h^1/h^0

Let X be a topos and $\underbrace{E^0 \xrightarrow{d} E^1}_{\shortparallel \atop E^\bullet}$ a morphism of abelian sheaves on X

Via d, we get the action $\begin{array}{ccc} E^0 \times E^1 & \longrightarrow & E^1 \\ (\mu, \nu) & \mapsto & d\mu + \nu \end{array}$ $\rightsquigarrow$ $h^1/h^0(E^\bullet) := [E^1/E^0]$.

stack-theoretic quotient.

For $U \in \mathrm{Ob}\,X$, $h^1/h^0(E^\bullet)(U) = \{(P, f)\}$ $\quad f : P \to E^1|_U$: E^0-equiv. morphism of sheaves.

E^0-torsor $/U$.

For a general complex, we define $h^1/h^0(E^\bullet) = h^1/h^0(\tau_{[0,1]} E^\bullet)$

$$\shortparallel$$
$$[\mathrm{coker}(E^{-1} \to E^0) \to \ker(E^1 \to E^2)].$$

Prop. Let $\varphi^\bullet : E^\bullet \to F^\bullet$ be a quasi-isom. Then $h^1/h^0(\varphi) : h^1/h^0(E^\bullet) \xrightarrow{\sim} h^1/h^0(F^\bullet)$,

$$(P, f) \mapsto (P \times_{E^0} F^0, \varphi^1(f))$$

where $\varphi^1(f)([p, \nu]) = \varphi^1(f(p)) + d\nu$.

pf. Define $\psi : E^\bullet \oplus F^0 \to F^\bullet$ by $\begin{cases} \psi^0(\mu, \nu) = \varphi^0(\mu) + \nu \\ \psi^1(\mu, \nu) = \varphi^1(\mu) + d\nu \end{cases}$

$$\Rightarrow \varphi : E^\bullet \xrightarrow{\mathrm{id} \oplus 0} E^\bullet \oplus F^0 \xrightarrow{\psi} F^\bullet$$

homotopy equiv. $\quad$ epimorphism.

Suffices to check. $\begin{cases} \varphi \sim \psi \Rightarrow h^1/h^0(\varphi) \simeq h^1/h^0(\psi) \\ \varphi \text{ epic. quasi-isom} \Rightarrow h^1/h^0(\varphi) \text{ isom.} \end{cases}$

Suppose $\varphi^0 - \psi^0 = kd$, $\varphi^1 - \psi^1 = dk$ for some $k : E^1 \to F^0$.

Define $\theta : h^1/h^0(\varphi) \to h^1/h^0(\psi)$ by $\theta(U)(P, f) : P \times_{E^0, \varphi^0} F^0 \longrightarrow P \times_{E^0, \psi^0} F^0$

$$[p, \nu] \longmapsto [p, kf(p) + \nu].$$

$$\Rightarrow \varphi^1(f) = \psi^1(f) \circ \theta(U)(P, f)$$

So $\theta(U)$ is a natural isomorphism $\Rightarrow$ θ is an isom.

Suppose φ epic. $\Rightarrow E^1 \to [F^1/F^0]$ epic.

Then $[E^1/E^0] \simeq [F^1/F^0] \Leftrightarrow$

$$\begin{tikzcd} E^0 \times E^1 \arrow[r, "d+id"] \arrow[d] \arrow[dr, phantom, "\square"] & E^1 \arrow[d] \\ E^1 \arrow[r] & {[F^1/F^0]} \end{tikzcd} \Leftrightarrow \begin{tikzcd} E^0 \times E^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E^1 \arrow[d] \\ F^0 \times E^1 \arrow[r] & F^1 \end{tikzcd} \quad \square$$

So we may define $h^1/h^0(\varphi)$ for any $\varphi : E^\bullet \to F^\bullet$ in $D(\mathcal{O}_X)$.

