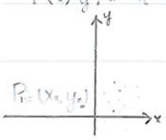
Subject : Chapter I.

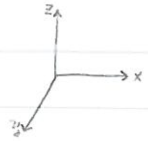
No. :

Date : 2011 / 2 / 22

§ 1.1 ~ 1.3 Functions of multivariable and continuity

$f(x, y) = x^2 - y^2$





· Limit of sequence of points

Def'n : " $\lim_{n\to\infty} P_n = Q$ " $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists N$ st $n \geq N$ $\Rightarrow |P_n - Q| < \varepsilon$

$P_n = (x_n, y_n)$, $Q = (a, b)$

equivalently , $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} y_n = b$

" $\Rightarrow$ " $[(x_n - a)^2 + (y_n - b)^2]^{\frac{1}{2}} < \varepsilon$

$|x_n - a| < \varepsilon$, $|y_n - b| < \varepsilon$

" $\Leftarrow$ " $\forall \varepsilon > 0$, $\exists N$ st $n \geq N$ $|x_n - a| < \varepsilon$ $|y_n - b| < \varepsilon$

$\Rightarrow |P_n - Q| < \sqrt{2}\,\varepsilon$ #Q.E.D.

S = Region ([illegible]) $\subset R^2$

$[a, b] \times [c, d] \subset R^2$

$\left\{ (x, y) \,\middle|\, \begin{matrix} a \leq x \leq b \\ c \leq y \leq d \end{matrix} \right\}$

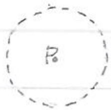
· ε - neiborhood

$(a - \varepsilon, a + \varepsilon) \times (b - \varepsilon, b + \varepsilon)$

open box

· $B_P(\varepsilon)$ = ε - ball

$\{ Q \in \mathbb{R}^2 \mid |Q - P| < \varepsilon \}$



$S = \{ (x, y) \mid x^2 + y^2 \leq 1$ and if $x = 0$ then $y < 0 \}$

· interior point

$S^\circ = \{ p \in R^2 \mid \exists\ \varepsilon$ - neiborhood of p contained in $S \}$

· exterior point

$= \{ p \in R^2 \mid \exists\ B_p(\varepsilon)$ st. $B_p(\varepsilon) \cap S = \phi \}$

S^c = complement of S
$= \mathbb{R}^2 \setminus S = \{ p \in \mathbb{R}^2 \mid p \notin S \}$

· boundary points

$\partial S = \{ p \in R^2 \mid \forall B_p(\varepsilon)$, $B_p(\varepsilon) \cap S \neq \phi$ $B_p(\varepsilon) \cap S^c \neq \phi \}$

disjoint

$R^2 = S^\circ \amalg S^e \amalg \partial S$

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Key terminology $\left[\begin{array}{l}\text{open set : } S = S^{\circ} \\ \text{close set : } S \subset \partial S\end{array}\right.$

· S is open $\Leftrightarrow$ S^c is close $[\partial S = \partial(S^c)]$

domain and range of a function

$f : S_{\subset R^2} \longrightarrow R$ $f(S)$ "image"

$\boxed{eg}$ $f(x, y) = \log(1 - x^2 - y^2)$ domain $S = B_0(1)$

$f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ we need to select a "principal branch" to make the function to be single valued.

Def'n = ① $\lim_{P_n \to P} f(P_n) = L$

"$\Leftrightarrow$" $\forall \varepsilon > 0$ $\exists N$, st $n \geq N$

$\Rightarrow |f(P_n) - L| < \varepsilon$

①' $\lim_{Q \to P} f(Q) = L$

$\forall \varepsilon > 0$, $\exists \delta > 0$

st. $|Q - P| < \delta \Rightarrow |f(Q) - L| < \varepsilon$

② $f(x, y)$ is continuous at $P = (a, b)$

$\overset{\text{def}}{\Longleftrightarrow}$ $\lim_{Q \to P} f(Q) = f(P)$

(Ex 1.) $f(x, y) = \frac{2xy}{x^2 + y^2}$ is conti outside $(0, 0)$

$\boxed{Q}$. Does $\lim_{Q \to 0} f(Q)$ exist?

along x-axis $\to f(x, 0) = 0$

y-axis $\to f(0, y) = 0$

$y = mx \to f(x, y) = \frac{2m}{1 + m^2}$ varies in m.

$\boxed{A}$: X

(Ex 2.) $f(x, y) = \frac{xy^2}{x^2 + y^2}$

along x-axis $\to f(x, 0) = 0$

y-axis $\to f(0, y) = 0$

$y = mx \to f(x, mx) = \frac{m^2}{1 + m^2} x$

$|f(x, y)| \leq y/2$

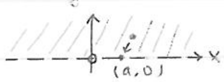
$f(x, y) = \frac{xy^2}{x^2 + y^4}$

$f(x, mx) = \frac{m^2 x^3}{x^2(1 + m^4 x^2)}$

along $y^2 = mx \Rightarrow f = \frac{m}{1 + m^2}$

(Ex 3) Continuous extension of a function to ∂S of its domain S

$f(x, y) = e^{-x/y}$ with $S = \{(x, y) \in R^2 \mid y > 0\}$ → (upper half plane)



∂S = x-axis

$\boxed{\lim_{P \to (a, 0)} e^{-x/y}}$

$\left[\begin{array}{l} a \neq 0, \ \lim_{P \to (a, 0)} \\ a = 0, \ \text{let } y = mx^2 \to f(x, y) = e^{-x/mx^2} = e^{-1/m} \to \text{constant}\end{array}\right.$

Recall

$$f(x,y) = \frac{xy^2}{x^2+y^2} \quad \text{if } (x,y) \neq (0,0)$$
$$= 0 \quad \text{if } (x,y) = (0,0)$$

$$|f(x,y) - f(0,0)| \leq \left|\frac{xy}{x^2+y^2}\right| |y| \leq \frac{1}{2}|y|$$

$\hookrightarrow$ $f(x,y)$ is continuous $\Longleftrightarrow$ $\lim_{n\to\infty} f(x_n, y_n) = f(\lim_{n\to\infty}(x_n, y_n)) = f(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n)$

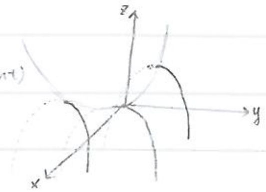
The order of a function "O" / "o"

$$\begin{cases} f = O(g) \Longleftrightarrow \left|\frac{f(h,k)}{g(h,k)}\right| \leq M \\ f = o(g) \Longleftrightarrow \lim_{(h,k)\to(0,0)} \frac{f(h,k)}{g(h,k)} = 0 \end{cases}$$

§1.4 Partial derivatives

$z = f(x,y) = x^2 - y^2$

"slicing"
($x \to$ constant)

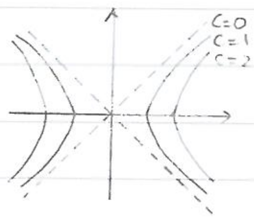


$x = 0, \; z = -y^2$

$x = C, \; z = -y^2 + C^2$

"level curves"
($z \to$ constant)

$x^2 - y^2 = C$



$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

$\downarrow$

$D_x(x_0, y_0)$, $D_1 f(x_0, y_0)$, $\partial_1 f(x_0, y_0)$, $f_x(x_0, y_0)$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

Higher partial derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = D_x D_x f$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx} = D_y D_x f$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = D_x D_y f$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = D_y D_y f$$

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(Ex 1) $f(x,y) = x^2 - y^2$

$f_x = 2x \quad , \quad f_y = -2y$

$f_{yx} = 0 \quad , \quad f_{xy} = 0$

(Ex 2) $f(x,y) = e^{x/y}$

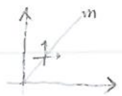
$f_x = e^{\frac{x}{y}} \cdot \frac{1}{y} \quad , \quad f_y = e^{\frac{x}{y}} \cdot \frac{-x}{y^2}$

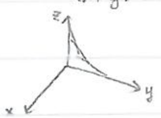
$f_{yx} = e^{\frac{x}{y}} \cdot \frac{-x}{y^2} \cdot \frac{1}{y} + e^{\frac{x}{y}} \cdot \frac{-1}{y^2}$

$f_{xy} = e^{\frac{x}{y}} \cdot \frac{1}{y} \cdot \frac{-x}{y^2} + e^{\frac{x}{y}} \cdot \frac{-1}{y^2}$

(Ex 3) f_x, f_y exist

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$



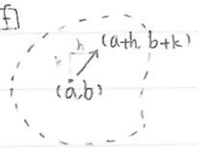


Thm: If f_x, f_y exists and $|f_x| \le M_1$, $|f_y| \le M_2$ on R then f is continuous.

$f_y = \frac{y(x^2+y^2) - xy \cdot 2x}{(x^2+y^2)^2} = \frac{y^3 - x^2 y}{(x^2+y^2)^2}$ is this bounded ?

$y = mx \quad f(x, mx) = \frac{1}{x} \cdot \frac{y^3 - x^2 y}{(1+m^2)^2}$

$\boxed{\text{pf}}$



$f(a+h, b+k) - f(a,b)$

$= \left(f(a+h, b+k) - f(a+h, b)\right) + \left(f(a+h, b) - f(a,b)\right)$

$= \underbrace{f_y(a+h, b+\theta k)}_{\le M_2} k + \underbrace{f_x(a+\theta' h, b)}_{\le M_1} h$

$|\Delta| \le M_2|k| + M_1|h|$

(Ex 4) $f(x,y,z) = \frac{1}{r} = (x^2+y^2+z^2)^{-1/2}$

Δ = Laplace operator $= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$f_x = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}} \cdot 2x$

$f_{xx} = \frac{3}{4}(x^2+y^2+z^2)^{-\frac{5}{2}} \cdot (2x)^2 - (x^2+y^2+z^2)^{-\frac{3}{2}}$

$\Delta f = \frac{3(x^2+y^2+z^2) - 3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$

(Ex 5) $f(x,t) = \frac{1}{\sqrt{t}} e^{-(x-a)^2/4t}$ satisfies $\frac{\partial f}{\partial t} = f_{xx}$

temperature

Newton's heat

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$\boxed{\text{Thm}}$: If f_{xy} and f_{yx} are continuous on an open set R_1

then $f_{xy} = f_{yx}$

$\boxed{\text{pf}}$

Let $A(h,k) := f(a+h, b+k) - f(a, b+k) - f(a+h, b) + f(a,b)$

- wrong method $A(h,k) = f_x(a+\theta_1 h, b+k) - f_x(a+\theta_2 h, b)$ still works.

Let $\phi(x) = f(x, y+k) - f(x,y)$

$A(h,k) = \phi(a+h) - \phi(a) = \phi'(a+\theta_1 h)h$

$= [f_x(a+\theta_1 h, b+k) - f_x(a+\theta_1 h, b)]h$

$= f_{yx}(a+\theta_1 h, b+\theta_2 k)hk$

$\lim_{(h,k)\to(0,0)} \frac{A(h,k)}{hk} = f_{yx}(a,b)$

- change the role of $x \leftrightarrow y$

$f_{xy}(a,b) = f_{yx}(a,b)$ #

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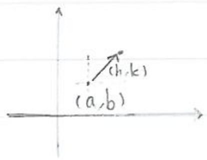
What is the noticing of "differentiability" of a multivariable function?

Def'n: A function $z = f(x,y)$ is differentiable at $(x,y) = (a,b)$

iff $f(a+\underset{\Delta x}{\underbrace{h}}, b+\underset{\Delta y}{\underbrace{k}}) = f(a,b) + Ah + Bk + o(\sqrt{h^2+k^2})$

$\hookrightarrow \varepsilon(h,k)\sqrt{h^2+k^2}$

with $\lim_{(h,k)\to(0,0)} \varepsilon = 0$



corollary: $A = \frac{\partial f}{\partial x}(a,b)$

sat $(h,k) = (h,0)$

$f(a+h,b) - f(a,b) = Ah + o(|h|)$

$B = \frac{\partial f}{\partial y}(a,b)$

$\boxed{\text{Thm}}$: If f_x, f_y exist and continuous in a nbd of (a,b)
then f is diff'ble at (a,b)

☆ nbd = neiborhood (i.e. open set)
diff'ble = differentiable

$\boxed{\text{pf}}$ $f(a+h,b+k) - f(a,b) = (f(a+h,b+k) - f(a+h,b)) + \underline{(f(a+h,b) - f(a,b))}$

$= f_y(a+h, b+\theta_1 k)k + f_x(a+\theta_2 h, b)h$

$f(a+h,b+k) - f(a,b) - (f_x(a,b)h + f_y(a,b)k)$

$= \underline{(f_x(a+\theta_2 h, b) - f_x(a,b))h} + (f_y(a+h, b+\theta_1 k) - f_y(a,b))k$

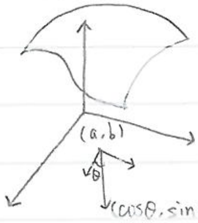
$= o(\sqrt{h^2+k^2})$ $\quad f(a+h,b) - f(a,b) - f_x(a,b)h = o(|h|)$

$\hookrightarrow$ by continuity of f_x & f_y ☆

Theorem': If $z = f(x_1, x_2, \dots, x_n)$ has at least $(n-1)$ of $\frac{\partial f}{\partial x_1}, \dots\dots, \frac{\partial f}{\partial x_n}$ to be conti., then f is diff'ble.

Def'n: $f \in C^k$ iff all partial derivatives of order $\le k$ are continuous.

Directional derivatives



$D_\theta f(a,b) = \lim_{r\to 0} \frac{f(a+r\cos\theta, b+r\sin\theta) - f(a,b)}{r}$

Assume that f is diff'ble at (a,b)

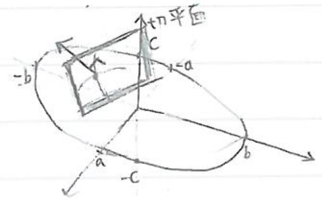
$= f_x(a,b)\cancel{r}\cos\theta + f_y(a,b)\cancel{r}\sin\theta + \cancel{\frac{o(\sqrt{h^2+k^2})}{r}}^{\,0}$

$= f_x\cos\theta + f_y\sin\theta$

Chain rule

$$u = f(x, y, z) \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

ellipsoid $\quad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$



$f(x, y, z) = C$ (constant)

$$\frac{\Delta f}{\Delta t} = f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} + f_z \frac{\Delta z}{\Delta t} + \frac{o(|\Delta \vec{x}|)}{\Delta t}$$

$\hookrightarrow \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$

$$\lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = f_x x' + f_y y' + f_z z' \qquad \pm \frac{o(|\Delta \vec{x}|)}{|\Delta \vec{x}|} \cdot \left|\frac{\Delta \vec{x}}{\Delta t}\right|$$

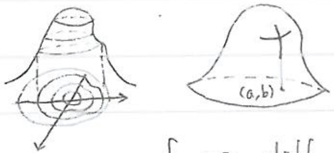
$$/\!/$$

$$\nabla f (x', y', z')$$

Def'n: $\nabla f = (f_x, f_y, f_z)$ called the gardient of f

[eg] $u = f(x, y, z) = 3x^2 + 2y^2 + z^2$ on the level surface $u = 1$.

the normal vector is given by $\nabla f = (6x, 4y, 2z)$



$$z = f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$\uparrow$ up to $o\left(\sqrt{(x-a)^2 + (y-b)^2}\right)$

f is diff. at $(a, b) \Leftrightarrow$ the notion of tangent plane exists.

Total differential 全微分

$df = f_x dx + f_y dy$

$$\text{"}d\text{"} = \underbrace{dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}}_{\substack{\text{operator} \\ \text{算子}}}$$

$(f \in C^2)$

$$d^2 f = d(f_x dx + f_y dy) = f_{xx} dx^2 + f_{xy} dy dx + f_{yx} dx dy + f_{yy} dy^2$$

$$= f_{xx} dx^2 + 2 f_{xy} dx dy + f_{yy} dy^2$$

§ 1.6 composite of functions

$u = f(\xi, \eta)$

$$\begin{tikzcd}
(\xi,\eta) \in R \subset \mathbb{R}^2 \arrow[r, "f"] & R \\
(x,y) \in S \subset \mathbb{R}^2 \arrow[u, "g=(\phi,\psi)"] \arrow[ur, "\underbrace{f\circ g}_{F}"'] &
\end{tikzcd}$$

$\xi = \phi(x,y)$; $\eta = \psi(x,y)$

$\boxed{\text{Thm}}$ f, g are differentiable $\Rightarrow$ $F := f \circ g$ is also differentiable & formula
(ϕ, ψ are both diff'ble)

$u = f(\xi, \eta) = f(\phi(x,y), \psi(x,y))$

$\Delta u = A\Delta x + B\Delta y + \underbrace{C}_{o(\sqrt{(\Delta x)^2+(\Delta y)^2})}$

$du = f_\xi \cdot d\xi + f_\eta \cdot d\eta$ $\quad \left[\begin{array}{l} d\xi = \phi_x dx + \phi_y dy \\ d\eta = \psi_x dx + \psi_y dy \end{array}\right.$

$= \underbrace{(f_\xi \phi_x + f_\eta \psi_x)}_{[\frac{\partial u}{\partial x}]} dx + \underbrace{(f_\xi \phi_y + f_\eta \psi_y)}_{[\frac{\partial u}{\partial y}]} dy$

$\Delta u = f_\xi \Delta \xi + f_\eta \Delta \eta + \varepsilon \sqrt{(\Delta\xi)^2 + (\Delta\eta)^2}$ $\quad \left[\begin{array}{l} \Delta\xi = \phi_x \Delta x + \phi_y \Delta y + \varepsilon_1 \cdot \sqrt{(\Delta x)^2+(\Delta y)^2} \\ \Delta\eta = \psi_x \Delta x + \psi_y \Delta y + \varepsilon_2 \cdot \sqrt{(\Delta x)^2+(\Delta y)^2} \end{array}\right.$

$= (f_\xi \phi_x + f_\eta \psi_x)\Delta x + (f_\xi \phi_y + f_\eta \psi_y)\Delta y + \underline{\left[\varepsilon \sqrt{(\Delta\xi)^2+(\Delta\eta)^2} + (f_\xi \varepsilon_1 + f_\eta \varepsilon_2) \cdot \sqrt{(\Delta x)^2+(\Delta y)^2}\right]}$

$(\Delta x, \Delta y) \to (0,0) \Rightarrow (\Delta\xi, \Delta\eta) \to (0,0)$

$\sqrt{(\Delta\xi)^2+(\Delta\eta)^2} \le |\Delta\xi| + |\Delta\eta| \le |\phi_x||\Delta x| + |\phi_y||\Delta y| + |\varepsilon| \sqrt{(\Delta x)^2+(\Delta y)^2}$

$\Rightarrow \dfrac{\sqrt{(\Delta\xi)^2+(\Delta\eta)^2}}{\sqrt{(\Delta x)^2+(\Delta y)^2}} \le |\phi_x| \dfrac{|\Delta x|}{\sqrt{(\Delta x)^2+(\Delta y)^2}} + |\phi_y| \dfrac{|\Delta y|}{\sqrt{(\Delta x)^2+(\Delta y)^2}} + |\varepsilon|$ is bounded ※

$u = f(x,y) = f(\underbrace{r\cos\theta}_{x(r,\theta)}, \underbrace{r\sin\theta}_{y(r,\theta)})$

$\dfrac{\partial u}{\partial r} = \underbrace{f_x}_{u_x} x_r + f_y y_r = f_x \cos\theta + f_y \sin\theta$

$\dfrac{\partial u}{\partial \theta} = u_x x_\theta + u_y y_\theta = -u_x r\sin\theta + u_y r\cos\theta$

$\Rightarrow (\Delta\xi, \Delta\eta) \to (0,0)$ $\quad (\Delta x, \Delta y) \to (0,0)$

$u = f(\xi, \eta)$, $\xi = \phi(x,y)$
$\eta = \psi(x,y)$

$u_x = u_\xi \xi_x + u_\eta \eta_x$

$u_{yx} = (u_\xi \xi_x)_y + (u_\eta \eta_x)_y = u_{\xi\xi} \xi_y \xi_x + u_{\eta\xi} \eta_y \xi_x + u_\xi \xi_{yx} +$
$u_{\xi\eta} \cdot \xi_y \eta_x + u_{\eta\eta} \eta_y \eta_x + u_\eta \eta_{yx}$

$u_{xx} = u_{\xi\xi}(\xi_x)^2 + u_{\eta\xi} \cdot \eta_x \xi_x + u_\xi \cdot \xi_{xx} + u_{\xi\eta} \cdot \xi_x \eta_x + u_{\eta\eta}(\eta_x)^2 + u_\eta \eta_{xx}$

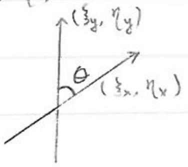
$u_{yy} =$ …… 依此類推

assume C^2

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_\xi(\xi_{xx} + \xi_{yy}) + u_\eta(\eta_{xx} + \eta_{yy}) + u_{\eta\eta}(\eta_x^2 + \eta_y^2) + u_{\xi\xi}(\xi_x^2 + \xi_y^2) + 2 \cdot u_{\xi\eta}(\xi_x \eta_x + \xi_y \eta_y)$$

