①

Cohomological field theory (CohFT)

Let $q: \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ be the map sewing p_{n+1} and p_{n+2}

$s: \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$ ($= g_1+g_2, n_1+n_2$) be the map sewing p_{n_1+1} on the 1st comp. and p_{n_2+1} on the 2nd comp.

$p: \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ forgetting p_{n+1}

Let $A \in \mathrm{FinVect}_{\mathbb{C}}$, $\mathbb{1} \in A$ nonzero

$\eta = \eta_{\mu\nu} e^\mu \otimes e^\nu$ sym. bilinear form on A, nondegenerate

A nodal CohFT of $(A, \eta, \mathbb{1})$ is a set of linear maps

$\overline{\Omega}_{g,n}: A^{\otimes n} \to H^\bullet(\overline{\mathcal{M}}_{g,n})$, $2g-2+n>0$ s.t.

1. $\overline{\Omega}_{g,n}$ is S_n-equivariant $\forall g, n$
2. $\overline{\Omega}_{0,3}(\mathbb{1} \otimes u \otimes v) = \eta(u,v)$ $\forall u, v \in A$
3. $q^* \overline{\Omega}_{g,n}(v_1 \otimes \cdots \otimes v_n) = \eta^{\mu\nu} \overline{\Omega}_{g-1,n+2}(v_1 \otimes \cdots \otimes v_n \otimes e_\mu \otimes e_\nu)$
4. $s^* \overline{\Omega}_{g,n}(v_1 \otimes \cdots \otimes v_n)$
$= \eta^{\mu\nu} \overline{\Omega}_{g_1,n_1+1}(v_1 \otimes \cdots \otimes v_{n_1} \otimes e_\mu) \times \overline{\Omega}_{g_2,n_2+1}(v_{n_1+1} \otimes \cdots \otimes v_n \otimes e_\nu)$
5. $p^* \overline{\Omega}_{g,n}(v_1 \otimes \cdots \otimes v_n) = \overline{\Omega}_{g,n+1}(v_1 \otimes \cdots \otimes v_n \otimes \mathbb{1})$

($=: \widetilde{\mathcal{M}}_{g,n}$ when $I = [n]$)

Let $\pi_I: \widetilde{\mathcal{M}}^I_{g,n} \to \overline{\mathcal{M}}_{g,n}$, $I \subset [n]$ be the torus bundle with fiber at $C \in \overline{\mathcal{M}}_{g,n}$ be $\prod_{i \in I} S(T_{p_i} C)$

$S(V) = (V \setminus \{0\}) / \mathbb{R}^+$

Denote $p: \widetilde{\mathcal{M}}_{g,n+1} \to \widetilde{\mathcal{M}}_{g,n}$ forgetting p_{n+1}

PAGE

DATE

(2)

We similarly define

$\tilde{s}: \widetilde{\mathcal{M}}^{I_1}_{g_1, n_1+1} \times \widetilde{\mathcal{M}}^{I_2}_{g_2, n_2+1} \to \widetilde{\overline{\mathcal{M}}}_{g,n}$, $I_i = [n_i]$

the map sewing p_{n_1+1} on the 1st comp

and p_{n_2+1} on the 2nd comp. ($\simeq$ normal bundle)

Let $\tilde{S} = \mathrm{im}(\tilde{s})$, $\nu: \tilde{N} \to \tilde{S}$ a tubular nbhd of $\tilde{S}$

also $\nu: \partial\tilde{N} \to \tilde{S}$ a circular nbhd of $\tilde{S}$

(a circle subbundle of $\tilde{N}$)

Prop. There's a sewing-smoothing map

$$\sigma: \widetilde{\mathcal{M}}_{g_1, n_1+1} \times \widetilde{\mathcal{M}}_{g_2, n_2+1} \to \widetilde{\mathcal{M}}_{g,n}$$

s.t. $\mathrm{Im}(\sigma) \subseteq \partial\tilde{N}$ and $\nu \circ \sigma = \tilde{s} \circ (\pi^{I_1} \times \pi^{I_2})$

pf. Define $\tilde{N} \to \widetilde{\mathcal{M}}_{g_1, n_1+1} \times_{S^1} \widetilde{\mathcal{M}}_{g_2, n_2+1}$ by

