge thm.

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linear algebra version

$\mathbb{R}^n$. 2 basis $\{v_1, \cdots, v_n\}$, $\{w_1, \cdots, w_n\}$ $\quad w_i = \sum_{j=1}^{n} a_{ij} v_j \quad A = (a_{ij})$

has the same orientation $\underset{def}{\Longleftrightarrow}$ $\det A > 0$

$\{\vec{v}_1, \cdots, \vec{v}_n\}$, $\{\vec{w}_1, \cdots, \vec{w}_n\}$

coordinates

$x_1, \cdots, x_n \qquad y_1, \cdots, y_n$

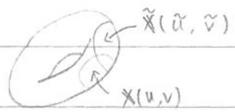
$\vec{v} = \sum_i x_i \vec{v}_i = \sum_i y_i \vec{w}_i$

$= \sum_{i,j} y_i a_{ij} \vec{v}_j$

$x_i = \sum_j y_j a_{ji}$

$\frac{\partial(x_1, \cdots, x_n)}{\partial(y_1, \cdots, y_n)} = (a_{ji}) = A^t$

What do we mean if a surface $S \subseteq \mathbb{R}^3$ is "orientable" (可定向)



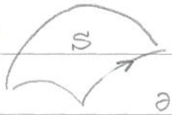
$T_pS \quad \{X_u, X_v\} \quad \{\tilde{X}_{\tilde{u}}, \tilde{X}_{\tilde{v}}\}$

$X_u = \tilde{X}_{\tilde{u}} \frac{\partial \tilde{u}}{\partial u} + \tilde{X}_{\tilde{v}} \frac{\partial \tilde{v}}{\partial u}$; $X_v = \tilde{X}_{\tilde{u}} \frac{\partial \tilde{u}}{\partial v} + \tilde{X}_{\tilde{v}} \frac{\partial \tilde{v}}{\partial v}$

法向量只能在少一維的情況定義 $\quad |X_u, X_v, X_u \times X_v| > 0$

Möbius band $\quad |\tilde{X}_{\tilde{u}}, \tilde{X}_{\tilde{v}}, \tilde{X}_u \times \tilde{X}_v| > 0$

①



{orientation on ∂S + inner normal vector} = orientation on S

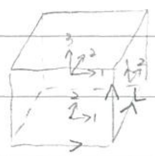
induced orientation

step 1. eventually, we only need to consider the case of a cube.

$\Omega = [0,1]^n \subset \mathbb{R}^n$ with orientation given by $e_1, \cdots, e_n$

$\partial\Omega$ = the $2n$ "boundary pieces"

as a set $\partial\Omega = \{(x_i)_{i=1}^n \in \Omega \mid x_i = 0 \text{ or } 1 \text{ for some } i\}$



$\partial\Omega = A_1 \cup \cdots \cup A_{2n}$

$\partial(\partial\Omega) = \partial A_1 \cup \cdots \cup \partial A_{2n} = 0$

closed surface: a surface without boundary ($\partial\Omega$)

$\partial^2 = 0 \longleftrightarrow d^2 = 0$

$\vec{F} \cdot d\vec{x} \rightarrow P dx + Q dy + R dz$

$\vec{F} \cdot \vec{dS} \rightarrow P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy$

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I. Point Set Topology

II. Limit

$\underbrace{\sup_{\theta} |f(x+r\cos\theta, y+r\sin\theta) - L|}_{g(r)} < \varepsilon \quad \forall r<\delta$

<Def> $\lim_{(x,y)\to(a,b)} f(x,y) = L \iff \forall \varepsilon>0,\ \exists \delta>0$ st. $|f(x,y)-L|<\varepsilon \quad \forall |(x,y)-(a,b)|<\delta$

