A NOTE FOR THE CHANGE OF VARIABLE FORMULA IN ARC SPACES

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Let $\phi : Y \to X$ be a proper birational morphism between two n-dimensional complex smooth varieties. Let $S_k \subset \mathcal{L}(Y)$ be the subset of formal arcs such that $\mathrm{ord}_t J(\phi) = k$. What is the structure of $\phi_* : S_k \to \mathcal{L}(X)$? We will see later that in the set level ϕ_* is almost an one to one map. However, the important observation which leads to the change of variable formula in motivic integration is that this map is indeed a piece-wise trivial $\mathbb{C}^k$ fibration onto its image when one considers it in formal arcs of certain finite level.

More precisely, instead of working on the infinite dimensional spaces $\mathcal{L}(Y)$ and $\mathcal{L}(X)$, one may consider the following diagram about truncations to discuss only algebraic varieties $\mathcal{L}_m(Y)$ and $\mathcal{L}_m(X)$:

$$\begin{tikzcd} \mathcal{L}(Y) \arrow[r, "\pi_m"] \arrow[d, "\phi"'] & \mathcal{L}_m(Y) \arrow[d, "\phi_m"] \\ \mathcal{L}(X) \arrow[r, "\pi_m"'] & \mathcal{L}_m(X) \end{tikzcd}$$

The map $\phi_0 = \phi$ is the original map. The map ϕ_1 is the tangent map $T_Y \to T_X$ since the first order arc is nothing but the tangent space.

Denef and Loeser showed that

Theorem 0.1. *For each $k \in \mathbb{N}$, there exists $m_k \in \mathbb{N}$ such that for $m \geq m_k$, $\pi_m S_k$ is a union of fibers of ϕ_m. Moreover, $\phi_m|_{\pi_m S_k}$ is a piece-wise trivial $\mathbb{C}^k$ fibration onto its image.*

This was also proved by them for singular variety X, with suitable modifications on the set S_k. Here we concern only the smooth case. In the smooth case π_m are all surjective (every finite arc can be lifted), so we may also define S_k directly on certain large enough level m.

The purpose of this note is to demonstrate a key step in the proof of Denef and Loeser's result. We will first do it for a simple blowing-up at one smooth point in §1, then in §2 we give the proof for the general case using the inverse function theorem (or Hensel's lemma). §2 is independent of §1, however §1 gives a down-to-the-earth treatment which I feel also helpful in understanding the real content of change of variable formula.

1. A Simple Blowing-Up Example

Consider a blowing-up $\phi : Y := \tilde{\mathbb{C}}^n \to X := \mathbb{C}^n$ at one smooth point $0 \in \mathbb{C}^n$. For the affine open set U_1 of Y with coordinates $(y_1, \cdots, y_n)$, the map ϕ takes the

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form

$$x_1 = y_1, \quad x_2 = y_1 y_2, \quad \cdots \quad x_n = y_1 y_n,$$

where $(x_1, \cdots, x_n)$ are coordinates of X. The Jacobian $J := J(\phi) = (y_1^{n-1})$ which corresponds to the divisor $(n-1)E$ which appears in the holomorphic change of variable formula $K_Y = \phi^* K_X + (n-1)E$, where $E = \phi^{-1}(0)$ is the exceptional divisor. In this affine chart it is defined by $y_1 = 0$.

A formal arc $\gamma \in \mathcal{L}(Y)$ is represented by $\gamma(t) = (y_1(t), \cdots y_n(t))$. Then $\mathrm{ord}_t J(\gamma) = \mathrm{ord}_t(y_1(t)^{n-1}) = s(n-1)$ if $y_1(t) = a_{1s}t^s + \cdots$ with $a_s \neq 0$. That is, $S_k \neq \emptyset$ only for k being of the form $k = s(n-1)$. The first (and trivial) case is $k = 0$ hence $s = 0$ too (i.e. $a_{10} \neq 0$). In this case, (y_i) is unique solvable by (x_i). By putting together all n affine open sets $U_0, \ldots, U_n$ of the standard covering of Y we simply get $S_0 = Y \backslash E \cong X \backslash 0$, a trivial $\mathbb{C}^0$ fibration. In this case m_0 can be any non-negative integer.

