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finite quot. sing. weil div. / ~

Generator of $\text{Pic}(\bar{M}) \otimes \mathbb{Q} \cong \text{Am}_{n-1}(\bar{M}) \otimes \mathbb{Q}$

$$\bar{M} = \bar{M}_{0,n}(\mathbb{P}^r, d), \quad \dim \bar{M} = m = rd + d + r + n - 3$$

$$L_i := \text{ev}_i^*(\mathcal{O}_{\mathbb{P}^r}(1))$$

(Assume $d > 0$, since $d=0$ reduces to $\bar{M}_{0,n} \times \mathbb{P}^r$)

$$\Delta := \{(A \cup B, d_A, d_B)\} \quad \begin{cases} (1) A, B \text{ partite } I = [n] \\ (2) d_A + d_B = d \end{cases}$$

$$\text{the set of bdy of } \bar{M} \quad \begin{cases} (3) \text{ If } d_A = 0 \text{ then } |A| \geq 2 \end{cases}$$

$$(H=0 \text{ if } d=0)$$

$$(d_B=0) \quad (|B| \geq 2)$$

 $H :=$ locus of maps meeting a fixed $n-2$ dim'l subspace of $\bar{M}$
Lemma. (1) If $n \geq 3$ then $\Delta \cup \{H\}$ gen. $\text{Pic}(\bar{M}) \otimes \mathbb{Q}$ (2) If $n=2$ then $\Delta \cup \{L_1, L_2\}$ gen. "(3) If $n=1$ then $\Delta \cup \{L_1, H\}$ gen. "(4) If $n=0$ then $\Delta \cup \{H\}$ gen. "Proof. (1) Consider the family $\mathbb{P}^1 \times W \times U \rightarrow \mathbb{P}^r$ (basepoint free) $U \subset \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(d)))^{(r+1)}$ the subset defining a deg d map $\mathbb{P}^1 \rightarrow \mathbb{P}^r$ $W = (\mathbb{P}^1)^{n-3} \setminus \{D_{ij}, S_{0,i}, S_{1,i}, S_{00,i}\}$ the marked pts other than 0, 1, 00 $(D_{ij} = \text{Diagonal of } i\text{th, } j\text{th comp.}, S_{p,i} = \text{The } i\text{th comp. takes value } p)$ By universal prop. this induces an inj. $W \times U \hookrightarrow \bar{M}$ $\bar{M} \setminus W \times U$ is the bdy of $\bar{M}$, thus(Note $\mathbb{P}(U) \cup U$ $\text{Am}_{n-1}(\bar{M})$ is gen. by Δ and $\text{Am}_{n-1}(W \times U)$ has codim ≥ 2)Since $W \times U \subset (\mathbb{P}^1)^{n-3} \times \mathbb{P}(U)$, $\text{Am}_{n-1}(W \times U)$ is gen. by $\mathcal{O}_{\mathbb{P}(U)}(1)$ (Pullback of $\mathcal{O}(1)$ of $\mathbb{P}^1$'s are trivial since $S_{0,i}$ is removed) $H|_{W \times U}$ is a pullback of a hypersurface in $\mathbb{P}(U)$ defined by a resultantThus $H|_{W \times U}$ is a multiple of $\mathcal{O}_{\mathbb{P}(U)}(1)$

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(2) Denote $\bar{U} = \bar{M}_{0,3}(P^r, d)$, $\bar{M} = \bar{M}_{0,2}(P^r, d)$

$X = \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{p} P$, $H_3 \subset P^r$ a fix hyperplane

$p: X \rightarrow \bar{M}$ omitting the 3rd point and contracting, is surj.

