GEOMETRY FINAL EXAM

There are **2 pages** with **6 problems**.
Each problem deserves **20 points**.
Show your answers/computations/proofs in details.
You may work on each part independently.

1. Let $S \subset \mathbb{R}^3$ be a parametrized regular surface defined by $\mathbf{x}(u, v)$.

(a) Derive the Codazzi equations:

$$e_2 - f_1 = e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2,$$
$$g_1 - f_2 = g\Gamma_{12}^2 + f(\Gamma_{12}^1 - \Gamma_{22}^2) - e\Gamma_{22}^1.$$

(b) If the coordinate curves are lines of curvature, simplify the equations to

$$e_2 = \frac{E_2}{2}\Big(\frac{e}{E} + \frac{g}{G}\Big), \qquad g_1 = \frac{G_1}{2}\Big(\frac{e}{E} + \frac{g}{G}\Big).$$

(c) Is there a surface with $E = G = 1$, $F = 0$ and $e = 1$, $g = -1$, $f = 0$? How about $E = 1$, $F = 0$, $G = \cos^2 u$ and $e = \cos^2 u$, $f = 0$, $g = 1$? You may use Bonnet's theorem, and the formula for K without proving it that for $F = 0$,

$$K = -\frac{1}{2\sqrt{EG}}\Big[\Big(\frac{E_2}{\sqrt{EG}}\Big)_2 + \Big(\frac{G_1}{\sqrt{EG}}\Big)_1\Big].$$

2. Let $\gamma(t) = \mathbf{x}(u(t), v(t))$ be a curve on S.

(a) Define the notion for γ to be a geodesic and derive its equations in terms of $(u(t), v(t))$.

(b) If S is a Liouville surface, namely $E = G = U(u) + V(v)$, $F = 0$. Show that any geodesic γ satisfies $U \sin^2\theta - V\cos^2\theta = c$ where $\theta = \angle(\gamma', \mathbf{x}_1)$ and c is a constant.

3. (a) Compute $\operatorname{ind}_p v$ at $p = (0, 0)$ in the following cases: (i) $v(x, y) = (x^2 - y^2, -2xy)$, (ii) $v(x, y) = (x^3 - 3xy^2, y^3 - 3x^2y)$.

(b) Can it happen that $\operatorname{ind}_p v = 0$ for p a singular point of v? If so, give an example.

(c) Let $C \subset S^2$ be a regular closed curve, v a vector field on S whose trajectories are never tangent to C. Prove that each region R with $\partial R = C$ contains some singular point of v.

4. Using geodesic polar coordinates to prove:

(a) Any two surfaces with the same constant curvature K are locally isometric.

(b) Let $A(r)$ be the area of the geodesic ball $B_r(p)$, then

$$K(p) = \frac{12}{\pi}\lim_{r\to 0}\frac{\pi r^2 - A(r)}{r^4}.$$

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5. (Poincaré models for hyperbolic geometry) Let $\mathbb{H} = \{w \mid \operatorname{Im} w > 0\}$ with $ds^2 = |dw|^2/(\operatorname{Im} w)^2$ and $\mathbb{D} = \{z \mid |z| < 1\}$ with $ds^2 = 4|dz|^2/(1-|z|^2)^2$.

(a) Show that $\mathbb{H} \to \mathbb{D}$, $w \mapsto z = (w-i)/(w+i)$ is an isometry.

(b) Determine all geodesics in $\mathbb{D}$.

(c) Let Ω be the region bounded by the 4 unit circles centered at $(\pm 1, \pm 1)$. Compute

$$\int_\Omega \frac{4\,dx\,dy}{(1-x^2-y^2)^2}.$$

(You may use Gauss–Bonnet theorem or do it directly.)

6. State and prove, as complete as possible, the Gauss–Bonnet theorem. (A very good solution to this problem may get some extra credits.)

* If you prefer to write proofs, you may replace **up to 2** problems from **1** to **5**, but not **6**, by (stating and proving) the following global surface theorems:

(i) Any $S \subset \mathbb{R}^3$ with constant K must be a sphere.

(ii) 2nd variation formula for normal variations and Bonnet's theorem on $d \le \pi/\sqrt{k}$.

(iii) Fenchel's theorem and Fary–Milnor's theorem on $\int k\,ds$.

(iv) Hilbert's theorem on complete surface S with $K = -1$.

Label your solution by n^* if it is for the n-th problem. Notice that you will still get at most 20 points for that problem.