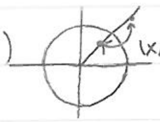
$(x,y) \longmapsto (\xi, \eta)$

保角變換



$\xi_x \xi_y + \eta_x \eta_y = 0$

$\xi_x^2 + \eta_x^2 = \xi_y^2 + \eta_y^2$

(Ex)  $(x,y) \mapsto \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ $\longleftarrow$ $f(\xi,\eta) = f\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$

Inversion

$\xi_x = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$ $\eta_x = \frac{-2xy}{(x^2+y^2)^2}$

$\xi_y = \frac{-2yx}{(x^2+y^2)^2}$ $\eta_y = \frac{x^2-y^2}{(x^2+y^2)^2}$

$\xi_x \eta_x + \xi_y \eta_y = 0$, $\xi_x^2 + \xi_y^2 = \eta_x^2 + \eta_y^2$

$\xi_{xx} = \frac{-2x(x^2+y^2)^2 - (-x^2+y^2) \cdot 2 \cdot (x^2+y^2) \cdot 2x}{(x^2+y^2)^4} = \frac{2x^3 - 6xy^2}{(x^2+y^2)^3}$, $\xi_{yy} = \frac{-2x^3+6xy^2}{(x^2+y^2)^3}$

$\Rightarrow \xi_{xx} + \xi_{yy} = 0 = \eta_{xx} + \eta_{yy}$

$u = f(x,y) = f(r\cos\theta, r\sin\theta)$ $r(\cos\theta + \sin\theta)$

$\Delta u = u_{xx} + u_{yy}$ $\begin{cases} u_r = u_x x_r + u_y y_r \\ u_\theta = u_x x_\theta + u_y y_\theta \end{cases}$ $\begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}$

$-r\sin\theta + r\cos\theta$

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Polar coordinate

$u = u(x,y)$ $\quad x = r\cos\theta,\ y = r\sin\theta$

$\Delta u = u_{xx} + u_{yy}$

$u_x = u_r r_x + u_\theta \theta_x$

$\quad = u_r \cdot \frac{x}{r} + u_\theta \frac{-y}{r^2}$

$u_y = u_r \cdot \frac{y}{r} + u_\theta \frac{x}{r^2}$

$r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1} y/x$

$r_x = \frac{2x}{2\sqrt{x^2+y^2}}$, $r_y = \frac{y}{x^2+y^2}$

$\theta_x = \frac{-y/x^2}{1+(y/x)^2} = \frac{-y}{x^2+y^2}$, $\theta_y = \frac{x}{x^2+y^2}$

$\boxed{\text{Rmk}}$ 非保角变换 $(r_x, \theta_x) \not\perp (r_y, \theta_y)$, So it's not a conformal change of coordinates

[Assume C^2]

$u_{xx} = (u_r \frac{x}{r} - u_\theta \frac{y}{r^2})_x$

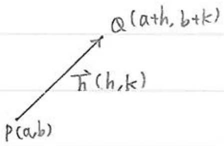
$\quad = u_{rr} \frac{x^2}{r^2} + u_{r\theta} \frac{x}{r} \cdot \frac{-y}{r^2} + u_r \underbrace{\frac{y^2}{r^3}}_{(\frac{x}{r})_x} - u_{\theta r} \frac{y}{r^2} \frac{x}{r} + u_{\theta\theta} (\frac{y}{r^2})^2 + u_\theta \underbrace{\frac{2yx}{r^4}}_{(\frac{y}{r^2})_x}$

$\quad = u_{rr} \frac{x^2}{r^2} + u_{\theta\theta} \frac{y^2}{r^4} - 2u_{r\theta} \frac{xy}{r^3} + u_r \frac{y^2}{r^3} + u_\theta \frac{2xy}{r^4}$

$u_{yy} = u_{rr} \frac{y^2}{r^2} + u_{\theta\theta} \frac{x^2}{r^4} + 2u_{r\theta} \frac{yx}{r^3} + u_r \frac{x^2}{r^3} - u_\theta \frac{2xy}{r^4}$

$\rightarrow \Delta u = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r$

§1.7 Mean Value Theorem & Taylor expansion



$f(Q) - f(P) = f(P+\vec{h}) - f(P)$

[Let $g(t) = f(P+t\vec{h})$] $= g(1) - g(0) = g'(\underset{}{\overset{\in(0,1)}{\theta}}) \times (1-0) = \nabla f(P+\theta\vec{h}) \cdot \vec{h}$

$g'(t) = \frac{d}{dt} f(P+t\vec{h}) = f_x h + f_y k = \nabla f \cdot (h,k) = \nabla f \cdot \vec{h}$

Taylor expansion

$f(x,y) = \underbrace{f(a,b)}_{a_{00}} + \underbrace{a_{10}}_{f_x(a,b)}(x-a) + \underbrace{a_{01}}_{f_y(a,b)}(y-b) + \underbrace{a_{20}}_{\frac{1}{2}f_{xx}(a,b)}(x-a)^2 + \underbrace{a_{11}}_{f_{xy}(a,b)}(x-a)(y-b) + \underbrace{a_{02}}_{\frac{1}{2}f_{yy}(a,b)}(y-b)^2 + \cdots\cdots$

$f_x(x,y) = a_{10} + 2a_{20}(x-a) + a_{11}(y-b) \cdots\cdots$

$\Rightarrow \underline{f_x(a,b) = a_{10}}$

* $f(x,y) = f(a,b) + a_{10}(x-a) + a_{01}(y-b) + a_{20}(x-a)^2 + a_{11}(x-a)(y-b) + a_{02}(y-b)^2 + \cdots\cdots$

$g(t) = f(P + t\vec{h})$

$g'(t) = \nabla f \cdot \vec{h} = f_x h + f_y k = (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}) f = df$ [differential operator 微分算子]

$g''(t) = (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^2 f$

$\quad \frac{d}{dt}(f_x h + f_y k) = f_{xx}h^2 + f_{xy}hk + f_{yx}kh + f_{yy}k^2 = (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^2 f$

$\Rightarrow$ If $f \in C^k$, the $g^{(k)}(t) = d^k f = (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^k f$

$g(1) = g(0) + g'(0) + \frac{g''(0)}{2!} + \cdots\cdots + \frac{g^{(n)}(0)}{n!} + R_n$

Assume that $g \in C^{n+1} \longrightarrow R_n = \frac{g^{(n+1)}(0)}{(n+1)!}$

$g^{(k)}(0) = h^k \frac{\partial^k f}{\partial x^k}(a,b) + \cdots\cdots + C^k_i h^i k^{k-i} \frac{\partial^k f}{\partial x^i \partial y^{k-i}}(a,b) + \cdots\cdots$

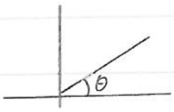
$\Rightarrow f(x,y) = f(a,b) + f_x(a,b)\cdot(x-a) + f_y(a,b)\cdot(y-b)$

$\quad + \frac{1}{2!}[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)\cdot(x-a)(x-b) + f_{yy}(a,b)\cdot(x-b)^2]$

$\quad + \frac{1}{3!}[\cdots]$

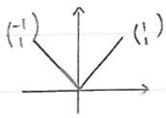
conformal mapping

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$ $\qquad R \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\qquad \left[\begin{matrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \\ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \end{matrix} \right.$



$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} a \\ c \end{pmatrix}$ $\qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} a+b \\ c+d \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} b \\ d \end{pmatrix}$ $\qquad \begin{pmatrix} -1 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} -a+b \\ -c+d \end{pmatrix}$



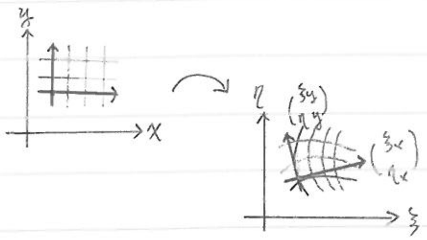
$ab + cd = 0 \Rightarrow ab = -cd$

$b^2 - a^2 + d^2 - c^2 = 0 \Rightarrow b^2 - a^2 = c^2 - d^2$

$\Rightarrow b^4 - 2a^2b^2 + a^4 = c^4 - 2c^2d^2 + d^2$

(since $ab = -cd$) $\Rightarrow b^4 + 2a^2b^2 + a^4 = c^4 + 2c^2d^2 + d^2 \Rightarrow (a^2+b^2)^2 = (c^2+d^2)^2$

$\begin{cases} b^2 + a^2 = c^2 + d^2 \\ b^2 - a^2 = c^2 - d^2 \end{cases} \Rightarrow \begin{cases} b^2 = c^2 \\ a^2 = d^2 \end{cases}$

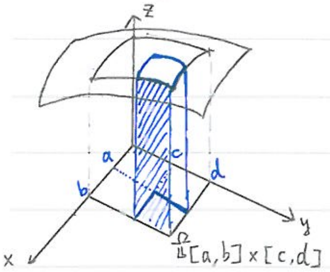


$u = f(\xi, \eta)$ $\qquad \begin{pmatrix} \xi(x,y) \\ \eta(x,y) \end{pmatrix}$

$\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}$ Jaccobian

① $\begin{cases} \xi_x = \eta_y \\ \xi_y = -\eta_x \end{cases}$ $\qquad$ ② $\begin{cases} \xi_x = -\eta_y \\ \xi_y = \eta_x \end{cases}$

§1.8 Integrals of functions with parameters



$\int_c^d f(x,y)\,dy$

parameter

$\to$ Let $f(x,y)$ be continuous

$\boxed{eg}$ $\int_a^b y^{\frac{1}{x}}\,dy = \frac{1}{1/x+1}\, y^{\frac{1}{x}+1} \Big]_a^b$

① $g(x)$ is continuous

$|g(x+h)-g(x)| = \left| \int (f(x+h,y) - f(x,y))\,dy \right| \le \varepsilon\,(c-d)$

uniform continuity for continuous function with motivations

Assume that $f_x(x,y)$ exists and is continuous

$g(x) = \int_c^d f(x,y)\,dy$

$g'(x) \overset{?}{=} \int_c^d f_x(x,y)\,dy$

$\downarrow$

Ans: Yes !

$\boxed{eg}$ $g(k) = \int_0^1 (x-1)\,\frac{x^k}{\log x}\,dx$ $\quad [k>-1,\ k \in \mathbb{R}]$

$\frac{d}{dk}\, g(k) = \int_0^1 \frac{x-1}{\log x}\, \frac{d}{dk}(x^k)\,dx$

$\qquad\qquad \| \ \log x \cdot x^k$

$\downarrow$

$x^k = e^{\log x^k} = e^{k \log x}$

$\boxed{pf}$ $g'(x) = \lim_{h\to 0} \frac{1}{h}\left[g(x+h) - g(x) \right]$

$= \lim_{h\to 0} \frac{1}{h} \int_c^d \underbrace{\left(f(x+h,y) - f(x,y) \right)}_{\| \ h\cdot f_x(x+\theta h,\,y)}\,dy \overset{?}{=} \int_c^d f_x(x,y)\,dy$ (P.75)

$\to \int_c^d f_x(x+\theta h, y) - \int_c^d f_x(x,y)\,dy = \int_c^d \left(f_x(x+\theta h, y) - f_x(x,y) \right) dy$

$\le \varepsilon\,(d-c) \quad$ when $|h| < \delta$ #

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f_x conti , ϕ_1, ϕ_2 diff'ble

$$g(x) = \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy$$

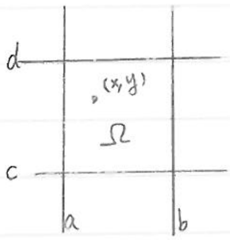
$$g'(x) = f(x,\phi_2(x))\cdot\phi_2'(x) - f(x,\phi_1(x))\cdot\phi_1'(x) + \int_{\phi_1(x)}^{\phi_2(x)} f_x(x,y)\,dy$$

$F(x,u,v) = \int_u^v f(x,y)\,dy$ $\qquad$ $g(x) = F(x,\phi_1(x),\phi_2(x))$ ↓

$$g'(x) = \underbrace{F_x}_{\frac{\partial F}{\partial x}} \frac{dx}{dx} + F_u \frac{du}{dx} + F_v \frac{dv}{dx} = \int_u^v f_x(x,y)\,dy - f(x,u)\cdot\phi_1'(x) + f(x,v)\cdot\phi_2'(x)$$ (p.76)

subsitute $u=\phi_1(x)$, $V=\phi_2(x)$

$$\boxed{\begin{aligned}&\tilde F(x,u) = \int_a^u f(x,y)\,dy\\ &g(x) = \tilde F(x,\phi_2(x)) - \tilde F(x,\phi_1(x))\end{aligned}}$$



$$\int_a^b dx \int_c^d f(x,y)\,dy \overset{?}{=} \int_c^d dy \int_a^b f(x,y)\,dx$$

Ans: Yes, if $f(x,y)$ is continuous <Fubini therrem>

$V(x,y) = \int_c^y f(x,\eta)\,d\eta \rightarrow V_y = f(x,y)$ conti.

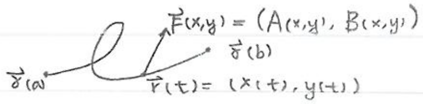
$u(x,y) = \int_a^x V(\xi,y)\,d\xi = \int_a^x d\xi \int_c^y f(\xi,\eta)\,d\eta$

$u_y(x,y) = \int_a^x V_y(\xi,y)\,d\xi = \int_a^x f(\xi,y)\,d\xi = \int_a^x f(\xi,y)\,d\xi$

$\underbrace{u(x,y)}_{\displaystyle \| \atop \displaystyle \int_a^x d\xi \int_c^y f(\xi,\eta)\,d\eta} - \underbrace{u(x,c)}_{0} = \int_c^y d\eta \int_a^x f(\xi,\eta)\,d\xi$ #

§1.9 Differential and Line integrals

<Recall> arc length, work



$$\vec{r}: [a,b] \to \mathbb{R}^2$$

$$\int_a^b \vec{F}\cdot d\vec{r} = \int_a^b \left(A(x,y)\frac{dx}{dt} + B(x,y)\frac{dy}{dt} \right) dt = \int_\Gamma A\,dx + B\,dy$$

$$\vec{r}'(t)dt = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)dt$$

⇒ this is independent of the parameter "t" as long as the orientation of Γ is preserved.

This is a general form of "1-differential form" 一次微分形式

$\boxed{\text{eg}}$ Total differential $df = f_x dx + f_y dy$

* usually (in this book), we denote by Γ^* a curve with a fixed orientation

* Γ with the reverge orientation is denoted by $-\Gamma^*$
$\left(\int_{-\Gamma^*} = -\int_{\Gamma^*} \right)$

Let $L := A\,dx + B\,dy + C\,dz$

A, B, C are functions in $x, y, z \in C^1$

If $L = df$ for some f

i.e. $A = f_x$, $B = f_y$, $C = f_z$ [or equivalently $(A, B, C) = \nabla f$]

then $\int_\Gamma L = \int_\Gamma df = \int_a^b \text{"}\frac{df}{dt}\text{"} dt = f(x(t), y(t), z(t))\Big|_a^b = f(Q) - f(P)$

$$\text{"}\frac{df}{dt}\text{"} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$$

If such an f does exist,

then we must have $A \overset{\text{"}}{=} f_x$, $B \overset{\text{"}}{=} f_y$, $C \overset{\text{"}}{=} f_z$ [$f \in C^2$]

$$\left[\begin{array}{l} A_y = B_x \quad (\text{both} = f_{xy}) \\ B_z = C_y \ ;\ C_x = A_z \end{array} \right] \ \maltese$$

Q: Does condition $\maltese$ imply the existence of f?

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(Examples) ① $L = y\,dx + z\,dy + x\,dz$

$A_y = 1 \neq B_x = 0$

② $L = yz\,dx + zx\,dy + xy\,dz = d(xyz)$; $f = xyz$

$A_y = z = B_x$; —— ; ——

③ Fake total differential

$$\text{“}d\theta\text{”} = d\tan^{-1}(y/x) = \frac{-y\,dx + x\,dy}{x^2+y^2} \longrightarrow \left(\tan^{-1}\left(\frac{y}{x}\right)\right)' = \frac{y'\frac{1}{x} + y\frac{-x'}{x^2}}{1+\left(\frac{y}{x}\right)^2}$$

$$= \frac{-yx' + xy'}{x^2+y^2}$$

$A = \frac{-y}{x^2+y^2}$, $B = \frac{x}{x^2+y^2}$

$$A_y = \frac{-(x^2+y^2)-(-y)2y}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$B_x = \frac{-(-y^2+x^2)}{(x^2+y^2)^2} = A_y$$


Γ^* = unit circle

$(x(t), y(t)) = (\cos\theta, \sin\theta)$
$\langle r=1 \rangle$

but $\int_{\Gamma^*} \frac{-y\,dx + x\,dy}{x^2+y^2}$

$$= \int_0^{2\pi} -\sin\theta(-\sin\theta)\,d\theta + \cos\theta\cos\theta\,d\theta$$

$$= \int_0^{2\pi} d\theta = 2\pi \neq 0$$

* Next time we will show the condition ※ is sufficient if the domain $U \subset \mathbb{R}^2$ of $\vec{F} = (a, b)$ is simply connected (單連通)

(Example 2) $\vec{F}(x,y,z) = \left(\overset{\parallel A}{\frac{x}{\sqrt{x^2+y^2+z^2}}}, \overset{\parallel B}{\frac{y}{\sqrt{x^2+y^2+z^2}}}, \overset{\parallel C}{\frac{z}{\sqrt{x^2+y^2+z^2}}}\right)$

$\overset{\parallel}{\nabla f}$ $\qquad f(x,y,z) = \frac{-1}{\sqrt{x^2+y^2+z^2}}$

$\Rightarrow A_y = B_x$, $B_z = C_y$, $C_x = A_z$

(Example 3) $\mathbb{R}^3 \setminus (0,0,0)$ is simply connected

$\mathbb{R}^3 \setminus \mathbb{R}$ is <u>not</u> simply connected

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$\boxed{\text{Fact}}$: The line integral $\int L$ is independent of path $\Longleftrightarrow$ $L = df$ for some f

$\boxed{\text{pf}}$ " $\Leftarrow$ " ok.

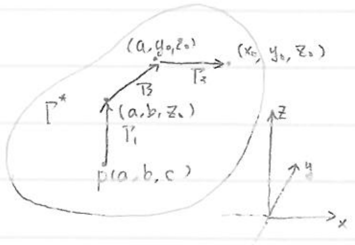
" $\Rightarrow$ " we have to "define f" first

$f(x,y,z) := \int_{\Gamma} L = \int_{\Gamma^*} A\,dx + B\,dy + C\,dz$

$\uparrow$ any piece-wise C' curve connecting P and (x,y,z)

fix $p \in \mathbb{R}^3$

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0,z_0)} = \frac{\partial}{\partial x}\left(\int_{\Gamma_1} + \int_{\Gamma_2} + \int_{(a,y_0,z_0)}^{(x_0,y_0,z_0)} A\,dx + B\,\cancel{dy} + C\,\cancel{dz} \right) = A(x_0,y_0,z_0)$$



similarly, $B = f_y$, $C = f_z$

Line integral

[1-form]

Γ^* oriented curve

$$\int_{\Gamma^*} \underbrace{L}_{A\,dx + B\,dy + C\,dz} = \int_a^b \Big(\underbrace{A}_{A(x(t),\, y(t),\, z(t))} \frac{dx}{dt} + B\frac{dy}{dt} + C\frac{dz}{dt} \Big)\, dt$$

last time: $\int_{\Gamma^*} L$ is independent of path $\Leftrightarrow$ $L = df = f_x dx + f_y dy + f_z dz$

for some f (potantial function)

$\vec{F} = \nabla f$

$f \qquad \nabla f = \text{grad } f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f$

$\vec{F} = (A, B, C) \qquad \text{div } f = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}$ (divergence 散度) $= \nabla \cdot \vec{F}$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A & B & C \end{vmatrix} = (C_y - B_z,\ A_z - C_x,\ B_x - A_y)$$

(旋度)

$? \Uparrow \quad \Downarrow$

compatibility condition. coming from $f_{x_i x_j} = f_{x_j x_i}$

$L = A\,dx + B\,dy + C\,dz \longleftrightarrow \vec{F} = (A, B, C)$

Elie Cartan's d-operator

$\hookrightarrow dA \wedge dx + dB \wedge dy + dC \wedge dz$

$dx\,dy \qquad \Delta x \Delta y$

$dx \wedge dy = -dy \wedge dx$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} b & a \\ d & c \end{vmatrix}$

$$\begin{aligned} &= (\cancel{A_x dx} + A_y dy + A_z dz) \wedge dx \\ &\quad + (B_x dx + \cancel{B_y dy} + B_z dz) \wedge dy \\ &\quad + (C_x dx + C_y dy + \cancel{C_z dz}) \wedge dz \\ &= (B_x - A_y)\, dx \wedge dy + (C_y - B_z)\, dy \wedge dz + (A_z - C_x)\, dz \wedge dx \end{aligned}$$

$L = \sum_{i=1}^{n} A_i\, dx^i \qquad dL = \sum_{i=1}^{n} \underbrace{dA_i}_{} \wedge dx^i = 0$

$$\Big(\sum_{j=1}^{n} \frac{\partial A_i}{\partial x_j} dx^j\Big) \wedge dx^i = \sum_{i<j} \Big(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x_j}\Big) dx^i \wedge dx^j$$

[closed 1-form]

$\boxed{\text{Theorem}}$

Let L be an C^1 one-form defined on a simply connected open set $U \subseteq \mathbb{R}^n$

$dL = 0 \Leftrightarrow L = df$ for some f

closed 1-form exact.