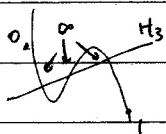
and generically finite. Let $Z = M \setminus (\mathbb{A}^1 \times \mathbb{A}^1 \times H_3) \subset \bar{M}$

then $p^{-1}(Z) \rightarrow Z$ is finite projective

It suffices to show $\text{Am}_{-1}(p^{-1}(Z)) = 0$ (so $\text{Am}_1(Z)$ is torsion)

In fact $p^{-1}(Z) = U \cap L_{\infty}(H_3) \setminus \{L_0(H_3), L_1(H_3)\}$

U as defined in (1) (Note $W \rightarrow pL$)



$L_p(H_i)$ the subset sending p to H_i

Thus $U \cap L_{\infty}(H_3)$ is gen. by hyperplane class in $L_{\infty}(H_3)$

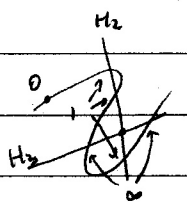
$\Rightarrow \text{Am}_{-1}(p^{-1}(Z)) = 0$ since $p^{-1}(Z) \subset U \cap L_{\infty}(H_3)$ is comp.

of hyperplanes.

(3) Denote $\bar{N} = \bar{M}_{0,3}(P^r, d)$, $\bar{M} = \bar{M}_{0,1}(P^r, d)$

$X = \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{p} P$ $H_2, H_3 \subset P^r$ fixed hyperplanes

$p: X \rightarrow \bar{M}$ omitting 2nd and 3rd points is surj. and generically finite.



Let $Z = M \setminus (\mathbb{A}^1 \times \mathbb{A}^1 \times H_2) \cup (\mathbb{A}^1 \times \mathbb{A}^1 \times H_3) \cup D_{2,3}$,

$D_{2,3}$ the Weil divisor of maps passing through $H_2 \cap H_3$

$p^{-1}(Z) \rightarrow Z$ is finite proj. and we show $\text{Am}_{-1}(p^{-1}(Z)) = 0$.

$p^{-1}(Z) = U \cap L_1(H_2) \cap L_{\infty}(H_3) \setminus S$ meeting $H_2 \cap H_3$

where $S = L_0(H_2) \cup L_0(H_3) \cup \{\text{resultant hypersurface of maps}\}$

thus $\text{Am}_{-1}(p^{-1}(Z)) = 0 \Rightarrow \text{Am}_{-1}(Z)$ torsion as before

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(c) Denote $\bar{N} = \bar{M}_{0,3}(\mathbb{P}^r, d)$, $\bar{M} = \bar{M}_{0,0}(\mathbb{P}^r, d)$,

$$X = \nu_1^{-1}(H_1) \cap \nu_2^{-1}(H_2) \cap \nu_3^{-1}(H_3), \quad H_i: \text{hyperplanes}$$

$p: X \rightarrow \bar{M}$ omitting marked pts is surj. generically finite

Let $Z = \bar{M} \setminus (D_{1,2} \cup D_{1,3} \cup D_{2,3})$, $p^{-1}(Z) \rightarrow Z$ finite

$$p^{-1}(Z) = U \cap L_0(H_1) \cap L_1(H_2) \cap L_\infty(H_3) \setminus S$$

$$\text{where } S = \{L_0(H_2) \cup L_0(H_3) \cup L_1(H_1) \cup L_3(H_3) \cup L_\infty(H_1) \cup L_\infty(H_3)\}$$

thus $\bar{M}^{-1}(Z)$ is torsion as before

Relations in $\text{Pic}(\bar{M}) \otimes \mathbb{Q}$

Write $\Delta = \coprod_{i=0}^{n-1} \Delta_i$ where Δ_i is the subset s.t. the partition

$$|A| + |B| = n \text{ is equal to } i + (n-i) = n$$

Lemma. (1) $H \not\equiv 0 \pmod{\Delta}$, moreover if $n=1$, then $\{H, L_1\}$ are indep mod Δ ; if $n=2$, then $\{L_1, L_2\}$ are indep mod Δ .