The next case is $k = n-1$ and $s = 1$. That is, $y_1(t) = a_{11}t^1 + \cdots$ with $a_{11} \neq 0$. From

$$x_1(t) = y_1(t), \quad x_2(t) = y_1(t)y_2(t), \quad \cdots \quad x_n(t) = y_1(t)y_n(t),$$

we see that $t | x_i(t)$ for all i. Moreover, this is the only condition needs to be satisfied for $x_i(t)$, $2 \leq i \leq n$. The image of ϕ_* consists of all $x(t) = tv(t) = t(v_1(t), \cdots, v_n(t))$ with $t \not| v_1(t)$. By gluing together over the standard affine covering, we see the image consists of all $x(t) = tv(t)$ with $t \not| v_i(t)$ for some i (i.e. $v(0) \neq 0$). Also from the formula we see that $y(t) \in \mathcal{L}(Y)$ is uniquely solvable for any such $x(t)$. This shows that $\phi_* : S_{n-1} \to \mathcal{L}(X)$ is one to one. The similar argument also shows that $\phi_* : S_k \to \mathcal{L}(X)$ is one to one for any $k = s(n-1)$.

In order to look at finite truncations, we first claim

Lemma 1.1. *Let $\phi_* y(t) = x(t)$ and $y(t) \in S_k$. Then for $\tilde{x}(t) = x(t) + t^\ell v$ with $\ell \geq k$ and $t \not| v$, there exists an unique $\tilde{y}(t) = y(t) + t^{\ell-k}u$ such that $\phi_* \tilde{y}(t) = \tilde{x}(t)$.*

Proof. Since $v(0) \neq 0$, by reordering the variables we may assume that $v_1(t) \neq 0$. Then we will perform the computations in the chart U_1 on Y.

For $\tilde{x}_1(t) = \tilde{y}_1(t)$, we need to solve $x_1(t) + t^\ell v_1 = y_1(t) + t^{\ell-k}u_1$. We may simply set $u_1 = t^k v_1$.

For $2 \leq i \leq n$, to solve $\tilde{x}_i(t) = \tilde{y}_1(t)\tilde{y}_i(t)$ we need to solve

$$\begin{aligned} x_i(t) + t^\ell v_i &= (y_1(t) + t^\ell v_1)(y_i(t) + t^{\ell-k}u_i) \\ &= y_1(t)y_i(t) + t^\ell v_1 y_i(t) + t^{\ell-k}(y_1(t) + t^\ell v_1)u_i. \end{aligned}$$

This is equivalent to solve u_i from the equation

$$t^{\ell-k}(y_1(t) + t^\ell v_1)u_i = t^\ell(v_i - v_1 y_i(t)).$$

The condition $y(t) \in S_k$ means that $\mathrm{ord}_t\, y_1(t)^{n-1} = k$. That is, $\mathrm{ord}_t\, y_1(t) = k/(n-1)$. In particular, the order in t in the LHS is $\ell - k + k/(n-1) \leq \ell$. Since the order in the RHS is at least ℓ, we see that u_i can be uniquely solved. $\square$

Proof of the theorem in our special case. Suppose that we are given the equation $\phi_*(y(t)) = x(t)$ with $y(t) \in S_k$, $k = s(n-1)$. Notice that in order for the solution $\tilde{y}(t)$ in the above lemma to be in S_k, we need to require that $\ell \geq k/(n-1) + 1$. It would be sufficient to require that $\ell \geq k+1$. (Indeed, for general birational morphisms ϕ, the number $n-1$ appears as the codimension (of the blowing-up center) minus one, so in general it can take the value 1.) We will show that $m_k := k+1$ will be enough for the theorem to be true.

