Intro to Donaldson's theory

* M cpt $\pi_1(M) = 0$, 4-dim, top mfd

$\Rightarrow$ M is orientable

$H^2(M, \mathbb{Z})$ is free abelian

intersection form: q_M as quad form $/\mathbb{Z}$

$$H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \longrightarrow \mathbb{Z}$$

via cup product $(\alpha, \beta) \mapsto (\alpha \cup \beta)[M]$

when M smooth, q_M may be computed from

- diff form $\int_M \alpha \wedge \beta$
- intersection, repr α, β by "surfaces" A, B, may assume $A \pitchfork B$

Poincaré duality $\Rightarrow$ q_M is unimodular

i.e. $\det(\text{matrix } q_M) = \pm 1$.

Theorem (Whitehead 1949)

for M satisfies *, the homotopy type of M is determined by q_M.

Theorem (Freedman 1981)

the homeomorphism type of M is det. by q_M if q_M is even. Up to 2 choices if q_M is odd. Every q is realizable.

Here: q_M is even iff $q_M(\alpha,\alpha) \in 2\mathbb{Z}$

i.e. diagonal entries are all even.

q_M is odd if otherwise.

Examples:

① $M = S^4$, $H^2(S^4, \mathbb{Z}) = 0$, $q_M = 0$. (so even)

Freedman's thm $\Rightarrow$ 4-dim'l Poincaré conj
(topological version)

(for dim ≥ 5, this is due to Smale)

② $M = S^2 \times S^2 = \mathbb{C}P^1 \times \mathbb{C}P^1$

$H_2(M, \mathbb{Z}) \simeq \mathbb{Z}^2$, gen by $a = S^2 \times pt$, $b = pt \times S^2$

(This follows from Künneth formula

$$H_2(S^2 \times S^2) = H_2(S^2) \otimes H_0(S^2) \oplus H_1(S^2) \otimes H_1(S^2) \oplus H_0(S^2) \otimes H_2(S^2) \oplus \text{torsion}.)$$

clearly $a^2 = 0$, $ab = 1$, $b^2 = 0$

so $q_M \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Notice: $q_M \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ over $\mathbb{R}$, but not $\mathbb{Z}$!
in fact over $\mathbb{Q}$ is enough

③ $M = \mathbb{C}P^2$, $H_2(M, \mathbb{Z}) = \mathbb{Z}$, $q_M = (1)$.

let $\overline{\mathbb{C}P^2}$ be the "$\mathbb{C}P^2$" with reverse orientation

then $q_M = (-1)$.

Fact: $q_{M_1 \# M_2} = q_{M_1} \oplus q_{M_2}$

so for $M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, get $q_M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

but this is NOT homotopy equiv to $S^2 \times S^2$.

Hard Question:

If $q_M \simeq q_1 \oplus q_2$ as quad form $/\mathbb{Z}$

can one find mfds M_1, M_2 st $M = M_1 \# M_2$?

④ K3 surface, consider the "Kummer surface"

$$M = K_3 = \{ [z] \in \mathbb{C}P^3 \mid \sum_{i=0}^{3} z_i^4 = 0 \}$$

- $\dim_{\mathbb{C}} M = 3-1 = 2$, so M is real 4-dim.
- in general for M_d a degree d hyp. surface in $\mathbb{P}^n$

$$M_d \overset{i}{\hookrightarrow} \mathbb{P}^n$$

$$0 \to T_{M_d} \to i^* T_{\mathbb{P}^n} \to \underset{\overset{\wr\wr}{[dH]|_{M_d}}}{N_{M_d}} \to 0$$

$\Rightarrow i^* c(\mathbb{P}^n) = c(M_d) \cdot (1 + dH|_{M_d})$.

i.e. $c(M_d) = i^* c(\mathbb{P}^n) \cdot (1 + d \cdot i^* H)^{-1}$

Fact: $c(\mathbb{P}^n) = (1+H)^{n+1}$

from these we get all chern classes of M_d.

For $M = M_4 = K_3$: let $h = i^* H$:

$$1 + c_1(K_3) + c_2(K_3) = (1 + 4h + 6h^2) \cdot (1 - 4h + 4^2 h^2)$$
$$= 1 + 0 \cdot h + 6h^2$$

i.e. $c_1(K_3) = 0$ (Calabi-Yau condition)

$$c_2(K_3) = 6h^2 = 6\, i^*(H)^2.$$

By Gauss-Bonnet:

$$\chi(K_3) = \int_{K_3} c_2(K_3) = \int_{K_3} 6 i^* H^2 = 6 H^2 \cdot (4H) = 24.$$

Since $\chi = \underset{1}{\underset{\shortparallel}{h^0}} - \underset{0}{\underset{\shortparallel}{h^1}} + h^2 - h^3 + h^4$ by Lefschetz since $H^1(\mathbb{P}^3) = 0$

$\Rightarrow H^2(K_3, \mathbb{Z}) \simeq \mathbb{Z}^{22}$. indeed $\pi_1 = 0$

How to determine the "ring str" q_{K_3}?

classification theory of unimodular quad form over $\mathbb{Z}$:

- q indefinite then q is uniquely det. by rank, signature and type

$$\sigma = \sigma_+ - \sigma_- \qquad \text{even or odd}$$

- q definite $\Rightarrow$ no easy classification
- when q is even, then $8 \mid \sigma(q)$.

for K_3: By Hirzebruch signature formula

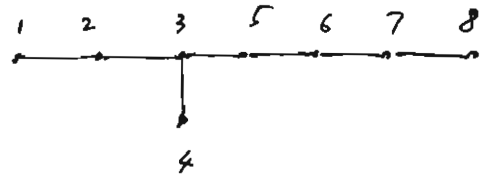
$$\sigma = \frac{p_1}{3} \; ; \quad p_1 = (-1)^1 c_2(TM \otimes \mathbb{C})$$
$$= -\left(c_2(T) + c_1(T)\cdot c_1(\bar{T}) + c_2(\bar{T})\right)$$
$$\Rightarrow \sigma(K_3) = -16. \qquad = -2c_2(K_3) = -48$$

(This can also be proved using Hodge index thm)

Exercise: Show that q_{K_3} is even.