(2) If $n \leq 1$, then $\Delta_0 = \Delta$ is a set of indep. elements

If $n \geq 2$, then $\Delta_0 \cup \Delta_1$ "

and $\Delta' = \Delta \setminus (\Delta_0 \cup \Delta_1)$ has no relation with $\Delta_0 \cup \Delta_1$,

(3) Relations among Δ' are pullback from $\bar{M}_{0,n}$

via $\eta: \bar{M}_{0,n}(\mathbb{P}^r, d) \rightarrow \bar{M}_{0,n}$ for $n \geq 4$.

$$(\eta^{-1}((A \cup B))) = \bigcup_{d_A + d_B = d} (A \cup B, d_A, d_B)$$

Proof. (1) Consider $S = \mathbb{P}^1 \times C \rightarrow C$ with n (disjoint) trivial sections
 $N \rightarrow S$ line bundle of type (d, k) for some large k (degree)
 $\exists \rho \in H^0(S, N)$ $\rho = 0, \dots, r$ determines a rational map
 $\mu: S \dashrightarrow \mathbb{P}^r$, since $r \geq 2$ we can make it basepoint free.

This determines a curve C on $\bar{M}$. We have

$$C \cdot K = 0 \quad \forall K \in \Delta \quad \text{since disjoint}$$

$$C \cdot H = N \cdot N = 2dk \neq 0$$

Thus H indep of Δ

If $n=1$, consider $S = \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$

S a trivial section, s' diagonal section

$$\mu: S \rightarrow \mathbb{P}^r \text{ bpf of type } (d, k)$$

S, s' induces two curves $\mathbb{P}^1, (\mathbb{P}^1)'$ on $\bar{M}_{0,1}(\mathbb{P}^r, d)$

$$\mathbb{P}^1 \cdot H = 2dk, \quad \mathbb{P}^1 \cdot L_1 = k, \quad \mathbb{P}^1 \cdot K = 0 \quad \forall K \in \Delta$$

$$(\mathbb{P}^1)' \cdot H = 2dk, \quad (\mathbb{P}^1)' \cdot L_1 = d+k, \quad (\mathbb{P}^1)' \cdot K = 0 \quad \forall K \in \Delta$$

Thus $\{H, L_1\}$ indep mod Δ

If $n=2$, consider $S(a, b) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)) \rightarrow \mathbb{P}^1$

s_1, s_2 sections determined by $\mathcal{O}(a), \mathcal{O}(b)$,

$$\mu = \mathcal{O}_{\mathbb{P}^1}(d) \oplus \pi^*(\mathcal{O}_{\mathbb{P}^1}(k)) \rightarrow S(a, b) \text{ with } k \text{ large}$$

$\mu: S(a, b) \rightarrow \mathbb{P}^r$ basepoint free map, inducing

$$\mathbb{P}^1 \rightarrow \bar{M}_{0,2}(\mathbb{P}^r, d), \text{ then}$$

$$\mathbb{P}^1 \cdot L_1 = -ad+k, \quad \mathbb{P}^1 \cdot L_2 = -bd+k, \quad \mathbb{P}^1 \cdot K = 0 \quad \forall K \in \Delta$$

Thus $\{L_1, L_2\}$ indep mod Δ

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(2) Let $\pi: S = \mathbb{P}^1 \times C \rightarrow C$ with n trivial sections S_i

$N \rightarrow S$ line bundle of type (d, k) .

For large k we can assign a simple bspt of $\mu: S \rightarrow \mathbb{P}^r$

s.t. blowing up at the bspt resolves μ and gives exceptional divisor of degree $1 \leq e \leq d$. Denote $\bar{S}$ blowup of S at p

$\bar{\mu}: \bar{S} \rightarrow \mathbb{P}^r$ determines a curve $C \subset \bar{M}$

then $C \cdot K = 1$ if $K \in \Delta$ is of type $\{A \cup B, e, d-e\}$

$A = \emptyset$ if $p \notin S_i$; $A = \{i\}$ if $p \in S_i$

$C \cdot K = 0$ o.w., thus we have the desired indep.

