Complex Analysis 2014 Fall

Intro to Elliptic & theta functions

Appendix $\zeta(z)$ "$:=$" $-\int^z \wp(w)\,dw$ well-defined since $\int \frac{dw}{w^2} = 0$ but $\zeta(z+\omega_i) =: \zeta(z) + \eta_i$ not periodic

$= \frac{1}{z} + \sum_{\omega\in\Lambda}' \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right)$ quasi-periode $\eta_i = 2\,\zeta\left(\frac{\omega_i}{2}\right)$

or $= \frac{1}{z} - E_4 z^3 - E_6 z^5 - \cdots$ odd function (no const. term)

Rmk: If we consider "$Z(z)$" $:= \zeta(t\omega_1 + s\omega_2) - t\eta_1 - s\eta_2$ for $z = t\omega_1 + s\omega_2$ then Z is doubly periodic, with simple pole at $z \in \Lambda$, but Z is NOT holomorphic.

$\sigma(z)$ "$:=$" $e^{\int^z \zeta(w)\,dw} = \underset{\substack{\parallel \\ z}}{e^{\log z}} \cdot e^{-\frac{1}{4}E_4 z^4 - \frac{1}{6}E_6 z^6 - \cdots}$ well-defined as an entire function

$= e^{\log z} \prod_{\omega\in\Lambda}' e^{\log(z-\omega) - \log(-\omega) + \frac{z}{\omega} + \frac{1}{2}\frac{z^2}{\omega^2}}$ convergence issue ? At least $\sigma(-z) = -\sigma(z)$ odd function

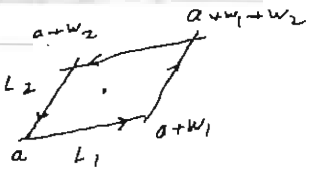
$\rightsquigarrow \frac{z-w}{-w} = \left(1 - \frac{z}{w}\right)$ oK

i.e. $(\log\sigma)' = \zeta$, so $\left(\log \frac{\sigma(z+\omega_i)}{\sigma(z)}\right)' = \zeta(z+\omega_i) - \zeta(z) = \eta_i \Rightarrow \frac{\sigma(z+\omega_i)}{\sigma(z)} = e^{\eta_i z + C_i}$

set $z = -\frac{\omega_i}{2} \Rightarrow -1 = e^{-\frac{1}{2}\eta_i\omega_i + C_i}$ i.e. $C_i = e^{\pi i + \frac{1}{2}\eta_i\omega_i}$; or $\sigma(z+\omega_i) = -\sigma(z)\, e^{\eta_i\left(z + \frac{\omega_i}{2}\right)}$

The correct product expansion is $\sigma(z) = z\prod'\left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\frac{z^2}{\omega^2}}$

Thm (Legendre period relation) $\begin{vmatrix} \eta_1 & \eta_2 \\ \omega_1 & \omega_2 \end{vmatrix} = 2\pi i$ pf: $2\pi i = \int_{\partial P} \zeta(w)\,dw$ in



✲

Digression: $e_1 = \wp\left(\frac{\omega_1}{2}\right) = \wp\left(\frac{1}{2}\right) = 4 + \sum{}'\left(\frac{1}{\left(\frac{1}{2}-(m+n\tau)\right)^2} - \frac{1}{(m+n\tau)^2}\right)$

$$\eta_1 = 2\zeta\left(\frac{\omega_1}{2}\right) = 2\zeta\left(\frac{1}{2}\right) = 2\left(2 + \sum{}'\left(\frac{1}{\frac{1}{2}-(m+n\tau)} + \frac{1}{m+n\tau} + \frac{\frac{1}{2}}{(m+n\tau)^2}\right)\right)$$

$$\Rightarrow e_1+\eta_1 = 8 + \sum{}'\left\{\frac{1}{\left(\frac{1}{2}-(m+n\tau)\right)^2} + \underbrace{\frac{2}{\frac{1}{2}-(m+n\tau)} + \frac{2}{m+n\tau}}_{\frac{1}{\left[\frac{1}{2}-(m+n\tau)\right](m+n\tau)}}\right\}$$

Key: Needs to be careful why does this converge (absolutely)! Answer: cubic structure $\Rightarrow$ abs. conv.