E_8: the 1st non-trivial positive def form

$$\sim \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & -1 & & & \\ & & -1 & 2 & 0 & & & \\ & & -1 & 0 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}$$



consequence: $q_{K_3} \sim (-E_8) \oplus (-E_8) \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Theorem (Donaldson 1982)

q_M definite $\Rightarrow$ q_M is diagonalizable over $\mathbb{Z}$.

in particular, any positive even form, e.g. E_8, $E_8 \oplus E_8$ all do not exist smoothly.

Existence of Fake $\mathbb{R}^4$!

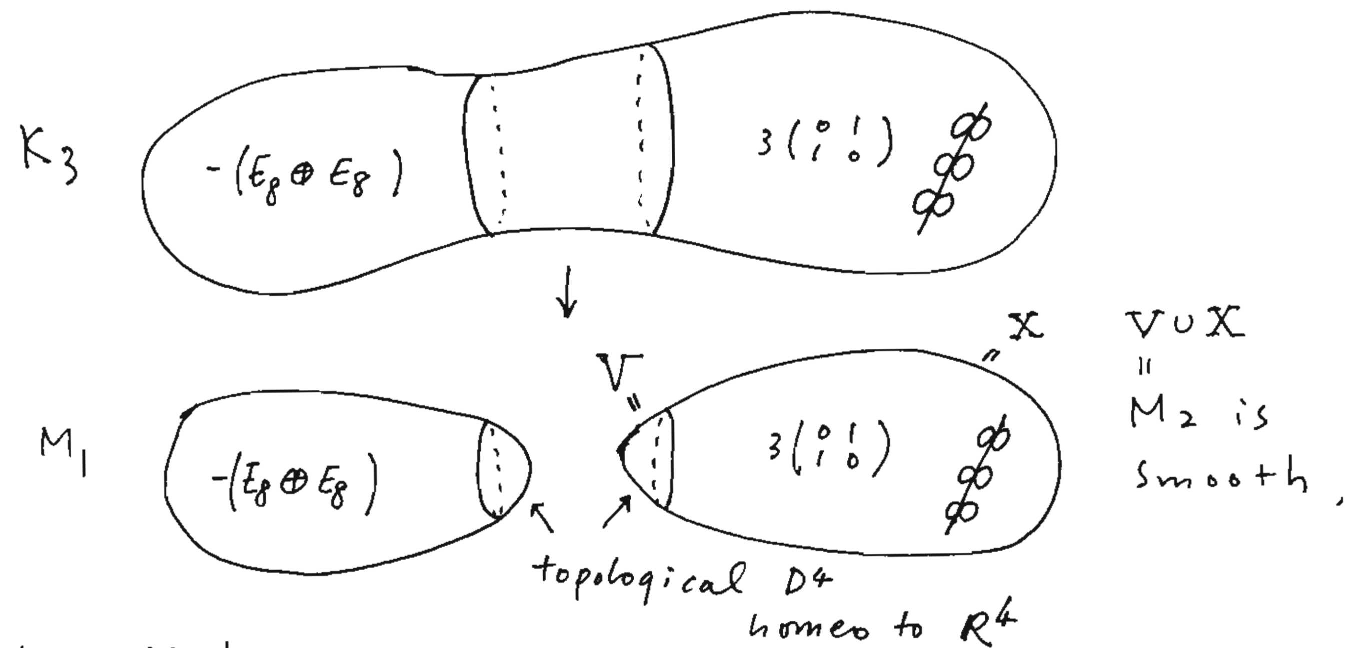
Freedman $\Rightarrow$ $\exists$ topological surgery $K_3 = M_1 \# M_2$

$$q_{M_1} = (-E_8) \oplus (-E_8) \; ; \; q_{M_2} = 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

indeed $M_2 = 3(S^2 \times S^2)$.

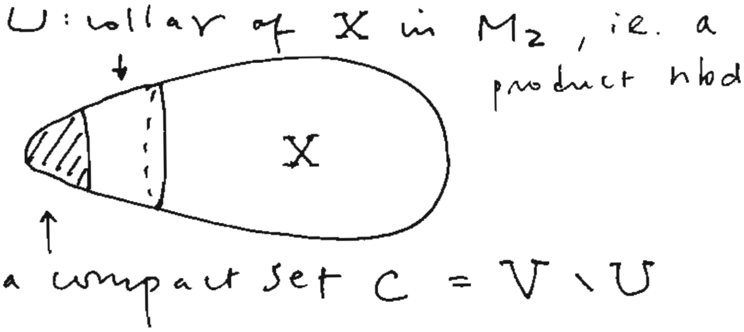
Donaldson $\Rightarrow$ can't do this smoothly.

