HONORED ADVANCED CALCULUS
MID-TERM EXAM
9:10 – 12:40, 11/08, 2011
A COURSE BY CHIN-LUNG WANG

1. (15 points) Let S be a set. Show that the cardinality of the power set $P(S)$ is strictly bigger than S.

2. (15 points) Let (S,d) be a metric space. Show that (S,d) is compact $\iff$ (S,d) is sequentially compact. (Do "$\Rightarrow$" first. For "$\Leftarrow$", first show that (S,d) is separable.)

3. (15 points) Let $\alpha \in BV[a,b]$. Show that $V(x) := V_\alpha(a,x) \in BV[a,b]$ and α is continuous at x if and only if V is continuous at x. Finally show that if $\alpha \in C[a,b] \cap BV[a,b]$ then α can be written as the difference of two strictly monotone continuous functions.

4. (15 points) Let $\alpha \in BV[a,b]$. Show that $f \in R(\alpha) \Rightarrow f \in R(V)$. Based on this, show that $f, g \in R(\alpha) \Rightarrow fg \in R(\alpha)$. Moreover, show that $G(x) := \int_a^x g\,d\alpha \in BV[a,b]$ and $f \in R(G)$ with $\int_a^b f\,dG = \int_a^b fg\,d\alpha$.

5. (15 points) Show that $f \in R(\alpha)$ on $[a,b]$ if and only if it satisfies the Cauchy criterion: For any $\epsilon > 0$, there exists $P_\epsilon \in \mathscr{P}[a,b]$ such that $|S(P,f,\alpha) - S(P',f,\alpha)| < \epsilon$ for all $P, P' \supset P_\epsilon$. Use this to show that $f \in R(\alpha)$ on $[a,b] \Rightarrow f \in R(\alpha)$ on any $[c,d] \subset [a,b]$.

6. (15 points) Let $\sum_{n=0}^\infty a_n \to A$ absolutely and $\sum_{n=0}^\infty b_n \to B$. Show that $\sum_{n=0}^\infty c_n \to AB$ where $c_n = \sum_{k=0}^n a_k b_{n-k}$ is the Cauchy product.

7. Let f be defined and bounded on $[a,b]$. If $T \subset [a,b]$, we define the oscillation of f on T as $\Omega_f(T) = \sup\{f(x) - f(y) : x, y \in T\}$. The oscillation of f at x is defined to be the number $\omega_f(x) = \lim_{h \to 0^+} \Omega_f(B(x,h) \cap [a,b])$.

(a) (5 points) Let $\varepsilon > 0$ be given. Assume that $\omega_f(x) < \varepsilon$ for all $x \in [a,b]$, show that there exists a $\delta > 0$ (depending only on ε) such that for every closed subinterval $T \subset [a,b]$, we have $\Omega_f(T) < \varepsilon$ whenever the length of T is less than δ.

(b) (5 points) For any $\varepsilon > 0$, show that the set $J_\varepsilon = \{x \in [a,b] : \omega_f(x) \geq \varepsilon\}$ is compact.

(c) (Bonus problem with 10 points) Let D denote the set of discontinuity of f in $[a,b]$. Prove that f is Riemann-integrable on $[a,b]$ if and only if D has measure zero.