Let $q = e^{\pi i \tau}$

$$\frac{2}{[(1-2m)-n(2\tau)]^2(m+n\tau)} = \frac{4}{[(1-2m)-n(2\tau)]^2} + \frac{2}{[(1-2m)-n(2\tau)]}\frac{1}{(m+n\tau)}$$

Similar idea can prove Dedekind $\eta(\tau) := q^{1/12}\prod_{n=1}^{\infty}(1-q^{2n})$ is modular of wt $1/2$.

i.e. to prove $F(\tau) - \hat{F}(\tau) = 2\pi i/\tau$ (motivated by Serre's book)

$$\sum_m{}'\left(\sum_n{}'\frac{1}{(n+m\tau)^2}\right) - \sum_n{}'\left(\sum_m{}'\frac{1}{(n+m\tau)^2}\right) = \sum_m{}'\sum_n{}'\left(\frac{1}{(n+m\tau)^2} - \frac{1}{(n-1+m\tau)(n+m\tau)}\right) - \sum_n{}'\sum_m{}'\left(\frac{1}{(n+m\tau)^2} - \frac{1}{(n-1+m\tau)(n+m\tau)}\right)$$

$$+ \sum_m{}'\sum_n{}'\left(\frac{1}{n-1+m\tau} - \frac{1}{n+m\tau}\right) - \sum_n{}'\sum_m{}'\left(\frac{1}{n-1+m\tau} - \frac{1}{n+m\tau}\right) \quad \text{hence} = \frac{2\pi i}{\tau} \; \cancel{*}$$

computing gaps $\mapsto$ $= 2$ $\qquad = ?$ $\qquad$ Ans. $2 - \frac{2\pi i}{\tau}$

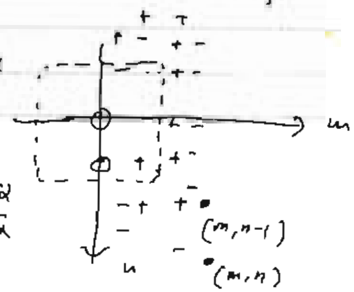
$(m,n) \neq (0,0), (0,1)$

"2" comes from the same reason in bounded sum

get $\sum_n \frac{1}{\tau}\left\{\pi\cot\left(\pi\frac{n-1}{\tau}\right) - \pi\cot\left(\pi\frac{n}{\tau}\right)\right\} = -2\pi i/\tau$

$\cot\alpha = i\,\frac{e^{i\alpha}+e^{-i\alpha}}{e^{i\alpha}-e^{-i\alpha}}$

$(-1)-1 = -2$



Rmk: $q := e^{\pi i\tau}$ in Stein's book, but $q := e^{2\pi i\tau}$ in Serre's book (A course in Arithmetic)

12/30. Addition Law

$$\wp(a+b)+\wp(a)+\wp(b) = \frac{1}{4}\left(\frac{\wp'(a)-\wp'(b)}{\wp(a)-\wp(b)}\right)^2$$

as elliptic function of a, has pole at $-b$, order 2, at 0, order 2 ; No others

LHS :

$$\wp(a+b) = \wp(b)+\wp'(b)a+\frac{\wp''(b)}{2!}a^2+\cdots$$

$$\wp(a) = \frac{1}{a^2}+3E_4a^2+\cdots$$

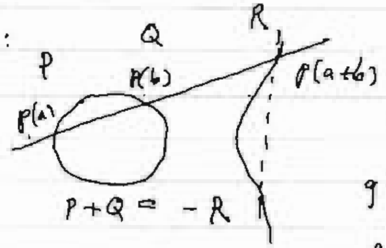
i.e. principal part : $\frac{1}{a^2}+2\wp(b)$

RHS : $\frac{1}{4}\left[\left(\frac{-2}{a^3}-\wp'(b)\right)\frac{1}{\frac{1}{a^2}-\wp(b)}\right]^2$

$$\frac{1}{4}\left(\frac{-2}{a}-\wp'(b)a^2\right)^2\left(1+\wp(b)a^2\right)^2$$

$$\frac{1}{4}\left(\frac{+4}{a^2}+4\wp'(a)a\right)\left(1+2\wp(b)a^2+\cdots\right) = \frac{1}{a^2}+2\wp(b)$$ ✫

geometric meaning :



real picture

e.g. $e_1<e_2<e_3$.

$$y = \frac{\wp'(b)-\wp'(a)}{\wp(b)-\wp(a)}\left(x-\wp(a)\right)+\wp'(a)$$

$$y^2 = 4x^3-g_2x-g_3$$

get a degree 3 poly in x, has roots $\wp(a)$, $\wp(b)$, $\wp(a+b)$

hence must have $\wp(a)+\wp(b)+$ "$\wp(a+b)$" $= \frac{1}{4}\left(\frac{\wp'(b)-\wp'(a)}{\wp(b)-\wp(a)}\right)^2$

the expected addition point.

Ex. why "$-R$" instead of R ?