(Recall) $\Leftarrow$ trivial

$\Rightarrow$ is reduced to prove that the line integral is independent of path.

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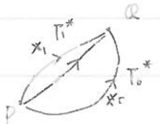
<Reality>

Simply connectedness 單連通



$x_0 : [0,1] \to \mathbb{R}^3$ $\quad x_0(0) = P,\ x_0(1) = Q$

$x_1 : [0,1] \to \mathbb{R}^3$ $\quad x_1(0) = P,\ x_1(1) = Q$

i.e. $x\ \underset{t}{[0,1]} \times \underset{s}{[0,1]} \to U$ conti.

Def'n: U is $\overset{\text{(simply)}}{1}$-connected if any 2 curves with the same end points can be deformed to each other continuously.

$x(t,s)$ s.t. $x(t,0) = x_0(t)$, $x(t,1) = x_1(t)$

Ball: $B_p(r)$

$x(t,s) = (1-s)\,x_0(t) + s\,x_1(t)$

Fact: Any convex set is 1-connected

$$\int_{\Gamma_1^*} L - \int_{\Gamma_0^*} L = \int_0^1 \left(A\frac{\partial x}{\partial t} + B\frac{\partial y}{\partial t} + C\frac{\partial z}{\partial t}\right)\Big|_{s=1} dt + \int_0^1 \left(A\frac{\partial x}{\partial t} + B\frac{\partial y}{\partial t} + C\frac{\partial z}{\partial t}\right)\Big|_{s=0} dt$$

$$A\frac{\partial x}{\partial t}\Big|_{s=1} - A\frac{\partial x}{\partial t}\Big|_{s=0} = \int_0^1 \frac{\partial}{\partial s}\left(A\frac{\partial x}{\partial t}\right) ds$$

$$\longrightarrow \left(A_x\frac{\partial x}{\partial s} + A_y\frac{\partial y}{\partial s} + A_z\frac{\partial z}{\partial s}\right)\frac{\partial x}{\partial t} + A\frac{\partial^2 x}{\partial s\partial t}$$

$$\int_0^1 dt \int_0^1 \boxed{\begin{array}{l}(A_x x_s + A_y y_s + A_z z_s)x_t + A x_{st} \\ +(B_x x_s + B_y y_s + B_z z_s)y_t + B y_{st} \\ +(C_x x_s + C_y y_s + C_z z_s)z_t + C z_{st}\end{array}}\ ds$$

$$\overset{\shortparallel}{(A x_s + B y_s + C z_s)_t} \qquad B_x = A_y,\ B_z = C_y,\ A_z = C_x$$

$$= \int_0^1 ds \int_0^1 (A x_s + B y_s + C z_s)_t\, dt$$

$$= \int_0^1 ds\, (A x_s + B y_s + C z_s)\Big|_{t=1} - \Big|_{t=0} = 0$$

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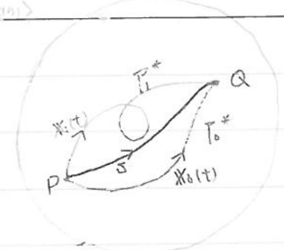
<Dream>



$u \subset \mathbb{R}^n$ open set simply connected

in $\mathbb{R}^n \to (x^1, x^2, \dots, x^n)$

$\int_{\Gamma_1^*} L$

$$L = \sum_{i=1}^{n} A_i\, dx^i$$

$$dL = \sum_{i=1}^{n} dA_i \wedge dx^i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial A_i}{\partial x^j} dx^j \right) \wedge dx^i$$

$$= \sum_{j<i} \left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \right) dx^i \wedge dx^j$$

The nececary condition for the line integral to be independent of path

is $dL = 0 \Rightarrow \dfrac{\partial A_j}{\partial x^i} = \dfrac{\partial A_i}{\partial x^j}$

$\dfrac{\partial^2 f}{\partial x^i \partial x^j}$ [indep. of path $\equiv \exists f$ s.t. $L = df$] $\leftarrow$ i.e. $A_i = \dfrac{\partial f}{\partial x^i} = f_i$

$(*)$: $\int_{\Gamma_1^*} L = \int_{\Gamma_0^*} L$

$\exists\ X : \overset{\ni t}{[0,1]} \times \overset{\ni s}{[0,1]} \xrightarrow{\text{conti}} U$

$X(t,s) = (x^1(t,s), x^2(t,s), \dots, x^n(t,s))$

s.t. $X(t,0) = X_0(t)$, $X(t,1) = X_1(t)$

$$(*) = \int_0^1 dt \sum_{i=1}^{n} \Big(\underbrace{A_i(X) \frac{\partial X}{\partial t}\Big|_{s=1}}_{\frac{dX_1}{dt}} - \underbrace{A_i(X) \frac{\partial X}{\partial t}\Big|_{s=0}}_{\frac{dX_0}{dt}} \Big)$$

$$= \int_0^1 dt \int_0^1 ds \sum_{i=1}^{n} \frac{\partial}{\partial s}\left(A_i \frac{\partial x^i}{\partial t}\right)$$

$$\frac{\partial A_i}{\partial x^j} \frac{\partial x^j}{\partial s} \frac{\partial x^i}{\partial t} + A_i \frac{\partial^2 x^i}{\partial t \partial s}$$

$$= \int_0^1 dt \int_0^1 ds \sum_{i=1}^{n} \frac{\partial}{\partial t}\left(A_i \frac{\partial x^i}{\partial s}\right)$$

$$= \int_0^1 ds \int_0^1 dt \sum_{i=1}^{n} \frac{\partial}{\partial t}\left(A_i \frac{\partial x^i}{\partial s}\right)$$

$$= \int_0^1 ds \left(\sum_{i=1}^{n} A_i \frac{\partial x^i}{\partial s} \Big|_{t=0}^{t=1} \right) = 0$$

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$\mathbb{X}(t,s) = (x^i(t,s))_{i=1}^n$

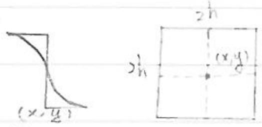
$\uparrow$ continuous in t, s

problem: $f(x,y)$ conti. in x, y

we want to approximate f by a C^2 function [h fixed small number]

$f_h(x,y) = \frac{1}{4h^2}\int_{x-h}^{x+h} d\xi \int_{y-h}^{y+h} d\eta\ f(\xi,\eta)$

$\boxed{\text{blur}}$



$\Rightarrow$ $\frac{\partial}{\partial x} f_h$, $\frac{\partial}{\partial y} f_h$, $\frac{\partial^2 f_h}{\partial x \partial y} = \frac{\partial^2 f_h}{\partial y \partial x}$ $\rightarrow$ good

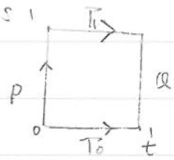
check the text book $\rightarrow$ $= [u(x+h, y+h) - u(x+h, y-h) - u(x-h, y+h) + u(x-h, y-h)]/4h^2$

$|f_h(x,y) - f(x,y)| = \frac{1}{4h^2}\int_{x-h}^{x+h}\left\{\int_{y-h}^{y+h}\underbrace{(f(\xi,\eta) - f(x,y))}_{(if\ <\ \varepsilon)} d\eta\right\} d\xi < \varepsilon$

"Homotopy"

$x^i(t,s)$ is now approximated by $\bar{x}^i(t,s)$, $i = 1, 2, \dots, n$

$\uparrow$ conti. $\qquad$ $\uparrow$ good function $\hookrightarrow U$



$\tilde{x}(t,s) = \bar{x}(t,s) - (1-s)(\bar{x}(t,s) - x_0(t)) - s(\bar{x}(t,s) - x_1(t))$

$- (1-t)(\bar{x}(t,s) - x_0(0)) - t(\bar{x}(t,s) - \underset{x_0(1)}{\underset{\shortparallel}{x_1(1)}})$

$+ (1-t)(1-s)(\bar{x}(0,0) - x_0(0))$

$+ (1-t)s(\bar{x}(0,1) - x_1(0))$

$+ (1-s)t(\bar{x}(1,0) - x_0(1))$

$+ st(\bar{x}(1,1) - x_1(1))$

Double A

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§ Appendix

$\mathbb{R}^1$ $\qquad$ $\mathbb{R}^n$

$[a,b]$ $\qquad$ $[a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n]$

closed and bounded subset

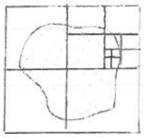
- bounded means $S \subset B_0(R)$ for some R
- closed means S contains all its boundary point ∂S

Theorem (Bolzano – Weirstrass)

Let $S \subset \mathbb{R}^n$ be a closed and bounded set,

then any sequence $P_n \in S$ $(n=1,2,\dots)$ has a convergent subspace in S

pf.



devided into 2^n subcube

$\exists$ a subcube which still contain $\frac{1}{2}\infty$-many

Def'n: A set $S \subset \mathbb{R}^n$ is called (sequentially) compact iff it is closed and bounded

$\boxed{\text{Thm}}$ Let $f: \underset{\mathbb{R}^n}{\underset{\cap}{S}} \to \mathbb{R}$ be continuous with S being compact,

then $\exists\, p \in S$ s.t. $f(p) = \max f$

pf.: claim $\exists M$ s.t. $f \le M$,

if not, $\forall n \in \mathbb{N}$, $\exists P_n \in S$ s.t. $f(p_n) > n$

Let P_{n_i} be a convergent subsequence $\lim P_{n_i} = q \in S$

$f(P_{n_i}) > n_i$

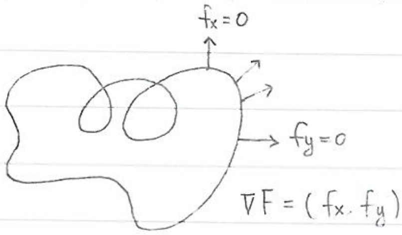
$\lim_{i\to\infty} f(P_{n_i}) \to \infty$ $\longrightarrow\!\!\!\times\!\!\!-$

$\quad \| $

$f(\lim_{i\to\infty} P_{n_i}) = f(q)$

Double A

Implicit functions 隱函數



(Example)

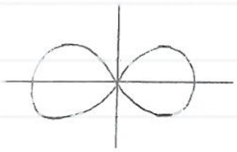
$F(x,y) \to (x^2+y^2)^2 - 2x^2(x^2-y^2) = 0$

[find the maximal value of y,
[regard y is a function in x, "$y = f(x)$"

Apply $\frac{d}{dx}$ to it: $2(x^2+y^2)(2x+2yf') - 2a^2(2x-2yf') = 0$

$f' = \ldots\ldots$, Here we consider $f(x) = 0$

$\Rightarrow (x^2+y^2)2x - a^2 - 2x = 0$



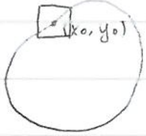
$2x(x^2+y^2-a^2) = 0 \longrightarrow x = 0$ or $x^2+y^2 = a^2$

$\Downarrow$ $y^4 - 2a^2y^2 = 0$ $\quad$ $\Downarrow$ $a^4 - 2a^2(a^2-2y^2) = 0$

(Inverse)
Theorem (Implicit function Theorem)

Let $F(x,y)$ be C^1, and $F(x_0,y_0) = 0$, $F_y(x_0,y_0) = m \neq 0$,

then $\exists$ nbd of (x_0,y_0) s.t. $\exists!\ y = f(x)$ s.t. $F(x,f(x)) = 0$ and $f \in C^1$

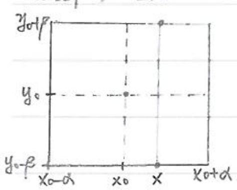


$F(x,y) = 0$ $\qquad$ $F_x + F_y \frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y}$$

$\boxed{pf}$:

<step1> "$\exists!$"



$\exists R = [x_0-\alpha, x_0+\alpha] \times [y_0-\beta, y_0+\beta]$

s.t. $F_y > \frac{m}{2}$ on R and let $|F_x| \le M$ on R

Notice that $F(x,y) \nearrow$ in y for any fixed x.

$|F(x,y_0) - \underbrace{F(x_0,y_0)}_{0}| = |F_x(\xi, \xi_0)| \cdot |x-x_0| \le M|x-x_0|$

$$F(x,y_0+\beta) = (\underbrace{F(x,y_0+\beta) - F(x,y_0)}_{=F_y(x,\eta)\beta}) + F(x,y_0) > \frac{m\beta}{2} - M|x-x_0| \rightsquigarrow > 0$$

$$F(x,y_0-\beta) = (\underbrace{F(x,y_0-\beta) - F(x,y_0)}_{=F_y(x,\eta')\cdot(-\beta)}) + F(x,y_0) < -\frac{m\beta}{2} + M|x-x_0| \rightsquigarrow < 0$$

we just require that $|x-x_0| < \frac{m\beta}{2M} =: \delta$

$\Rightarrow \exists!\ y$ s.t. $F(x,y) = 0$

call this $x \mapsto y$ by $y = f(x)$

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<step2> "$f \in C^1$" Let x be fixed

$f(x+h) - \overset{y}{f(x)} = k$

$0 = F(x+h, \underbrace{f(x+h)}_{y+k}) - F(x, \underbrace{f(x)}_{y}) = \underbrace{F_x}_{\le M|h|}(x+\theta h, y+\overset{(0,1)}{\theta} k)h + \underbrace{F_y}_{> \frac{m}{2}}(x+\theta h, y+\theta k)k$

$\Rightarrow |k| \le \frac{2M}{m}|h| \Rightarrow f$ is Lip. continuous

$\frac{f(x+h)-f(x)}{h} = \frac{k}{h} = -\frac{F_x(x+\theta h, y+\theta k)}{F_y(x+\theta h, y+\theta k)}$

$\swarrow$

$-\frac{F_x(x,y)}{F_y(x,y)}$ # Q.E.D.

$\boxed{\text{Multi-variable case}}$

$F(x^1, x^2, \ldots, x^n, y)$ s.t. $F(\vec{x}_0, y_0) = 0$; $F_y(\vec{x}_0, y) = m > 0$

<step1> change x into $\vec{x}$

<step2> $0 = F(\vec{x}+\vec{h}, f(\vec{x}+\vec{h})) - F(\vec{x}, f(\vec{x}))$

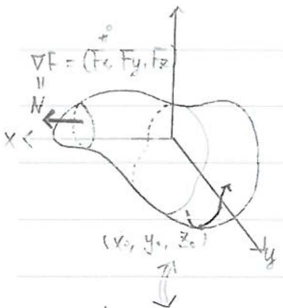
$= \sum_{i=1}^{n} F_{x^i}(\vec{x}+\theta\vec{h}, y+\theta k)h^i + F_y(\vec{x}+\theta\vec{h}, y+\theta k)k$

$\boxed{\frac{\partial f}{\partial x^i} = \frac{F_{x^i}}{F_y}}$ $f(x,y) = 0$, $f' = -\frac{F_x}{F_y}$, $f'' = \frac{F_{xx}F_y + F_y^2(-\frac{F_x}{F_y}) - F_xF_{yx} + F_{yy}(-\frac{F_x}{F_y})}{-F_y^2}$

Q.E.D.

$u = F(x, y, z)$

level set , $u = \text{constant} = 0$



$\subset \mathbb{R}^3$ "surface"

$F(x_0, y_0, z_0) = 0$, $F_x(x_0, y_0, z_0) \neq 0$

$\Rightarrow \exists!$ implicit function $x = f(y, z)$ near (x_0, y_0, z_0)

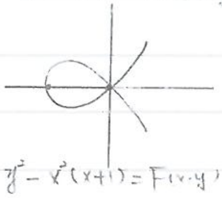
s.t. $(f(y, z), y, z) = 0$

* What happens if $\nabla f(x_0, y_0, z_0) = 0$

consider any curve through (x_0, y_0, z_0) on $F = 0$

$F(x(t), y(t), z(t)) = 0$

$0 = \nabla F \cdot (x'(t), y'(t), z'(t)) \big|_{t=0}$

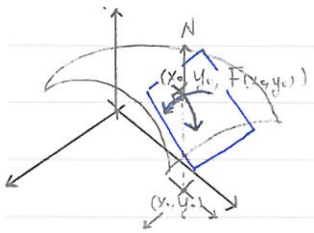


Def'n : A point $p \in \{\vec{x} \mid F(\vec{x}) = 0\}$ is a singular (奇異) point if the tangent vectors of p span the whole space ($\Leftrightarrow \nabla F(p) = 0$)

$y^2 - x^2(x+1) = F(x, y)$

$\nabla F = (-3x^2, 2y)$

(Example 1) : $z = f(x, y)$



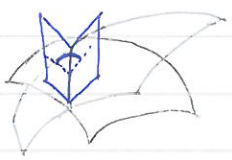
① By definition $z - z_0 = f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0)$

② $(x, y, f(x, y))$ $\frac{\partial}{\partial x}$ $(1, 0, f_x)$, $\frac{\partial}{\partial y}$ $(0, 1, f_y)$ $\Big\rangle$ $(-f_x, -f_y, 1)$

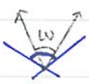
$(x - x_0, y - y_0, z - z_0) \cdot (-f_x, -f_y, 1) = 0$

③ $F(x, y, z) := z - f(x, y) = 0$; $F = 0$ level set $\Rightarrow \nabla F = (-f_x, -f_y, 1)$

(Example 2) $F = 0$, $G = 0$ in $\mathbb{R}^3$



$\nabla F = N_1$ $\nabla G = N_2$



$$\cos w = \frac{\nabla F \cdot \nabla G}{|\nabla F| \cdot |\nabla G|} = \frac{F_x G_x + F_y G_y + F_z G_z}{\sqrt{F_x^2 + F_y^2 + F_z^2} \cdot \sqrt{G_x^2 + G_y^2 + G_z^2}}$$

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$\boxed{\text{Trivial linear model}}$

$u = a_1x_1 + a_2x_2 + \cdots + a_nx_n + by$

when $b \neq 0$, $u = 0$, $u \leftrightarrow y$

$y = -\frac{a_1}{b}x_1 - \frac{a_2}{b}x_2 - \cdots\cdots - \frac{a_n}{b}x_n + \frac{u}{b}$

☆ $\boxed{\text{Non-linear version}}$

$u = F(\vec{x}, y)$ for a fixed $(\vec{x}_0, y_0)$, $\frac{\partial u}{\partial y} \equiv F_y \neq 0$

$\Rightarrow y = G(\vec{x}, u)$ s.t. $u = F(\vec{x}, G(\vec{x}, u))$

<Cor.> set $u=0$, get $y = g(\vec{x})$ implicit function

Actually, ☆ follows from the $u=0$ case

$\boxed{\text{pf}}$ Let $H(\vec{x}, u, y) := u - F(\vec{x}, y)$

$H_y = -F_y \neq 0 \quad \Rightarrow y = G(\vec{x}, u)$

s.t. $\underbrace{H(\vec{x}, u, G(\vec{x}, u))}_{u - F(\vec{x}, G(\vec{x}, u))} = 0$ ※Q.E.D.

$\boxed{\text{Theorem}}$ General form of IFT (implicit (inverse) function theorem)

$u = F(\vec{x}, y, z)$

$v = G(\vec{x}, y, z)$

with $D := \begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix} \neq 0$ at $(\vec{x}_0, y_0, z_0)$

Then in a nbd. of $(\vec{x}_0, y_0, z_0)$, $\exists\ y = A(\vec{x}, u, v)$, $z = B(\vec{x}, u, v)$

s.t. $u = F(\vec{x}, A(\vec{x}, u, v), B(\vec{x}, u, v))$

$v = G(\vec{x}, A(\vec{x}, u, v), B(\vec{x}, u, v))$

$\boxed{\text{pf}}$ May assume that $F_y \neq 0$ (otherwise change $y \leftrightarrow z$)

$\frac{\partial u}{\partial y} = F_y \neq 0 \overset{☆}{\Rightarrow} \exists\ y = \Phi(\vec{x}, u, z)$ s.t. $u = F(\vec{x}, \Phi(x, u, z), z)$

Now, the variables are u, z, and $\vec{x}$

$V = G(\vec{x}, \overbrace{\Phi(\vec{x}, u, z)}^{y}, z)$, Need to compute $\frac{\partial V}{\partial z} = G_y\Phi_z + G_z$

$0 = \frac{\partial u}{\partial z} = F_y\Phi_z + F_z \Rightarrow \frac{\partial V}{\partial z} \neq 0 \Rightarrow \exists\ z = \Psi(\vec{x}, u, v)$

s.t. $u = F(\vec{x}, \Phi(x, u, \Psi(\vec{x}, u, v)), \Psi(\vec{x}, u, v))$

$v = G(\vec{x}, \Phi(x, u, \Psi(\vec{x}, u, v)), \Psi(\vec{x}, u, v))$

for any $(\vec{x}, u, v)$ in the nbd.