$(C = \tilde{s}(C', C''), [v_1], \dots, [v_n], v' \otimes v'')$

$\to ((C', [v_1], \dots [v_{n_1}], [v']), (C'', [v_{n_1+1}], \dots, [v_n], [v'']))$

(Note $\tilde{N} \cong [T_{p_{n_1+1}} C' \otimes T_{p_{n_2+1}} C'']$ (fiber at C),

S^1 act on $\widetilde{\mathcal{M}}_{g_1, n_1+1} \times \widetilde{\mathcal{M}}_{g_2, n_2+1}$ by $z \cdot (\dots, [v']), (\dots, [v''])$

$= (\dots, [zv']), (\dots, [z^{-1}v''])$)

This becomes isom. when restrict to $\partial\tilde{N}$

and we define $\sigma: \widetilde{\mathcal{M}}_{g_1, n_1+1} \times \widetilde{\mathcal{M}}_{g_2, n_2+1} \xrightarrow[\text{quotient}]{} \widetilde{\mathcal{M}}_{g_1, n_1+1} \times_{S^1} \widetilde{\mathcal{M}}_{g_2, n_2+1} \cong \partial\tilde{N} \subset \widetilde{\mathcal{M}}_{g,n}$

Similarly there's $\tau: \widetilde{\mathcal{M}}_{g-1, n+2} \to \widetilde{\mathcal{M}}_{g,n}$

s.t. $\mathrm{Im}(\tau) \subseteq \partial\tilde{P}$ and $\nu_q \circ \tau = \tilde{q} \circ \pi^{J}$, $J = [n]$

where $\nu_q: \partial\tilde{P} \to \tilde{T}$ the circular nbhd of $\tilde{T} = \mathrm{im}\,\tilde{q}$

$\tilde{q}: \widetilde{\mathcal{M}}^{J}_{g-1, n+2} \to \widetilde{\overline{\mathcal{M}}}_{g,n}$ sewing p_{n+1} and p_{n+2}, ③

A fixed bdry CohFT of $(A, \eta, \mathbb{1})$ is a set of linear maps

$$\tilde{\Omega}_{g,n} : A^{\otimes n} \to H^{\cdot}(\widetilde{M}_{g,n}), \quad 2g-2+n>0, \text{ s.t.}$$

1. $\tilde{\Omega}_{g,n}$ is S_n-equivariant $\forall g,n$
2. $\tilde{\Omega}_{0,3}(\mathbb{1} \otimes u \otimes v) = \eta(u,v) \in H^0(\widetilde{M}_{0,3})$
3. $\tau^* \tilde{\Omega}_{g,n}(v_1 \otimes \cdots \otimes v_n) = \eta^{\mu\nu} \tilde{\Omega}_{g-1,n+2}(v_1 \otimes \cdots \otimes v_n \otimes e_\mu \otimes e_\nu)$
4. $\sigma^* \tilde{\Omega}_{g,n}(v_1 \otimes \cdots \otimes v_n) = \eta^{\mu\nu} \tilde{\Omega}_{g_1, n_1+1}(v_1 \otimes \cdots \otimes v_{n_1} \otimes e_\mu)$
$\times \tilde{\Omega}_{g_2, n_2+1}(v_{n_1+1} \otimes \cdots v_n \otimes e_\nu)$
5. $p^* \tilde{\Omega}_{g,n}(v_1 \otimes \cdots \otimes v_n) = \tilde{\Omega}_{g,n+1}(v_1 \otimes \cdots \otimes v_n \otimes \mathbb{1})$