<u>Intrinsic normal cone</u>

Let X be a DM stack, loc of fin type $/k$. $L_X^\bullet$ cotangent cpx of X (relative to k).

$$\Rightarrow L_X^\bullet \in D(\mathcal{O}_{X_{\acute{e}t}}), \quad \begin{cases} h^i(L^\bullet) = 0 \text{ for } i > 0 \\ h^i(L^\bullet) \text{ is coherent for } i = 0, -1. \end{cases} \quad (\star)$$

The intrinsic normal sheaf $\mathcal{N}_X := h^1/h^0\left((L_X^\bullet)_{fl}^\vee\right)$

sheaf / big fppf-site $(X_{fl} \to X_{\acute{e}t})$

Def. A local embedding of X is a diagram:

affine k-scheme of fin. type

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "i : \text{étale}"] & M \\ X & \end{tikzcd}$$

loc. immersion smooth

$$\rightsquigarrow \varphi : L_X^\bullet|_U \to \underset{\underset{L_U^\bullet}{\parallel}}{[I/I^2 \to f^*\Omega_M]} \text{ in } D(\mathcal{O}_{U_{\acute{e}t}})$$

conormal sheaf of U in M.

Claim. $\varphi^\vee : h^1/h^0\left((L_U^\bullet)_{fl}^\vee\right) \to h^1/h^0\left((L_X^\bullet)_{fl}^\vee\right)$ is an isom.

In fact, we have:

Prop. For any $\varphi : E^\bullet \to L^\bullet$ in $D(\mathcal{O}_{X_{\acute{e}t}})$, with $E^\bullet, L^\bullet$ satisfy $(\star)$.

(i) $\varphi^\vee$ is a closed immer. $\Leftrightarrow h^0(\varphi)$ isom $+ h^{-1}(\varphi)$ surj.

(ii) $\varphi^\vee$ is an isom $\Leftrightarrow h^0(\varphi), h^{-1}(\varphi)$ isom.

pf. This question is local in X $\Rightarrow$ may assume $E^{\cdot}, L^{\cdot}$ are cpx of free $\mathcal{O}_X$-modules, $E^i = L^i = 0$ for $i > 0$, E^i, L^i fin rk for $i = 0, -1$.

Consider
$$\begin{tikzcd}
C^{-1}(E) \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E^0 \arrow[d] \\
& C^{-1}(L) \arrow[r] & L^0
\end{tikzcd}$$
$C^{-1}(L) = \operatorname{coker}(L^{-2} \to L^{-1})$

$\varphi^\vee$ closed immer $\Leftrightarrow$ $C(F) \to Z^1(E^{\vee\cdot})$ is surj.; $Z^1(E^{\vee\cdot}) = \ker(E_1 \to E_2)$

$\Leftrightarrow$ $C^{-1}(E) \to F$ surj. ; $F = C^{-1}(L) \times_{L^0} E^0$ equalizer.

$\Leftrightarrow$ $h^0(\varphi)$ isom $+$ $h^{-1}(\varphi)$ surj.

$\varphi^\vee$ isom $\Leftrightarrow$ $C^{-1}(E) \to F$ isom. $\Leftrightarrow$ $h^0(\varphi)$ isom $+$ $h^{-1}(\varphi)$ isom. $\square$.

$\rightsquigarrow$ $\varphi^\vee : [N_{U/M} / f^* T_M] \xrightarrow{\sim} i^* \mathfrak{N}_X$, i.e., $N_{U/M}$ is a local presentation of $\mathfrak{N}_X$.

If $\chi : (U', M') \to (U, M)$ $\left(\Leftrightarrow \begin{tikzcd} U' \arrow[r, "f'"] \arrow[d, "\chi_U"'] & M' \arrow[d, "\chi_M"] \\ U \arrow[r, "f"] & M \end{tikzcd} \text{ with } \begin{cases} \chi_U & \text{étale,} \\ \chi_M & \text{smooth.} \end{cases} \right)$

$$\Rightarrow \begin{tikzcd}
I/I^2|_{U'} \arrow[r] \arrow[d] & f^*\Omega_M|_{U'} \arrow[d] \\
I'/I'^2 \arrow[r] & f'^*\Omega_{M'}
\end{tikzcd}, \quad \tilde{\chi} : [I/I^2 \to f^*\Omega_M]|_{U'} \to [I'/I'^2 \to f'^*\Omega_M]$$

$\tilde{\chi} \circ \varphi|_{U'} = \varphi'$ $\Rightarrow$ $\tilde{\chi}^\vee : [N_{U'/M'} / f'^* T_{M'}] \xrightarrow{\sim} [N_{U/M} / f^* T_M]|_{U'}$

$$\begin{tikzcd}
{[N_{U'/M'} / f'^* T_{M'}]} \arrow[rr, "\tilde{\chi}^\vee"] \arrow[dr, "\varphi'^\vee"'] & \circlearrowright & {[N_{U/M} / f^* T_M]|_{U'}} \arrow[dl, "\varphi^\vee|_{U'}"] \\
& i'^* \mathfrak{N}_X &
\end{tikzcd}$$

$$\begin{tikzcd}
f'^* T_{M'}\text{-cone} : C_{U'/M'} \arrow[r, hook] \arrow[d] & N_{U'/M'} \arrow[d] \\
f^* T_M|_{U'}\text{-cone} : C_{U/M}|_{U'} \arrow[r, hook] & N_{U/M}|_{U'}
\end{tikzcd}$$