$\iff \forall \{(x_n,y_n)\}_{n=1}^{\infty}$ with $(x_n,y_n)\to(a,b)$, we have $\lim_{n\to\infty} f(x_n,y_n) = L$

eg. $\lim_{(x,y)\to(0,0)} \sqrt{1+e^{xy}} = \sqrt{2}$

eg. $\lim_{(x,y)\to(0,0)} |x|^{y}$
(1) $(0,y) \to \lim_{(x,y)\to(0,0)} 0^{y} = 0$ → 不存在
(2) $(x,x) \to \lim_{(x,y)\to(0,0)} |x|^{x} = 1$

eg. $\lim_{(x,y)\to(0,0)} |x|^{\frac{1}{|y|}} = 0$

$0 \le |x|^{\frac{1}{|y|}} \le |x|^{2} \to 0$ as $(x,y)\to(0,0)$

$\lim_{(x,y)\to(0,0)} \frac{\sin(x^4+y^4)}{x^2+y^2} = 0$

pf. $0 \le \left|\frac{\sin(x^4+y^4)}{x^2+y^2}\right| \le \frac{x^4+y^4}{x^2+y^2} \le \frac{(x^2+y^2)^2}{x^2+y^2} = x^2+y^2 \longrightarrow 0$ as $(x,y)\to(0,0)$

$\lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4} = 0$

pf. $\sup_{\theta} \frac{e^{-\frac{1}{r^2}}}{r^4(\cos^4\theta+\sin^4\theta)} = 2\frac{e^{-\frac{1}{r^2}}}{r^4} \to 0$ as $r\to 0$

$\cos^4\theta+\sin^4\theta = 1-\frac{1}{2}\sin^2 2\theta$

$\lim_{r\to 0^+} |f(a+r\cos\theta, b+r\sin\theta) - L| = 0,\ \forall\theta \not\Rightarrow f(x,y)\to L$ as $(x,y)\to(a,b)$

$C^1 \Rightarrow$ diff. $\Rightarrow$ conti. $\Rightarrow$ limit exists

f_x, f_y is conti.

diff. ⇓ f_x or f_y 存在, 方向 deri. 存在

eg. $\lim_{(x,y)\to(0,0)} \frac{x^6+y^6}{x^6+y^6+(y-x^2)^2}$

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<Definition> f is differentiable at (a,b) if there exists $(A,B)\in\mathbb{R}^2$ st.

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k)-f(a,b)-Ah-Bk}{\sqrt{h^2+k^2}} = 0$$

Chuyu culture

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How to check differentiability?

(1) Test the limit $\forall (A,B) \in \mathbb{R}^2$ → impossible

<Lemma>

If such (A,B) exists, then $A = f_x(a,b)$, $B = f_y(a,b)$

Moreover, $D_{(\theta)} f(a,b) = f_x(a,b)\cos\theta + f_y(a,b)\sin\theta$

∴ we need only to check the limit for $A = f_x(a,b)$ and $B = f_y(a,b)$

(2) Calculate $f_x(x,y)$ and $f_y(x,y)$ near (a,b)

<Lemma>

f is C^1 at $(a,b) \Rightarrow f$ is differentiable at (a,b)

$$\varphi(x,t) = \begin{cases} x, & 0 \le x \le \sqrt{t},\ t \ge 0 \\ -x + 2\sqrt{t}, & \sqrt{t} \le x \le 2\sqrt{t},\ t \ge 0 \\ -x, & 0 \le x \le \sqrt{-t},\ t \le 0 \\ x - 2\sqrt{-t}, & \sqrt{-t} \le x \le 2\sqrt{-t},\ t \le 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\int_{-1}^{1} \frac{\partial}{\partial t}\varphi(x,0)\,dx = 0 \neq 1 = \frac{d}{dt}\int_{-1}^{1} \frac{\partial \varphi}{\partial t}(x,0)\,dx \Big|_{t=0}$$

<Thm> $L = A\,dx + B\,dy + C\,dz$ is conservative on $D \subset \mathbb{R}^3$

$\Leftrightarrow \exists \phi$ st. $\nabla\phi = (A, B, C)$ on D

$\Leftrightarrow \int_\Gamma L$ is independent of path $\forall \Gamma \subset D$

$\Leftarrow dL = 0$ on D and D is simply connected

$$\int_\Gamma (9x^2y + 8xy^2 + 5y^3)\,dx + (3x^3 + 8x^2y + 15xy^2)\,dy$$

$\Gamma : \{ (x,y) \mid y = (\pi x)^3 \sin\frac{1}{x},\ 0 \le x \le \frac{1}{2\pi} \}$