We can also define smooth theory without tangent vectors

Denote $q : M_{g-1,n+2} \to \overline{M}_{g,n}$, $\nu_q : N_q \to \operatorname{im} q = T$

$s : M_{g_1,n_1+1} \times M_{g_2,n_2+1} \to \overline{M}_{g,n}$, $\nu_s : N_S \to \operatorname{im} s = S$

the tubular nbhd (normal bundle) of self-sewing map / sewing map, resply. (Note q is a 2-sheeted covering)

We use the same notation for circular nbhd (circle subbundle)

then a free bdry CohFT of $(A, \eta, \mathbb{1})$ is

$$\Omega_{g,n} : A^{\otimes n} \to H^{\cdot}(\Omega_{g,n}), \quad 2g-2+n>0, \text{ s.t.}$$

1. $\Omega_{g,n}$ is S_n-equivar. $\forall g,n$
2. $\Omega_{0,3}(\mathbb{1} \otimes u \otimes u) = \eta(u,v)$
3. $\Omega_{g,n}(v_1 \otimes \cdots \otimes v_n)|_{\partial N_q} = \nu_q^* (\tfrac{1}{2} q_* \eta^{\mu\nu} \Omega_{g-1,n+2}(v_1 \otimes \cdots \otimes v_n \otimes e_\mu \otimes e_\nu))$ ← autom.
4. $\Omega_{g,n}(v_1 \otimes \cdots \otimes v_n)|_{\partial N_s} = \nu_s^* \eta^{\mu\nu} \Omega_{g_1,n_1+1}(v_1 \otimes \cdots \otimes v_{n_1} \otimes e_\mu)$
$\times \Omega_{g_2,n_2+1}(v_{n_1+1} \otimes \cdots \otimes v_n \otimes e_\nu)$
5. $p^* \Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) = \Omega_{g,n+1}(v_1 \otimes \cdots \otimes v_n \otimes \mathbb{1})$

④

The CohFTs are related by

Prop. Let $(\Omega_{g,n})_{g,n}$ be a free bdry CohFT

$\pi : \widetilde{M}_{g,n} \to M_{g,n}$ forgetting tangents

then $(\pi^* \Omega_{g,n})_{g,n}$ is a fixed bdry CohFT

Prop. Let $(\overline{\Omega}_{g,n})_{g,n}$ be a nodal CohFT

then the restriction of $\overline{\Omega}_{g,n}$ to $M_{g,n}$ is a free bdry CohFT.

Let $R(z) = \mathrm{Id} + R_1 z + R_2 z^2 + \cdots \in \mathrm{End}(A)[[z]]$ be symplectic*

Givental's R-matrix action:

Let $C \in S \subset \overline{M}_{g,n}$ (open stratum), the dual graph Γ_C (or Γ_S) of C consists of

$\widehat{C'} \to C$ normalization,

① vertices V, ② edges E, ③ legs L

corr. to ① comp. of C' (with genus $g(v)$), ② pairs of points on C' mapping to a node on C, ③ points on C' mapping to a marked pt resply.

Define $\mathrm{Cont}_{\Gamma_S} : A^{\otimes n} \to H^*(\overline{M}_{g,n})$ by

1. Put $\sum_{m \geq 0} \frac{1}{m!} (p_m)^* \overline{\Omega}_{g(v), n(v)+m} (\cdot \otimes T(\psi_{n(v)+1}) \otimes \cdots \otimes T(\psi_{n(v)+m}))$ at vertex v, where $p_m : \overline{M}_{g,n+m} \to \overline{M}_{g,n}$ forgetful map
2. Insert $R^{-1}(\psi_\ell) a$ for each $\ell \in L$ ($a \in A$ of corr. marked pt)
3. Insert $(\eta^{-1} - R^{-1}(\psi_e') \eta^{-1} R^{-1}(\psi_e'')^t) / \psi_e' + \psi_e''$ for each $e \in E$
4. Then we have a map

$$A^{\otimes n} \longrightarrow \prod_{v \in V} H^*(\overline{M}_{g(v), n(v)}) \xrightarrow{\text{push forward}} H^*(\overline{M}_{g,n})$$

Define $(R\overline{\Omega})_{g,n} = \sum_{S \in \overline{M}_{g,n}} \frac{1}{|\mathrm{Aut}(\Gamma_S)|} \mathrm{Cont}_{\Gamma_S}$

Prop. If $\overline{\Omega}$ is a nodal CohFT, then so is $R\overline{\Omega}$.