<2 special case>

(I.) set $u = v = 0$, get $y(\vec{x}), z(\vec{x})$

s.t. $F(\vec{x}, y(\vec{x}), z(\vec{x})) = 0$

$G(\vec{x}, y(\vec{x}), z(\vec{x})) = 0$

(II.) let $n = 0$ (i.e. no $\vec{x}$) ($x^1, x^2, \ldots, x^n$)

$u = F(y, z) \quad v = G(y, z)$

if $D = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} \neq 0$ $\longrightarrow$ $u = F(\overset{y}{A(u,v)}, \overset{z}{B(u,v)})$

$v = G(A(u,v), B(u,v))$

$\begin{pmatrix} y \\ z \end{pmatrix} \underset{A.B}{\overset{F.G}{\rightleftarrows}} \begin{pmatrix} u \\ v \end{pmatrix}$

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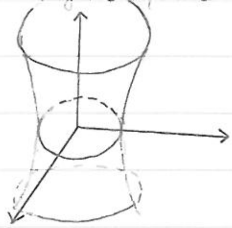
(Eg) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

$N = \nabla F = \left(\frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{-2z_0}{c^2} \right)$ at P

tangent plane at $P = (x_0, y_0, z_0)$



$N_P \cdot (x - x_0, y - y_0, z - z_0) = 0$

$\Rightarrow \frac{x_0(x - x_0)}{a^2} + \frac{y_0(y - y_0)}{b^2} - \frac{z_0(z - z_0)}{c^2} = 0$

$\Rightarrow \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - \frac{z_0 z}{c^2} = 1$

$u = F(x, y, z)$, $v = G(x, y, z)$, C^1

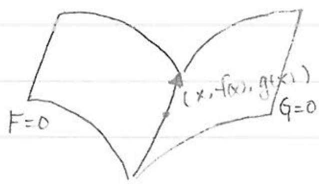
condition $D = \begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix} \neq 0$ at (x_0, y_0, z_0)

(Eg). $u = v = 0$

$\Rightarrow y = f(x, u, v)$ $z = g(x, u, v)$ $\Rightarrow$ get $y = f(x)$, $z = g(x)$

s.t. $F(x, f(x), g(x)) = 0$

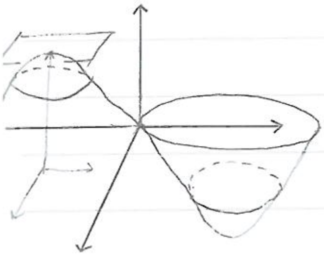
$G(x, f(x), g(x)) = 0$



§ 3.7 Maxima and minima problem

$y = f(x)$, $f'(x) = 0$.

(Eg.) $f(x,y) = (ax^2+by^2)\,e^{-(x^2+y^2)}$ $[a, b \neq 0]$



(x_0, y_0) is an extremal point

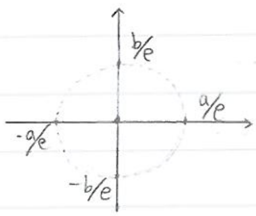
$\Rightarrow f_x(x_0,y_0) = f_y(x_0,y_0) = 0$

$\nabla f = 0 \leftarrow N = (-\nabla f, 1)$

$f_x = 2ax \cdot e^{-(x^2+y^2)} + (ax^2+by^2)\,e^{-(x^2+y^2)}(-2x) = e^{-(x^2+y^2)} \cdot 2x(a-ax^2-by^2)$

$f_y = 2by \cdot e^{-(x^2+y^2)} + (ax^2+by^2)\,e^{-(x^2+y^2)}(-2y) = e^{-(x^2+y^2)}\, 2y(b-ax^2-by^2)$

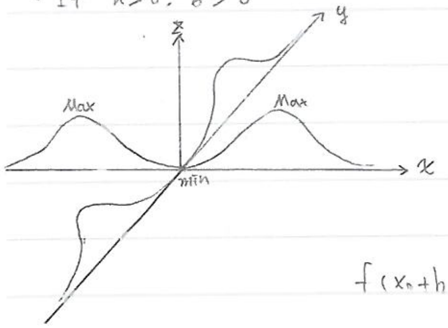
$$\nabla f = 0 \Leftrightarrow \begin{cases} x = 0 & \begin{cases} y = 0 \\ y \neq 0 \Rightarrow b(1-y^2) = 0 \Rightarrow y = \pm 1 \end{cases} \\ x \neq 0 \Rightarrow a - ax^2 - by^2 = 0 & \begin{cases} y = 0 \Rightarrow x = \pm 1 \\ y \neq 0 \Rightarrow \begin{cases} b - ax^2 - by^2 = 0 \\ a - ax^2 - by^2 = 0 \end{cases} \end{cases} \end{cases}$$



$\Downarrow$

$a = b$ $\begin{cases} \text{if } a \neq b, \text{ no such case} \\ \text{if } a = b, \ x^2 + y^2 = 1 \end{cases}$ $[f(x,y) = ar^2 e^{-r^2}]$

- If $a > 0$, $b > 0$



Taylar expansion

$f(x,y) = f(x_0,y_0) + \frac{1}{2}\left[f_{xx}(x-x_0)^2 + 2f_{xy}(x-x_0)(y-y_0) + f_{yy}(y-y_0)^2\right]$

$f(x_0+h, y_0+k) = f(x_0,y_0) + \frac{1}{2}(h,k)\begin{pmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} + \cdots\cdots$

$\vec{x}^t A \vec{x}$ quadratic form

symmetic matrix at (x_0, y_0)

$\|$

$\vec{y}^t \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \vec{y}$ ▨

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Next, we consider min/max problem with side conditions.

$\boxed{\text{eg.}}$ $\frac{x^2+y^2+z^2}{3} \geq \sqrt[3]{x^2y^2z^2}$

Think as the min/max problem for $f(x,y,z)=x^2+y^2+z^2$ under $x^2+y^2+z^2=C$

* Lagrange multiplier (乘子)

$u = f(x,y,z)$

$v = g(x,y,z) = \text{constant}$

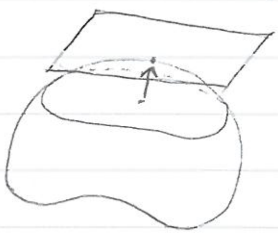
$\nabla f = \lambda \nabla g$ for some λ $\quad \left[\begin{array}{l} (f_x, f_y, f_z) = \lambda(g_x, g_y, g_z) \\ g = \text{constant} \end{array}\right.$

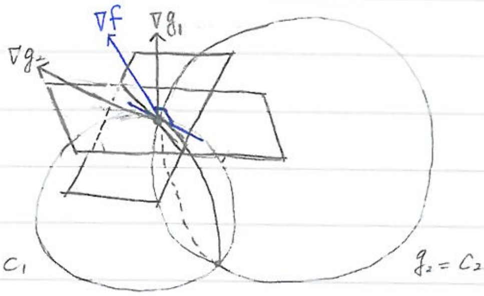
(f_x, f_y, f_z)

$u = f(\vec{x}, y, z)$

$g_1(\vec{x}, y, z) = 0, \quad g_2(\vec{x}, y, z) = 0$

$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \quad \left\{\begin{array}{l} \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 = 0 \\ g_2 = 0 \end{array}\right.$





$u = f(x,y,z) \qquad = f(x, y, h(x,y))$

$h(x,y)$

$v = g(x,y,z) = \text{constant}$

$\left[\begin{array}{l} 0 = \frac{\partial u}{\partial x} = f_x + f_z h_x \\ 0 = \frac{\partial u}{\partial y} = f_y + f_z h_y \end{array}\right. \quad \Bigg\| \quad \begin{array}{l} g_x + g_z h_x = 0 \\ g_y + g_z h_y = 0 \end{array}$

$u = f(\vec{x}, h(\vec{x}), k(\vec{x})) \quad \left\{\begin{array}{l} g_1(\vec{x}, h(\vec{x}), k(\vec{x})) = 0 \\ g_2(\vec{x}, h(\vec{x}), k(\vec{x})) = 0 \end{array}\right.$

$\left[\begin{array}{l} 0 = \frac{\partial u}{\partial x^i} = f_{x^i} + f_y h_{x^i} + f_z k_{x^i} \\ \qquad \forall\, i = 1, 2, \cdots, n \\ 0 = g_{1x^i} + g_{1y} h_{x^i} + g_{1z} h_{x^i} \\ 0 = g_{2x_i} + g_{2y} h_{x^i} + g_{2z} h_{x^i} \end{array}\right.$

(Ex1) $\dfrac{x_1+x_2+\cdots+x_n}{n} \geq \sqrt[n]{x_1x_2\cdots x_n}$

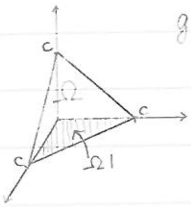
method① Let $x_1+\cdots+x_n = c > 0$ fixed constant

$f(x_1,\dots,x_n) = x_1\cdot x_2\cdot\cdots\cdot x_n$

equiv. to concider $g(x_1,\dots,x_{n-1}) = x_1\cdot\cdots\cdot x_{n-1}\cdot(c-x_1-x_2-\cdots-x_{n-1})$

for simplicity we write out the case $n=3$

$g(x_1,x_2) = x_1x_2(c-x_1-x_2)$



$x_1+x_2+x_3 = c \qquad x_i \geq 0$

$f\equiv 0$ on $\partial\Omega$, the maxima of f exist and is in
the maxima of f exists and is in the interior of Ω

$g\equiv 0$ on $\partial\Omega_1$,
the maxima of g exists and is in the interior of Ω_1

$\dfrac{\partial g}{\partial x_1} = x_2(c-x_1-x_2)-x_1x_2 = x_2(c-2x_1-x_2) \overset{\checkmark}{=} 0$

$\dfrac{\partial g}{\partial x_2} = x_1(c-x_1-2x_2) \overset{\checkmark}{=} 0$

$\Rightarrow \begin{cases} 2x_1+x_2 = c \\ x_1+2x_2 = c \end{cases} \Rightarrow x_1 = x_2$ in this case ※

method② Let $h(x) = x_1+x_2+\cdots+x_n - c$

under $h(x)=0$, solve maxima of f

$\left(\frac{f}{x_1},\cdots,\frac{f}{x_n}\right) = \nabla f = \lambda\nabla h = (1,1,\dots,1) \Rightarrow x_1=\cdots=x_n$ ※

(Ex2) Hölder inequality

$uv \leq \frac{1}{\alpha}u^\alpha + \frac{1}{\beta}v^\beta \leftarrow f(u,v) \qquad \langle u,v\geq 0,\ \frac{1}{\alpha}+\frac{1}{\beta}=1\rangle$

fix $h(u,v) = uv = c > 0$

$(u^{\alpha-1}, v^{\beta-1}) = \underline{\nabla f} = \lambda\underline{\nabla h} = (v,u) \Rightarrow u^\alpha = v^\beta$

$v^\beta = u^\alpha = (\lambda v)u = \lambda uv = \lambda c \quad \begin{cases} u = (\lambda c)^{1/\alpha} \\ v = (\lambda c)^{1/\beta} \end{cases} \begin{array}{l} \Rightarrow \overset{c}{\overset{\shortparallel}{uv}} = (\lambda c)^{\frac{1}{\alpha}+\frac{1}{\beta}} = \lambda c \\ \Rightarrow \lambda = 1 \rightarrow (\frac{1}{\alpha}+\frac{1}{\beta})\lambda c = c \end{array}$

"=" holds

[general form] $\sum_{i=1}^{n} u_iv_i = \big(\underbrace{\sum u_i^\alpha}_{A}\big)^{1/\alpha}\cdot\big(\underbrace{\sum v_i^\beta}_{B}\big)^{1/\beta}$ ※ $\alpha=\beta=2 \Rightarrow$ cauchy

[pf] Let $u = \frac{u_i}{A}$, $v = \frac{v_i}{B}$

$\dfrac{u_iv_i}{AB} = uv \leq \dfrac{1}{\alpha}\dfrac{u_i^\alpha}{\sum u_i^\alpha} + \dfrac{1}{\beta}\dfrac{v_i^\beta}{\sum v_i^\beta}$

"$\sum_{i=1}^{n}$" $\longmapsto \dfrac{\sum_{i=1}^{n} u_iv_i}{AB} \leq \dfrac{1}{\alpha}+\dfrac{1}{\beta} = 1$ ※ Q.E.D.

(Ex 3) More constanaints

$u = f(\vec{x}, y, z)$ $\qquad g(\vec{x}, y, z) = 0$, $h(\vec{x}, y, z) = 0$

assume that $\begin{vmatrix} g_y & g_z \\ h_y & h_z \end{vmatrix} \neq 0$ at the extremed point

$\Rightarrow$ get $y = A(x)$, $z = B(x)$

$$\Rightarrow \quad 0 = \frac{\partial u}{\partial x_i} = f_{x^i} + f_y A_{x^i} + f_z B_{x^i} \qquad \forall i = 1, 2, \dots, n$$
$$0 = g_{x^i} + g_y A_{x^i} + g_z B_{x^i}$$
$$0 = h_{x^i} + h_y A_{x^i} + h_z B_{x^i}$$

$\Rightarrow \nabla f - \lambda_1 \nabla g + \lambda_2 \nabla h$

$$\begin{pmatrix} f_{x^i} & f_y & f_z \\ g_{x^i} & g_y & g_z \\ h_{x^i} & h_y & h_z \end{pmatrix} \overset{\neq 0}{\begin{pmatrix} 1 \\ A_{x^i} \\ B_{x^i} \end{pmatrix}} = 0$$

$\Rightarrow \det(\quad) = 0 \quad \Rightarrow \begin{cases} (f_x, f_y, f_z)(g_x, g_y, g_z)(h_x, h_y, h_z) \text{ linear dependent} \\ g, h \text{ linear independent } \left(\because \begin{vmatrix} g_y & g_z \\ h_y & h_z \end{vmatrix} \neq 0\right) \end{cases}$

$\Rightarrow (f_{x^i}, f_y, f_z) = \lambda_1 (g_{x^i}, g_y, g_z) + \lambda_2 (h_{x^i}, h_y, h_z)$

λ_1, λ_2 are uniquely determined by the (y, z) components

(indep of $i = 1, 2, \dots, n$)

$\Rightarrow \nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h$ #

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§3.3 systems of functions, transformation and mappings

we had proved the inverse function theorem by a composition of <u>primitive</u> mappings.

i.e. replace one variable each time

$\mathbb{R}^2 \to \mathbb{R}^2$

$F = \begin{pmatrix} f \\ g \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$

under the assumption that

F is C^1 and $\begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} \neq 0$ at (x_0, y_0)

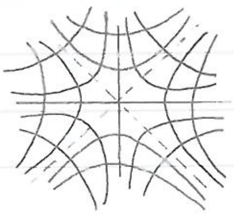
(Ex 1) curvilinear coordinates

$$\xi = f(x,y) = x^2 - y^2$$
$$\eta = g(x,y) = 2xy$$

$D \equiv J \equiv \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2) \neq 0$ unless $(x,y) = (0,0)$

Jacobian

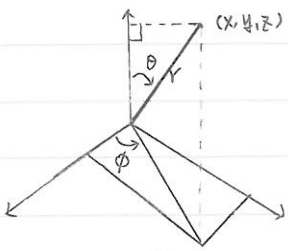
$(x,y)\ (-x,-y)$ give the same (ξ, η)



$\xi + i\eta = x^2 - y^2 + 2ixy = (x+iy)^2 = z^2$

$\underset{(u,v)}{\mathbb{R}^2} \xrightarrow{\mathbb{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}} \underset{(x,y,z)}{\mathbb{R}^3} \qquad \mathbb{X}(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$

(Ex 2) spherical coordinates



$r = \sqrt{x^2+y^2+z^2}$	$x = r\sin\theta\cos\phi$
$\theta =$	$y = r\sin\theta\sin\phi$
$\phi =$	$z = r\cos\theta$

$\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} \overset{\mathbb{X}}{\longmapsto} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$J = \det[\mathbb{X}_r, \mathbb{X}_\theta, \mathbb{X}_\phi]$$
$$= \begin{vmatrix} \sin\theta\cdot\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\cdot\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

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matrix notation & chain rule

$$\begin{cases} dx = x_u du + x_v dv \\ dy = y_u du + y_v dv \end{cases}$$

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}$$

$$d\mathbf{x} = \underbrace{[\mathbf{x}_u, \mathbf{x}_v]}_{\mathbf{x}' = D\mathbf{x}} \, d\underbrace{\vec{u}}_{\binom{u}{v}}$$

$$\mathbf{x}(p+\vec{h}) - \mathbf{x}(p) = \underbrace{A}_{\mathbf{x}'(p)} \vec{h} + o(|\vec{h}|)$$

$$\overset{\vec{x}}{\begin{pmatrix} x \\ y \end{pmatrix}} \xrightarrow[G \, \binom{\phi}{\psi}]{} \overset{\vec{u}}{\begin{pmatrix} u \\ v \end{pmatrix}} \xrightarrow[F \, \binom{\Phi}{\Psi}]{} \overset{\vec{\xi}}{\begin{pmatrix} \xi \\ \eta \end{pmatrix}}$$

$$d\vec{u} = G' dx$$

$$d\vec{\xi} = F' du = F' G' d\vec{x} \overset{\text{By Def.}}{=} (F \circ G)' d\vec{x}$$

chain rule: $(F \circ G)'(p) = F'(G(p)) \, G'(p)$

If $F = G^{-1}$ i.e. $F \circ G = id.$ $[\, (id)' = id \,]$

$$F'(G(\vec{x})) \cdot G'(\vec{x}) = I_n$$

(Ex I)

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \frac{1}{D} \begin{pmatrix} v_y & -u_y \\ -u_x & u_x \end{pmatrix}$$

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§ 3.3 continued

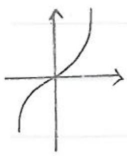
Dependent functions

$$D = \begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix} = 0$$

(IFT)

$D \neq 0$ at $(x_0, y_0) \Rightarrow \exists$ inverse locally near (x_0, y_0)

$\boxed{\text{eg1}}$ $y = f(x) = x^3$



$\frac{dy}{dx} = 3x^2 = 0$ at $x=0$

$x = y^{1/3}$ still exists, though it's not differentiable

$u = x^3$
$v = y$

$$D = \begin{vmatrix} 3x^2 & 0 \\ 0 & 1 \end{vmatrix} = 3x^2$$

$D = 0$ along the y-axis

$\boxed{\text{eg2}}$

$$\begin{cases} u = x + y + z \\ v = x^2 + y^2 + z^2 \\ w = xy + yz + zx \end{cases} \Rightarrow v + 2w = u^2$$

$$D = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix} = 0$$

(condition):

If $u = \phi(x,y)$, $v = \psi(x,y)$ satisfies $\begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix} \equiv 0$

If $\phi_x \equiv 0 \equiv \phi_y$, then ϕ = constant

otherwise, we may assume that $\phi_x \neq 0$ in $u \to (x_0, y_0)$

Then, we may solve $x = \Phi(u, y)$ s.t. $u = \phi(\Phi(x,y), y)$ for any (u, y)

$v = \psi(\Phi(x,y), y) = \chi(u,y)$

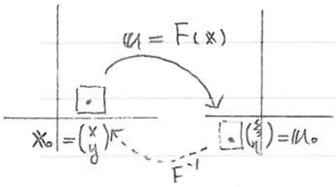
$$\frac{\partial v}{\partial y} = \psi_x \Phi_y + \psi_y = \psi_x \left(-\frac{\phi_y}{\phi_x}\right) + \psi_y = \frac{D}{\phi_x} = 0$$

$$= \frac{\partial u}{\partial y} = \phi_x \Phi_y + \phi_y \qquad \phi_x \psi_y - \phi_y \psi_x \equiv 0$$

(u, y) indep. variables

$\Rightarrow v = \chi(u, y) = \chi(u)$ is indep. of y

i.e. $\psi(\Phi(u,y), y) = \chi(u) = \chi(\phi(\Phi(x,y), y))$



If $\dfrac{d(\xi,\eta)}{d(x,y)} = D(x_0,y_0) = \det F'(x_0,y_0) \neq 0$

$\exists F^{-1}$ locally

$F' = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix}$

Solve the inverse mapping near $u_0 = F(x_0)$

Assumption : $F \in C'$, $F'(x_0)$ is invertible as a matrix

Given u near u_0, we want to solve $x = G(x) := x - A(u - F(x))$

expect to hold for some x

$x \longmapsto G(x)$ is a dynamical system

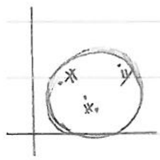
fixed point : $x = G(x) \Leftrightarrow u = F(x)$

$G' = I - AF'(x)$ [pick $F'(x_0)^{-1} = A$, then $G'(x_0) = 0$]

$< \frac{1}{2} = n^2 \cdot \frac{1}{2n^2}$

$|G(y) - G(x)| \le \sum_{i=1}^{n} |\nabla g_i(x + \theta_i(y-x)) \cdot (y-x)| \le \left(\sum_{i=1}^{n} \max_{x\,y} |\nabla g_i|\right) |y - x|$

$\begin{pmatrix} g_1(\tilde{x},\tilde{y}) \\ \vdots \\ g_n(\tilde{x},\tilde{y}) \end{pmatrix} - \begin{pmatrix} g_1(x,y) \\ \vdots \\ g_n(x,y) \end{pmatrix}$ pick $\delta > 0$ small, s.t.



