2022 FALL - LIE GROUPS AND LIE ALGEBRAS

MIDTERM EXAM

A COURSE BY CHIN-LUNG WANG AT NTU

We work over the ground field $F = \overline{F}$ with char $F = 0$.

1. Let L be a semisimpe Lie algebra over F.
  (a) Show that $\operatorname{ad} L = \operatorname{Der} L$. Use it and properties of $\operatorname{Der} L \subset \operatorname{End} L$ to define the abstract Jordan decomposition $x = s + n$ for $x \in L$.
  (b) For any f.d. representation $\phi : L \to \mathfrak{gl}(V)$, show that $\phi(x) = \phi(s) + \phi(n)$ is the standard (usual) Jordan decomposition.

2. Let $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be the standard basis of $L = \mathfrak{sl}(2, F)$.
  (a) Classify f.d. irreducible L-modules as $V(m)$'s (of highest weight $m \geq 0$).
  (b) Let $\phi : L \to \mathfrak{gl}(V(m))$ with $m > 0$. Show that $e^{\phi(x)} \in \operatorname{Aut}(V(m))$ and $\tau := e^{\phi(x)} e^{-\phi(y)} e^{\phi(x)}$ interchanges positive and negative weight spaces.

3. Let Φ be a root system with base $\Delta = \{ \alpha_1, \ldots, \alpha_\ell \}$ and Weyl group $\mathscr{W} = \langle \sigma_\alpha \mid \alpha \in \Phi \rangle$.
  (a) Show that $\Delta^\vee = \{ \alpha_1^\vee, \ldots, \alpha_\ell^\vee \}$ is a base of $\Phi^\vee := \{ \alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)} \mid \alpha \in \Phi \}$.
  (b) For $\lambda = \sum k_i \alpha_i$ with $k_i \geq 0$, show that either λ is a multiple of a root or there exists $\sigma \in \mathscr{W}$ such that $\sigma(\lambda) = \sum k_i' \alpha_i$ with some $k_i' > 0$ and some $k_i' < 0$.

4. Let L be a semisimple Lie algebra with $x \in L$ semisimple. Recall that x is regular if $C_L(x)$ is a maximal toral subalgebra.
  (a) Show that x is regular if and only if x lies in exactly one CSA.
  (b) For any two Borel subalgebras B and B', show that $B \cap B'$ contains a CSA of L.

5. Let L be a semisimple Lie algebra, $\{h_i\}$ and $\{k_j\}$ be bases of H with $\kappa_H(h_i, k_j) = \delta_{ij}$, $x_\alpha \in L_\alpha$, $z_\alpha \in L_{-\alpha}$ with $\kappa(x_\alpha, z_\alpha) = 1$.
  (a) Show that the universal Casimir element $c_L := \sum_{i=1}^{\ell} h_i k_i + \sum_{\alpha \in \Phi} x_\alpha z_\alpha \in U(L)$ is independent of the various choices and it lies in the center of $U(L)$.
  (b) For $\lambda \in \Lambda^+$, show that $c_L = (\lambda + \delta, \lambda + \delta) - (\delta, \delta)$ on $V(\lambda)$.

6. (Bonus) Present ONE essential topic in Lie algebras which you have well-prepared but not listed above. (E.g. the proof of a major theorem.)

Date: November 14, 2022. Time and place: 10:20 – 12:50 at AMB 202.

Each problem in **1—5** is of 20 points. The bonus depends on the depth and completeness of the presentation on the chosen topic. Be sure to show your answers/computations/proofs in details.