GEOMETRY MIDTERM EXAM

1. (15 pt) Prove the iso-perimetric inequality $\ell^2 \geq 4\pi A$ for piecewise C^1 simple closed plane curves, and the equality holds if and only if the curve is a circle.

2. (20 pt) Let $\alpha : I \to \mathbb{R}^3$ be a regular curve parametrized by arc length with $0 \in I$, $\kappa > 0$.
(a) Derive the local canonical form of $\alpha(s)$ at $\alpha(0)$ in terms of the Frenet frame at $s = 0$, and sketch the projection curves on the **TN**, **TB**, **NB** planes respectively.
(b) Assume that $\kappa, \kappa', \tau \neq 0$ and denote by $R = 1/\kappa$, $T = 1/\tau$. Show that α lies in a sphere of radius r if and only

$$R^2 + (R'T)^2 = r^2.$$

3. (20 pt) Let $\alpha : [0, \ell] \to \mathbb{R}^3$ be a regular curve parametrized by arc length so that the Frenet frame is defined, and let S be the tubular surface along α of radius $r > 0$.
(a) Determine the range of r so that S is locally a regular surface, (i.e. an immersion).
(b) Compute its area.
(c) If α is moreover simple (injective), show that S is a regular surface for small r.

4. (20 pt) Let $N : S \to S^2$ be the Gauss map with $(U, \mathbf{x})$ a coordinate chart for $p \in S$.
(a) Show that the matrix for dN_p with respect to the bases $\mathbf{x}_u, \mathbf{x}_v$ of T_pS is given by

$$\frac{1}{EG - F^2} \begin{pmatrix} fF - eG & eF - fE \\ gF - fG & fF - gE \end{pmatrix}.$$

(b) Derive the differential equation for lines of curvature, and show that the coordinate curves are lines of curvature if and only if $F = f = 0$.
(c) Derive the equation for asymptotic curves. Solve them for Enneper's surface

$$\mathbf{x}(u, v) = (u - \tfrac{1}{3}u^3 + uv^2, v - \tfrac{1}{3}v^3 + vu^2, u^2 - v^2).$$

5. (20 pt) Let S be a surface of revolution on a curve α in the xz plane.
(a) For α being parametrized by arc length s, compute E, F, G, e, f, g and K, H.
(b) Determine all such regular surfaces S with $K = 1$. When is S compact?
(c) Determine all minimal surfaces of revolution.

6. (15 pt) Let S be the graph defined by $z = f(x, y)$ over a compact domain $D \subset \mathbb{R}^2$, with ∂D a smooth curve. Let h be C^∞ on D with $h|_{\partial D} = 0$. Consider variations of surfaces S_t defined by $z = f + th$ with $A(t)$ being its area.
(a) Compute H for S, and show that $A'(0) = 0$ for all h if and only if $H = 0$.
(b) For S a minimal graph, show that $A''(0) > 0$ for any $h \not\equiv 0$.

7. (10 pt) Let S be a minimal surface. Construct isothermal coordinates near any non-planar points. (Show first that $\langle dN_p(v), dN_p(w)\rangle = -K(p)\langle v, w\rangle$.)

Date: 12:30 – 15:20, November 1, 2013, A course by Chin-Lung Wang at NTU..
Show your answers/computations/proofs in details. You may work on each part independently.