Analysis on the failure :



Let V be equipped with the differentiable structure inherited from M_2.

$\Rightarrow$ $\nexists$ smoothly embeded $S^3 \hookrightarrow U$, otherwise the surgery can be done smoothly.



$\Rightarrow$ V is homeo to $\mathbb{R}^4$ but not diffeo to $\mathbb{R}^4$ (standard C^∞ str)

since in $\mathbb{R}^4_{std}$ any cpt set is contained in some sphere with large radius. $\square$

Idea of Proof of Donaldson's theorem :

$$\begin{array}{l} E \quad G\text{-bundle}, \quad G \text{ cpt Lie group} \\ \downarrow \qquad\qquad\qquad\qquad\qquad \text{eg. } G = SU(N) \\ M \quad \text{cpt 4-fold} \qquad\qquad\qquad N = 2. \\ \quad \pi_1(M) = 0 \end{array}$$

notations :
Λ^i : bundle
Ω^i : C^∞ sections of Λ^i.

$\mathfrak{g}$ = Lie algebra of G

eg. $\mathfrak{g} = su(N) \subseteq End(\mathbb{C}^N)$

with bi-G invariant inner product

$$\langle A, B \rangle = -\operatorname{tr} AB \quad \left(= \operatorname{tr} A\bar{B}^t \right)$$

- G-connections : $A = d + \theta_\alpha$ (on U_α open)

  $\theta_\alpha \in \Omega^1(\mathfrak{g}_E) \subset \Omega^1(End\, E)$ conn. matrix of 1-forms

  $\mathfrak{g}_E$: bundle of Lie algebras (associated to adjoint repr.)

  $End\, E = E \otimes E^*$

  eg. trace-free, skew-adjoint endo on E.

- curvature : $F_A = d\theta_\alpha + \theta_\alpha \wedge \theta_\alpha$

  Recall that $F_A(\sigma) := A^2(\sigma)$, F_A is a tensor,

  $$F_A \in \Omega^2(\mathfrak{g}_E) \subset \Omega^2(End\, E)$$

extension d_A : $\cdots \to \Omega^1(\mathfrak{g}_E) \xrightarrow{d_A} \Omega^2(\mathfrak{g}_E) \to \cdots$

Basic Fact (Bianchi identity) : $d_A F_A = 0$:

for $S \in \Omega^p(End\, E)$

$$d_A(S\sigma) = (d_A S)\sigma + (-1)^p S d_A\sigma$$

$$\Rightarrow \quad (d_A S)\sigma = (-1)^p S\, d_A\sigma + d_A S\sigma$$

ie. $d_A S = [d_A, S]$ as operators.

(This $\ni$ $d_A s = ds + [\theta_\alpha, s]$ in local frame)

where $[T, s] = Ts - (-1)^{|T|\cdot|s|} sT$ is super-bracket.

Pf of Bianchi: $d_A F_A = [d_A, F_A] = [d_A, d_A \circ d_A] = 0$.

- Yang-Mills functional: let (M, g) Riemannian

$$\underset{\substack{\text{space of}\\ \text{connections}}}{\mathcal{A}} \longrightarrow \mathbb{R}^+ \; ; \; A \longmapsto \int_M |F_A|^2 d_g vol = \| F_A \|^2$$

critical point of YM: let $a \in \Omega^1(\mathfrak{g}_E)$,

$$\begin{aligned} F_{A+ta}\, \sigma &= (d_A + ta)(d_A + ta)\sigma \\ &= d_A^2 \sigma + t\big(d_A(a\sigma) + a \wedge d_A \sigma\big) + t^2 a \wedge a(\sigma) \\ &= \big(\underbrace{d_A^2}_{= F_A} + t\, d_A a + t^2 a \wedge a\big)\sigma \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \| F_{A+ta} \|^2_{t=0} &= \frac{d}{dt} \int_M \left| F_A + t d_A a + t^2 a \wedge a \right|^2_{t=0} \\ &= 2 \int_M \langle d_A a, F_A \rangle \\ &= 2 \langle d_A a, F_A \rangle = 2 \langle a, d_A^* F_A \rangle \end{aligned}$$

this is 0 $\forall\, a \in \Omega^1(\mathfrak{g}_E) \Leftrightarrow \underbrace{d_A^* F_A = 0}$

2nd order Yang-Mills Eq'n

4-dim'l case: $\Lambda^2(M) = \Lambda^2_+(M) \oplus \Lambda^2_-(M)$

since $*^2 = + id$,

$\Lambda^2_+(M) = \{\alpha : *\alpha = \alpha\}$ self-dual 2-forms

$\Lambda^2_-(M) = \{\alpha : *\alpha = -\alpha\}$ ASD 2-forms

This applies to any bundle V, $\Lambda^2(V) = \Lambda^2_+(V) \oplus \Lambda^2_-(V)$.

in particular, to $\Lambda^2(\mathfrak{g}_E) = \Lambda^2_+(\mathfrak{g}_E) \overset{\perp}{\oplus} \Lambda^2_-(\mathfrak{g}_E)$

so $F_A = F_A^+ + F_A^-$ orthogonal decomposition

$\| F_A \|^2 = \| F_A^+ \|^2 + \| F_A^- \|^2$

• <u>characteristic classes consideration:</u>

$c_1(E) = \left[\frac{\sqrt{-1}}{2\pi} \operatorname{tr} F_A\right] = 0$ since $\mathfrak{g} = su(N)$