$0 \to f'^* T_{M'/M} \to f'^* T_{M'} \to f^* T_M|_{U'}$

$f'^* T_{M'/M} \to C_{U'/M'} \to C_{U'/M}$ are exact ($\because$ f', $f \circ \chi_U$ are immer.)

$\Rightarrow$ $(C_{U/M} \hookrightarrow N_{U/M})|_{U'}$ is the quotient $(C_{U'/M'} \hookrightarrow N_{U'/M'}) / f'^* T_{M'/M}$

So
$$\begin{tikzcd}
\tilde{\chi}^\vee : [N_{U'/M'}/f'^*T_{M'}] \arrow[r, "\simeq"] & {[N_{U/M}/f^*T_M]|_{U'}} & & \mathcal{N}_X \\
{[C_{U'/M'}/f'^*T_{M'}]} \arrow[u, hook] \arrow[r, "\simeq"] & {[C_{U/M}/f^*T_M]|_{U'}} \arrow[u, hook] & \rightsquigarrow & \mathcal{C}_X \arrow[u, hook]
\end{tikzcd}$$

$N_{U/M} = \operatorname{Spec}\operatorname{Sym}(I/I^2)$, $C_{U/M} = \operatorname{Spec}\left(\bigoplus I^k/I^{k+1}\right)$

$\Rightarrow$ $N_{U/M}$ is the abelianization of $C_{U/M}$ $\Rightarrow$ $\mathcal{N}_X$ is the abelianization of $\mathcal{C}_X$

<u>Virtual fundamental class.</u>

Let $E^\bullet \xrightarrow{\varphi} L_X^\bullet$ be an obstruction theory ($h^0(\varphi)$ isom + $h^{-1}(\varphi)$ surj.)

$\Rightarrow$ $\varphi^\vee : \mathcal{N}_X \to h^1/h^0((E^\bullet)^\vee_{fl})$ is a closed immersion.

$E^\bullet$ is perfect if $E^\bullet \simeq [\underset{\uparrow}{F^{-1}} \to \underset{\uparrow}{F^0}] = F^\bullet$ in $D(\mathcal{O}_{X_{ét}})$.

loc. free sheaves.

Let $C(F) = \mathcal{C}_X \times_{h^1/h^0(E^\vee)} F_1 \subseteq F_1$ $\quad$ ($\mathcal{C}_X \subseteq h^1/h^0(E^\vee)$).

Define virtual fundamental class $[X, E] := 0^![C(F)]$, where $0 : X \to F_1$ is the zero section.

(well-definedness) Suppose $G^\bullet \to E^\bullet$ is another res.

May assume $F^\bullet \xrightarrow{\varphi} E^\bullet$, $G^\bullet \xrightarrow{\psi} E^\bullet$ are in $\mathrm{Kom}(\mathcal{O}_{X_{ét}})$.

$\rightsquigarrow$ $F^\bullet \oplus G^\bullet \to E^\bullet$. Let
$$\begin{tikzcd}
H^{-1} = E^{-1} \times_{E^0} (F^0 \oplus G^0) \arrow[r] \arrow[d, "x^{-1}"'] \arrow[dr, phantom, "\square"] & F^0 \oplus G^0 = H^0 \arrow[d, "x^0"] \\
E^{-1} \arrow[r] & E^0
\end{tikzcd}$$

Then $F^\bullet \to H^\bullet$, $G^\bullet \to H^\bullet$ are monic.

Consider the diagram
$$\begin{tikzcd}
X \arrow[r, "0"] \arrow[d, equal] \arrow[dr, phantom, "\square"] & C(H) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}_X \arrow[d] \\
X \arrow[r, "0"'] & H_1 \arrow[r, two heads, "\alpha"] & F_1 \arrow[r] & h^1/h^0(E^\vee)
\end{tikzcd}$$

α smooth

$$\Rightarrow (\alpha \circ 0)^![C(F)] = 0^!\alpha^![C(F)] \overset{\checkmark}{=} 0^![C(H)] \qquad \square$$

$\mathcal{M}$ alg. stack, loc. of fin. type, pure-dim $/k$.