① $|x - x_0| < \delta \Rightarrow \dfrac{\partial g_i}{\partial x_j}(x) < \dfrac{1}{2n}$ $\forall i, j$

② $|G(x_0) - x_0| < \dfrac{\delta}{2}$, i.e. $|u - F(x_0)| < \dfrac{\delta}{2|A|}$

(ok since $G'(x_0) = 0$)

Let $x_{n+1} = G(x_n)$ $n = 0, 1, 2, \cdots$

(*) $|x_{n+1} - x_0| \le |x_{n+1} - x_1| + |x_1 - x_0|$

$\frac{1}{2}|x_n - x_0| \ge |G(x_n) - G(x_0)|$ $G(x_0) - x_0$

claim : $x_n \in B_\delta(x_0)$ $\forall n$

If $n = 0$, $|G(x_0) - x_0| = |A(u - F(x))| \le |A| \cdot |u - F(x)| < \frac{\delta}{2}$

by induction,

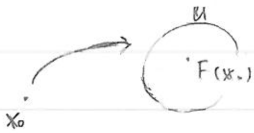
(*) $\Rightarrow |x_{n+1} - x_0| \le \frac{1}{2}\underbrace{|x_n - x_0|}_{\frac{1}{2}\delta} + \frac{1}{2}|x_1 - x_0| < \delta$

$x = \lim_{n\to\infty} x_n$

$= x_0 + (x_1 - x_0) + (x_2 - x_1) + \cdots\cdots < \delta$

$|x_{k+1} - x_k| \le \frac{1}{2}|x_k - x_{k-1}|$

$< \cdots\cdots < \frac{1}{2^n}|x_1 - x_0|$



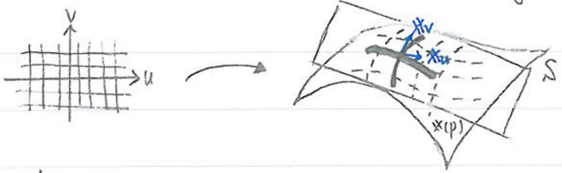
$x_{n+1} = G(x_n)$

$\lim_{n\to\infty} x_n \Rightarrow x = G(x_n)$

§ 3.4 Geometric applications :

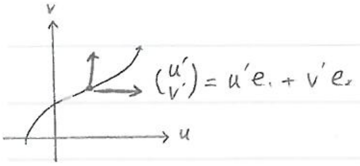
- To describe surface in $\mathbb{R}^3$

$(u,v) \in U \subset \mathbb{R}^2 \longmapsto \mathbb{R}^3 \ni (x(u,v)\ y(u,v),\ z(u,v))^t = \mathbf{x}(u,v)$



$\mathbf{x}_u, \mathbf{x}_v$ form a basis of $T_{\mathbf{x}(p)} S$

$\frac{d}{dt}\mathbf{x}(u(t), v(t)) = \mathbf{x}_u u'(t) + \mathbf{x}_v v'(t) = \mathbf{x}'\begin{pmatrix} u' \\ v' \end{pmatrix}$



$$\left(\frac{ds}{dt}\right)^2 = \frac{d\mathbf{x}}{dt}\cdot\frac{d\mathbf{x}}{dt}$$

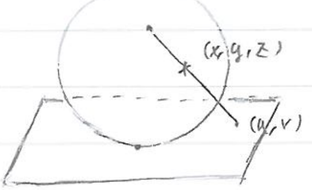
$$= \left(\mathbf{x}_u \frac{du}{dt} + \mathbf{x}_v \frac{dv}{dt}\right)\left(\mathbf{x}_u \frac{du}{dt} + \mathbf{x}_v \frac{dv}{dt}\right)$$

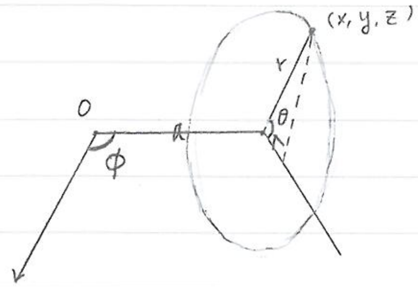
$$= \underbrace{(\mathbf{x}_u \cdot \mathbf{x}_u)}_{E}\left(\frac{du}{dt}\right)^2 + 2\underbrace{(\mathbf{x}_u \cdot \mathbf{x}_v)}_{F}\frac{dv}{dt}\frac{du}{dt} + \underbrace{(\mathbf{x}_v \cdot \mathbf{x}_v)}_{G}\left(\frac{dv}{dt}\right)^2$$

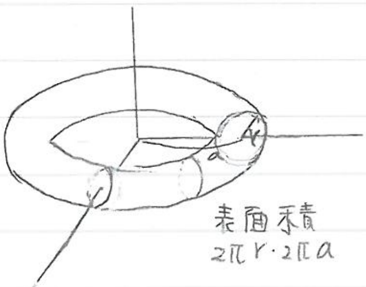
$ds^2 = E\,du^2 + 2F\,dv\,du + G\,dv^2$ ⇐ 1^{st} fundemental form

$$(du\ dv)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$$

$$ds^2 = \sum_{i=1}^{2} g_{ij}\, du^i dv^j$$





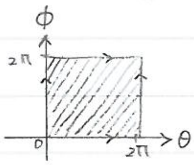


表面積
$2\pi r \cdot 2\pi a$

$x = (a + r\cos\theta)\cos\phi$

$y = (a + r\cos\theta)\sin\phi$

$z = r\sin\theta$



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Chapter 4 : Multiple integrals

· definition of area , volumn ...... etc

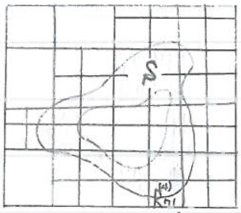
$A \subset \mathbb{R}^2$. $|A| \equiv$ area A

· for (closed) rectangle

$R = [a,b] \times [c,d]$. $|R| = (b-a) \times (c-d)$

· If $S_1, S_2, \dots, S_n$ are disjoint sets

and $|S_i|$ is defined , then $|\bigcup S_i| = \sum_{i=1}^{n} |S_i|$



$\# R_{ij}^{(n)} \leftarrow$ closed

(recall on $\mathbb{R} \supset \mathbb{Q}$ does not have length)

"Jordan measurable set"

outer measure $A_n^+(S) \searrow$

the sum of area of those sub. rectangles containing points of S.

inner measure $A_n^-(S) \nearrow$

the sum of $R_{ij}^{(n)} \subset S$

S is Jordan measure if $A^+(S) = A^-(S)$ < $\mathbb{Q}$ is not >

· But it has lebesgnl measure $= 0$

$\mathbb{Q}$ is countible $\mathbb{Q}_1, \mathbb{Q}_2, \dots$

Fact : S has a Jordan measure $\Leftrightarrow |\partial S| = 0$

$\boxed{\text{pf}}$ "$\Leftarrow$" $0 \le A_n^+(S) - A_n^-(S) \le A_n^+(\partial S)$ $\langle A_n^-(\partial S) = 0 \rangle$

we may have $R_{ij}^{(n)} \subset S$ but ...

$n \to \infty$. get $A^+(S) = A^-(S)$ $R_{ij}^{(n)} \cap \partial S \neq \phi$

"$\Rightarrow$" $\boxed{3^2}(A_n^+(S) - A_n^-(S)) \ngeq A_n^+(\partial S)$ $n \to \infty$ done # Q.E.D.

Tarski's paradox

$\exists$ partition $S^2 = \coprod_{i=1}^{N} A_i$

$\exists\ T_i \in SO(3)$ $T_i^t T_i = I_3$

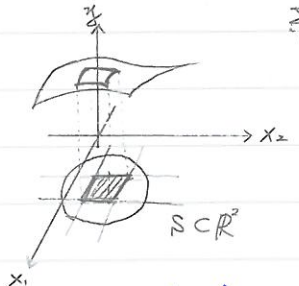
$\bigcup_{i=1}^{N}$ "$T_i(A_i)$" $= S^2 \amalg S^2$

(Example) Definition of Riemann integral over a Jordan measurable set

$S \subset \mathbb{R}^n$ s.t. $|S|$ exists

$f : S \to \mathbb{R}$ continuous function

$\int_S \underbrace{f(x^1, x^2, \dots, x^n)}_{\text{高 height}} \underbrace{dx^1 \cdots x^n}_{\Delta x^1 \cdots \Delta x^n = |R_{ij}^{(n)}|}$



$\to$ graph of f

$y = f(x^1, x^2)$

$\boxed{pf}$: 見掌変もV

Theorem (Fubini) <the simple form>

$R = [a,b] \times [c,d]$

$$\int_R f(x,y)\,dxdy = \int_c^d \Big[\underbrace{\int_a^b f(x,y)\,dx}_{\to \phi(y)}\Big] dy$$

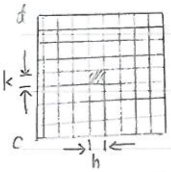
(Example) $I = \int_0^\infty \frac{e^{-ax} + e^{-bx}}{x} dx = \log b/a$

$$\left(-\frac{e^{-xy}}{y} \Big|_0^T = \frac{1 - e^{-Ty}}{y} \right)$$

$$\lim_{T\to\infty} \int_0^T dx \int_a^b e^{-yx} dy = \int_a^b dy \int_0^T e^{-xy} dx$$

$$= \log y \Big|_a^b - \left(\int_a^b \frac{e^{-Ty}}{y} dy \right) \to 0 \ (T \to \infty) \quad = \log b/a$$

$\boxed{pf}$ $LHS = \lim_{m,n\to\infty} \sum_{\nu=1}^{n} \sum_{\mu=1}^{m} f(a+\mu h, c+\nu k)\,hk$



$h = \frac{b-a}{n}$, $k = \frac{d-c}{n}$

$\forall \varepsilon > 0,\ \exists N$ s.t. $m,n \ge N \Rightarrow |\sum f(\ ,\ )hk - \int| < \varepsilon$

Denote by $\Phi_y = \sum_{\mu=1}^{m} f(a+\mu h, \underbrace{c+\nu k}_{\text{fix}})\,h$

$|\int_R f(x,y)\,dxdy - (\sum_{\nu=1}^{n} \Phi_\nu)k| < \varepsilon$

true $\forall\ m,n \ge N$

Let $m \to \infty$, $\lim_{m\to\infty} \Phi_y = \int_a^b f(x, c+\nu k)\,dx = \phi(c+\nu k)$

$|\int_R f(x,y)\,dxdy - \sum_{\nu=1}^{m} \phi(c+\nu k)k| \le \varepsilon$

$\int_R f(x,y)\,dxdy = \int_c^d dy \int_a^b f(x,y)\,dx = \int_c^d \phi(y)\,dy$

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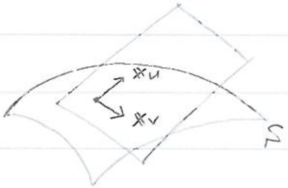
$\boxed{\text{Review}}$ Conformal mapping

$\mathbf{x} : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$(u,v)^t \qquad (x,y,z)^t$

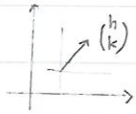


$\mathbf{x}' = [\mathbf{x}_u, \mathbf{x}_v]$

$\mathbf{x}'(u,v)\binom{h}{k} = \mathbf{x}_u \cdot h + \mathbf{x}_v \cdot k$

$\uparrow$

$T_{\mathbf{x}(u,v)} S$



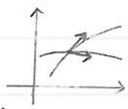
Theorem : $\mathbf{x}$ is conformal $\iff ds^2 = E(du^2 + dv^2)$ i.e. $E = G$ & $F = 0$

$\boxed{\text{pf}}$ " $\Rightarrow$ " $e_1 = \binom{1}{0} \longmapsto \mathbf{x}_u$, $e_2 = \binom{0}{1} \longmapsto \mathbf{x}_v$

$e_1 \perp e_2$, $F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$

$e_1 = \binom{1}{1}$, $e_2 = \binom{1}{-1}$ $\Rightarrow (\mathbf{x}_u + \mathbf{x}_v)(-\mathbf{x}_u + \mathbf{x}_v) = 0 = G - E$

" $\Leftarrow$ "



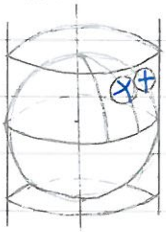
$(u_1(t), v_1(t)) \rightsquigarrow \mathbf{x}_u u_1' + \mathbf{x}_v v_1'$

$(u_2(t), v_2(t)) \rightsquigarrow \mathbf{x}_u u_2' + \mathbf{x}_v v_2'$

$$\cos\theta = \frac{(\mathbf{x}_u u_1' + \mathbf{x}_v v_1')(\mathbf{x}_u u_2' + \mathbf{x}_v v_2')}{\|\mathbf{x}_u u_1' + \mathbf{x}_v v_1'\| \cdot \|\mathbf{x}_u u_2' + \mathbf{x}_v v_2'\|}$$

$$= \frac{\cancel{E}(u_1'u_2' + v_1'v_2')}{\sqrt{\cancel{E}(u_1'^2 + u_2'^2)}\sqrt{\cancel{E}(v_1'^2 + v_2'^2)}}$$ = cos of the original angle on (u,v) plane

保長 $A^t A = I$



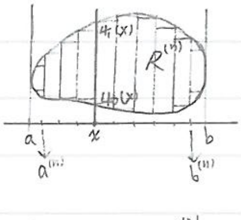
check the radios' cancel out to get 1

$z = \sqrt{1 - x^2 - y^2}$

- $R = [a,b] \times [c,d]$

$$\int_R f(x,y)\, \underbrace{dx\,dy}_{dA} = \int_c^d dy \int_a^b f(x,y)\, dx$$

- R is a convex set



$$\int_R \int dA = \int_a^b dx \int_{\psi_0}^{\psi_1} f(x,y)\, dy$$

$$\int_{R_1 \cup R_2} f\, dA = \int_{R_1} f\, dA + \int_{R_2} f\, dA$$

If $R_1^\circ \cap R_2^\circ = \phi$

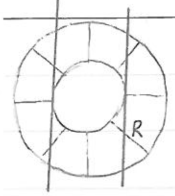
This holds for $R^{(n)}$

$$\int_{R^{(n)}} f\, dA = \int_{a^{(n)}}^{b^{(n)}} dx \int_{\psi_0^{(n)}}^{\psi_1^{(n)}} f(x,y)\, dy$$

(now let $n \to \infty$)

$$\int_R f\, dA = \int_a^b dx \int_{\psi_0}^{\psi_1} f(x,y)\, dy$$

$$\left(\begin{array}{c} \left|\int_{R^{(n)}} f\, dA - \int_R f\, dA\right| = M \cdot A_+^{(n)}(\partial R) \\ |f| \le M \end{array} \right)$$



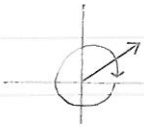
Mean Value Thm: $f \in C^0 \Rightarrow$

$$\underbrace{\frac{1}{|R|}}_{A(R)} \int_R f\, dA = f(p) \quad \text{for some } p \in R$$

(Example)

$I = \int_{-\infty}^{\infty} e^{-x^2} dx$

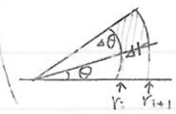
$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} \underbrace{dx\,dy}_{dA = r\,dr\,d\theta} = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta$$



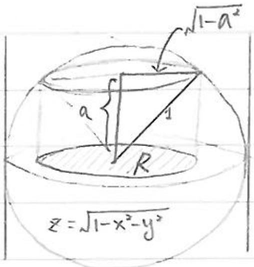
$r \in (0, \infty)$
$\theta \in (0, 2\pi)$

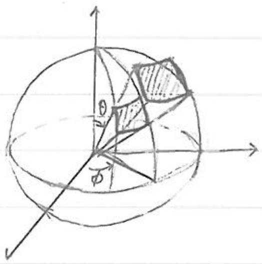
$$= 2\pi \left(-\frac{1}{2} e^{-r^2} \Big]_0^{\infty} \right) = \pi$$

$\Rightarrow I = \sqrt{\pi}$



$$\Delta A = \frac{1}{2}\left((r+\Delta r)^2 - r^2\right)\Delta\theta$$
$$= r\,\Delta r\,\Delta\theta + \frac{1}{2}\Delta r^2 \Delta\theta$$
$$\Delta A = \frac{1}{2}\left(r_{i+1}^2 - r_i^2\right)\Delta\theta = \frac{r_{i+1}+r_i}{2}\Delta r\,\Delta\theta$$





$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$$\int_{\underset{\{x^2+y^2 \le 1-a^2\}}{R}} \left(\sqrt{1-x^2-y^2} - a\right) dx\,dy = \int_0^{2\pi}\int_0^{\sqrt{1-a^2}} \left(\sqrt{1-r^2} - a\right) r\,dr\,d\theta$$

$(dr)(r\,d\theta)(r\sin\theta\,d\phi) = r^2\sin\theta\,dr\,d\theta\,d\phi$

<HomeWork>

$$\int_R \frac{dx\,dy}{(1-x^2-y^2)^2}$$

$v_1, v_2, v_3 \in \mathbb{R}^5$

area $\rightarrow |v| = \sqrt{a_1^2 + \cdots + a_5^2} = \sqrt{v \cdot v} = \sqrt{v^t v}$

$|v_1 \wedge v_2|$ "wage", the parallelgram spanned by v_1 & v_2

i.e. $v_1 \wedge v_2 = \{ t_1 v_1 + t_2 v_2 \mid 0 \le t_i \le 1 \}$

$$V = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \quad \underbrace{\sqrt{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2}}_{\sqrt{\det(V^tV)}}$$

$\mathbb{R}^n \quad V = [v_1, \cdots, v_m]$

体積 $= \sqrt{\det(V^tV)}$

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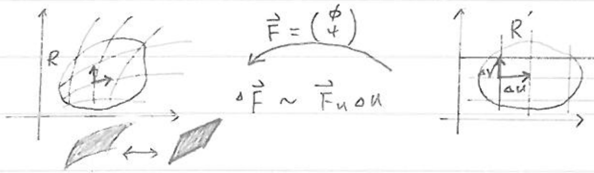
change of variable formula for multiple integrals

$$\int_R f(x,y)\,dx\,dy = \int_{R'} f(\phi(u,v),\ \psi(u,v)) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Assume $\begin{cases} x = \phi(u,v) \\ y = \psi(u,v) \end{cases}$ $D = \dfrac{\partial(x,y)}{\partial(u,v)}$ is a C^1 mapping from R' to R with $D \neq 0$



$$|\vec{F}_u \Delta u,\ \vec{F}_v \Delta v| = \text{"}\frac{\partial(x,y)}{\partial(u,v)}\text{"}\,\Delta u\,\Delta v$$

↑ at some point

primitive mapping

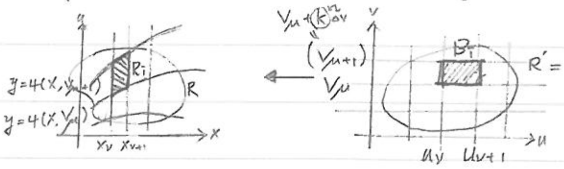
$\boxed{\text{e.g.}}$ $\vec{x} = \vec{u} \in R^{n-1}$, $y = \psi(\vec{u}, v)$

$$\frac{\partial(\vec{x}, y)}{\partial(\vec{u}, v)} = \begin{vmatrix} I_{n-1} & \psi_u \\ 0 & \psi_v \end{vmatrix} = \psi_v$$

$\nabla_{n-1}\psi_u = \dfrac{\partial\psi}{\partial u_1} \cdots \dfrac{\partial\psi}{\partial u_{n-1}}$ in R_i'

* proof of CVF for primitive mapping

Assume $\begin{aligned} x &= u \\ y &= \psi(u,v) \end{aligned}$ gives a 1-1 C^1 mapping between



, which has $\psi_v > 0$

Area of R_i

↑

$$\begin{aligned} \Delta R_i &= \int_{x_\nu}^{x_\nu + h} \left(\psi(x, v_\mu + k) - \psi(x, v_\mu)\right) dx \\ &= h \cdot \left(\psi(\tilde{x}_\nu, v_\mu + k) - \psi(\tilde{x}_\nu, v_\mu)\right) \\ &= hk \cdot \psi_v\underbrace{(\tilde{x}_\nu, \tilde{v}_\mu)}_{P_i} \end{aligned}$$

$\begin{cases} \tilde{x}_\nu \in [x_\nu, x_\nu + h] \\ \tilde{v}_\mu \in [v_\mu, v_\mu + k] \end{cases}$

Riemann Sum

$$\sum f(\tilde{x}_\nu, \psi(\tilde{x}_\nu, \tilde{v}_\mu))\,\Delta R_i = \sum_i f(\tilde{x}_\nu, \psi(\tilde{x}_\nu, \tilde{v}_\mu)) \cdot \psi_v(P_i)\, h \cdot k$$

<let $(h, k) \to (0, 0)$>

$$\rightarrow \lim_{h \to 0, k \to 0} \sum f(\tilde{x}_\nu, \psi(\tilde{x}_\nu, \tilde{v}_\mu))\,\Delta R_i = \int f(x,y)\,dx\,dy$$

$$\lim_{h \to 0, k \to 0} \sum f(\tilde{x}_\nu, \psi(\tilde{x}_\nu, \tilde{v}_\mu)) \cdot \psi_v(P_i)\, h\, k = \int \underset{\substack{\downarrow \\ f(u,v)}}{f \cdot F} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

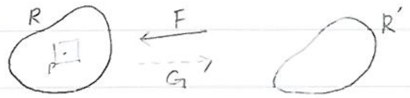
Double A

Lemma : If CVF holds for $(x, y) = F_1(u, v)$
$(u, v) = F_2(\xi, \eta)$

Then it holds for $(x, y) = (F_1 \circ F_2)(\xi, \eta) \cdot \left|\frac{\partial(x, y)}{\partial(\xi, \eta)}\right|$

$\boxed{\text{pf:}}$ $\int_R f(x, y)\, dx\, dy = \int_{R'} (f \circ F_1 \circ F_2)(\xi, \eta) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| F_2(\xi, \eta) \left|\frac{\partial(u, v)}{\partial(\xi, \eta)}\right| d\xi\, d\eta$

<for general case :>



$\forall\, p \in R$, $\exists$ cube $C_p(r)$

s.t. F can be decomposed into primitive maps.