(Note the stability of C is automatically true)

(3) Let $\pi: S = \mathbb{P}^1 \times C \rightarrow C$ with n section s_i (with intersections)

$N \rightarrow S$ line bundle of type (d, k)

For large k we can assign simple bspts so that

the bspts are at the intersections of s_i with given degree

Moreover, 1) On any fiber of π there's only one intersection of s_i

2) Sections through an intersection have distinct tangent dir.

3) An intersection pt lies on at most $(n-2)$ sections

Denote $\bar{S}$ the blowup of S at the intersections,

$\bar{\mu}: \bar{S} \rightarrow \mathbb{P}^r$ determines a curve $C \subset \bar{M}$

then 1) $\Rightarrow C \nparallel K, K \in \Delta$

2) $\Rightarrow$ strict transforms of the sections are disjoint

3) $\Rightarrow$ stability

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Now if $\sum_{K \in \Delta'} c_K K = 0$ is a relation in $\text{Pic}(\bar{M}) \otimes \mathbb{Q}$
 then $\sum_{K \in \Delta'} c_K (C \cdot K) = 0$ and c_K is indep. of the
 degree of bspts we assigned, thus $\sum_{K \in \Delta'} c_K K = 0$ is a pullback
 of relations in $\bar{M}_{0,n}$.

Picard Number.

Thm. For $n \geq 0$, $\dim_{\mathbb{Q}} \text{Pic}(\bar{M}) \otimes \mathbb{Q} = \lfloor \frac{d}{2} \rfloor + 1$

$$n \geq 1, \dim_{\mathbb{Q}} \text{Pic}(\bar{M}) \otimes \mathbb{Q} = (d+1)2^{n-1} - \binom{n}{2}$$

(this holds also for $d=0$)

Proof. For $n=0$, $\Delta \cup \{H\}$ is a basis by above lemmas.

For $1 \leq n \leq 3$, $\Delta = \Delta_0 \cup \Delta_1$ are indep., then by above lemmas

$$(n=1: \Delta \cup \{L_1, H\}, n=2: \Delta \cup \{L_1, L_2\},$$

$$n=3: \Delta \cup \{H\})$$

For $n \geq 4$, $|\Delta'| = (d+1)(2^{n-1} - 1 - n)$ generators

with $\dim \text{Pic}(\bar{M}_{0,n}) = \#(\text{bdry comp. of } \bar{M}_{0,n})$

$$= (2^{n-1} - \binom{n-1}{2} - n) - \left(\frac{2^n - 2 - 2n}{2} \right)$$

$$= \binom{n-1}{2} - 1 \text{ indep. relations, b/w } \Delta' \text{ (Pull back of } \bar{M}_{0,n} \text{)}$$

$$\text{and } |\Delta_0 \cup \Delta_1 \cup \{H\}| = d + d_n + 1$$

The above arguments, with $\text{Pic}(\bar{M}_{0,n}) \cong \text{Num}(\bar{M}_{0,n})$,

also implies $\text{Pic}(\bar{M}_{0,n}(P^r, d)) \otimes \mathbb{Q} \cong \text{Num}(\bar{M}_{0,n}(P^r, d)) \otimes \mathbb{Q}$

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Intersection with bdry comp.

Let $K = (A \cup B, d_A, d_B) \in \Delta$,

write $\bar{M}_A = \bar{M}_{0, |A|+1}(\mathbb{P}^r, d_A)$, $\bar{M}_B = \bar{M}_{0, |B|+1}(\mathbb{P}^r, d_B)$

with the extra marked pt P_A, P_B resp. by, denote ev_A, ev_B
eval. at P_A, P_B
then $\tilde{E} = \bar{M}_A \times_{\mathbb{P}^r} \bar{M}_B \subset \bar{M}_A \times \bar{M}_B \xrightarrow{\tau_A} \bar{M}_A$
 $\xrightarrow{\tau_B} \bar{M}_B$
 $(ev_A \times ev_B)^*(D)$, D diagonal in $\mathbb{P}^r \times \mathbb{P}^r$

has class in $\bar{M}_A \times \bar{M}_B$

$$[\tilde{K}] = \sum_{i,j} \tau_A^*(c_i(\mathcal{L}_A)^i) \cdot \tau_B^*(c_j(\mathcal{L}_B)^{j-i})$$