$\Rightarrow \; -2c_2(E) = \left[\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \operatorname{tr} F_A^2\right]$ this is $c_1^2 - 2c_2$ in general.

$$= \int \frac{-1}{4\pi^2} \operatorname{tr} F_A \wedge F_A$$

$$= \int \frac{-1}{4\pi^2} \left(\operatorname{tr} F_A^+ \wedge F_A^+ + \operatorname{tr} F_A^- \wedge F_A^-\right)$$

$$= \frac{-1}{4\pi^2} \int \operatorname{tr}\left(F_A^+ \wedge * F_A^+\right) - \left(\operatorname{tr} F_A^- \wedge * F_A^-\right)$$

get $\overset{k}{\overset{\shortparallel}{c_2(E)}} = \frac{1}{8\pi^2}\left(\| F_A^- \|^2 - \| F_A^+ \|^2\right) \in \underset{H^4(M,\mathbb{Z})}{\overset{\shortparallel}{\mathbb{Z}}}$

this is called the "charge" of the YM field.

$k > 0$ the the absolute minimum of

$\| F_A \|^2$ is $8\pi^2 k = 8\pi^2 c_2(E)$, which occurs

$\Leftrightarrow \; F_A^+ \equiv 0$ ie. $* F_A = - F_A$

ASD connections.

$k < 0$, min $= 8\pi^2(-c_2(E))$, $\Leftrightarrow F_A^- \equiv 0$, SD.

We consider $k > 0$ case:

$F_A^+ = 0$ is a 1st order non-linear PDE.

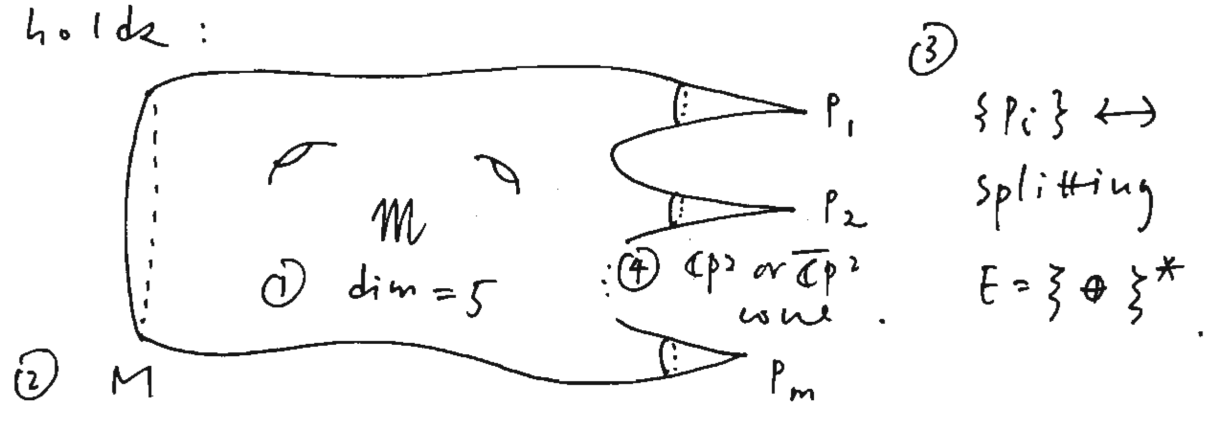
Donaldson consider

$$\begin{array}{l} E \quad \text{rk } 2,\ SU(2) \text{ bundle with} \\ \downarrow \qquad\qquad k = c_2(E) = 1 \quad (c_1 = 0) \\ M^4 \end{array}$$

- On 4 manifold, $SU(2)$ bundle $\overset{1-1}{\longleftrightarrow}$ $k \in \mathbb{Z}$:
  since $SU(2) \simeq \mathbb{H}^\times$, such v.b are $\mathbb{H}$ line bundle with classifying space $\mathbb{HP}^\infty$
  so 1-1 corr to homotopy classes
  $[M, \mathbb{HP}^\infty] = [M, S^4] \simeq \mathbb{Z}$ (degree map, $= c_2$)
  by the CW cpx str of $\mathbb{HP}^\infty$: $\mathbb{HP}^0 \subset \underset{S^4}{\mathbb{HP}^1} \subset \mathbb{HP}^2 \subset \dots$
  and cellular approximation thm.
- for E with $c_2(E) = k$,
  each splitting $E \simeq \xi \oplus \xi^*$ 1-1 corr. to solutions of $q_M(a,a) = -k$, $a \in H^2(M, \mathbb{Z})$ up to sign $\pm a$
  with $c_1(\xi) = \pm a$.

$\rightsquigarrow$ (since $\pi_1(M) = 0$, line bundle $\longleftrightarrow$ $H^2(M, \mathbb{Z})$.)

- <u>Theorem (Donaldson, 1982)</u> let g generic, let $k=1$, $\mathcal{M}$ the "moduli space" of $F_A^+ = 0$ assume that q_M negative def. Then ① – ④ holds:



⑤. And then ①–④ $\Rightarrow$ $q_M \sim h^2 \cdot (-1)$.

proof of ⑤: Since M is cobordant to disjoint union of $m(\pm\mathbb{C}P^2)$'s,

and the <u>Signature is cobordism inv</u>.

$$\Rightarrow \quad h^2(M) = -\sigma(q_M) \leq m\,\sigma(\mathbb{C}P^2) = m.$$

However, ~~by the process of diagonalization of integral quad form~~, we must have $m \leq h^2$

So $m = h^2$ and this $\Rightarrow$ q_M is diagonalisable

to $\begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$ $\square$.