* Other addition theorem :

$$\zeta(a+b)-\zeta(a)-\zeta(b) = \frac{1}{2}\,\frac{\wp'(a)-\wp'(b)}{\wp(a)-\wp(b)}$$ (symmetrized version of)

$$\left[\zeta(a+b)-\zeta(a)-\zeta(b)\right]^2 = \wp(a+b)+\wp(a)+\wp(b).$$

$$\wp(z)-\wp(u) = -\frac{\sigma(z-u)\,\sigma(z+u)}{\sigma^2(z)\,\sigma^2(u)}$$ (starting) cf. Ahlfors

$\Downarrow$ log '

$$\frac{\wp'(z)}{\wp(z)-\wp(u)} = \zeta(z+u)+\zeta(z-u)-2\zeta(z)$$

1/6 (2015) Last class — Jacobi Theta functions

$\Theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau} e^{2\pi i n z}$; $\theta(\tau) = \Theta(0,\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$

$e^{\pi i n^2\tau} e^{2\pi i n z} = q^{n^2} e^{2\pi i n z}$

$\theta(\tau)^2 = \sum_{n=0}^{\infty} r_2(n)\, q^n \overset{?}{=} 1+\sum_{n=1}^{\infty} 4\left(d_1(n)-d_3(n)\right) q^n$

$d_1(n)$: number of 4k+1 divisors etc

Thm: Sum of 2 squares

$$\theta(\tau)^2 = C(\tau)$$

i.e. $1+4\sum_{n=1}^{\infty}\left(\frac{q^n}{1-q^{4n}}-\frac{q^{3n}}{1-q^{4n}}\right)$

$= 1+4\sum_{n=1}^{\infty}\frac{(1-q^{2n})\,q^n}{(1-q^{2n})(1+q^{2n})} = \sum_{n=-\infty}^{\infty}\frac{2}{q^n+q^{-n}}$

$= \sum_{n=-\infty}^{\infty}\frac{1}{\cos(n\pi\tau)} =: C(\tau)$.

at $\tau = i$, $q = e^{-\pi}$

$C(i) = \sum \frac{2}{e^{\pi}+e^{-\pi}} \neq 0$

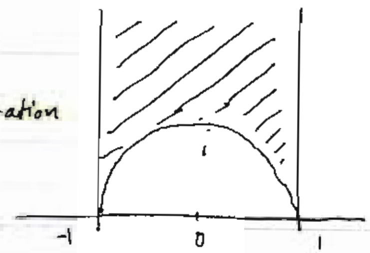
idea of proof: $\tau \mapsto -1/\tau$

1) both are modular wrt. $\langle T_2, S\rangle$ of wt. 1. (T_2: $\tau \mapsto \tau+2$)
2) same asymptotic behavior at cusps. (1 & ∞)
3) there is only 1-dim'l modular form of this form (ie dimension formula)

Rmk:

1) for $\theta(\tau)^2$ follows from the case $\vartheta(t) := \theta(it)$ Poisson Summation and identity principle for hol. functions.

for $C(\tau)$, Also by Poisson summation formula.



$\vartheta(1/t) = t^{1/2}\vartheta(t)$,

i.e. $\theta(-1/\tau) = \sqrt{\frac{\tau}{i}}\,\theta(\tau)$.

For sum of 4 squares: $\sigma_1^*(n) := \sum_{4\nmid d|n} d$, **Thm:** $r_4(n) = 8\sigma_1^*(n)$ (≥ 8, $\forall n\in\mathbb{N}$).

equivalently, $\theta(\tau)^4 = \frac{-1}{\pi^2}\left(F\left(\frac{\tau}{2}\right) - 4F(2\tau)\right)$, where

$F(\tau) := \sum_m \left(\sum_n \frac{1}{(m\tau+n)^2}\right)$ $= E_2^*(\tau)$

reason : $\sigma_1^*(n) = \sigma_1(n)$ if $4 \nmid n$

$= \sigma_1(n) - 4\,\sigma_1(n/4)$ if $4 \mid n$

Recall that $F(\tau) = \frac{\pi^2}{3} - 8\pi^2 \sum_{k=1}^{\infty} \sigma_1(k)\, e^{2\pi i k \tau}$; hence $E_2^*(\tau) = -\pi^2 \left(1 + 8 \sum_{k=1}^{\infty} \sigma_1^*(k)\, e^{\pi i k \tau}\right)$.

the corresponding 1)' is now

$$\begin{cases} E_2^*(\tau+2) = E_2^*(\tau) & \text{ok.} \\ E_2^*(-1/\tau) = -\tau^2 E^*(\tau) & \text{: need} \end{cases}$$

in Stein's book

"Lemma 3.9" Let $\tilde{F}(\tau) = \sum_n \left(\sum_m \frac{1}{(m\tau+n)^2} \right)$, then

