HONORED ADVANCED CALCULUS
MID-TERM EXAM
9:10 – 12:40, 4/17, 2012
A COURSE BY CHIN-LUNG WANG

1. (10 pts) Let f be a bounded function on $Q = [a,b] \times [c,d] \subset \mathbb{R}^2$. Assume that $f(x,y)$ is increasing in x for any fixed y, and decreasing in y for any fixed x. Prove that $f \in R(Q)$.

2. (15 pts) Let $S \subset \mathbb{R}^{m+n}$ and $S_x = \{y \in \mathbb{R}^n : (x,y) \in S\}$.
(a) Prove that S has measure zero if and only if there exists a countable collection of intervals $\{I_k\}_{k=1}^{\infty}$ with $\sum_{k=1}^{\infty} \mu(I_k) < \infty$ such that for any $x \in S$, $x \in I_k$ for infinitely many k's.
(b) Prove that if S has ($(m+n)$-dimensional) measure zero, then S_x has (n-dimensional) measure zero for almost all x.

3. (15 pts) Let $E \subset \mathbb{R}^n$ be open and $F \in C^1(E, \mathbb{R}^n)$ such that $F(0) = 0$ and $F'(0)$ is invertible.
(a) Prove that there exists open $U \ni 0$ such that $F'(0)^{-1}F(x) = G_n \circ G_{n-1} \circ \cdots \circ G_1(x)$ on U, where each G_i is primitive C^1 with $G_i(0) = 0$ and $G_i'(0)$ invertible.
(b) State and prove the change of variable formula for $f \in C(\mathbb{R}^n)$ with compact support.

4. (10 pts) Let $\omega \in \Omega^1(E)$, where E is an open set in $\mathbb{R}^n$. Suppose that $\int_\gamma \omega = 0$ for all C^1 closed curves γ in E, show that ω is exact on E.

5. (15 pts) Let $\bar{c}(S)$ and $\underline{c}(S)$ be the outer/inner Jordan content for $S \subset \mathbb{R}^n$.
(a) Show that S is Jordan measurable (i.e. $c(S) := \bar{c}(S) = \underline{c}(S)$) if and only if $c(\partial S) = 0$. (Hint: Show that $\bar{c}(S) - \underline{c}(S) = \bar{c}(\partial S)$.)
(b) Show that the Jordan content is finitely additive but not σ-additive.

6. (15 pts) Let μ be a non-negative, additive, finite and regular set function on the collection of elementary sets $\mathcal{E} = \{A \subset \mathbb{R}^n : A = \bigcup_{j=1}^k I_j,$ where I_j's are bounded intervals.$\}$.
(a) Define the outer measure μ^* for all subsets of $\mathbb{R}^n$ and construct the collection of finitely μ-measurable sets $\mathfrak{M}_F(\mu)$.
(b) Construct the collection of measurable sets $\mathfrak{M}(\mu)$ and show that it is a σ-algebra on which μ^* is σ-additive.

7. (10 pts) Show that $C[a,b]$ is dense in $L^p[a,b]$ for any $p > 0$.

8. (a) (5 pts) Consider $F = \{f_k\}_{k=1}^{\infty} \subset L^2([-\pi,\pi])$ where $f_k(x) = \sin kx$. Show that F is a closed and bounded subset, but not compact.
(b) (5 pts) Let $n_k \in \mathbb{N}$, $k = 1, 2, \cdots$ be a strictly increasing sequence. Show that the set $E = \{x \in [-\pi,\pi] \mid \lim_{k\to\infty} \sin n_k x \text{ converges}\}$ has $m(E) = 0$.

9. (Bonus: 10 pts) Prove the change of variable formula for $f \in L(g(T))$ where $T \subset \mathbb{R}^n$ is open, $g : T \to \mathbb{R}^n$ is C^1, one to one, and $\det g'(t) \neq 0$ for all $t \in T$. (Hint: Use **3.** (b) or Fubini.)