$\mathcal{L}_A, \mathcal{L}_B$ line bundles induced by P_A, P_B (on $\bar{M}_A, \bar{M}_B$)

and there's natural map $\psi: \tilde{K} \rightarrow K$ gluing the marked pts

To calculate the intersection with K , we pullback by ψ

For $T = (A' \cup B', d_{A'}, d_{B'}) \in \Delta$, T intersects K if

(i) There's $C \subset A$ and d_C s.t.

$$T = ((A \setminus C) \cup (B \cup C), d_A - d_C, d_B + d_C) \text{ , or}$$

(ii) There's $C \subset B$ and d_C s.t.

$$T = ((A \cup C) \cup (B \setminus C), d_A + d_C, d_B - d_C)$$

$$\text{and } \psi^*(T) = \sum_{C, d_C} \tau_A^*(A \cup (C \cup \{P_A\}), d_A, d_C)|_{\tilde{K}} \\ + \sum_{C, d_C} \tau_B^*(B \cup (C \cup \{P_B\}), d_B, d_C)|_{\tilde{K}}$$

sum over those C satisfying (i) and (ii)

This utilized the fact that distinct, bdry, divisors
intersect transversally

We also want to find $\psi^*(K)$

$$(\text{others: } \psi^*(H) = (\tau_A^*(H_A) + \tau_B^*(H_B))|_{\tilde{K}},$$

$$\psi^*(L_i) = \tau_A^*(L_i)|_{\tilde{K}} \text{ if } i \in A) \\ \tau_B^*(L_i)|_{\tilde{K}} \text{ if } i \in B)$$

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conics in $\mathbb{P}^2$ (known)

Lemma. Locus of stable map with nontrivial autom.

has $\text{codim} \geq 2$ except for $\bar{M}_{0,0}(\mathbb{P}^2, 2)$ ($g=0, r \geq 2$)

Pf. Let $A \subset \bar{M}$ be the locus, If $\mu: \mathbb{P}^1 \rightarrow \mathbb{P}^r$ with n marked pts has nontrivial autom, then μ must be a $k=1$ map, $2 \leq k \leq d$

such map moves in a family of $\dim \leq$

$$(r+1) \cdot \left(\frac{d}{k} + 1\right) - 1 - 3 + (2k-2)$$

(Recall $\dim \bar{M} = rd + r + d + n - 3$) ← ramifications as marked pts

$$\text{Thus } \dim \bar{M} - \dim A \geq (rd + d) \left(1 - \frac{1}{k}\right) + n - 2k + 2$$

$$= rd + d + n + 2 - \frac{rd + d + 2k^2}{k}$$

If $k=2$ then

attain min on $k=2$ or d

$$rd + d - \frac{rd + d + 8}{2} = (r+1)\frac{d}{2} - 4 \geq 0 \text{ unless } r=2, d=2$$

If $k=d$ then

$$rd + d - \frac{rd + d + 2d^2}{d} = (r+1)(d-1) - 2 \geq 0 \text{ unless } r=2, d=2$$

For $(g, n, r, d) \neq (0, 0, 2, 2)$, let $\bar{M}^* \subset \bar{M}$ be the autom. free locus

$\pi: U^* \rightarrow \bar{M}^*$ universal family with sections $S_i, i=1, \dots, n$

$M = U^* \rightarrow \mathbb{P}^r$, and ω_π rel. dualizing sheaf of π

We consider the following elements in $A_{n-1}(\bar{M})$:

$$\pi_* (C, (\omega_\pi)^2), \quad \pi_* (S_i^2)$$

$$\text{Lemma. } 1) \quad \pi_* (C, (\omega_\pi)^2) = - \sum_{K \in \Delta} K \quad \text{in } \text{Pic}(\bar{M}) \otimes \mathbb{Q}$$

$$2) \quad \pi_* (S_i^2) = -\frac{1}{d} H + \frac{2}{d} L - \sum_{j=0}^d \frac{(d-j)^2}{d^2} K^{1,j}$$

$$\text{where } K^{1,j} = \sum_{\substack{1 \leq A \\ d_A = j}} (A \cup B, d_A, d_B)$$