(a) $F(-1/\tau) = \tau^2 \tilde{F}(\tau)$ ok by definition

(b) $F(\tau) - \tilde{F}(\tau) = \frac{2\pi i}{\tau}$ crucial part

(c) $(a)+(b) \Rightarrow F(-1/\tau) = \tau^2 F(\tau) - 2\pi i \tau$

Hence $E_2^*(-1/\tau) = F(-1/2\tau) - 4F(-2/\tau) = 4\tau^2 F(2\tau) - 2\pi i \frac{1}{2\tau} - 4\frac{\tau^2}{4} F(\frac{\tau}{2}) - 2\pi i \frac{4}{\tau/2} = -\tau^2 E_2^*(\tau)$

done

The crucial (b). Recall Dedekind eta $\eta(\tau) := q^{1/12} \prod_{n=1}^{\infty} (1 - q^{2n})$; $q = e^{\pi i \tau}$

then $\frac{\eta'}{\eta} = \frac{\pi i}{12} - 2\pi i \sum_{n=1}^{\infty} \frac{n q^{2n}}{1 - q^{2n}} = \frac{i}{4\pi} F(\tau)$

Now use the fact "$\eta(-1/\tau) = \sqrt{\tau/i}\; \eta(\tau)$" $\Rightarrow$ $\frac{\eta'(-1/\tau)}{\eta(-1/\tau)} \cdot \frac{1}{\tau^2} = \frac{1}{2\tau} + \frac{\eta'(\tau)}{\eta(\tau)} = \frac{1}{2\tau} + \frac{i}{4\pi} F(\tau)$

Stein proves this via product formula of (H).

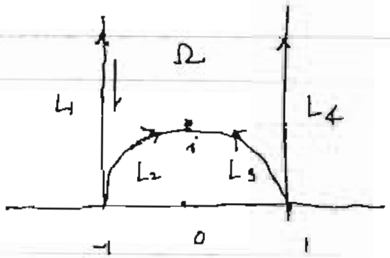
" $\frac{i}{4\pi} F(-1/\tau) \frac{1}{\tau^2} = \frac{i}{4\pi} \tilde{F}(\tau)$ by (a), get (b)

but in fact (b) can be proved directly (c.f. Serre : A course in Arithmetic)

hence $\exists$ "η ..." as well

see p 49 for the proof

About ③ : Dimension Formula of modular forms of weight k in the current case : let $f = C/\theta^2$ wt=0 or $= E_2^*/\theta^4$



$f(-1/\tau) = \tau^k f(\tau)$ $\quad f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$

only defined on $\mathbb{H}$ hence is not entire (θ has no zero)

$$I = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f'(\tau)}{f(\tau)}\, d\tau = \sum_{p\neq i} v_p(f) + \frac{1}{2} v_i(f)$$

$\tau = e^{i\theta}$; $-1/\tau = -e^{-i\theta} = e^{\pi - i\theta}$

hence $I = \frac{1}{2\pi i}\cdot\frac{k\pi i}{2} = \frac{k}{4}$

$f(-1/\tau) = \tau^k f(\tau)$

$f'(-1/\tau)\,\frac{1}{\tau^2} = k\tau^{k-1} f(\tau) + \tau^k f'(\tau)$

$\Rightarrow \frac{f'(-1/\tau)}{f(-1/\tau)}\,\frac{1}{\tau^2} = \frac{k}{\tau} + \frac{f'(\tau)}{f(\tau)}$

i.e. $\frac{f'(-1/\tau)}{f(-1/\tau)}\, d(-1/\tau) = \frac{k}{\tau}\, d\tau + \frac{f'(\tau)}{f(\tau)} d\tau$

But now $wt(f) = 0$, hence f has no zeros

if we take $f - c$ (say with $c \in f(\mathbb{H}^*)$), then we must have

$f = c$ is just a constant.

So the only remaining part to complete the pf is ② : ie asymptotic behavior at cusps " 1 & ∞ "

Rmk: the group $\langle T_2, S\rangle \subset SL(2,\mathbb{Z})$ is known as $\Gamma_0(2)$, and $[SL(2,\mathbb{Z}) : \Gamma_0(2)] = 3$ (in general $p+1$ for $\Gamma_0(p)$, p: prime)

in general the modular form formula for $SL(2,\mathbb{Z})$ is $\sum \frac{1}{e_p} v_p(f) = \frac{k}{12}$.