(Problem : r could vary when p varies)

$\prod_{i=1}^{n} [P_i - r, P_i + r]$

- Assume R is compact (closed & bounded) and Jordan measurable, Then $\{ C_p^{\circ}(r_p) \}$ is an open cover of R
  ($^{\circ}$ interior)

Heire - Borel Theorem

Any open cover of a compact set admit a finite sub cover

$\bigcup_{i \in N} U_i \supset R \Rightarrow \exists\, i_1, \dots, i_N$ (finite) s.t. $U_{i_1} \cup \dots \cup U_{i_N} \supset R$

By Heire-Borel, $\exists\, p^{(1)}, \dots, p^{(N)} \in R$

s.t. $R \subset C^{\circ}_{p^{(1)}}(r_{p^{(1)}}) \cup \dots \cup C^{\circ}_{p^{(N)}}(r_{p^{(N)}})$

pick $r = \min(r_{p^{(i)}})$, $1 \le i \le N$

Note the "1st step" can applied to $R^{(i)} \cap C^{\circ}_{p^{(j)}}(r_{p^{(j)}})$
$j = 1, 2, \dots, N$

$\downarrow$

when F is decomposable into primitive mappings.

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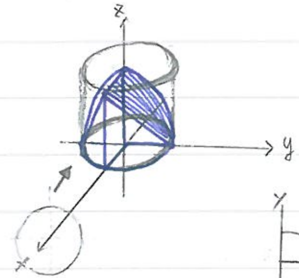
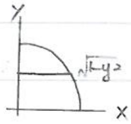
find the volumn of the intersection of 2 cylinders in $\mathbb{R}^3$

$\{x^2+y^2\le 1\}$ and $\{y^2+z^2\le 1\}$



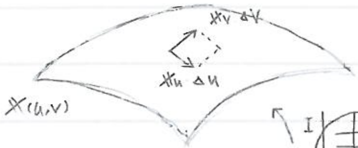
$8\times\int_R \sqrt{1-y^2}\,dx\,dy$

R^2 ⇒ $\begin{cases} x^2+y^2\le 1 \\ x\ge 0,\ y\ge 0\end{cases}$



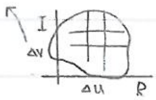
$= 8\int_0^1 \sqrt{1-y^2}\left(\int_0^{\sqrt{1-y^2}} dx\right) dy$

$= 8\int_0^1 (1-y^2)\,dy = 8\left(y-\frac{1}{3}y^3\right)\Big]_0^1 = \frac{16}{3}$ ※

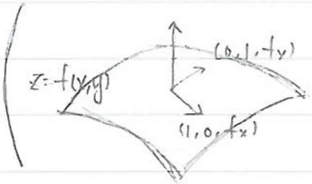


$|X_u\wedge X_v|\,\Delta u\,\Delta v$

$\sqrt{EG-F^2}$
⇑

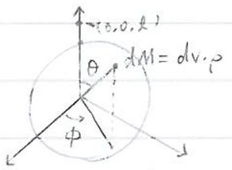


$A(S)=A(X(R)) \overset{Def}{=} \int_R |X_u\wedge X_v|\,du\,dv$



$\sqrt{1+f_x^2+f_y^2}\;dx\,dy$

$EG-F^2=(1+f_x^2)(1+f_y^2)-f_x^2f_y^2$



$\begin{bmatrix} x=r\sin\theta\cos\phi \\ y=r\sin\theta\sin\phi \\ z=r\cos\theta\end{bmatrix}$

$\vec{F}=-\frac{GMm}{r^3}\vec{r}$ x, y components integrate to 0

$2\pi\times Gm\rho\cdot\int_B \frac{(r\cos\theta-\ell)r^2\sin\theta}{\sqrt{\ell^2+r^2-2r\cos\theta}}\,dr\,d\theta\,\cancel{d\phi}$

$r^2\sin^2\theta+(r\cos\theta-\ell)^2$
$= r^2-2r\ell\cos\theta+\ell^2$

[we integrate θ first =: u $du=2r\ell\sin\theta\,d\theta$]

$=\int \frac{r}{2\ell}\cdot\frac{1}{2\ell}(r^2-\ell^2-u)\,u^{-\frac{3}{2}}\,du\,dr$ $\left[\cos\theta=\frac{r^2+\ell^2-u}{2r\ell}\right]$

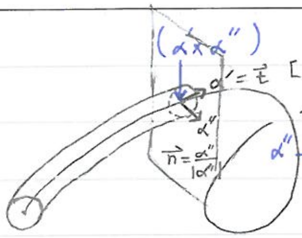
$=\frac{1}{4\ell^2}\int r\,dr\int (r^2-\ell^2)u^{-\frac{3}{2}}-u^{-\frac{1}{2}}\,du \quad -2(r^2-\ell^2)u^{-\frac{1}{2}}\Big]_{\theta=0}^{\theta=\pi} - 2u^{\frac{1}{2}}\Big]_{\theta=0}^{\theta=\pi}$

⇓ $-2(r^2-\ell^2)\left(\frac{1}{\ell+r}-\frac{1}{\ell-r}\right)$ ⇓ $-2(\ell+r-(\ell-r))$

$=2(\ell-r-(\ell+r))=-4r$ $=-4r$

$=-\frac{2}{\ell^2}\int_0^R r^2\,dr=-\frac{2}{3}\frac{R^3}{\ell^2}$

⇒ $-Gm\rho\frac{4\pi}{3}R^3=\frac{-GmM}{\ell^2}$



moving frame

$\alpha' \cdot \alpha' = 1$

$2\alpha'' \cdot \alpha' = 0$

C: curve

$\alpha(s)$, $s \in [0, l]$

$\uparrow$

any length $l =$ length (C)

$$\mathbb{X}(s,\theta) = \alpha(s) + r(\cos\theta\,\vec{n} + \sin\theta\,\vec{b})$$

$$\{\vec{t}, \vec{n}, \vec{b}\}$$

$$\alpha'' \quad \overset{\shortparallel}{\frac{\alpha''}{|\alpha''|}} \quad \overset{\shortparallel}{\vec{t}\times\vec{n}}$$

$$\int_0^{2\pi}\int_0^{l} |\mathbb{X}_s \times \mathbb{X}_\theta|\, ds\, d\theta$$

$$\mathbb{X}_s = \vec{t} + r(\cos\theta\,\vec{n}' + \sin\theta\,\underset{}{\overset{\vec{t}\times\vec{n}'}{\vec{b}'}})$$

$$\mathbb{X}_\theta = r(-\sin\theta\,\vec{n} + \cos\theta\,\vec{b})$$

$$\begin{pmatrix} \vec{b} = \vec{t}\times\vec{n} \\ \vec{b}' = \vec{t}'\times\vec{n} + \vec{t}\times\vec{n}' = \vec{t}\times\vec{n}' \end{pmatrix}$$

$$\mathbb{X}_s \times \mathbb{X}_\theta = r(-\sin\theta\,\vec{b} - \cos\theta\,\vec{n}) + \cdots\cdots$$

$$= -r(\cos\theta\,\vec{n} + \sin\theta\,\vec{b}) + \cdots\cdots$$

$$\alpha = \vec{b}'\cdot\vec{t} = (\vec{b}\cdot\vec{t})' - \vec{b}\cdot\vec{t}' = 0$$

$$\begin{pmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{pmatrix}' = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}\begin{pmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{pmatrix}$$

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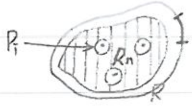
§ 4.7 Improper integrals

① $A(R) < \infty$



f could be not continuous or even $\nearrow \infty$ at some points $P_i \in R$

$\int_R \frac{dx\,dy}{(1-x^2-y^2)^2}$

② $A(R) = \infty$

$R = \mathbb{R}^2$ $\qquad \int_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy$

$\boxed{\text{Theorem}}$ Let R be bounded with area (i.e. R is Jordan − m)

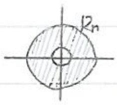
(i) $R_n \nearrow R$ s.t. f is conti on R_n $\forall n$

└ closed set, i.e. $R_n \subset R_{n+1}$ & $A(R_n) \to A(R)$

(ii) and $\int_{R_n} |f| \le M$ $\forall n$ $\qquad (*)$

Then $I = \lim_{n\to\infty} \int_{R_n} f$ exists and independent of choises of $\{R_n\}$ in (i)

(Ex 1)



$\int_{R = B_0(1)} \frac{dv}{|\vec{r}|^\alpha}$ $\quad dv \leftarrow dx\,dy\,dz$ $\qquad [\ \vec{r} = (x, y, z)\ ]$

"=" $\lim_{n\to\infty} \int_{R_n} r^{-\alpha} \cdot r^2 \sin\theta \, dr\, d\theta\, d\phi = 2\pi \cdot -\cos\theta \Big]_\pi^0 \cdot \int_{\frac{1}{n}}^1 r^{2-\alpha}\,dr$

$R_n = \{\vec{r} \mid \frac{1}{n} \le \vec{r} \le 1\}$

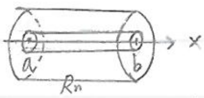
↓ require $2-\alpha > -1$

$\Rightarrow$ i.e. $\alpha < 3$

Q: on $\mathbb{R}^2$, $1-\alpha > -1$ $\Rightarrow$ i.e. $\alpha < 2$

How about $\mathbb{R}^n$?

(Ex 2)



if $f(x, y, z) \le \frac{M}{\sqrt{y^2+z^2}}$

$R = [a, b] \times$ "$B_0(1)$"

on yz - plane

need only $\alpha < 2$

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$\boxed{\text{pf}}$ "step 1" $I_n^+ := \int_{R_n} |f| \nearrow$ bounded by μ

$\Rightarrow I_n^+ \to I^+$ exists. (and is a cauchy sequence)

$\Rightarrow I_n : \int_{R_n} f$ is also a Cauchy sequence

because $|I_n - I_m| = |\int_{R_n} f - \int_{R_m} f| = |\int_{R_n \setminus R_m} f|$ $(n>m)$

$\le \int_{R_n \setminus R_m} |f| = I_n^+ - I_m^+ < \varepsilon$ for $n, m > N(\varepsilon)$

$\Rightarrow I = \lim_{n\to\infty} I_n$ exists.

"step 2" Now, for any $S \subset R$, closed, J-m, $f|_S$ is conti.

Need to check $(*)$: "$\int_S |f| \le \mu$"

$|\int_S f - \int_{S\cap R_n} f| \le A(S\setminus R_n)\cdot \sup|f| \xrightarrow{n\to\infty} 0$ $(**)$

similarly to $|f|$, get $\int_S |f| = \lim_{n\to\infty} \int_{S\cap R_n} |f| \le \mu$

S 不屬於 R_n 還是符合 (ii)

$|\int_S f - \int_{S\cap R_n} f| = \lim_{m\to\infty} |\int_{S\cap R_m} f - \int_{S\cap R_n} f| \le \lim_{m\to\infty} \int_{R_m \setminus R_n} |f| < \varepsilon$ for $m, n > N(\varepsilon)$ indep of S

$\hookrightarrow \int_{S\cap(R_m - R_n)}$

now, for another sequence

$S_m \nearrow R$ satisfying (i) then (ii) is also satisfied $(*)$

so $J = \lim_{m\to\infty} \int_{S_m} f$ exists.

$|J - \int_{S_m \cap R_n} f| \le |J - \int_{S_m} f| + |\int_{S_m} f - \int_{S_m\cap R_n} f| \le 2\varepsilon$

as long as $m, n > N(\varepsilon)$

similarly, $|I - \int_{S_m \cap R_n} f| \le 2\varepsilon \Rightarrow |I - J| \le 4\varepsilon$

$\Rightarrow I = J$ #Q.E.D.

"$\lim_{n\to\infty} \int_{R_n} f = \lim_{m\to\infty} \int_{S_m} f$"

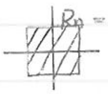
* The case with unbounded R

(i) $R_n \nearrow R$ now requires the "exhausion condition"

(cpt. J-m) (every compaction must be contained in R_n for n large) CPT

(ii) $\int_{R_n} |f| \le \mu$ $\forall n$

Then $I := \lim_{n\to\infty} \int_{R_n} f$ exists and indep. of the choices of $\{R_n\}$



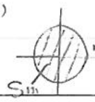
$R_n = [-n, n] \times [-n, n]$

(Ex 3)

Gauss' integral

$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$\lim_{n\to\infty} \int_{R_n} e^{-(x^2+y^2)} dx dy = \lim_{m\to\infty} \int_{S_m = B_0(m)}$



Double A

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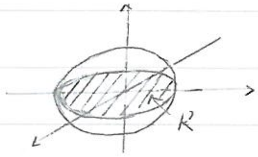
§ 4.8 more geometric applications

(Ex 4)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1 \qquad \begin{cases} x = ar\cos\theta \\ y = br\sin\theta \\ z = z \end{cases} \Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = abr$$



$$V = 2\int_R c\sqrt{1-\left(\tfrac{x}{a}\right)^2-\left(\tfrac{y}{b}\right)^2}\,dx\,dy$$

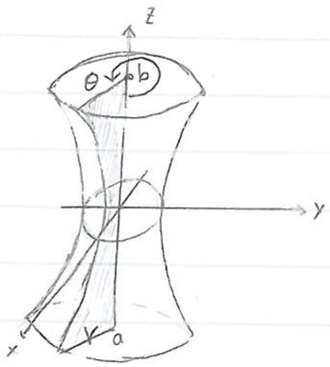
$$= 2abc\int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\,r\,dr\,d\theta$$

$$= \frac{4}{3}\pi\,abc \quad /\!/$$

(Ex 5) Cylindrical coordinates

$(x, y, z) \longleftrightarrow (r, \theta, z)$

surface (solid) of revalution



$$V = \int_R dv = \int_a^b dz\,\underbrace{\int_0^{2\pi} d\theta \int_0^{\phi(z)} r\,dr}_{2\pi\cdot\frac{1}{2}r^2}$$

$$= \pi\int_a^b \phi^2(z)\,dz$$

$$S = \int_a^b \underbrace{dz}\cdot\sqrt{1-\phi'(z)^2}\cdot 2\pi\cdot\phi(z)$$

in fact, $x(z,\theta) = (\phi(z)\cos\theta,\ \phi(z)\sin\theta,\ z)$

$$x_z = (\phi'\cos\theta,\ \phi'\sin\theta,\ 1) \qquad F = x_z\cdot x_\theta = 0$$
$$x_\theta = (-\phi\sin\theta,\ \phi\cos\theta,\ 0)$$

$$E = |x_\theta|^2 = 1+(\phi')^2, \qquad G = |x_\theta|^2 = \phi^2$$

$$dA = \sqrt{EG-F^2}\,d\theta\,dz = \sqrt{1-\phi'^2}\,\phi\,d\theta\,dz$$

$$S = 2\pi\int_a^b \sqrt{1-\phi'^2}\,\phi\,dz$$

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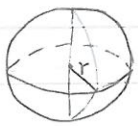
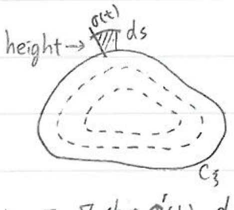
§ 4.10 Multiple integrals in Curvilinear coordinates.

(等高線積分法)



$V(r) = \frac{4}{3}\pi r^3$

$A(r) = 4\pi r^2$



$\phi(x,y) = \xi \rightarrow$ constant

$\phi(\underset{\parallel}{\sigma(t)}) = \xi$, $\sigma(t) = (x(t), y(t))$

$d\xi = (\phi_x \frac{dx}{dt} + \phi_y \frac{dy}{dt})dt = \nabla\phi \cdot \underbrace{\sigma'(t)\, dt}_{d\vec{\sigma}}$

$\sigma'(t) = (\frac{dx}{dt}, \frac{dy}{dt})$

$\int f(x,y)\, ds \boxed{}$

$\frac{d\xi}{|\nabla\phi|} = \frac{d\xi}{\sqrt{\phi_x^2+\phi_y^2}}$

The more rigorous deduction of the "co-Area formula"

$\boxed{pf}$: concider $\begin{cases} \xi = \phi(x,y) \\ \eta = y \end{cases}$

$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \phi_x & \phi_y \\ 0 & 1 \end{vmatrix} = \phi_x \Rightarrow \frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{1}{\phi_x}$

$\int_R f\, dxdy = \int f \frac{d\xi d\eta}{|\phi_x|} = \int f \frac{d\xi}{\sqrt{\phi_x^2+\phi_y^2}} \left(\frac{\sqrt{\phi_x^2+\phi_y^2}}{|\phi_x|} dy \right)$

$y = f(x)$

$ds = \sqrt{1+f_x^2}\, dx$

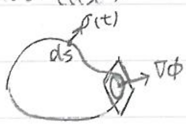
$\phi(x,y) = \xi \leftarrow$ fixed

$\phi_x + \phi_y f_x = 0 \rightarrow f_x = -\frac{\phi_x}{\phi_y}$

$ds = \sqrt{1+\frac{\phi_x^2}{\phi_y^2}}\, dx = \frac{\sqrt{\phi_x^2+\phi_y^2}}{|\phi_y|} dx$

#Q.E.D.

<3D case>



$\phi(x,y,z) = \xi$

$\int f\, dv = \int \frac{f}{|\nabla\phi|}\, ds\, d\xi$

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(Example) $f \equiv 1$

$\phi(x, y, z) = r$

$|\nabla \phi| = |\nabla r| = 1$

$\downarrow$

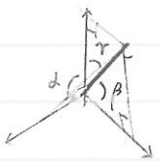
$\left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{\cdots}}, \frac{z}{\sqrt{\cdots}} \right)$

$$\begin{cases} \xi = \phi(x, y, z) \\ y = y \\ z = z \end{cases} \Rightarrow \frac{\partial(\xi, y, z)}{\partial(x, y, z)} = \begin{vmatrix} \phi_x & \phi_y & \phi_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \phi_x$$

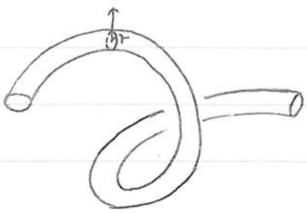
$$\Rightarrow \frac{\partial(x, y, z)}{\partial(\xi, y, z)} = \frac{1}{\phi_x}$$

$$\int_R f \, dx\,dy\,dz = \int f \frac{d\xi\, dy\, dz}{\phi_x} = \int f \frac{d\xi}{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}} \underbrace{\left(\frac{\sqrt{}}{|\phi_x|} \right) dy\,dz}_{ds} \quad \overset{\frac{1}{\cos\alpha}}{\curvearrowleft}$$

$\cdot\, ds = \sqrt{1+f_y^2+f_z^2}\; dy\,dz \qquad \cdot\, ds = |X_u \wedge X_v|\, du\, dv = \sqrt{EG-F^2}\, du\, dv$



$$\frac{(\phi_x, \phi_y, \phi_z)}{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}} = \frac{\nabla\phi}{|\nabla\phi|} = (\cos\alpha, \cos\beta, \cos\gamma)$$



$V^n(1) \cdot R^n \qquad \curvearrowleft \{ \vec{x} \in R^n \mid |\vec{x}| \le R \}$

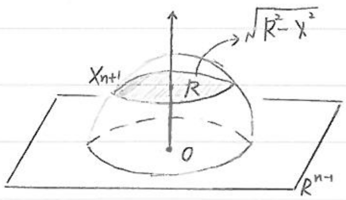
$\parallel$

$V^n(R) = |B_0^n(R)|$

$A^{n-1}(R) = |S_0^{n-1}(R)|$

$\uparrow$

$\{ \vec{x} \in R^n \mid |\vec{x}| = R \}$



$V(R) = 2R$

$$\underset{V^n(R)}{\overset{\parallel}{2\int_0^R}} \underline{V^{n-1}\left(\sqrt{R^2-x^2}\right)} dx$$

$\downarrow$

$$2V^{n-1}(1) \int_0^R (R^2-x^2)^{\frac{n}{2}}\, dx$$

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$V^n(R) = n\text{-dim'l volumn of } B_0(R) = V^{(n)}(1) \cdot R^n$

$A^{n-1}(R) = (n-1)\text{-dim'l area of } S_0^{n-1}(R) = A^{n-1}(1) \cdot R^{n-1}$

