HONORED ADVANCED CALCULUS
FINAL EXAM
PM 6:30 – 9:30
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1. Let $\alpha \in BV[a,b]$ and $f_n \in R(\alpha)$ on $[a,b]$ for $n \in \mathbb{N}$. Assume that $f_n \to f$ uniformly on $[a,b]$. Show that $f \in R(\alpha)$ and
$$\int_a^x f_n\,d\alpha \to \int_a^x f\,d\alpha \quad \text{uniformly on } x \in [a,b].$$

2. Prove Levi's theorem for step functions. Namely for $s_n \in S(I)$ and increasing, if $\lim_{n\to\infty} \int_I s_n = A$ exists then $s_n \to f \in U(I)$ a.e. on I with $\int_I f = A$.

3. Prove Fatou's lemma: Suppose that $f_n \in L(I)$ and $f_n \geq 0$ a.e. for all $n \in \mathbb{N}$.
(a) Prove that $\inf_n f_n \in L(I)$.
(b) If $\liminf_{n\to\infty} \int_I f_n < \infty$, prove that $\liminf_{n\to\infty} f_n \in L(I)$ and
$$\int_I \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_I f_n.$$
(c) State Lebesgue's dominated convergent theorem and prove it using (b).

4. Let $f \in L([0,2\pi])$ with its n-th partial Fourier series being $s_n(x)$.
(a) Prove Fejér's theorem: If f is continuous and $f(0) = f(2\pi)$, show that $\sigma_n(x) := \frac{1}{n}\sum_{i=0}^{n-1} s_i(x) \to f(x)$ uniformly, and Parseval's formula holds.
(b) If $f \in C^1$, $f(0) = f(2\pi)$, and $\int_0^{2\pi} f = 0$, show that $\|f'\| \geq \|f\|$ with equality holds if and only if $f(x) = a\cos x + b\sin x$.

5. Using Fourier series to deduce $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, and $\zeta(6) = \pi^6/945$.

6. Show that $D_{12}f = D_{21}f$ at $p = (a,b)$ in the following two cases:
(a) Both $D_1 f$, $D_2 f$ exist in a neighborhood of p and are differentiable at p.
(b) $D_1 f$, $D_2 f$, $D_{12} f$ exist in a neighborhood of p and continuous at p. Show that $D_{21} f(p)$ exists and equals $D_{12} f(p)$.

7. Assuming the inverse function theorem for C^1 maps on $\mathbb{R}^n$.
(a) State and prove the implicit function theorem.
(b) Let $f, g_1, \ldots, g_n : (U \subset \mathbb{R}^{n+m}) \to \mathbb{R}$ be C^1 functions and p be a smooth point in the level set $S = \{x \in U \mid g_k(x) = c_k,\ k = 1, \ldots, n\}$. If f takes its local maximum or minimum at p, show that there are constants $\lambda_1, \ldots, \lambda_n$ such that $\nabla f = \lambda_1 \nabla g_1 + \cdots + \lambda_n \nabla g_n$ at p.

8. Using the axiom of choice to construct a non-measurable subset $E \subset [0,1]$.