$X \to \mathcal{M}$ morphism of relative DM type $\left(\begin{array}{l} \Delta : X \to X \times_{\mathcal{M}} X \text{ is unramified} \\ \Leftrightarrow \; X \times_{\mathcal{M}} T \text{ is DM}, \; \forall T \to \mathcal{M} \end{array} \right)$

$L^{\bullet}_{X/\mathcal{M}}$: cotangent complex of X relative to $\mathcal{M}$.

$\mathcal{N}_{X/\mathcal{M}} := h^1/h^0((L^{\bullet}_{X/\mathcal{M}})^{\vee}) \rightsquigarrow C_{X/\mathcal{M}} \subseteq \mathcal{N}_{X/\mathcal{M}}$.

$E^{\bullet}$ is a <u>relative</u> obstruction theory for X over $\mathcal{M}$ if there is a homomorphism $\phi : E^{\bullet} \to L^{\bullet}_{X/\mathcal{M}}$ in $D(\mathcal{O}_{X_{ét}})$ with $\begin{cases} h^0(\phi) \text{ isom.} \\ h^{-1}(\phi) \text{ surj.} \end{cases}$

$E^{\bullet}$ perfect $\rightsquigarrow$ $[X, E]$

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr] & & Y \arrow[dl] \\ & \mathcal{M} & \end{tikzcd}$$

$E^{\bullet}, F^{\bullet}$ perfect rel. ob. theory for $X, Y / \mathcal{M}$, resp.

Suppose $\exists$
$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "p"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "q"] \\ Z \arrow[r, "v"] & W \end{tikzcd}$$
local complete intersection morphism of rel DM type.

E and F are called compatible over v if

$$\exists \begin{tikzcd} u^*F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & p^*L_{Z/W} \arrow[r] \arrow[d] & u^*F[1] \arrow[d] \\ u^*L_{Y/\mathcal{M}} \arrow[r] & L_{X/\mathcal{M}} \arrow[r] & L_{X/Y} \arrow[r] & u^*L_{Y/\mathcal{M}}[1] \end{tikzcd}$$

morphism of distinguished Δ in $D(\mathcal{O}_{X_{ét}})$

Thm. If E and F are compatible $/v$, then $v^![Y, F] = [X, E]$.

Prop. Let $X \xrightarrow{i} Y \xrightarrow{j} Z$ be morphisms of rel DM type.

Then $\mathcal{N}_{X \times \mathbb{P}^1 / M^{\circ}_{Y/Z}} \simeq h^1/h^0(C(f)^{\vee})$, where

$M^{\circ}_{Y/Z}$ = deformation to the normal stack $= Bl_{Y \times \{0\}}(Z \times \mathbb{P}^1) \setminus \widetilde{Z \times \{0\}}$

for closed immersion

$f = (\mathrm{id} \cdot x^0, \mathrm{can} \cdot x^1) : i^* L_{Y/Z} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \to i^* L_{Y/Z} \oplus L_{X/Z}$.

pf of Thm. Let $\mathcal{N} = g^*\mathcal{N}_{Z/W}$. Consider

$$\underbrace{\begin{aligned} \rho &: h^1/h^0(E^\vee) \to X \\ \sigma &: \mathcal{N} \oplus u^*(h^1/h^0(F^\vee)) \to X \\ \pi &: h^1/h^0(F^\vee) \to Y \end{aligned}}_{\text{v.b. stacks}}$$

By homotopy inv. for vector bundle,

$$[X,E] = (\rho^*)^{-1}[C_{X/m}], \quad [Y,F] = (\pi^*)^{-1}[\overset{\overset{C_0}{\shortparallel}}{C_{Y/m}}].$$

Then

$$\begin{aligned} C_{X/C_0} &\hookrightarrow \mathcal{N} \xrightarrow{r} X \\ \shortparallel & \\ C_{X/Y}\times_X u^*C_0 &\to u^*(h^1/h^0(F^\vee)) \end{aligned} \rightsquigarrow \begin{tikzcd} C_{X/C_0} \arrow[r, hook] \arrow[dr] & \mathcal{N}\oplus u^*(h^1/h^0(F^\vee)) \arrow[d, "\sigma"] \\ & X \end{tikzcd}$$