Let $m \geq m_k$. By applying the lemma to the case $\ell = m+1$, we see that in order to solve $\phi_m(\tilde{y}(t) \mod t^{m+1}) = x(t) \mod t^{m+1}$, we may assume that $\tilde{y}(t) = y(t) + t^{m-1+k}u$. The solutions will be the residue classes $\bar{u} = u \mod t^k$. This space is $\mathbb{C}^{kn}$, but we will show that only a k-dimensional subspace will give rise to solutions.

From $x_1(t) = y_1(t) + t^{m+1-k}u_1 \mod t^{m+1}$ we see that $t^k | u_1$ hence that $\bar{u}_1 = 0$. For other i with $2 \leq i \leq n$, the equation

$$x_i(t) = y_1(t)(y_i(t) + t^{m+1-k}u_i) \mod t^{m+1}$$

implies that $y_1(t)t^{m+1-k}u_i = 0 \mod t^{m+1}$. That is, $t^k | y_1(t)u_i$. Since $\mathrm{ord}_t y_1(t) = s$, we get $\mathrm{ord}_t u_i \geq k-s$. Equivalently the solutions of u_i has dimension $k-(k-s) = s$ by counting the number of coefficients of $u_i(t) \mod t^k$. Since there are $n-1$ such i's, we see the total solutions have dimension $s(n-1) = k$. $\square$

Remark 1.2. By adding a few extra coordinates, the same argument also proves the theorem in the case that $\phi : Y \to X$ is the blowing-up of a smooth variety X along a smooth center.

2. The General Case for Smooth Varieties

Now for a birational morphism $\phi : Y \to X$ with Y and X smooth, ϕ naturally induces a map $\phi_* : \mathcal{L}(Y) \to \mathcal{L}(X)$. If $K_Y = \phi^* K_X + E$ with E a normal crossing divisor, the change of variable formula of Denef and Loeser states that

$$\int_S \mathbb{L}^{-f} d\mu_X = \int_{\phi^{-1}(S)} \mathbb{L}^{-f\circ\phi_* - \mathrm{ord}_t J\phi} d\mu_Y.$$

Here $J\phi := \mathcal{O}_Y(-E)$ is the ideal sheaf generated by the holomorphic Jacobian factor, $\mathrm{ord_t}\, \mathcal{I} : \mathcal{L}(X) \to \mathbb{N} \cup \{0\}$ for any ideal sheaf $\mathcal{I}$ is the function of minimal degree in t. Namely for $\gamma \in \mathcal{L}(X)$, $\mathrm{ord_t}\, \mathcal{I}(\gamma) := \min_{g\in\mathcal{I}} \deg_t g \circ \gamma(t)$.

We do not define the motivic integration here. We only remark that this formula follows from Theorem 0.1 once we know the definition of integration. Also we will only prove that the fiber of $\phi_m|_{S_k}$ is $\mathbb{C}^k$ and ignore the piece-wise-trivial-fibration statement since it involves other tools in doing so (Boolean algebra and semi-algebraic geometry).

The proof is indeed an application of the inverse function theorem over power series rings which traces carefully the orders in t. Let $\phi : Y \to X$ be the birational morphism with $E_{\mathrm{red}} \subset Y$ and $Z \subset X$ be the exceptional loci in Y and X respectively.

For each $k \in \mathbb{N} \cup \{0\}$ let $S_k \subset \mathcal{L}(Y)$ be the subset $\gamma \in \mathcal{L}(Y)$ such that $\mathrm{ord}_t\, J\phi(\gamma) = k$. By the inverse function theorem, the map $\phi_* : \mathcal{L}(Y) \to \mathcal{L}(X)$ is a bijection between $\mathcal{L}(Y)^\times := \mathcal{L}(Y)\backslash\mathcal{L}(E_{\mathrm{red}})$ and $\mathcal{L}(X)^\times := \mathcal{L}(X)\backslash\mathcal{L}(Z)$, thus there is no interesting geometry on the map $\phi_*|_{S_k} : S_k \to \mathcal{L}(X)^\times$. However, the important observation by Denef and Loeser is that when one takes finite truncations in