$\frac{d}{dR}(V^n(R)) = A^{n-1}(R) \quad \Rightarrow \quad nV^n(1) = A^{n-1}(1)$

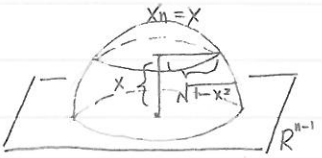
$V^n(1) = 2\int_0^1 dx \; \underbrace{V^{n-1}(\sqrt{1-x^2})}_{= V^{n-1}(1)\cdot(1-x^2)^{\frac{n-1}{2}}}$

$= V^{n-1}(1) \cdot 2\int_0^1 (1-x^2)^{\frac{n-1}{2}} \, dx \qquad$ [Let $x = \sin\theta$]

$= V^{n-1}(1) \cdot \underbrace{2\int_0^{\frac{\pi}{2}} \cos^n\theta \, d\theta}$

$(n \in \text{even}) \quad \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdots \cdot \frac{1}{2} \cdot \pi \quad \leftarrow 2 \times \frac{\pi}{2}$

$(n \in \text{odd}) \quad \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdots \cdot \frac{2}{3} \cdot 2 \quad \leftarrow 2 \times 1$



< for $n = 2k$ >

$V^{2k}(1) = V^{2k-1}(1) \cdot \frac{2k-1}{2k} \cdot \cdots \cdot \frac{1}{2} \times \pi$

$= V^{2k-2}(1) \cdot \frac{1}{2k} \cdot 2\pi = \frac{\pi^k}{k!}$ #

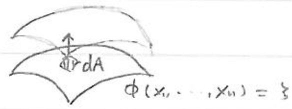
$\omega_n \overset{def}{:=} A^{n-1}(1)$

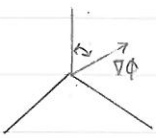
co-area formula :

$\int_R f(x_1, \cdots, x_n) \, dx_1 \cdots dx_n$

$= \int_R \frac{f}{|\nabla\phi|} \, d\xi \, \underbrace{dA}$

$\hookrightarrow \quad dA = \left(\frac{\sqrt{\phi_{x_1}^2 + \cdots + \phi_{x_n}^2}}{|\phi_{x_n}|} \right) \underbrace{dx_1 \cdots dx^{n-1}}_{x_n = f(x_1, \cdots, x_{n-1})}$





$dA = \sqrt{1 + f_{x_1}^2 + \cdots + f_{x_{n-1}}^2} \; dx_1 \cdots dx_{n-1}$

$r = \xi = \phi = \sqrt{x_1^2 + \cdots + x_n^2}, \quad |\nabla\phi| = 1$

$\int_{R^n} f \, dV = \int_{R^n} f \, dr \, \underbrace{\text{"}dA\text{"}} \leftarrow$ area element on $S_0^{n-1}(r)$

pick : $f = e^{-(x_1^2 + \cdots + x_n^2)} = e^{-r^2}$

$\pi^{\frac{n}{2}} = \prod_{i=1}^{n} \int_{-\infty}^{\infty} e^{-x_i^2} \, dx_i = A^{n-1}(1) \cdot \int_0^\infty \underbrace{e^{-r^2} r^{n-1} \, dr}$

[let $s = r^2$, $ds = 2r\,dr$]

$\rightarrow \frac{1}{2}\int_0^\infty e^{-s} \cdot s^{\frac{n-1}{2} - \frac{1}{2}} \, ds = \frac{1}{2}\Gamma\left(\frac{n}{2}\right)$

$\uparrow$

$* \; \Gamma(s+1) = s\Gamma(s)$

$\Rightarrow A^{n-1}(1) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$

Integration / Differentiation of improper integrals with a parameter

$F(x) = \int_0^\infty \overset{C^0}{f(x,y)}\, dy \qquad x \in [a,b]$

converges uniformly is $\forall \varepsilon > 0,\ \exists A$ s.t. $\left|\int_B^\infty f(x,y)\,dy\right| < \varepsilon \quad \forall B \geq A$

simple test:

"$|f(x,y)| < \frac{M}{y^\alpha}$ for $y \geq y_0$" $\Rightarrow$ unif. conv. , since $\alpha > 1$

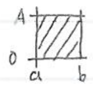
$\boxed{\text{Thm}}$ unif. conv $\Rightarrow$ $f(x)$ is conti on $[a,b]$

$\boxed{\text{pf.}}$ Given $\varepsilon > 0$

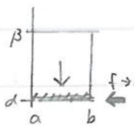
$|F(x+h) - F(x)| \underset{\text{choose } A}{<} \left|\int_0^A (f(x+h,y) - f(x+h,y))\,dy\right| + \underset{\left|\int_A^\infty\right| + \left|\int_A^\infty\right|}{2\varepsilon} < 3\varepsilon$ ※ Q.E.D.

choose h small (depend on A)

s.t. $|f(x+h,y) - f(x,y)| < \frac{\varepsilon}{A}$



similarly, $f(x) = \int_\alpha^\beta f(x,y)\,dy$



But $y \to \alpha$ has ∞-discontinuity

unif. conv. $\underset{\text{given } \varepsilon>0,\ \exists K}{\Longleftrightarrow}$ s.t. $\left|\int_\alpha^{\alpha+h} f(x,y)\,dy\right| < \varepsilon \quad \forall h \leq k$

Test: $|f(x,y)| < \frac{M}{(y-\alpha)^\nu} \quad (\nu < 1)$

$\boxed{\text{Integrals}}$ $\int_\alpha^\beta dx \underbrace{\int_0^\infty}_{\int_0^A + \boxed{\int_A^\infty}} f(x,y)\,dy \overset{?}{=} \underbrace{\int_0^\infty}_{\int_0^A + \boxed{\int_A^\infty}} dy \int_\alpha^\beta f(x,y)\,dx$

bdd. by ε

$\Rightarrow \int_\alpha^\beta dx \int_0^A f(x,y)\,dy = \int_0^A dy \int_\alpha^\beta f(x,y)\,dx$

$A \to \infty$ ※ Q.E.D.

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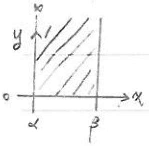

$\int_0^\infty f(x,y)\,dy = \int_0^B f(x,y)\,dy + R_B(x)$

$\underset{F(x)}{\parallel}$

uniformly convergent $\Rightarrow$ $F(x)$ continuous

$\forall \varepsilon > 0,\ \exists A$ s.t. $\forall B \geq A \Rightarrow |R_B(x)| < \varepsilon$

$\int_\alpha^\beta dx \int_0^\infty f(x,y)\,dy \overset{?}{=} \int_0^\infty dy \int_\alpha^\beta f(x,y)\,dx$

Yes, By Theorem (需要均勻收斂)

$\left|\int_\alpha^\beta R_B(x)\,dx\right| \leq (\beta-\alpha)\,\varepsilon$

$\int_\alpha^\beta dx \int_0^B f(x,y)\,dy + \int_\alpha^\beta R_B(x)\,dx = \int_0^B dy \int_\alpha^\beta f(x,y)\,dx + \int_\alpha^\beta R_B(x)\,dx$

(let $B \to \infty$) This "=" holds ※

· for exchange of integral $\int_0^\infty dy \int_0^\infty dx\, f(x,y)$

so far, we only know that it holds if $\int_{\mathbb{R}^2} |f(x,y)|\,dx\,dy$ exists.

< Differentiation >

suppose that $f_x(x,y)$ is piece-wise continuous on $x \in [\alpha, \beta]$

and $F(x) = \int_0^\infty f(x,y)\,dy$, $G(x) = \int_0^\infty f_x(x,y)\,dy$ exist & uniformly

Then $F'(x) = G(x)$

① $F_B'(x) = G_B(x)$ ② let $B \to \infty$, $F_B(x) = \int_0^B f(x,y)\,dy$

$\boxed{\text{pf}}$ $\int_\alpha^\xi G(x)\,dx = \int_\alpha^\xi dx \int_0^\infty f_x(x,y)\,dy$

$= \int_0^\infty dy \int_\alpha^\xi f_x(x,y)\,dx = \int_0^\infty [f(\xi,y) - f(\alpha,y)]\,dy$

$= F(\xi) - F(\alpha) \Rightarrow F'(x) = G(x)$ ※ Q.E.D.

(Remark) $\frac{d}{dx}\left(\int_{\phi(x)}^\infty f(x,y)\,dy\right) = \int_3^\infty f(x,y)\,dy + \int_{\phi(x)}^3 f(x,y)\,dy$

(Example 1)

$\int_0^\infty e^{-xy}\,dy = -\frac{e^{-xy}}{x}\Big]_0^\infty$ $(x>0)$

unif. conv.

diff' in x:

$\int_0^\infty \underline{y e^{-xy}}\,dy = \frac{1}{x^2}$

require unif. conv of ye^{-xy}

$\int_0^\infty y^n \cdot e^{-\frac{xy}{2}} \cdot e^{-\frac{xy}{2}} = \frac{n!}{x^{n+1}}$

$M^{?}$

unif. conv.

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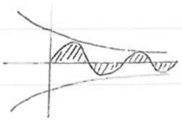
(Example 2)

$$\int_0^\infty \frac{\sin y}{y}\, dy \overset{\text{Thm}}{=} \frac{\pi}{2}$$



$\hat{e^{-xy}}$ (丟進去 → unif. conv)　　$\int |\cdots| \le \int e^{-xy} = \left.\frac{-e^{-xy}}{x}\right|_0^\infty$

Let $f(x) = \int_0^\infty e^{-xy} \frac{\sin y}{y}\, dy$　　$(x \ge 0)$

$f'(x) = -\int_0^\infty e^{-xy} \sin y\, dy = \frac{-1}{1+x^2}$　　unif. conv. $(x \ge \varepsilon)$

$f(x) = C - \tan^{-1} x$　　Let $x \to \infty$

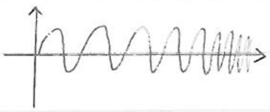
$0 = C - \frac{\pi}{2} \Rightarrow F(0) = C = \frac{\pi}{2}$

(Example 3)

Fresrel's integral

$$F_1 = 2\int_0^\infty \sin(x^2)\, dx = \int_1^\infty \frac{\sin t}{\sqrt{t}}\, dt$$

(Let $t = x^2$, $dt = 2x\,dx$)

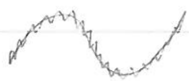


HW: completed yourself

Fourier transpose

$f(x) \mapsto \hat{f}(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ixy}\, dx$

oscilation mode (phase



$f(x) \longmapsto \hat{f}(y) \longmapsto f(-x)$

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\hat{f}(y)}_{(iy)^3}\, e^{+iyx}\, dy$

$\left(\frac{d}{dx}\right)^3$

Dream

$v_1, \cdots, v_m \in \mathbb{R}^n$,　$v_1 \wedge \cdots \wedge v_m = \{ \sum_{i=1}^m t_i v_i \mid 0 \le t_i \le 1 \}$

$V = [v_1 \cdots v_m]_{n\times m}$　　$V^t V = [v_i^t \cdot v_j]_{m\times m}$

$\begin{bmatrix} v_1^t \\ \vdots \\ v_m^t \end{bmatrix} [v_1 \cdots v_m]$　　m-dim'l area $= \sqrt{\det V^t V}$

$T^{\sim v_1 \cdots v_m}$,　$TV = \begin{pmatrix} \square \\ 0 \end{pmatrix}_{m\times m}$　　$\mathbb{R}^m \subset \mathbb{R}^n$

$(a_1, \cdots, a_m, 0, \cdots, 0)$

$(TV)^t (TV) = V^t T^t T V = V^t V \iff (\square\, 0)\begin{pmatrix} \square \\ 0 \end{pmatrix} = \square^t \square$

$\det V^t V = \det \square^t \square = (\det \square)^2$

$u_1 \cdots u_m$

$\mathbb{R}^m \xrightarrow{X} \mathbb{R}^n$

$X(u_1, \cdots, u_m)$

$$\begin{vmatrix} X_1 \cdot X_1 & X_1 \cdot X_2 \\ X_2 \cdot X_1 & X_2 \cdot X_2 \end{vmatrix}^{1/2} = \begin{vmatrix} E & F \\ F & G \end{vmatrix}^{1/2} = \sqrt{EG - F^2}$$

Double A

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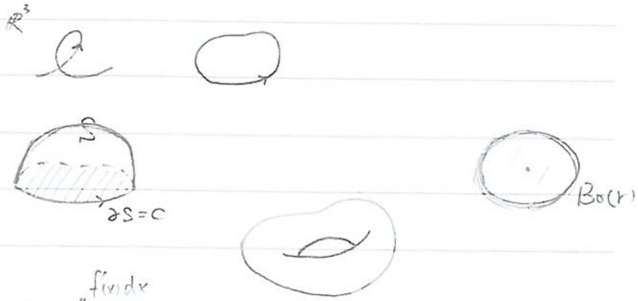
"vector caculus"

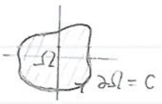
- Green's theorem
- Gauss' theorem (divergence thm)
- stoke's theorem



⇒ integration by parts

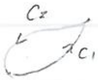
$\int_a^b f\,dg = fg - \int_a^b g\,df$ ← (let $f = g \cdot h$) $\int_a^b df = f\big|_a^b$ $f'(x)dx$

Green's theorem



$$\int_C P\,dx + Q\,dy = \int_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\vec{F} \cdot dx = (P, Q) \cdot (dx, dy)$$



$C = C_1 \cup (-C_2)$

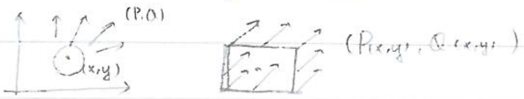
on $\mathbb{R}^2$ $\left[\begin{array}{l} \omega = P\,dx + Q\,dy \\ C = \partial\Omega \end{array}\right.$ 1-form

$\boxed{\int_{\partial\Omega} \omega = \int_\Omega d\omega}$ $\left(\begin{array}{l} \Omega = [a,b] \\ \omega = f \quad (0\text{-form}) \\ f\big|_a^b = \int_a^b f'\,dx \end{array}\right)$

$$d\omega = dP \wedge dx + dQ \wedge dy = (P_x\,dx + P_y\,dy) \wedge dx + (Q_x\,dx + Q_y\,dy) \wedge dy$$
$$= (Q_x - P_y)\,dx \wedge dy$$

$\left(\begin{array}{l} dx \wedge dy = -dy \wedge dx \\ df = \sum_i \frac{\partial f}{\partial x_i} dx_i \end{array}\right)$
total diff'

* An equivalent form   2-dim'l divergence thm



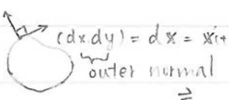
$\vec{F} = (P, Q)$

$\boxed{\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}}$ 散度

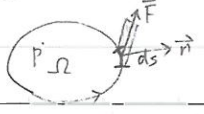
[假定已有 $\int_C P\,dx + Q\,dy = \int_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$]

↳ $\int_\Omega (P_x + Q_y)\,dx\,dy = \int_{\partial\Omega} -Q\,dx + P\,dy$

$$\int_\Omega \operatorname{div}\vec{F}\,dx\,dy = \int_\Omega (P_x + Q_y)\,dx\,dy = \int_{\partial\Omega} (-Q\,dx + P\,dy)$$
$$= \int_{\partial\Omega} \vec{F} \cdot (dy, -dx) = \int_{\partial\Omega} \vec{F} \cdot \vec{n}\,ds$$



$\underbrace{(dy, -dx)}$ 法向量 $(y'(t), -x'(t))dt$, length $= |x'(t)|dt = ds$



$\Rightarrow \lim_{\Omega \to P} \frac{1}{|\Omega|} \int_{\partial\Omega} \vec{F} \cdot \vec{n}\,ds$
$= \frac{1}{|\Omega|} \int_\Omega \operatorname{div}\vec{F}\,dA = \operatorname{div}\vec{F}$ at P

↑ 通量 flux

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$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$

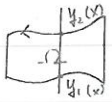
$* \int_{\partial\Omega} P\,dx + Q\,dy = \int_{\Omega} (Q_x - P_y)\, dx\, dy$

for any two C^1 functions P, Q in Ω $\qquad \cdot\ \partial\Omega = \cup C_i$, $C_i \in PC^1$ curve

It's enough to prove * for P & Q separately

$\boxed{\text{pf}}$ Green's thm for P

< case 1 > Ω is a region of the form

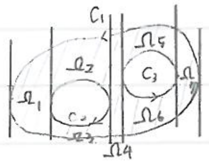


$RHS = \int_{\Omega} -P_y\, dx dy = -\int_a^b dx\ P(x,y)\Big|_{y=y_1(x)}^{y=y_2(x)}$

$= \int_a^b P(x, y_1(x))\, dx - \int_a^b P(x, y_2(x))\, dx = \int_{\partial\Omega} P\, dx = LHS$

< case 2 > Devide Ω into subregions Ω_i

s.t. exist Ω_i is of the form in step 1



$\partial\Omega = C_1 - C_2 - C_3$ (orientation)

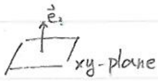
Then $RHS = \int_{\Omega} -P_y\, dx = \sum_{i=1}^{N} \int_{\Omega_i} -P_y\, dx$

similar pf works for Q, but with step 1 modified to be 

Q.E.D.

① $\int_{\Omega} \operatorname{div} \vec{F}\, dA = \int_{\partial\Omega} \vec{F} \cdot \vec{n}\, ds$

② $\int_{\Omega} \underbrace{(\operatorname{curl} \vec{F})\, \vec{e}_3}_{Q_x - P_y}\, dA = \int_{\partial\Omega} \vec{F}\, dx$



$\vec{F} = \nabla f$ (grad f) $\qquad \underset{(\nabla\cdot)}{\operatorname{div}} \nabla f = (f_x)_x + (f_y)_y = \Delta f$

①: $\int_{\Omega} \Delta f\, dA = \int_{\partial\Omega} \nabla f \cdot \vec{n}\, ds$

" $\frac{\partial f}{\partial n}$ " special notation for normal derivating

$\vec{F} = g \nabla f$, $\operatorname{div} \vec{F} = \nabla \cdot (g \nabla f) = \nabla g\, \nabla f + g \Delta f$

Green's formula

$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \cdot (P, Q)$

$\int_{\Omega} \nabla g \cdot \nabla f + g \Delta f = \int_{\Omega} g \frac{\partial f}{\partial n}\, ds$

$- \int_{\Omega} \nabla f \cdot \nabla g + f \Delta g = \int_{\Omega} f \frac{\partial g}{\partial n}\, ds$

$\int_{\Omega} f \Delta g - g \Delta f = \int_{\partial\Omega} (f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n})\, ds$

Stokes' Thm

$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$

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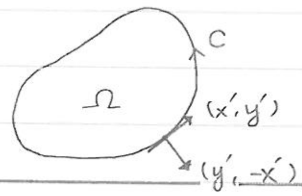
$$\int_C \overset{\text{1 form}}{\omega} = \int_{\underset{\text{closed curve}}{C=\partial\Omega}} P\,dx + Q\,dy \overset{\text{Green's thm}}{=\!=\!=} \int_\Omega (Q_x - P_y)\,dx\,dy$$

$\leftarrow P(x,y)\ \&\ Q(x,y) \in C^1$

$\oint_C \vec{F}\cdot d\mathbf{x}$ circulation $\quad \mathbf{x}(t) = (x(t), y(t))$, $\vec{F} = (P, Q)$

$\oint_C \vec{F}\cdot\vec{n}\,ds$ flux integral $\quad \vec{F} = (Q, -P)$



1. change of variable formula (CVF)

$I = \int_R f\,dxdy \qquad [\,x = x(u,v),\ y = y(u,v) \in C^2\,]$

<step 1> Let $f = Q_x$ for some Q

單変数積分 — change of variable

<step 2> $I = \int_R Q_x\,dxdy = \int_C Q\frac{dy}{dt}\,dt = \int_C Q\,(y_u du + y_v dv)$

$= \int_{C'} (Qy_u)\,du + (Q\,y_v)\,dv$

(apply Green's thm) $= \int_{R'} \big[\underbrace{(Qy_v)_u - (Qy_u)_v}\big]\,dudv$

$(Q_x x_u + Q_y y_u) y_v + \cancel{Q y_{uv}} - (Q_x x_v + Q_y y_v) y_u - \cancel{Q y_{uv}}$

$= \int_{R'} \overset{f}{Q_x} \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}$

* problem: y must be C^2 !

#

2. the area enclosed by a curve C

$A = -\int_C y\,dx = \int_C x\,dy = \frac{1}{2}\int_C x\,dy - y\,dx$

3. isoperimetric inequality 等周不等式

$\boxed{Q}$ Fix a length of a curve, find the maximal area enclosed by all such curve

$\boxed{A}$ $4\pi A \le \ell^2$

"=" holds iff C is a circle

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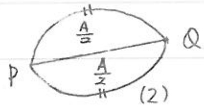
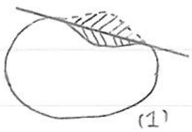
$\boxed{\text{pf 1 (古希臘)}}$ If C bounds Ω with the maximal area

<step 1> Ω is convex





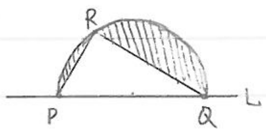
若否則鏡射處理

<step 2> for any $\widehat{PQ} = \frac{\ell}{2}$

$\widehat{PQ}$ devide Ω into 2 pieces of equal area

<step 3> only need to concide $P, Q \in L$

$\angle PRQ = \frac{\pi}{2}$ $\forall R \in \widehat{PQ}$ [with $|\widehat{PQ}| = \frac{\ell}{2}$ fixed]



fixed the shadow with P.Q allowed to more $\Rightarrow$ $\widehat{PQ}$ is a half circle

* problem: maxima Ω exist or not

※ Q.E.D.