$\phi(x,y) = \int (9x^2y + 8xy^2 + 5y^3)\,dx = 3x^3y + 4x^2y^2 + 5y^3x + C(y)$

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$f \in C^1 \Rightarrow f$ differentiable $\Rightarrow f_{x_i}$ exists

$f(a+h, b+k) = f(a,b) + f_x \cdot h + f_y \cdot b + o(\sqrt{h^2+k^2}) \Rightarrow D_{(\theta)} = f_x\cos\theta + f_y\sin\theta$

$D_{(\theta)} \equiv \lim_{r \to 0} \frac{f(x + r\cos\theta, y + r\sin\theta) - f(x,y)}{r}$

MVT. $f(x+h, y+k) - f(x,y) = h f_x(x+\theta h, y+\theta k) + k f_y(x+\theta h, y+\theta k)$

convex set, $f_x, f_y \in C^1 \Rightarrow$ MVT

convex set, $f_x, f_y \in C^1$, $|f_x|, |f_y| \le M \Rightarrow f$ Lipschitz conti. $\Rightarrow$ uniformly conti.

$f(x,y)$ uniformly conti. on $U \subset \mathbb{R}^2 \Rightarrow \forall \varepsilon\ \exists \delta$ st. $|f(x_1,y_1) - f(x_2,y_2)| < \varepsilon\ \forall \|(x_1,y_1)-(x_2,y_2)\| < \delta$

$(x_1,y_1), (x_2,y_2) \in U$

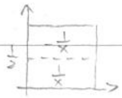
$f(x,y)$ conti. on closed bounded set $S \Rightarrow f$ uniformly conti.

☆ $f(x,y)$, f_x conti. on $[a,b]\times[c,d] \Rightarrow \frac{d}{dx}\int_c^d f(x,y)\,dy = \int_c^d f_x(x,y)\,dy$ conti.

$\Rightarrow$ g_{xy}, g_y conti., g_x exists $\Rightarrow g_{xy} = g_{yx}$ (g_{xy}, g_{yx} conti. $\Rightarrow g_{xy} = g_{yx}$)

$\Rightarrow f$ conti. on $[a,b]\times[c,d] \Rightarrow \int_a^b\int_c^d f\,dy\,dx = \int_c^d\int_a^b f\,dx\,dy$

反例: $(0,1)\times(0,1)$



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Recall IFT: If $F(x,y,u,v) \in C^1$, $G(x,y,u,v) \in C^1$ and $\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} \neq 0$ at (u_0, v_0)

$x = F(u,v)$ $y = G(u,v)$

then $F=0$, $G=0$ can be solve as $u = u(x,y)$, $v = v(x,y)$ near (u_0, v_0)

$x = F(u,v)$, $y = G(u,v)$

(1) 中間值定理

(2) Dynamic system:

Goal: Find $\delta > 0$ st. $\forall (x,y) \in B_\delta(F(u_0,v_0), G(u_0,v_0))\ \exists! (u,v)$ st. $x = F(u,v)$, $y = G(u,v)$

Key: Consider $\varphi\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} u \\ v \end{pmatrix} + A\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} F(u,v) \\ G(u,v) \end{pmatrix}\right)$

A ↑ 事後選擇的; $\begin{pmatrix} x \\ y \end{pmatrix}$ ↑ fixed $\begin{pmatrix} x \\ y \end{pmatrix} \in B_\delta(F(u_0,v_0), G(u_0,v_0))$

Geometry:

Surface theory $\mathbb{X}: U \subset \mathbb{R}^2 \to \mathbb{R}^3$

<1> 切平面: Given a curve $\{(x(t), y(t))\} \subset U$ $x(0) = x_0$, $y(0) = y_0$

$\vec{r}(t) = (x(t), y(t), z(x(t), y(t)))$

$\frac{d\vec{r}(t)}{dt} = \left(x'(t), y'(t), \frac{\partial z}{\partial x}x'(t) + \frac{\partial z}{\partial y}y'(t)\right)$

