Introduction to Calabi-Yau Manifolds

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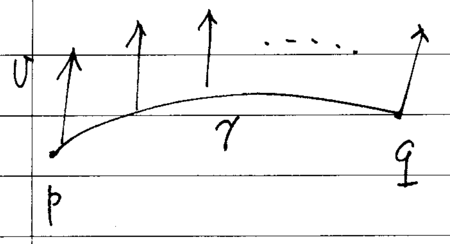
Calabi-Yau Manifolds

Levi-Civita connection, Riem curv.

(M,g) Riem. mfd (connected). ∇, $R = \nabla^2$



$\varphi_\gamma : T_pM \longrightarrow T_qM$ parallel. $\Rightarrow$ isometry (i.e. $\nabla_{\dot\gamma} V = 0$) called "holonomy" along γ.

$\Omega(p)$: loop group at p.

$\varphi : \Omega(p) \longrightarrow O(T_pM)$ holonomy representation

$H := \operatorname{Im}\varphi$ is called the "Holonomy Group".

$H \subset O(n)$. let H_0 = conn. component $\subset SO(n)$

If M orientable, then $H \subset SO(n)$.

de Rham decomposition thm:

If (M,g) complete Riem. $\pi_1(M) = 0$ (i.e. any loop can be shrinked to a point), then

$$M \cong M_1 \times \cdots \times M_k \quad \text{isometric}$$

if $H \cong H_1 \times \cdots \times H_k$.

So, (M,g) irreducible $\Longleftrightarrow$ H irreducible.

H is "smaller" $\longleftrightarrow$ M is "simpler"

H is special $\longleftrightarrow$ M is special

Berger Classification thm:

If (M,g) is not locally symmetric (l.s. $\Leftrightarrow \nabla R = 0$)

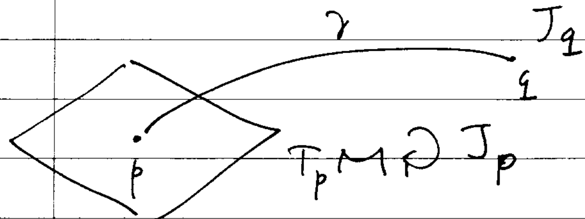
then $H_0 \subset SO(n)$ is one of the following:

$SO(n)$	$Sp(r)$ $\}$ $n = 4r$	$Spin(9)$	$n = 16$
$U(m)$ $\}$ $n = 2m$	$Sp(r) \times Sp(1)$ $\}$ $n = 4r$	$Spin(7)$	$n = 8$
$SU(m)$ $\}$ $n = 2m$		G_2	$n = 7$

Explaination:

- $H = SO(n)$ "General" Riem mfd (M, g) should has this holonomy group.
- $H = U(m) \subset SO(2m)$ as the subgroup preserving the standard cpx structure J on $\mathbb{R}^{2m}$ (i.e. $\mathbb{R}^{2m} \cong \mathbb{R}^m \oplus i\mathbb{R}^m \cong \mathbb{C}^m$, J = mult. by i)

Holonomy Principle:



May parallel translate the tensor J_p to any point q along any γ get well defined J_q ($\exists!$)

$\Rightarrow$ J is a global parallel tensor field ($\nabla J = 0$)

Lemma: $\nabla J = 0 \Leftrightarrow (M, g)$ is a Kähler mfd.

Def. A Kähler mfd is a complex mfd (M, J) with a hermitian metric g (i.e. g is J-inv) s.t. the fundamental 2 form $\omega(X, Y) := g(X, JY)$ is closed: $d\omega = 0$.