$v^![Y,F]$ is rep by $[C_{X/C_0}]$ in $\mathcal{N}\oplus u^*(h^1/h^0(F^\vee))$:

$$\begin{tikzcd}
 & C_{X/C_0} \arrow[d, phantom, "\cap"] & & \pi^*V \sim C_0 \arrow[d, phantom, "\cap"] \arrow[dl] \arrow[dr, phantom, "\square"] & C(F) \arrow[l] \arrow[d, phantom, "\cap"] \\
 & \mathcal{N}\oplus u^*(h^1/h^0(F^\vee)) \arrow[dd, "\sigma"] & & h^1/h^0(F^\vee) \arrow[dd, "\pi"] & F_1 \arrow[l] \\
V \arrow[rr] \arrow[dr] & & V \arrow[dr] & & \\
 & X \arrow[rr, "u"] \arrow[d, "p"] \arrow[drr, phantom, "\square"] & & Y \arrow[d, "g"] \arrow[uur, "0"] & \\
 & Z \arrow[rr, "v"] & & W &
\end{tikzcd}$$

From the def. $[C_{X/Y}] = r^*v^![Y]$

$[Y,F] = 0^!C(F)$

Replace $[C_0]$ by $\overset{\overset{Y}{\cup}}{\pi^*V}$, $[V] = [Y,F]$.

By push-forward, may replace V by Y

Reduce to $[C_{X/(h^1/h^0(F^\vee))}] = v^![Y]$.

So $v^![Y,F] = [X,E] \Leftrightarrow (\sigma^*)^{-1}[C_{X/C_0}] = (\rho^*)^{-1}[C_{X/m}]$ in $A_*(X)$.

Consider

$$\begin{array}{ccc} M^\circ_{X\times\mathbb{P}^1/M^\circ_{Y/m}} & \to & \mathbb{P}^1\times\mathbb{P}^1 \\ \cup\!\shortmid & & \cup\!\shortmid \\ C_{X\times\mathbb{P}^1/M^\circ_{Y/m}} & \to & \{0\}\times\mathbb{P}^1 \end{array} \qquad \begin{tikzcd} X\times\mathbb{P}^1 \arrow[r] \arrow[dr] & M^\circ_{Y/m} \arrow[d] & \overset{\overset{C_0}{\shortparallel}}{C_{Y/m}} \arrow[d] & m \arrow[d] \\ & \mathbb{P}^1 & \{0\} & \{\infty\} \end{tikzcd}$$

$$\mathbb{P}^1\times\{0\} \sim \mathbb{P}^1\times\{\infty\} \xrightarrow{\cdot C_{X\times\mathbb{P}^1/M^\circ_{Y/m}}} [C_{X/C_0}] \sim [C_{X/m}] \text{ on } C_{X\times\mathbb{P}^1/M^\circ_{Y/m}}$$

Abelianization of $\mathcal{C}_{X\times\mathbb{P}^1/M^\circ_{Y/m}}$ is $\mathcal{N}_{X\times\mathbb{P}^1/M^\circ_{Y/m}} \simeq h^1/h^0(C(f)^\vee)$

Prop. $(Z = \mathfrak{M})$.

$$\begin{tikzcd}
u^*L_{Y/m}(-1) \arrow[r, "f = (\mathrm{id}\cdot x^0,\ \mathrm{can}\cdot x^1)"] \arrow[dr, phantom, "\curvearrowright"] & u^*L_{Y/m} \oplus L_{X/m} \arrow[r] & C(f) \arrow[r] & u^*L_{Y/m}(-1)[1] \\
u^*F(-1) \arrow[u] \arrow[r, "g = (\mathrm{id}\cdot x^0,\ \varphi\cdot x^1)"'] & u^*F \oplus E \arrow[u] \arrow[r] & C(g) \arrow[u, dashed] \arrow[r] & u^*F(-1)[1] \arrow[u]
\end{tikzcd}$$

($\because$ v compatible) over $X\times\mathbb{P}^1$.

$u^*F \xrightarrow{\varphi} E \longrightarrow p^*L_{Z/W} \longrightarrow u^*F[1]$. over X.