$$\begin{array}{ccc} \mathcal{L}(Y) & \xrightarrow{\pi_m} & \mathcal{L}_m(Y) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi_m} \\ \mathcal{L}(X) & \xrightarrow[\pi_m]{} & \mathcal{L}_m(X) \end{array}$$

for all large enough m the induced map $\phi_m|_{\pi_m(S_k)} : \pi_m(S_k) \to \mathcal{L}_m(X)$ is indeed a piece-wise trivial $\mathbb{C}^k$ fibration over its image. Together with the fact that $\mathcal{L}(Z)$ is of *measure zero* in $\mathcal{L}(X)$, this will imply the change of variable formula.

To investigate the fibration structure near one arc $\gamma \in \mathcal{L}(Y)$, it is enough to restrict the map to formal neighborhoods $\phi : \hat{\mathbb{C}}^n_{(0)} \to \hat{\mathbb{C}}^n_{(0)}$. Or equivalently to represent ϕ by an algebraic map (still called ϕ) on power series $\phi : \mathbb{C}[[t]]^n \to \mathbb{C}[[t]]^n$ with $\phi(0) = 0$. Let $\phi(y(t)) = x(t)$ with $y(t) \in S_k$ and let $\ell \geq 2k+1$. We first notice that for each $v \in \mathbb{C}[[t]]^n$, there is a unique solution $u \in \mathbb{C}[[t]]^n$ of the equation

$$\phi(y(t) + t^{\ell-k}u) = x(t) + t^{\ell}v.$$

Indeed by Taylor's expansion

$$\phi(y(t) + t^{\ell-k}u) = \phi(y(t)) + D\phi(y(t))t^{\ell-k}u + t^{2(\ell-k)}R(t,u).$$

Let $A = D\phi(y(t))$. The equation becomes $Au + R(t,u)t^{\ell-k} = t^k v$. That is,

$$u = (\det A)^{-1}t^k A^*(v - R(t,u)t^{\ell-2k}).$$

Here A^* is the adjoint matrix of A. Since $\operatorname{ord}_t \det A = \operatorname{ord}_t J\phi(y(t)) = k$, the term $(\det A)^{-1}t^k$ has order zero. Also since $\ell - 2k \geq 1$, by repeated substitutions this relation solves u as a vector in formal power series.

Now let $m \geq 2k$ and let $\ell = m+1$. The above discussion shows that in order to find all solutions of $\phi(\tilde{y}(t) \mod t^{m+1}) = x(t) \mod t^{m+1}$, we may assume that $\tilde{y}(t) = y(t) + t^{m+1-k}u$. Notice that the residue classes $\bar{u} = u \mod t^k$ form a linear space isomorphic to $\mathbb{C}^{nk}$. By Hensel's lemma, in order to count the solutions we may simply consider the equation $At^{m+1-k}\bar{u} = 0 \mod t^{m+1}$. That is, $A\bar{u} = 0 \mod t^k$. Since $\operatorname{ord}_t \det(A^*) = (n-1)k$, the solution space of $\bar{u}$ has dimension $nk - (n-1)k = k$ as expected.

This verifies that $\phi_m^{-1}\bar{x}(t) \cong \mathbb{C}^k$. The piece-wise triviality needs other tools to prove it, which will not be reported here. For the complete details the readers are referred to the original paper.

Remark 2.1. For $S = \mathcal{L}(X)$ and $E = \sum_{i=1}^n e_i E_i$ a normal crossing, the change of variable formula gives

$$[X] = \int_{\mathcal{L}(X)} \mathbb{L}^0 d\mu_X = \sum\nolimits_{I\subset\{1,\ldots,n\}} [E_I^\circ] \prod\nolimits_{i\in I} \frac{\mathbb{L}-1}{\mathbb{L}^{e_i+1}-1}.$$

Since $\mathbb{L}^{e+1} - 1 = (\mathbb{L}-1)[\mathbb{P}^e]$, this lives in the localization of $K_0(\mathrm{Var}_{\mathbb{C}})$ in projective spaces.