$\frac{d\vec{r}(t)}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix}\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$

(matrix: $\mathbb{X}'$; vector: 切向量)

$\mathbb{X}'$: 線性算子，將路徑切向量 → 曲面切向量

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$\frac{x^2}{1^2}+\frac{y^2}{2^2}+\frac{z^2}{3^2}=1$ $x+y+z=1$ 上半部體積

sol. let $x=x^*,\ y=2y^*,\ z=3z^*$ $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}\begin{pmatrix}x^*\\y^*\\z^*\end{pmatrix}$

$\iiint_R dxdydz=\iiint_{R^*}6\,dxdydz$ $R^*:\ x^{*2}+y^{*2}+z^{*2}=1$ $x^*+2y^*+3z^*=1$

$\rightarrow z^{**}=\frac{1}{\sqrt{14}}$

ex. $\iiint_{x^2+y^2+z^2\le 1}\cos(x+y+z)\,dxdydz$

$\iiint_{x^2+y^2+z^2\le 1}\sin(x+y+z)\,dxdydz=0$

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Recall: line integral In $\mathbb{R}^2$, $C:\vec{r}(t)=(x(t),y(t),z(t))$, $t\in I=[a,b]$, $\varphi:\mathbb{R}^2\to\mathbb{R}$

(1) $\int_C\varphi\,\underset{\text{弧長元}}{ds}=\int_a^b\varphi(x(t),y(t))\sqrt{x'^2+y'^2}\,dt$ 算質量(φ:密度)、弧長(φ:1)

(2) $\int_a^b\frac{d}{dt}(\varphi\circ\vec{r})\,dt=\varphi\circ\vec{r}(b)-\varphi\circ\vec{r}(a)$

$\int_a^b(\varphi_x\cdot x'+\varphi_y\cdot y')\,dt=\int\varphi_x dx+\varphi_y dy$

(2') $\int_C w=\int_C Pdx+Qdy$ 1-form 只能在曲線上積分

$\Rightarrow$ Green's theorem $\oint_C Pdx+Qdy=\iint_D(Q_x-P_y)\,dxdy$

$\oint_C Pdx+Qdy+Rdz=$ Stoke's theorem

$\hookrightarrow \iint_{S_1}\text{curl}\,\vec{F}\cdot\vec{n}\,ds=\iint_{S_2}\text{curl}\,\vec{F}\cdot\vec{n}\,ds$

Surface integral In $\mathbb{R}^3$ $S:\vec{r}(u,v)=(x(u,v),y(u,v),z(u,v))$

$\varphi:\mathbb{R}^3\to\mathbb{R}$

(1) $\iint_S\varphi\,\underset{\text{面元}}{dS}=\iint\varphi(x(u,v),y(u,v),z(u,v))\sqrt{EG-F^2}\,dudv$

$\sqrt{EG-F^2}=\sqrt{\|\vec{r_u}\|^2\|\vec{r_v}\|^2-(\vec{r_u}\cdot\vec{r_v})^2}$

~~(2')~~ $\underset{\text{2-form}}{w}=Pdx\wedge dy+Qdy\wedge dz+Rdz\wedge dx$ 2-form 只能在曲面上積分

$\int_S w=\int_S Pdx\wedge dy+Qdy\wedge dz+Rdz\wedge dx\triangleq\int P(x_u\cdot du+x_v\cdot dv)\wedge(y_u du+y_v dv)+\cdots$

$=\int P(x_uy_v\,du\wedge dv+x_vy_u\,dv\wedge du)+\cdots=\int P\begin{vmatrix}x_u&x_v\\y_u&y_v\end{vmatrix}du\wedge dv+\cdots$

$=\int(P,Q,R)\cdot(|\ \ |,|\ \ |,|\ \ |)\,dudv$ ($\vec{r_u}\times\vec{r_v}$)

$=\iint_S(P,Q,R)\cdot\vec{n}\,dS$

$\overrightarrow{\text{Gauss' thm}}$

Chryv culture