Rmk: In cpx coor. $z_1, \ldots, z_m$

$$g = \sum g_{\alpha\bar\beta}\, dz^\alpha \otimes d\bar z^\beta$$

$$\omega = \sum g_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta$$

and $d\omega = 0 \Leftrightarrow \partial_\gamma g_{\alpha\bar\beta} = \partial_\alpha g_{\gamma\bar\beta}$; $\partial_{\bar\gamma} g_{\alpha\bar\beta} = \partial_{\bar\beta} g_{\alpha\bar\gamma}$.

e.g. any submfd $X \subset \mathbb{P}^n$ with the Fubini-Study metric $\omega_{FS}|_X$

$$\omega_{FS} = \frac{i}{2\pi} \partial\bar\partial \log |Z|^2 \quad ; \quad Z = (z_0 : \ldots : z_n)$$

homog. coor. in $\mathbb{P}^n$.

No.

- $H = SU(m) \subset U(m) \subset SO(2m)$

  (M, g) is a Kähler manifold, moreover since $SU(m) = \{ A \in U(m) \mid \det A = 1 \}$ may parallel the volume form $e_1^* \wedge \cdots \wedge e_m^*$ at p to get a "no-where vanishing" m form of type $(m, 0)$ Ω: $\Omega(z) = f(z)\, dz_1 \wedge \cdots \wedge dz_m$ which is parallel: $D\Omega = 0$, hence holomorphic.

$$\left(\underset{\text{parallel}}{D\Omega = 0} \Rightarrow \underset{\text{closed}}{d\Omega = 0} \Rightarrow \underset{\text{holomophic}}{\bar\partial \Omega = 0} \right)$$

$\Omega \in \Gamma(X, \Omega_M^m)$, $\Omega_M^m = K_M$ canonical (line) bundle

$D\Omega = 0 \iff K_X$ with metric g is a flat bundle

$\iff$ Ricci Curvature $Ric(M, g) \equiv 0$.

We have shown that
(M, g) has $H \subseteq SU(m) \iff$ Kähler + Ricci flat $Ric(g) = 0$
and a necessary condition is that K_M is trivial.

Yau's theorem on Calabi's Conjecture:
let M be a compact cpx mfd with K_M trivial ($\Leftrightarrow c_1(M) = 0$) then there exists an unique Ricci flat Kähler metric in each Kähler class.

Definition:
Such M's are called "Calabi-Yau Manifolds".

Examples:

$M = \mathbb{C}^m / \Lambda$ flat tori

$M =$ Smooth hypersurface of degree $(n+1)$ in $\mathbb{P}^n$ or, complete intersections of degree $(d_1, \cdots, d_r)$ in $\mathbb{P}^n$ st. $\sum d_i = n+1$.

Paper House

No.

CY mfd with $H \equiv SU(m)$ are also called "unitary" CY mfd.

- $H = Sp(r) \subset SU(2r) \subset SO(4r)$:

These are "symplectic CY mfds" or "hyperkähler mfds".
Since $Sp(r) = \mathrm{Aut}_{\mathbb{H}}(\mathbb{H}^r)$ preserving a form g

$\mathbb{H}$ = quaternionian numbers $= \mathbb{R} + \mathbb{R}I + \mathbb{R}J + \mathbb{R}K$

g is an alternating $\mathbb{C}$-linear 2 form
$\Rightarrow \exists$ parallel (hence holomorphic) 2 form g on M which is nondegenerate
TM has $\mathbb{H}$-structure, hence a set of cpx structures $\{aI + bJ + cK \mid a^2+b^2+c^2 = 1\} \cong S^2$.
The corr. family is called the twistor of (M, I).

$$\begin{array}{ccc} \mathcal{M} & \supset & M \\ \downarrow & & \downarrow \\ S^2 & \ni & (1,0,0) = I \end{array}$$

Theorem: Any cpt Kähler mfd of $\dim M = 4r$ and a nondegenerate holomorphic 2 form g is hyperkähler.

Sketch: in fact such g gives $g^r \in \Gamma(M, K_M) \Rightarrow K_M$ trivial
so $\exists$ Ricci flat Kähler metric ω (Yau's thm)
Now Bochner principle $\Rightarrow$ any holomorphic forms are parallel, hence $\nabla\varphi = 0$.
but non-deg 2 form $\longleftrightarrow$ almost cpx structure
hence

$$\varphi + \bar{\varphi} \longleftrightarrow J$$

$$\nabla(\varphi + \bar{\varphi}) = 0 \Leftrightarrow J \text{ is integrable \& Kähler}$$

and $i(\varphi - \bar{\varphi}) \longleftrightarrow K$ etc. $\square$

Rmk: The Ricci flat metric is the same for all twistor family.