- Disscussion of Proofs:

① If $\mathcal{M} = \{ A \text{ su}(2) \text{ conn} \mid F_A^+ = 0 \} / \mathrm{Aut}(E) \neq \phi$

then $\dim \mathcal{M}$ = dim ker of linearized elliptic op. at A:

(T) $T_{[A]}\mathcal{M} : \Omega^1(\mathfrak{g}_E) \xrightarrow{d_A^+ + d_A^*} \Omega^2_+(\mathfrak{g}_E) \oplus \Omega^0(\mathfrak{g}_E)$

for generic metric g, coker $= \{0\}$

so $\dim \mathcal{M} = \text{index}(d^+ + d^*) = 8k - 3(1 - b_1 + b_2^+)$
$= 5$.

See later: non-trivial calculation via Atiyah-Singer index thm.

② M cpt $\Rightarrow$ any unbounded sequence A_i has a subsequence st A_{n_i} concentrates at a point $p \in M$ and flat outside p. (Because $k = 1$). ie. M is the natural boundary of $\mathcal{M}$ (Uhlenbeck compatification).

③ Singular pt $\leftrightarrow$ action of $\mathrm{Aut}(E) = \mathcal{G}$ has $\mathcal{G}_A \neq \{1\}$ (stabilizer)

In fact, if A is not flat ($F_A \neq 0$) then TFAE

(a) $\mathcal{G}_A / \mathbb{Z}_2 \simeq U(1)$

(b) $d_A : \Omega^0(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)$ has $\ker \neq \{0\}$

(c) A is a reducible connection

(d) $\mathcal{G}_A / \mathbb{Z}_2 \neq \{1\}$.

Pf: ~~(a) $\Rightarrow$ (b) : let $u \in \Omega^0(\mathfrak{g}_E)$ st $u \in \mathrm{Lie}\,\mathcal{G}_A$~~

then $e^{-tu} d_A e^{tu} = d_A \Rightarrow d_A \circ u - u \circ d_A = 0$

i.e. $d_A u = 0$.

~~(b) $\Rightarrow$ (c) : let $d_A u = 0$~~, u is skew-hermitian and $\mathrm{tr}\, u = 0$ at every point $p \in M$ (2×2 matrix)

so with eigenvalues $\pm i\lambda(p)$, function on M

consider open set $\subset M$ st $\lambda > 0$ (on U)

with eigenvector e, $ue = i\lambda e$, C^∞ on U

say with $|e| = 1$.

d_A : $\Rightarrow$

$$u\, d_A e = i(d\lambda)e + i\lambda\, d_A e$$

$$\Rightarrow i\, d\lambda \langle e, e\rangle = \langle u\, d_A e, e\rangle - i\lambda \langle d_A e, e\rangle$$

$$= \langle d_A e, \underset{-u}{u^*} e\rangle - i\lambda \langle d_A e, e\rangle$$

$= -i\lambda e$

$$= 0$$

i.e. $\lambda =$ constant

$\Rightarrow$ e is globally defined, also $d_A e \in i\lambda$-eigenspace

so $d_A e \subset \Omega^1(\langle e\rangle)$ hence a splitting $(E, A) = (E_1, A_1) \overset{\perp}{\oplus} (E_2, A_2)$

~~(c) ⇒ (d)~~ : If A is reducible conn.

then $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in \mathcal{G}_A$, $\forall \theta$ constant on M.

but $\{e^{i\theta}\} \cong S^1$ is abelian, whose action on A must be trivial (on (E_1, A_1) and (E_2, A_2))

E_i line bundles.

~~(d) ⇒ (a)~~ :

from (a) ⇒ (b) ⇒ (c) we get splitting.

and (c) ⇒ (d) get $\mathcal{G}_A \supset U(1) \cong S^1$

if $\mathcal{G}_A$ is larger then $U(1)$ then the str gp of E_i will be discrete, so E is flat *

④ The local model $\mathbb{C}P^2$-cone only holds for generic metric g st. $H^2_+ = 0$ in Ⓣ.

<u>idea is</u> : before we reach $\dim M = 5$

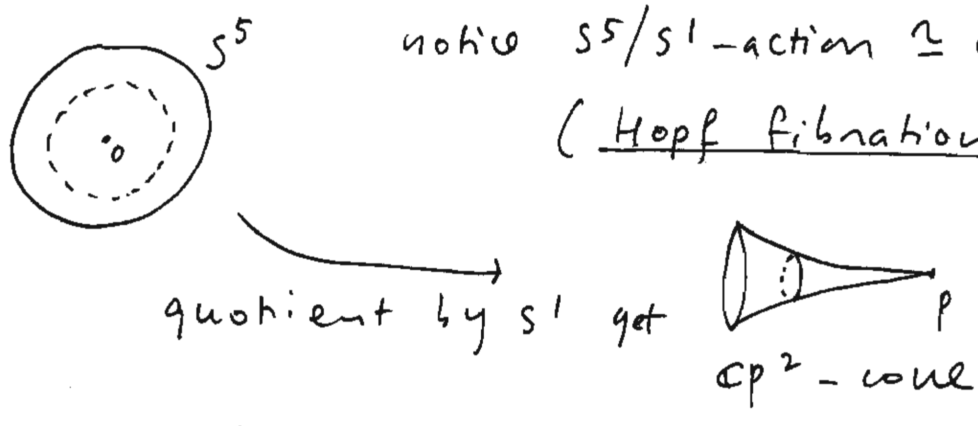
we have a 6-dim'l v.s V

with S^1-action, stabilizer of $0 \in V$

V :
S^1
$\mathbb{R}^6$

notice S^5/S^1-action $\cong \mathbb{C}P^2$

(<u>Hopf fibration</u>).