Over $X\times\{0\}$, $g = (0,\varphi) : u^*F \to u^*F \oplus E$.

$$\Rightarrow h^1/h^0(C(g)^\vee)\big|_{X\times\{0\}} \simeq u^*(h^1/h^0(F^\vee)) \oplus p^*(\underbrace{h^1/h^0(L_{Z/W}^\vee)}_{= \mathcal{N}_{Z/W}}) \xrightarrow{\sigma} X$$

Over $X\times\{\infty\}$. $g = (\mathrm{id},0) : u^*F \hookrightarrow u^*F \oplus E$.

$$\Rightarrow h^1/h^0(C(g)^\vee)\big|_{X\times\{\infty\}} \simeq h^1/h^0(E^\vee) \xrightarrow{\rho} X$$

So $(\sigma^*)^{-1}[C_{X/C_0}] \sim (\rho^*)^{-1}[C_{X/m}]$ □

Prop. Let $X \xrightarrow{i} Y \xrightarrow{j} Z$ be morphisms of rel DM type.

Then $\mathcal{N}_{X \times \mathbb{P}^1 / M^\circ_{Y/Z}} \simeq h^1/h^0(C(f)^\vee)$, where

$M^\circ_{Y/Z}$ = deformation to the normal cone stack,

$f = (\mathrm{id} \cdot x^0, \theta \cdot x^1) : i^* L_{Y/Z} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \to i^* L_{Y/Z} \oplus L_{X/Z}$.

pf. Case 1. $X \xrightarrow{i} Y$, $Y \xrightarrow{j} Z$ are closed immersions

$M^\circ_{Y/Z} = Bl_{Y \times \{0\}} Z \times \mathbb{P}^1 \setminus \tilde{Z}$. Let $\begin{cases} I = \text{ideal sheaf of } X \text{ in } Z, \\ J = \text{ideal sheaf of } Y \text{ in } Z. \end{cases}$

$\rightsquigarrow M^\circ_{Y/Z}\Big|_{Z \times (\mathbb{P}^1 \setminus \{0\})} = Z \times (\mathbb{P}^1 \setminus \{0\}) = \operatorname{Spec} \mathcal{O}_Z[U] \to \operatorname{Spec} \mathcal{O}_Z[U]$

$M^\circ_{Y/Z}\Big|_{Z \times \underset{\mathbb{A}^1}{\underset{\parallel}{(\mathbb{P}^1 \setminus \{\infty\})}}} = \operatorname{Spec}\left(\mathcal{O}_Z[T] \oplus \bigoplus_{k=1}^{\infty} \frac{J^k}{T^k}\right) \to \underset{Z \times \mathbb{A}^1}{\underset{\parallel}{\operatorname{Spec} \mathcal{O}_Z[T]}}$

$X \times \mathbb{P}^1 \hookrightarrow Y \times \mathbb{P}^1 \hookrightarrow M^\circ_{Y/Z} \rightsquigarrow$ ideal sheaf $\tilde{I}$

$\tilde{I}\big|_{X \times (\mathbb{P}^1 \setminus \{0\})} = I[U]$, $\tilde{I}\big|_{X \times \mathbb{A}^1} = I[T] \oplus \bigoplus_{k=1}^{\infty} \frac{J^k}{T^k}$ $(\ast)$

$\mathcal{N}_{X \times \mathbb{P}^1 / M^\circ_{Y/Z}} = \operatorname{Spec} \operatorname{Sym}(\tilde{I}/\tilde{I}^2)$, $h^1/h^0(C(f)^\vee) = \operatorname{Spec} \operatorname{Sym}(\operatorname{coker} h^{-1}(f))$,

$f = (\mathrm{id} \cdot T, \theta \cdot U) : J/IJ \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow J/IJ \oplus I/I^2$

$(\ast) \Rightarrow \tilde{I}/\tilde{I}^2\big|_{X \times \mathbb{A}^1} = (J/IJ)\, T^{-1} \oplus (I/I^2)[T]$.

On $X \times \mathbb{A}^1$,

$f : J/IJ \otimes \mathcal{O}_X[T] \xrightarrow{(T, \theta)} (J/IJ \oplus I/I^2) \otimes \mathcal{O}_X[T] \twoheadrightarrow (J/IJ)\, T^{-1} \oplus (I/I^2)[T]$

is exact. On $X \times \mathbb{P}^1 \setminus \{0\}$, $\updownarrow$ compatible.

$f : J/IJ \otimes \mathcal{O}_X[U] \xrightarrow{(1, \theta \cdot U)} (J/IJ \oplus I/I^2) \otimes \mathcal{O}_X[U] \twoheadrightarrow (I/I^2)[U]$.