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Proof. 1) Let $\pi: S \rightarrow C$ be a proj. bundle of rk 1

ω_π rel. dualizing sheaf

$\lambda^2 \in$ fiber class

$$\text{we have } \pi_* (c_1(\omega_\pi)^2) = \pi_* (-2C_0 + 2f) = 4E - 4E = 0$$

in notations of Hartshorne's AG.

in $\text{Num}(C)$

Let $\rho: \tilde{S} \rightarrow S$ blowing up k pts in distinct fibers of π

$$\tilde{\pi}: \tilde{S} \rightarrow C \quad \text{then } \omega_{\tilde{\pi}} = \rho^*(\omega_\pi) + \sum_{i=1}^k E_i \quad \text{exception div.}$$

$$\text{hence } \pi_* (c_1(\omega_{\tilde{\pi}})^2) = -k$$

by considering all curves $C \rightarrow \tilde{M}$ we have

$$\pi_* (c_1(\omega_\pi)^2) = - \sum_{k \in \mathbb{A}} k \quad \text{in } \text{Num}(M) \otimes \mathbb{Q}$$

2) Using the setup when proving relations on Δ'

$(\pi: S \cong \mathbb{P}^1 \times C \rightarrow C, \quad s_i \text{ sections, } i=1, \dots, n, \quad \mathcal{N} \text{ line bundle of type } (d, k) \text{ on } S$

$\mu: S \rightarrow \mathbb{P}^r, \quad \tilde{S} \rightarrow S \text{ blowup at intersections,}$

$\lambda: C \rightarrow \tilde{M} \text{ induced map})$

Assume s_i has type $(1, q)$ on S , we show both side

has the same intersection number with such C

Let $C \cdot K^{i,j} = x_i + y_j$, where

$x_j = \# \text{ blowup at a pt of deg } j \text{ in } s_i$

$y_j = \# \text{ " " " " " } d-j$

$$\text{then } C \cdot \pi_* (\tilde{S}_i^2) = 2q - \sum_{j=0}^d x_j$$

$$C \cdot \mathcal{H} = 2dk - \sum_{j=0}^d j^2 x_j - \sum_{j=0}^d (d-j)^2 y_j$$

$$C \cdot \mathcal{L}_i = dq + k - \sum_{j=0}^d jx_j$$

now we can check the equality.

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With these two calculations we have

$$\begin{aligned} \psi^*(\pi_*(c_1(w_\pi)^2)) &= \tau_A^*(\pi_{A*}((c_1(w_{\pi_A}) + s_{pA})^2)) \\ &\quad + \tau_B^*(\pi_{B*}((c_1(w_{\pi_B}) + s_{pB})^2)) \\ \Rightarrow -\sum_{T \in \Delta} \psi^*(\tau) &= \tau_A^*(\pi_{A*}(c_1(w_{\pi_A})^2 - s_{pA}^2)) \quad (\text{Since } c_1(w_{\pi_A}) \cdot s_{pA} \\ &\quad + \tau_B^*(\pi_{B*}(c_1(w_{\pi_B})^2 - s_{pB}^2)) \quad = -s_{pA}^2) \end{aligned}$$

which can be expressed in terms of $\{L_i\} \cup \Delta \cup \{H\}$
on $\bar{M}_A$ and $\bar{M}_B$.

Now all the top intersections of $\{L_i\} \cup \{H\}$ are known
by reconstruction theorem on $\mathbb{P}^n$,

and top intersections involving $K \in \Delta$ can be carried out on $\bar{K}$
thus every top intersections on $\bar{M}$ is determined