The actual analysis requires "Kuranshi"-technique in Kodaira-Spencer Theory.

• ~~Remark~~: Before we compute $\dim M$ by index of linearized eq'n, we need $M \neq \phi$,

This is via <u>Taubes' existence thm</u> + <u>S^4 case</u>.

~~Linearization of ASD Eq'n~~ : ~~let $F_A^+ = 0$~~

$F_{A+a} = F_A + d_A a + a \wedge a$; $a \in \Omega^1(\mathfrak{g}_E)$:

for $A + a(t)$ a family of conn. $a(0) = 0$, $a'(0) = \tau$

$$\frac{d}{dt}\Big|_{t=0} F^+_{A+a(t)} = d_A^+ \tau \quad \left(= \tfrac{1}{2}(d_A + * d_A)\tau \right)$$

if $A(t)$ is from ~~Gauge transform~~

$f_t \in \Omega^0(\underset{\text{Aut}(E,h)}{\mathcal{G}})$, $f_0 = id$

ie. $A(t) = f_t^* A = f_t^{-1} d f_t + f_t^{-1} A f_t$

$\Rightarrow \tau = A'(0) = d f'(0) - f'(0) A + A f'(0) = d_A f'(0)$

These $\Rightarrow T_{[A]} \mathcal{M} \simeq H^1(\mathfrak{g}_E^\bullet)$ in complex (AHS)

$$\mathfrak{g}_E^\bullet : \quad 0 \to \Omega^0(\mathfrak{g}_E) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_E) \xrightarrow{d_A^+} \Omega^2_+(\mathfrak{g}_E) \to 0$$

(it is a cpx since $d_A^+ d_A f'(0) = F_A^+ f'(0) = 0$.)

$\Rightarrow \; -\chi(\mathfrak{g}_E^\bullet) = -h^0 + h^1 - h^2 = \text{index}\,(d_A^+ + d_A^*)$ in (T):

$$\underset{\text{Hom}(S^-, S^+)}{\Gamma(\Lambda^1 \otimes \mathfrak{g}_E)} \xrightarrow{d_A^+ + d_A^*} \underset{\text{Hom}(S^-, S^-)}{\Gamma((\Lambda^0 \oplus \Lambda^2_+) \otimes \mathfrak{g}_E)}$$

Ex. By comparing: ~~Clifford action and dim.~~ $\Rightarrow$ get Dirac operator, $W = S^{-*} \otimes \mathfrak{g}_E$

$$d_A^+ + d_A^* \equiv D^W : \Gamma(S^+ \otimes W) \to \Gamma(S^- \otimes W)$$

$$\text{index}\, D^W = \hat{A}(M)\, \underset{2 + \cdots}{ch(S^{-*})}\, \underset{\dim G + c_1 + \frac{1}{2}(c_1^2 - 2c_2)}{ch(\mathfrak{g}_E)}\, [M]$$

$c_1 = 0$ for any End(·)

Sign? $\rightarrow \; = 2\cdot[-c_2(\mathfrak{g}_E)] + \dim G \cdot \hat{A}(M) \cdot ch(S^{-*})[M]$

$= 2c_2(\mathfrak{g}_E) + \dim G \,.\, \text{index}\, D$

So in fact we do not need to know the Clifford str!

for $\Lambda^1_{\mathbb{C}} \longrightarrow \Lambda^0_{\mathbb{C}} \oplus \Lambda^2_{+\mathbb{C}}$ no W.

clearly $\operatorname{index} D = -b_0 + b_1 - b_{2+}$

for $\alpha \in \Gamma(\Lambda^1)$, $(d^+ + d^*)\alpha = 0 \iff d^+\alpha = 0$, $d^*\alpha = 0$

$d^*\alpha = * d * \alpha$ i.e. $d*\alpha = 0$

$0 = d^+\alpha = \frac{1}{2}(d\alpha + \underbrace{d*\alpha}_{=0})$

$\Rightarrow d\alpha = 0$ too

hence $\alpha \in \mathcal{H}^1$ harmonic.

equiv. $\operatorname{index} D = \frac{1}{2}(-b_0 + b_1 - b_2^+$

$\qquad - b_2^+ + b_3 - b_4)$

$= \frac{1}{2}(-\chi + b_2^- - b_2^+) = \frac{-1}{2}(\chi + \sigma)$.

