2022 FALL - LIE GROUPS AND LIE ALGEBRAS

FINAL EXAM

A COURSE BY CHIN-LUNG WANG AT NTU

1. Let G be a connected Lie group. Show that
(a) G is generated by any neighborhood U of e.
(b) Any discrete normal subgroup H lies in the center $Z(G)$.
(c) The fundamental group $\pi_1(G)$ is abelian.

2. Let G be a compact Lie group with invariant measure dg such that $|G| = 1$.
(a) State and prove Schur orthogonality relations.
(b) For two finite dimensional representations V, W of G, show that

$$\langle \chi_V, \chi_W \rangle := \int_G \chi_V(g)\overline{\chi_W(g)}\, dg = \dim \operatorname{Hom}_G(V, W).$$

(c) Show that $V \cong W$ if and only if $\chi_V = \chi_W$.

3. Consider the spin representation $S = \bigwedge W$ of $\mathrm{Spin}(n)$ and half-spin representations $S^\pm = \bigwedge^\pm W$ when $n = 2m$, where $W \subset \mathbb{C}^n$ is maximally isotropic. Show that
(a) S is not a representation of $\mathrm{SO}(n)$.
(b) $T = \{g_\theta = (\cos\theta_1 + e_1e_2 \sin\theta_1)\cdots(\cos\theta_m + e_{2m-1}e_{2m}\sin\theta_m) \mid \theta_i \in \mathbb{R}^m\}$ is a maximal torus in $\mathrm{Spin}(2m)$ and in $\mathrm{Spin}(2m+1)$.
(c) For $n = 4$, find the joint eigenspaces of T on $S^\pm$ and calculate $\chi_{S^\pm}(g_{\theta_1,\theta_2})$.

4. Let $\mathcal{H}_m(\mathbb{R}^n)$ be the space of real harmonic polynomials of degree m. Show that
(a) $\mathcal{H}_m(\mathbb{R}^2)$ is $\mathrm{O}(2)$-irreducible but not $\mathrm{SO}(2)$-irreducible.
(b) Under the usual action $(gf)(v) = f(g^{-1}v)$,

$$L^2(S^{n-1}) = \widehat{\bigoplus_{m\in\mathbb{Z}_{\geq 0}}} \mathcal{H}_m(\mathbb{R}^n)|_{S^{n-1}}, \quad n \geq 2,$$

is the canonical decomposition of $L^2(S^{n-1})$ under $\mathrm{O}(n)$.

5. Let G be a compact connected Lie group, S a connected abelian Lie subgroup of G.
(a) Show that $Z_G(S)$ is the union of all maximal tori containing S.
(b) For $g \in G$, show that $Z_G(g)^\circ$ is the union of all maximal tori containing g.

6. (Bonus) Present ONE essential topic in Lie groups which you have well-prepared but not listed above. (E.g. the proof of a major theorem.)

Date: December 22, 2022. Time and place: 17:30 – 21:30 at AMB 102.

Each problem in **1**—**5** is of 20 points. The bonus depends on the depth and completeness of the presentation on the chosen topic. Be sure to show your answers/computations/proofs in details.