Examples: K3 surface: $X = (4) \subset \mathbb{P}^3$

eg. $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$

In this case K_X = trivial $\Rightarrow$ holo 2 form non-deg.
hence hyperkähler.

Hilbert scheme of r points in X:
$X^{[r]} \longrightarrow X^{(r)} = (X \times \cdots \times X)/S_r$ resolution
is again hyperkähler. (Beauville)

No other C-Y hypersurface can be Hyperkähler
since $H^2(X, \mathbb{Z}) = 1 \Rightarrow H^{2,0}(X) = H^0(X, \Omega_X^2) = 0$
by Lefschetz hyperplane section thm.

- $H = G_2$, Spin(7) special holonomy groups.
  (M, g) with such H are just constructed and studied recently by Joyce. (JDG)

Structure Theorem for Calabi-Yau Manifolds.
Let M be a cpt Calabi-Yau Manifold. Then $\exists$ finite covering $\tilde{M} \longrightarrow M$ st.

$$\tilde{M} = T \times \prod V_i \times \prod X_j$$

with $T = \mathbb{C}^k/\Lambda$ (tori),
V_i simply connected SU manifold
X_j simply connected Sp mfd.

Rmk: V may be characterized by $H^0(V, \Omega_X^p) = 0$ for all $0 < p < m$ ($\Rightarrow$ V projective)
X may be characterized by
- $\exists !$ holo. 2-form non-deg at any pt. $= \sigma$
- $H^0(X, \Omega_X^p) = 0$ p odd, $H^0(X, \Omega_X^{2q}) = \mathbb{C} \cdot \sigma^q$.

Main progress in the 80's:

- Periods of C-Y mfds (3-fold case) by Bryant / Griffiths

local Torelli thm:

M is locally distinguishable by its period point

$$M \longmapsto H^{3,0}(M) \subset \mathbb{P}(H^3(X,\mathbb{C}))$$

- Moduli Space of C-Y mfds by Bogomolov / Tian / Todorov:

  $\mathrm{Def}(M)$ is smooth of dimension $h^1(M,T) = h^{n-1,1}(M)$

- Weil-Petersson metric on Moduli Space. / Strominger "Special Geometry"

Main Problems in this area:

I. Finiteness conjecture of components of moduli

eg. for C-Y 3-folds as complete intersections

$$\sum_{i=1}^{r} d_i = n+1 \;;\; n = r+3$$

$$\Rightarrow \sum_{i=1}^{r}(d_i - 1) = 4 \qquad (d_i \geq 2)$$

$$\left(\begin{array}{llll} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & \\ 2 & 2 & & \\ 3 & 1 & & \\ 4 & & & \end{array}\right. \quad \Rightarrow \quad \begin{array}{l} (2,2,2,2) \subset \mathbb{P}^7 \\ (3,2,2) \subset \mathbb{P}^6 \\ (3,3) \;;\; (4,2) \subset \mathbb{P}^5 \end{array}$$

only 5 of them! $(5) \subset \mathbb{P}^4$

But $\exists \sim 10000$ CY 3folds in Toric varieties!

II. Connectedness Conjecture (M. Reid)

$$\begin{array}{c} \tilde{X} \\ \downarrow \\ X \rightsquigarrow X_0 \text{ degenerate} \end{array}$$

$\tilde{X}$ resolution of singularities st. $\tilde{X}$ is still C-Y

III. Mirror Symmetry Conjecture: (String theory)

eg. CY 3folds X, $\tilde{X}$ st.

$h^{1,1}(X) = h^{2,1}(\tilde{X})$; $h^{2,1}(\tilde{X}) = h^{1,1}(X)$ etc..

Strominger - Yau - Zaslow conjecture (1996). $\boxed{\text{End}}$