Finally we plug in $G = SU(2)$, E rk $= 2$:

since $E^* \otimes E = \mathfrak{g}_E \oplus \mathbb{C}$ (trivial line bundle corr to trace)

$c_2(\mathfrak{g}_E) = c_2(E^* \otimes E) = 4c_2(E)$, notice $c_1(E) = 0$

$\operatorname{ch}(E^* \otimes E) = \operatorname{ch}(E^*) \cdot \operatorname{ch}(E)$

$\operatorname{ch}(E^* \otimes E) = 4 - c_2(E^* \otimes E)$ since $c_1(E^* \otimes E) = 0$

$= \left[2 - c_1(E^*) + \frac{1}{2}\left(c_1^2(E^*) - 2c_2(E^*)\right)\right] \cdot \left[2 - c_1(E) + \frac{1}{2}\left(c_1^2(E) - 2c_2(E)\right)\right]$

$= (2 - c_2(E))(2 - c_2(E))$ ※

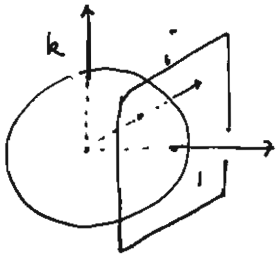
conclusion:

$\operatorname{index}(d_A^+ + d_A^*) = 2c_2(\mathfrak{g}_E) - \frac{1}{2}\dim G\,(\chi + \sigma)$ general case.

$= 8c_2(E) - 3(b_2^+(M) + 1)$; for $G = SU(2)$, $b_1(M) = 0$ case.

Rmk: For generic metric g, "Sard" $\Rightarrow$ coker $D = 0 \Rightarrow \mathcal{M}$ C^∞ ※

~~ADHM for k=1.~~



On $\mathbb{H} \cong \mathbb{R}^4$

unit gp $= Sp(1) \cong SU(2) \cong S^3 \cong Spin(3)$

Lie alg $= su(2) \cong \operatorname{Im}\mathbb{H} = \mathbb{R}\langle i,j,k\rangle$

On the Hopf bundle $\gamma^1_{\mathbb{H}} \to \mathbb{H}P^2 \simeq S^4 = \mathbb{R}^4 \cup \{\infty\}$,

it's known (easily) $c_1(\gamma_{\mathbb{C}}) = 0$, $c_2(\gamma_{\mathbb{C}}) = e(\gamma_{\mathbb{R}}) =: k = 1$.

Over the trivialization $\gamma^1_{\mathbb{H}}|_{\mathbb{R}^4}$

with section $\sigma(x) = \frac{(x,1)}{\sqrt{1+|x|^2}}$. ~~Define ASD conn A~~:

connection form ($su(2)$-valued) $\omega = \frac{\theta_1 i + \theta_2 j + \theta_3 k}{1+|x|^2} = \frac{\operatorname{Im}(\bar{x}dx)}{1+|x|^2}$.

$$\theta_1 = -x_2 dx_1 + x_1 dx_2 + x_4 dx_3 - x_3 dx_4$$

$$\theta_2 = -x_3 dx_1 + x_1 dx_3 - x_4 dx_2 + x_2 dx_4$$

$$\theta_3 = -x_4 dx_1 + x_1 dx_4 + x_3 dx_2 - x_2 dx_3$$

$$\Rightarrow F = \frac{d\theta_1 i + d\theta_2 j + d\theta_3 k}{(1+|x|^2)^2} = \frac{d\bar{x}\wedge dx}{(1+|x|^2)^2} \quad (\mathbb{H}\text{ notation})$$

~~HW~~

$$d\theta_1 = 2(dx_1\wedge dx_2 - dx_3\wedge dx_4)$$

$$d\theta_2 = 2(dx_1\wedge dx_3 + dx_2\wedge dx_4)$$

$$d\theta_3 = 2(dx_1\wedge dx_4 - dx_2\wedge dx_3)$$

are presicely basis of $\Lambda^2_-(\mathbb{R}^4)$

and then $c_2(\gamma_{\mathbb{C}}) = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \operatorname{tr} F^2 = 1$.

- Notice that ~~4D YM functional~~ is conformal inv.

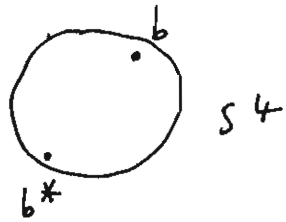
Under the conformal transf. In particular, for

$$T_{\lambda,b} : x \mapsto \lambda^{-1}(x-b) \quad ; \quad \lambda > 0,\ b \in S^4$$

$A_{\lambda,b} := T_{\lambda,b}^* A$ is a ASD conn with center b, scale λ.

we identify $T_{\lambda,b} \sim T_{\lambda^{-1},b^*}$, b^* = antipodal to b

May assume $\lambda \le 1$, by changing $b \leftrightarrow b^*$.



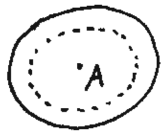
since $A_{\lambda,b} = \dfrac{\lambda^{-1}\,\mathrm{Im}\,\bar\lambda^{-1}(\overline{x-b})\,dx}{1+\bar\lambda^{-2}|x-b|^2} = \dfrac{\mathrm{Im}\,\bar x\,dx}{\lambda^2+|x-b|^2}$

$$f_{\lambda,b} = \frac{\bar\lambda^{-2}\,d\bar x\wedge dx}{(1+\bar\lambda^{-2}|x-b|^2)^2} = \frac{\lambda^2\,d\bar x\wedge dx}{(\lambda^2+|x-b|^2)^2}$$

by changing b to b^* may assume that $\lambda \le 1$

as $\lambda \to 0$, $A_{\lambda,b}$, $F_{\lambda b}$ concentrate on $B_b(\lambda)$.

