GEOMETRY — FINAL EXAM

January 14th, 2010, pm 6:10 - 9:10

A course given by Chin-Lung Wang at NTU

Important: Give your solutions in detail. Each problem deserves 20 points.

1. Denote by S_t, $t \in (-\epsilon, \epsilon)$ a normal variation of $S = \mathbf{x}(U)$ defined by $\mathbf{x}^t = \mathbf{x} + hN$ for some smooth function h, and let $A(t)$ be the area of S_t.
   (1) Show that S has $H \equiv 0$ (minimal surface) if and only if $A'(0) = 0$ for any such S_t.
   (2) For S being a minimal surface, show that $\langle dN_p(w_1), dN_p(w_2)\rangle = -K(p)\langle w_1, w_2\rangle$ for any $w_1, w_2 \in T_pS$.

2. Define the notion of geodesics on a regular surface and derive the differential equations of the geodesics $\alpha(t) = \mathbf{x}(u(t), v(t))$. For a surface of revolution $\mathbf{x}(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$, prove that $f\cos\theta$ takes constant value along geodesics, where θ is the angle between $\mathbf{x}_u$ and $\alpha'(t)$.

3. Use the Gauss-Bonnet theorem to prove Jacobi's theorem: If a closed regular curve in $\mathbf{R}^3$ has $k > 0$ and its principal normal $\mathbf{n}(s)$ form a curve γ on S^2 without self-intersections, then γ separates S^2 into two regions with equal area.

4. Use the Gauss-Bonnet theorem to show that
   (1) Let S be a regular surface such that the parallel transport between any two points in it is independent of the path, then $K = 0$ on S.
   (2) Let S be a regular surface homeomorphic to a cylinder with $K < 0$, then S has at most one simple closed geodesic.

5. Use the geodesic polar coordinates to show that
   (1) Any two surfaces with the same constant curvature K are locally isometric.
   (2) Let $A(r)$ be the area of the geodesic ball of radius r centered at $p \in S$, then

$$K(p) = \lim_{r\to 0} \frac{12}{\pi}\left(\frac{\pi r^2 - A(r)}{r^4}\right).$$

6. State and prove the Gauss-Bonnet theorem.