$\boxed{\text{Poincaré}}$ $\rightarrow \int_0^\ell f^2\,dx \overset{?}{=\!=} \int_0^\ell (f')^2\,dx$

suppose that $\int_0^\ell f\,dx = 0$

then, $\int_0^\ell f^2\,dx \le \int_0^\infty |f'|^2\,dx$; "=" holds iff $f = a\cos t + b\sin t$

$\boxed{\text{pf}}$ $f = (a_1\cos t + b_1\sin t) + (a_2\cos 2t + b_2\sin 2t) + \cdots\cdots$

$f' = (-a_1\sin t + b_1\cos t) + (-2a_2\sin t + 2b_2\cos t) + \cdots\cdots$

$\frac{1}{\pi}\int_0^{2\pi} |f|^2 = a_1^2 + b_1^2 + a_2^2 + b_2^2 + \cdots\cdots$

$\frac{1}{\pi}\int_0^{2\pi} |f'|^2 = a_1^2 + b_1^2 + 2^2(a_2^2 + b_2^2) + \cdots\cdots$

$\boxed{\text{pf 2}}$ $\ell^2 \ge 4\pi A$, For simply, Let's assume $\ell = 2\pi$

$$\begin{cases} A = \int_C xy'\,dx \\ 2\pi = \int_0^{2\pi} \underbrace{(x'^2 + y'^2)}_{1}\,ds \end{cases}$$ [use arc length s as parameter of C

$\Rightarrow 2(\pi - A) = \int_0^{2\pi} (x'^2 + y'^2 - 2xy')\,ds$

$= \underbrace{\int_0^{2\pi} (x'^2 - x^2)\,ds}_{\ge 0} + \underbrace{\int_0^{2\pi} (x - y')^2\,ds}_{\ge 0}$

Poincaré (Fourier)

$\pi \ge A$

"=" holds iff $\begin{cases} x(s) = a\cos s + b\sin s \\ y(s) = a\sin s - b\cos s + c \end{cases}$

※ Q.E.D.

Double A

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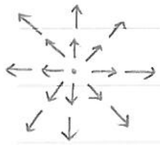
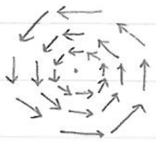
$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}\, ds = \int_{\Omega} \overset{\text{散度}}{\text{div}\,\vec{F}}\, dA$$

$$\int_{\partial\Omega} \vec{F}\cdot\vec{t}\, ds = \int_{\Omega} (\text{curl}\,\vec{F})_z\, dA$$

旋度 vorticity (vortex)

$$* \ \text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$





$$\Delta F = \text{div}\,\underset{\nabla F}{\underset{\parallel}{\vec{F}}} = 0$$

$\Delta F = 0$ homonic function

special case: $f = f(r)$

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

$$rf'' + f' = (rf')' = 0 \Rightarrow rf' = C$$

$$\Rightarrow f' = \frac{c}{r}$$

$$\Rightarrow f = c\log r + C_1$$

$$\vec{F} = \nabla f = \left(\frac{c}{2}\,\frac{2x}{x^2+y^2},\ \frac{c}{2}\,\frac{2y}{x^2+y^2} \right)$$

$(-y)$ (x)

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Green's theorem $\int_c P\,dx + Q\,dy$

→ Stoke's theorem $\int_c \vec{F}\cdot d\mathbb{x} = \int_\Omega (\mathrm{curl}\,\vec{F})_z\, dA$

(for dim = 2)

→ if $(\mathrm{curl}\,\vec{F})_z = 0$, then "locally" $\underset{(P,Q)}{\vec{F}} = \underset{(f_x, f_y)}{\nabla \vec{F}}$

$= Q_x - P_y$

→ Gauss' theorem $\int \vec{F}\cdot\vec{n}\, ds = \int_\Omega \mathrm{div}\,\vec{F}\, dA$ (flux integral)

→ when $\vec{F} = \nabla\vec{F} \Rightarrow \mathrm{div}\,\nabla\vec{F} = \Delta f$

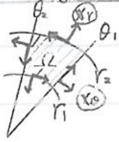
"$\Delta f = 0$" laplace equation (harmonic function)

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

⇒ Green's formula (integration by parts)

$$\int_\Omega \Delta f\, dA = \int_c \nabla f\cdot\vec{n}\, ds = \int_c \frac{\partial f}{\partial n}\, ds \qquad (\vec{F} = g\nabla f)$$

$$\int_\Omega (g\Delta f + \nabla g\cdot\nabla f)\, dA = \int_c g\frac{\partial f}{\partial n}\, ds$$



$$\int_\Omega \Delta f\, dA = \int_{\partial\Omega} \frac{\partial f}{\partial n}\, ds$$

$ds = r\,d\theta$

unit normal vector

$\because |\mathbb{x}_\theta| = r$

$$= \int_{\theta_1}^{\theta_2}\left(\frac{\partial f}{\partial r} r\Big|_{r=r_2} - \frac{\partial f}{\partial r} r\Big|_{r=r_1}\right) d\theta + \int_{r_1}^{r_2}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\Big|_{\theta=\theta_2} - \frac{1}{r}\frac{\partial f}{\partial\theta}\Big|_{\theta=\theta_1}\right) dr$$

$\mathbb{x}(r,\theta) = (x,y) = (r\cos\theta, r\sin\theta)$

$\mathbb{x}_r = (\cos\theta, \sin\theta) \quad \mathbb{x}_\theta = (-r\sin\theta, r\cos\theta)$

$$= \int_\Omega \left[\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right)\right] dr\,d\theta$$

$$= \int_\Omega \underbrace{\left\{\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(r\frac{\partial f}{\partial\theta}\right)\right]\right\}}_{\Delta f}\, dA \quad (r\,dr\,d\theta)$$

(let $\Omega \to$ a point) Ω_i

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* Divergence thm in $\mathbb{R}^3$

~~which can be partioned into a~~ finite union of simple regions Ω_i (x, y, z directions)

Let Ω be a bounded open set in $\mathbb{R}^3$ with $\partial\Omega$ be a surface S (which is C^1), and Let $\vec{F}$ be a C^1 vector field, then

$$\int_\Omega \operatorname{div}\vec{F}\, dV = \int_{\partial\Omega} \vec{F}\cdot\vec{n}\, dS$$

$$\int_\Omega (a_x + b_y + c_z)\, dx\,dy\,dz = \int_S a\, dy\,dz + b\, dz\,dx + c\, dx\,dy$$

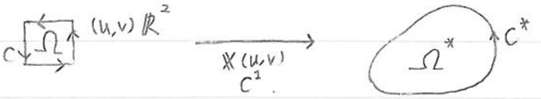
$\Omega \cup \partial\Omega = \bar{\Omega}$ closed set

In order to talk about the "boundary of surface" we need to consider the "induced topology" from $\mathbb{R}^3$ to S

$\boxed{\text{Definition}}$ A set $u \subset S$ is open iff $u = V \cap S$ for some open set $V \subset \mathbb{R}^3$

- a point $p \in S$ is an interior if $\exists$ open set $u \ni p$ s.t. u looks like a disk
- a point $p \in S$ is a boundary point of S if $\nexists\, u \ni p$ s.t. u looks like a disk

* 2-dim Green theorem



$$\int_{\Omega^*} (Q_x - P_y)\, dx\,dy \overset{?}{=} \int_{C^*} P\,dx + Q\,dy = \int_C P(x_u du + x_v dv) + Q(y_u du + y_v dv)$$

$$= \int_C (P x_u + Q y_u)\, du + (P x_v + Q y_v)\, dv$$

$$= \int_\Omega [\, (P x_v + Q y_v)_u - (P x_u + Q y_u)_v \,]\, du\,dv$$

$$= \int_\Omega \begin{pmatrix} P_u x_v + \cancel{P x_{uv}} - P_v x_u - \cancel{P x_{uv}} \\ + Q_u y_v + \cancel{Q y_{uv}} - Q_v y_u - \cancel{Q y_{uv}} \end{pmatrix} du\,dv$$

$$= \int_\Omega \underbrace{\begin{pmatrix} (\cancel{P_x x_u} + P_y y_u) x_v - (\cancel{P_x x_v} + P_y y_u) x_u \\ + (Q_x x_u + \cancel{Q_y y_u}) y_u - (Q_x x_v + \cancel{Q_y y_v}) y_u \end{pmatrix}}\, du\,dv$$

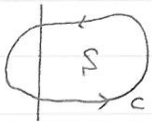
$$\hookrightarrow Q_x \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} - P_y \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

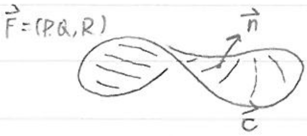
$$= (Q_x - P_y) \underbrace{\frac{\partial(x,y)}{\partial(u,v)}\, du\,dv}_{dx\,dy}$$

$$= \int_{\Omega^*} (Q_x - P_y)\, dx\,dy$$

* Stoke's thm in $\mathbb{R}^3$

$\int_C \vec{F}\, dx = \int_S \underbrace{(\text{curl}\,\vec{F})_z}_{\hookrightarrow\ \text{curl}\,\vec{F}\cdot\vec{n}}\, dA$





$S \subset \mathbb{R}^3$ be an oriented surface

(i.e. with a given continuously defined normal vector)

" $C=\partial S$ " has the inclosed (positive) orientation

Its primitive form is $\int_C P\,dx + Q\,dy + R\,dz$

$\nabla\times\vec{F} = \text{curl}\,\vec{F}$ $\quad\downarrow$

$\text{curl}\,\vec{F}\cdot\vec{n} = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ P & Q & R \end{vmatrix} \cdot (\cos\alpha, \cos\beta, \cos\gamma)$

$\downarrow$

$$= \int_S \begin{pmatrix} (R_y - Q_z)\cos\alpha\, ds \\ + (P_z - R_x)\cos\beta\, ds \\ + (Q_x - P_y)\cos\gamma\, ds \end{pmatrix}$$

$$= \int_S (R_y - Q_z)\,dy\,dz + (P_z - R_x)\,dz\,dx + (Q_x - P_y)\,dx\,dy$$

parameterise S by $X(u,v)$ $\quad \underset{\mathbb{R}^2}{\overset{}{\Omega}} \xrightarrow{X} \mathbb{R}^3$

$\partial S = C = C_1 \cup C_2$

$\boxed{pf}$ $\int_C P\,dx + Q\,dy + R\,dz$

$= \int_{C^*} (P x_u + Q y_u + R z_u)\,du + (P x_v + Q x_v + R x_v)\,dv$

Green's thm

$= \int_\Omega \left[(P x_v + Q y_v + R z_v)_u - (P x_u + Q y_u + R z_u)_v \right] du\,dv$

$= \int_\Omega \begin{pmatrix} P_u x_v + Q_u y_v + R_u z_v \\ - P_v x_v - Q_v y_u - R_v z_u \end{pmatrix} du\,dv$

$$= \int_\Omega \begin{pmatrix} P_y y_u x_v + P_z z_u x_v \\ - P_y y_v x_u - P_z z_v x_u \end{pmatrix} du\,dv + \begin{pmatrix} Q_x x_u y_v + Q_z z_u y_v \\ - Q_x x_v y_u - Q_z z_v y_u \end{pmatrix} du\,dv + \begin{pmatrix} R_x x_u z_v + R_y y_u z_v \\ - R_x x_v z_u - R_y y_v z_u \end{pmatrix} du\,dv$$

$$= \int_\Omega \begin{matrix} (Q_x - P_y) \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du\,dv & \rightarrow dx\,dy = \cos\gamma\, ds \\ (P_z - R_x) \begin{vmatrix} z_u & z_v \\ x_u & x_v \end{vmatrix} du\,dv & \rightarrow dz\,dx = \cos\beta \cdot ds \\ (R_y - Q_z) \begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix} du\,dv & \rightarrow dy\,dz = \cos\alpha \cdot \underline{ds} \end{matrix}$$

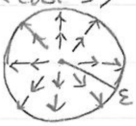
$\rightarrow ds = |X_u \times X_v|\, du\,dv$

$(\cos\alpha, \cos\beta, \cos\gamma) = \vec{n} = \dfrac{X_u \times X_v}{|X_u \times X_v|}$

$$\int_C \underbrace{P\,dx + Q\,dy + R\,dz}_{\omega} = \int_S (R_y - Q_z)\,dy\wedge dz + (P_z - R_x)\,dz\wedge dx + (Q_x - P_y)\,dx\wedge dy$$

$\boxed{Q.}$ what kind of f do we have?

<case 1> if $f = f(r)$



$$f'' + \frac{1}{r} f' = 0$$

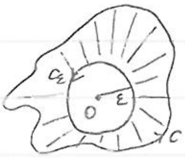
$$(rf')' = 0 \Rightarrow rf' = c \Rightarrow f = c \log r + c$$

$$\vec{F} = \nabla f = c\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right) \qquad r = (x^2+y^2)^{1/2}$$

$\boxed{c=1}$

$$\int_{C_\varepsilon} \vec{F}\cdot\vec{n}\, ds = \int_0^{2\pi} \frac{\sqrt{x^2+y^2}}{x^2+y^2}\, \varepsilon\, d\theta = \int_0^{2\pi} d\theta = 2\pi$$

↑ this is independent of ε!!



$$\int_C \vec{F}\cdot\vec{n}\, ds$$

$$\partial\Omega = C - C_\varepsilon$$

0 無旋度

$$\Rightarrow \int_{C-C_\varepsilon} \vec{F}\cdot\vec{n}\, ds = \int_\Omega \operatorname{div} \vec{F}\, dA = 0$$

(Remark)

$f(x(r,\theta))$

$\frac{\partial f}{\partial \theta} = \nabla f \cdot x_\theta$ ← not unit vector $|x_\theta| = r$

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)$$

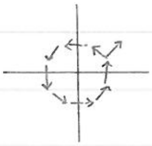
<case 2> if $f = f(\theta)$

$\Rightarrow f(\theta) = a\theta + b = a \tan^{-1}\frac{y}{x} + b$ (is not a well-defined function of (x,y))

$$\nabla f = a\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

$\int_C \vec{F}\cdot\vec{n}\, ds = 0$ ← (沒有 flux integral)

$$\int_C \vec{F}\cdot dx = f\Big|_{\theta=0}^{\theta=2\pi} = (a\theta + b)\Big|_{\theta=0}^{\theta=2\pi} = a\cdot 2\pi$$



Green's thm : $\int_{\partial\Omega} P\,dx + Q\,dy = \int_{\Omega} (Q_x - P_y)\,dxdy$

$$\int_{\partial\Omega} P\,dy - Q\,dx = \int_{\Omega} \underbrace{(P_x + Q_y)}_{\operatorname{div}\vec{F}}\,dxdy$$

$\vec{F} = (\overset{a(x,y,z)}{a}, b, c)$

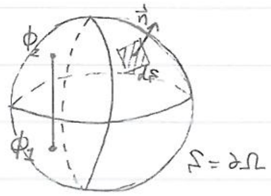
$\operatorname{div}\vec{F} = a_x + b_y + c_z$

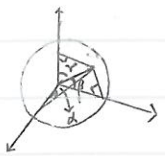
3D divergence theorem (Gauss' thm)

$$\int_{\Omega} \operatorname{div}\vec{F} \overset{dxdydz}{(dV)} = \int_{\partial\Omega} \vec{F}\cdot\vec{n}\, \underbrace{ds}_{\text{area on } \partial\Omega = S}$$

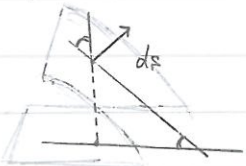
bounded any surface.

i.e. $\int_{\Omega} \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}\right) dxdydz$



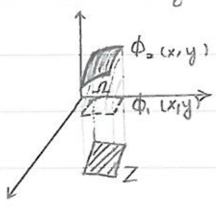


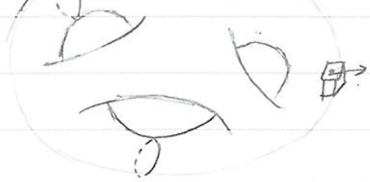
$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}\,ds = \int (a\cos\alpha + b\cos\beta + c\cos\gamma)\,ds$$
$$= \int_{\partial\Omega} a\,dydz + b\,dzdx + c\,dxdy$$



$\boxed{\text{pf}}$ of div. thm: (we prove the thm for "c")

<step 1> Ω is bounded by two graghs of functions $\phi_1(x,y)$ and $\phi_2(x,y)$ over some region in xy-plane





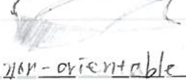
<step2> For g~~~~ Ω

Find a partition $\Omega = \cup \Omega_i$

each Ω_i is as in step 1 , $\partial\Omega = S$ #Q.E.D.

· Mobius band

不可定向曲面



non-orientable

$\boxed{\text{Thm}}$ Jordan curve theorem

Any "closed" surface S in $\mathbb{R}^3$ is orientable and $S = \partial\Omega$ for a bounded region Ω

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$R \xrightarrow[g]{(C^1)} R'$

$(x,y) \longmapsto (u,v)$

$\int_{R'} f(u,v)\, du dv = \int_R f \frac{\partial(u,v)}{\partial(x,y)}\, dx dy$

↳ ① if g is 1-1 and $J(g) = \frac{\partial(u,v)}{\partial(x,y)} > 0$

$= -\int_R f \frac{\partial(u,v)}{\partial(x,y)}\, dx dy$

↳ ② if g is 1-1 but $J(g) < 0$

If g is 1-1

Define $\varepsilon_R(u,v) = \begin{cases} 0 & (u,v) \notin \mathrm{Img} \\ \mathrm{sign}\left(\frac{\partial(u,v)}{\partial(x,y)}\right) & (u,v) = g(x,y) \end{cases}$

CVF : $\int f \varepsilon_R\, du dv =$ " $\int_R f \frac{\partial(u,v)}{\partial(x,y)}\, dx dy$ "

general case :

$R \xrightarrow{g} R'$ not 1-1, but $R = \bigcup_{i=1}^{m} R_i$ st. g is 1-1 on R_i

(Let $C_i = \partial R_i$)

$\int_R f \frac{\partial(u,v)}{\partial(x,y)}\, dx dy = \sum_{i=1}^{m} \int_{R_i} f \frac{\partial(u,v)}{\partial(x,y)}\, dx dy$

$= \sum_i \int_{\mathbb{R}^2} f \cdot \varepsilon_{R_i}\, du dv$

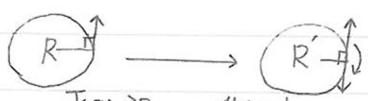
$= \int_{\mathbb{R}^2} f \underbrace{\sum_{i=1}^{m} \varepsilon_{R_i}}_{=: \chi_R(u,v)}\, du dv$

$=: \chi_R(u,v)$: degree of the mapping g at (u,v)

Identity : $(*)$ $\chi_R(u,v) = \mu_c(u,v)$

└ winding number of c' = image of c at point (u,v)

(i) $*$ is true if $R \to R'$ is 1-1



$J(g) > 0$ $\mu_c = 1$

$J(g) < 0$ $\mu_c = -1$

(ii) $*$ is additive

$R = \cup A_i = C$

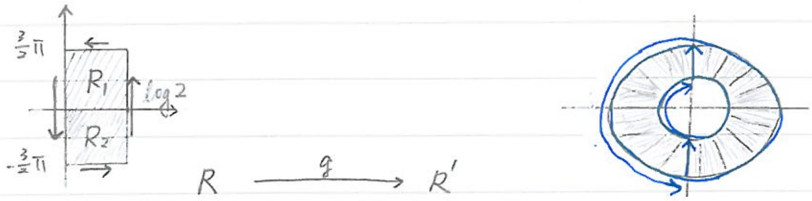
$\partial R = \cup \underset{C_i}{\partial A_i}$

$\chi_R(u,v) = \sum_i \chi_{A_i}(u,v)$

$= \sum_i \mu_{C_i}(u,v) = \mu_c(u,v)$

(Example) $u = e^x \cos y$ $\qquad [\, u + iv = e^x(\cos y + i \sin y) = e^{x+iy} \,]$

$v = e^x \sin y$



$R \xrightarrow{g} R'$

harmonic function

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$\Delta f = 0$ $\qquad$ " $f = f(r)$, $f = f(\theta)$ "

$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ Cauchy - Riemann eq^n

$$\begin{cases} u_x = v_y \\ v_x = -u_y \end{cases} \qquad \Delta u = u_{xx} + u_{yy} = v_{yx} + (-v_{xy}) = 0$$

$\vec{F} = \nabla f$

$w = z^3 = (x + iy)^3$, $\quad w = e^z$

in general we need " $f(x)$ to be analytic "

i.e. $f(x)$ = Taylar series $\qquad$ " $f(z)$ well defined "

$$w = x^3 + 3x^2 \cdot iy - 3xy^2 - iy^3$$
$$= \underbrace{(x^3 - 3xy^2)}_{u(x,y)} + i\underbrace{(3x^2 y - y^3)}_{v(x,y)}$$

$u_x = 3x^2 - 3y^2 = v_y$, $\quad u_y = -v_x$