Case 2. $X \to Y$, $Y \to Z$ are representable local embeddings:

May assume Z is a scheme. Then replace X, Y, Z by étale covers.

Case 3. $\exists\, X \to V$ s.t. $X \to Y \times V$ is a local embedding.

and $j : Y \underset{\text{loc. embedding}}{\to} Z' \underset{\text{smooth and repre.}}{\to} Z$:

$$\Rightarrow \quad M^\circ_{Y/Z'} \to M^\circ_{Y/Z} \quad \text{is smooth and repre.}$$

Let $Z'' = Z' \times_Z Z' \Rightarrow M^\circ_{Y/Z''} = M^\circ_{Y/Z'} \times_{M_{Y/Z}} M^\circ_{Y/Z'}$

Consider $X \times \mathbb{P}^1 \hookrightarrow M^\circ_{Y \times V/Z' \times V} \to M^\circ_{Y/Z}$.

$$\Rightarrow \quad \mathcal{N}_{X \times \mathbb{P}^1/M^\circ_{Y/Z}} = \Big[\mathcal{N}_{X \times \mathbb{P}^1/\underbrace{M^\circ_{Y \times V \times V/Z'' \times V \times V}}_{= M^\circ_{Y \times V/Z' \times V} \times_{M^\circ_{Y/Z}} M^\circ_{Y \times V/Z' \times V}}} \rightrightarrows \mathcal{N}_{X \times \mathbb{P}^1/M^\circ_{Y \times V/Z' \times V}} \Big]$$

$$= [\operatorname{Spec} \operatorname{Sym} \mathcal{D} \rightrightarrows \operatorname{Spec} \operatorname{Sym} \mathcal{C}],$$

where $\mathcal{C} = C(i^* L_{Y/Z'}(-1) \xrightarrow{f'} i^* L_{Y/Z'} \oplus L_{X/Z' \times V})$ $L_{Y \times V/Z' \times V} = L_{Y/Z'}$ on X

$\mathcal{D} = C(i^* L_{Y/Z''}(-1) \xrightarrow{f''} i^* L_{Y/Z''} \oplus L_{X/Z'' \times V \times V})$

$$f'' = f' \times_f f' \rightsquigarrow \mathcal{N}_{X \times \mathbb{P}^1/M^\circ_{Y/Z}} \simeq h^1/h^0 (C(f)^\vee)$$

Case 4. The general case. Choose a smooth atlas Z_0 for Z.

Y_0 an affine étale atlas for $Y \times_Z Z_0$. $\rightsquigarrow$ $Y_0 \underset{\text{loc. embedding.}}{\longrightarrow} \underset{= Z_0 \times \mathbb{A}^n}{Z_0'} \underset{\text{smooth repre.}}{\to} Z_0$

X_0 an affine étale atlas for $X \times_Y Y_0$

$$X_0 \xrightarrow{\text{smooth}} V$$

Define $X_1 = X_0 \times_X X_0$, $Y_1 = Y_0 \times_Y Y_0$, $Z_1 = Z_0 \times_Z Z_0$, $Z_1'' = Z_0' \times_Z Z_0'$

Similarly,

$$\mathfrak{N}_{X\times\mathbb{P}^1/M^\circ_{Y/Z}}\Big|_{X_1\times\mathbb{P}^1} = [\mathfrak{N}_{X_1\times\mathbb{P}^1/M^\circ_{Y_1\times V\times V/Z_1''\times V\times V}} \to \mathfrak{N}_{X_0\times\mathbb{P}^1/M^\circ_{Y_0\times V/Z'\times V}}]$$

$$= [\operatorname{Spec}\operatorname{Sym}\mathcal{D} \to \operatorname{Spec}\operatorname{Sym}\mathcal{C}] = h^1/h^0(C(f)^\vee)\Big|_{X_1\times\mathbb{P}^1}$$

$$\mathcal{C} = C(i^* L_{Y_0/Z_0'}(-1) \xrightarrow{f'} i^* L_{Y_0/Z_0'} \oplus L_{X_0/Z_0'\times V_0})$$

$$\mathcal{D} = C(i^* L_{Y_1/Z_1''}(-1) \xrightarrow{f''} i^* L_{Y_1/Z_1''} \oplus L_{X_1/Z_1''\times V_1\times V_1})$$

$f'' = f' \times_f f'$

$\square$.

