HONORED ADVANCED CALCULUS
FINAL EXAM
AM 9:10 – 12:40
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Each of 1, 2, 3 deserves 20 points, and each of 4, 5, 6, 7 deserves 15 points.

1. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous.
(a) If $f' = 0$ a.e. on $[a, b]$, show that f is a constant.
(b) Show that there exists $g \in L[a, b]$ such that $f(x) = f(a) + \int_a^x g$.

2. (a) Let $g \in L[0, 1]$. If $\exists\, C > 0$ and $1 \le p < \infty$ such that $|\int_0^1 fg| \le C\|f\|_p$ for any bounded $f \in M[0, 1]$, show that $g \in L^q[0, 1]$ and $\|g\|_q \le C$, where $\frac{1}{p} + \frac{1}{q} = 1$.
(b) Let $F : L^p[0, 1] \to \mathbb{R}$ be a continuous linear functional, $1 \le p < \infty$. Show that $\exists\, g \in L^q[0, 1]$ with $F(f) = \int_0^1 fg$ and $|F| = \|g\|_q$.

3. Let E and F be Banach spaces and U is an open set in E. If $f : U \to F$ is a C^k function, where $k \in \mathbb{N}$, and $f'(x_0) \in L(E, F)$ is an isomorphism, show that f is locally C^k invertible at x_0.

4. Let F be a Banach space and Ω be a compact topological space.
(a) Show that $L(E, F)$ and $C(\Omega, F)$ are also Banach spaces.
(b) If $U \subset F$ is open, show that $C(\Omega, U)$ is open in $C(\Omega, F)$.

5. Let E, E_1, E_2 be normed vector spaces, $U \subset E$ open and $f : U \to E_1$, $g : U \to E_2$ be two maps such that $f'(x)$, $g'(x)$ exists at a point $x \in U$. For any $(,) \in L(E_1, E_2; F)$. Show that $(f, g)'(x)$ exists and derive its formula.

6. Let $k \in \mathbb{Z}_{\ge 0}$, $s > k + \frac{m}{2}$. Show that $H_s \subset C^k(\mathbb{R}^m)$ and there is a constant $C > 0$ such that $|f|_{C^k} \le C|f|_s$ for all $f \in H_s$. (Hint: Consider the case $k = 0$ and $f \in \mathcal{S}$ first.)

7. Let $f_n \in \mathcal{S}$, $n \in \mathbb{N}$, with $\operatorname{supp}(f_n) \subset K$ for a compact set $K \subset \mathbb{R}^m$. Let $s > t$. If $|f_n|_s \le C$ for all n, show that there is a subsequence f_{n_k} which converges in H_t. Show that the conclusion may fail if the supports of f_n are not uniformly bounded.