GEOMETRY MID-TERM EXAM

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A COURSE GIVEN BY CHIN-LUNG WANG AT NTU

1. Consider the Helix $\alpha : \mathbb{R} \to \mathbb{R}^3$ given by $\alpha(t) = (a\cos t, a\sin t, bt)$.

(a) Parmetrizes the Helix by its arc length s.
(b) Show that both $\kappa(s)$ and $\tau(s)$ are constant in s.
(c) Determine all space curves with constant κ and τ.

2. Let $\alpha : I \to \mathbb{R}^3$ be a regular smooth curve parametrized by its arc length such that the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is well-defined on it.

(a) Prove the local canonical form near the point $\alpha(0)$:

$$\alpha(s) - \alpha(0)$$
$$= \left(s - \frac{\kappa(0)^2}{6}s^3\right)\mathbf{T}(0) + \left(\frac{\kappa(0)}{2}s^2 + \frac{\kappa'(0)}{6}s^3\right)\mathbf{N}(0) - \frac{\kappa(0)\tau(0)}{6}s^3\mathbf{B}(0) + R(s),$$

where $R(s)$ is the error term with $\lim_{s\to 0} R(s)/s^3 = 0$.

(b) Let $\pi : \mathbb{R}^3 \to E$ be the orthogonal projection onto the osculating plane E at $\alpha(0)$ spanned by $\mathbf{T}(0)$ and $\mathbf{N}(0)$. show that the curvature $\bar{\kappa}(0)$ of the plane curve $\bar{\alpha} := \pi \circ \alpha : I \to E \cong \mathbb{R}^2$ at $s = 0$ equals $\kappa(0)$.

3. Let $\alpha : [0, \ell] \to \mathbb{R}^3$ be a regular smooth curve parametrized by its arc length such that Frenet frame is well-defined on it.

(a) Let $S \subset \mathbb{R}^3$ be the tubular surface along α with a fixed radius $r > 0$. Find a parametrization of S using the Frenet frame.
(b) Show that S is a regular surface if r is small enough.
(c) Compute the first fundamental form of S.
(d) Show that the area of S equals $2\pi r\ell$.

4. Regular surfaces defined by level sets:

(a) Let $S = F^{-1}(a)$ be the level set of a smooth function $F : \mathbb{R}^3 \to \mathbb{R}$ with $a \in \mathbb{R}$ a regular value. Prove in detail that S is a regular surface.
(b) Show that T_pS is the plane orthogonal to the vector $\nabla F(p)$.
(c) Consider three surfaces S_1, S_2 and S_3 defined by $x^2 + y^2 + z^2 = ax$, $x^2 + y^2 + z^2 = by$ and $x^2 + y^2 + z^2 = cz$ respectively where $a, b, c \neq 0$. Show that they are all regular surfaces. Moreover, if $p \in S_1 \cap S_2 \cap S_3$, show that T_pS_1, T_pS_2 and T_pS_3 intersect each other orthogonally.

5. Let S be a regular surface in $\mathbb{R}^3$ with $N : S \to S^2$ the Gauss map. Consider a local parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ with coordinates $(u, v) \in U$.

(a) Show that the matrix representing dN_p with respect to the basis $\mathbf{x}_1$, $\mathbf{x}_2$ is given by
$$\frac{1}{EG-F^2}\begin{pmatrix} fF-eG & gF-fG \\ eF-fE & fF-gE \end{pmatrix}.$$
(b) Find the differential equation for a curve $\alpha(t)=\mathbf{x}(u(t),v(t))$ on S to be a line of curvature.
(c) Show that the coordinate curves are precisely the lines of curvature if and only if $F=0$ and $f=0$. Explain that such a coordinate system can be achieved in a neighborhood of p if p is not an umbilical point.
(d) Compute the two principal curvatures of the Enneper's surface given by
$$\mathbf{x}(u,v)=(u-u^3/3+uv^2, v-v^3/3+vu^2, u^2-v^2).$$

6. Let $\alpha(v)$ be a curve in the xz plane and let S be the surface of revolution of α in the z-axis.
(a) For $\alpha(v)=(\phi(v),\psi(v))$ with v being the arc length, compute K and H and determine all such S with $K\equiv 1$. If moreovee S is compact and regular, show that S must be the sphere.
(b) If the curve is given as a graph $\alpha(z)=(h(z),z)$, compute its two principal curvatures and find all $h(z)$ such that S has $H\equiv 0$.