Let X be a sm. proj var. $/\mathbb{C}$. $\beta\in H_2(X,\mathbb{Z})$. $\mathcal{U} = \overline{M}_{0,n+1}(X,\beta) \xrightarrow{e} X$

$$\begin{tikzcd} \mathcal{U} \arrow[r, "e"] \arrow[d, "\pi"] & X \\ \overline{M}_{0,n}(X,\beta) =: MX & \end{tikzcd}$$

$[\overline{M}_{0,n}(X,\beta)]^{vir} := [\overline{M}_{0,n}(X,\beta), (R\pi_* e^* T_X)^\vee]$

$$\begin{tikzcd} V \arrow[d] & & E_\beta = \pi_* e^* V \arrow[d] \\ Y = (s) \subseteq X \arrow[u, bend left, "s"] & & MX \arrow[u, bend left, "0_\beta"] \end{tikzcd}$$

V : convex bundle $(H^1(C, f^*V) = 0) \to$

For $\gamma\in H_2(Y,\mathbb{Z})$, we have $j_\gamma : \overline{M}_{0,n}(Y,\gamma) \hookrightarrow \overline{M}_{0,n}(X, i_*\gamma)$

Thm. For any $\beta\in H_2(X,\mathbb{Z})$,

$$0_\beta^! [\overline{M}_{0,n}(X,\beta)]^{vir} = \sum_{i_*\gamma=\beta} (j_\gamma)_* [\overline{M}_{0,n}(Y,\gamma)]^{vir}$$

$$\| $$

$$e(E_\beta) \cap [\overline{M}_{0,n}(X,\beta)]^{vir}$$

pf. Let $MY = \bigsqcup_{i_*\gamma=\beta} \overline{M}_{0,n}(Y,\gamma) \overset{r}{\hookrightarrow} MX$. $\mathfrak{M} = \mathfrak{M}_{0,n}$

$$\begin{tikzcd} r^*\mathcal{U} \arrow[r, "e'"] \arrow[d, "\pi'"] \arrow[dr] & Y \arrow[dr] & \\ MY \arrow[dr] & \mathcal{U} \arrow[r, "e"] \arrow[d, "\pi"] & X \\ & MX & \end{tikzcd}$$

$$\Rightarrow \begin{cases} [\overline{M}_{0,n}(X,\beta)]^{vir} = [MX, (R\pi_* e^* T_X)^\vee] \\ \sum_{i_*\gamma=\beta} (j_\gamma)_* [\overline{M}_{0,n}(Y,\gamma)]^{vir} = [MY, (R\pi'_* e'^* T_Y)^\vee] \end{cases}$$

From $0 \to T_Y \to i^* T_X \to \underset{\substack{\wr\| \\ i^*V}}{N_{Y/X}} \to 0$, we get

$$\rightsquigarrow R\pi'_* e'^* T_Y \to R\pi_* e^* T_X\big|_{MY} \to R\pi_* e^* V\big|_{MY} \to R\pi'_* e'^* T_Y [1]$$

$$R\pi_* e^* V\big|_{MY} = \pi_* e^* V = E_\beta \quad (\because V \text{ convex}).$$

Take dual, get

$$\begin{tikzcd}
r^*(R\pi_* e^* T_X)^\vee \arrow[r] \arrow[d] & (R\pi'_* e'^* T_Y)^\vee \arrow[r] \arrow[d] & r^* E_\beta^\vee [1] \arrow[r] \arrow[d] & r^*(R\pi_* e^* T_X)^\vee [1] \arrow[d] \\
r^* L_{MX/\mathfrak{m}} \arrow[r] & L_{MY/\mathfrak{m}} \arrow[r] & L_{MY/MX} \arrow[r] & r^* L_{MX/\mathfrak{m}} [1],
\end{tikzcd}$$

Apply Thm to

$$\begin{tikzcd}
MY \arrow[r, "r"] \arrow[d, "r"'] \arrow[dr, phantom, "\square"] & MX \arrow[d, "\tilde{s} = \pi_* e^* s"] \\
MX \arrow[r, "0_\beta"'] & E_\beta = \pi_* e^* V
\end{tikzcd}$$

$(E_\beta^\vee [1] = L_{MX/E_\beta})$

$$\rightsquigarrow [MY, (R\pi'_* e'^* T_Y)^\vee] = 0_\beta^! [MX, (R\pi_* e^* T_X)^\vee]. \qquad \square$$