This gives a " δ_b ASD connection " B^5



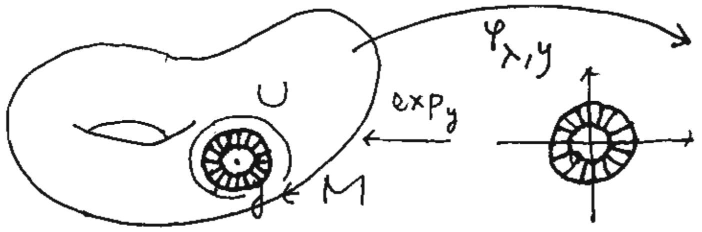
as well as the collar structure $[0,\lambda_0)\times S^4$.

~~HW~~: Show that $\mathrm{Conf}^+(S4)/SO(5) \simeq B^5 \simeq \mathcal{M}^0_{ASD,\,k=1}$

where $\mathrm{Conf}^+(S4)$ acts on $\mathcal{M}^0$ with $\mathrm{stab}_A \simeq SO(5)$.

(In fact $\mathcal{M}$ has only one comp, but this is hard)

~~Taubes' glueing procedure~~ via $\Phi_{\lambda,y} : M \longrightarrow S^4$ as $T_\lambda \circ \varphi_{\lambda,y}$:



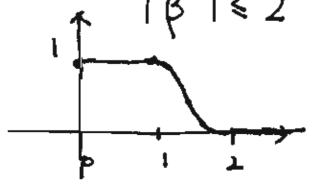
$\mathbb{R}^4 \hookrightarrow S^4 \xrightarrow{T_\lambda} S^4$

$B_{\sqrt\lambda} \Subset B_{2\sqrt\lambda}$, $\sqrt\lambda \le \dfrac{i(M)}{2}$

U : geod. normal ball at y

$$\varphi_{\lambda,y}(x) = \frac{\exp^{-1}(x)}{\beta(\exp^{-1}(x)/\sqrt\lambda)} \quad x\in U \; ; \quad \infty \text{ if } x\notin U$$

$|\beta'| \le 2$



where $\beta : [0,\infty] \to [0,1]$ C^∞ cut-off

$E = \Phi^*_{\lambda,y}\,\gamma^1_H$ is a $SU(2)$ bundle on M with $c_2(E) = k = 1$.

$$F_\lambda = \Phi^*_{\lambda,y}\,F \underset{\substack{\text{at } x\in U\\ \text{normal coor}}}{=} \frac{\lambda^2}{\lambda^2 + \dfrac{|x|^2}{\beta\left(\frac{x}{\sqrt\lambda}\right)^2}}\, d\left(\frac{\bar x}{\beta\left(\frac{x}{\sqrt\lambda}\right)}\right)\wedge d\left(\frac{x}{\beta\left(\frac{x}{\sqrt\lambda}\right)}\right)$$

~~Thm~~: F_λ has small, controllable, SD part :

for any $1 < p \le \infty$, $\exists$ const $c_1(p), c_2(p)$ indep of λ st.

- $\|F_\lambda\|_{L^p} \le c_1(p)\cdot\lambda^{\frac{4}{p}-2}$.   eg. $p = 2$.
- $\|F^+_\lambda\|_{L^p} \le c_2(p)\cdot\lambda^{\frac{2}{p}}$.

The pf is direct calculation. But it explains the choice of $\sqrt\lambda$.

Perturbation to get ASD conn.

For A almost ASD, consider eq'n for $a \in \Omega^1(\mathfrak{g}_E)$

$$0 = F_{A+a}^+ = F_A^+ + d_A^+ a + (a \wedge a)^+$$

As in linear case, need to fix the bundle auto (Gauge sym)

Write $D = d_A^+$; let $a = D^* u$, $u \in \Omega^2_+(\mathfrak{g}_E)$

get $Lu := DD^* u + (D^* u \wedge D^* u)^+ = -F_A^+$

Continuity method : $L u_t = -t F_A^+$ $(*)_t$ conf. inv. eq'n.

$t = 0$ OK : $u_0 \equiv 0$.

$I := \{ t \in [0,1] \mid (*)_t$ has a sol u_t with $\|D^* u_t\|_{L^4}$ small $\}$

notice : L^2 conf inv for 2 forms, L^4 for 1 forms

Openness of I : linearized eq'n at u_t is

$$L_t' \varphi := DD^* \varphi + 2(D^* u_t \wedge D^* \varphi)^+ = 0 .$$

enough to show L_0' invertible, ie. $\lambda_1 > 0$

this step requires already $g_M < 0$ and λ_1 indep of λ.

Closedness of I : Need a priori estimate for $u_{t_n} \to u_{t_0}$,

via Bochner formula : $\Delta_D = \frac{1}{2} \operatorname{tr} \nabla_A^2 + \frac{1}{2} \mathrm{Ric} - \iota(\cdot) F_A^-$
for 1-forms $\Omega^1(\mathfrak{g}_E)$

Trouble : Not possible to be unif in $\lambda \to 0$ since $\|F_\lambda\|_{L^\infty} = O(\lambda^{-2})$.

Possible solutions : (1) Tricky L^p iteration

(2) Blowing-Up the metric g on M.

Taubes' and Freed-Uhlenbeck develop idea (2)

conf inv $\Rightarrow$ can blow-up g on $M_y := M \setminus \{y\}$

st. M_y is almost a cylinder

st. estimates in cpt part + cylinder part.

and F becomes bounded.

(Prototype of long neck # argument)

• Uhlenbeck's Removable sing. thm and her compactness thm are the final steps to analyze $\mathcal{M}$ $\ast$