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* $R(z) R(-z)^* = \mathrm{id}$, where $R(z)^* = \eta^{-1} R^t(z) \eta$

PAGE
DATE

(5)

Frobenius Algebra structure on CohFT

$((A, \cdot, \mathbb{1}, \eta)$ s.t. $\eta : A \otimes A \to \mathbb{C}$ satisfies $\eta(a \cdot b, c) = \eta(a, b \cdot c)$

Given $(\Omega_{g,n})_{g,n}$ a nodal CohFT, A has a frob. structure,

s.t. $\Omega_{0,3}(v_1 \otimes v_2 \otimes v_3) = \eta(v_1 \cdot v_2, v_3)$

Pf. Identity: Define $i : A \to A^*$ by $v \mapsto \eta(v, \cdot)$

then $v \cdot \mathbb{1} = i^{-1}\Omega_{0,3}(v \otimes \mathbb{1} \otimes \cdot) \underset{\uparrow}{=} i^{-1}\Omega_{0,3}(\mathbb{1} \otimes v \otimes \cdot)$

$= i^{-1}\eta(v, \cdot) = v$ $\qquad S^3$-equivar.

"$\cdot$" Assoc.: $\Omega_{0,3}((v_1 \cdot v_2) \otimes v_3 \otimes v_4) = \Omega_{0,3}(v_1 \otimes (v_2 \cdot v_3) \otimes v_4)$

$\overset{*}{\Leftrightarrow}$ $S^*\Omega_{0,4}(v_3 \otimes v_4 \otimes v_1 \otimes v_2) = S^*\Omega_{0,4}(v_1 \otimes v_4 \otimes v_2 \otimes v_3)$

where $S : \bar{\mathcal{M}}_{0,3} \times \bar{\mathcal{M}}_{0,3} \to \bar{\mathcal{M}}_{0,4}$ is the sewing map

then by S^4-equivar.

$\eta(v_1 \cdot v_2, v_3) = \eta(v_1, v_2 \cdot v_3)$: By assoc.,

$\Omega_{0,3}((v_1 \cdot v_2) \otimes v_3 \otimes \mathbb{1}) = \Omega_{0,3}(v_1 \otimes (v_2 \cdot v_3) \otimes \mathbb{1})$ $\square$

The same holds for other CohFTs.

* (In general if $S : \bar{\mathcal{M}}_{g,n} \times \bar{\mathcal{M}}_{0,3} \to \bar{\mathcal{M}}_{g,n+1}$

then $S^* \bar{\Omega}_{g,n+1}(\cdots \otimes v_n \otimes v_{n+1}) = \eta^{\mu\nu} \bar{\Omega}_{g,n}(\cdots \otimes e_\nu) \times \bar{\Omega}_{0,3}(v_n \otimes v_{n+1} \otimes e_\mu)$

$= \eta^{\mu\nu}\eta(v_n \cdot v_{n+1}, e_\mu)\bar{\Omega}_{g,n}(\cdots \otimes e_\nu) = \bar{\Omega}_{g,n}(\cdots, v_n \cdot v_{n+1}))$

⑥

Euler class

Let $(A, \cdot, \mathbb{1}, \eta)$ be a frob. alg., then

$\alpha = \eta^{\mu\nu} e_\mu \cdot e_\nu \in A$ is callled the Euler class (well-def under choice of basis)

Prop. $(A, \cdot, \mathbb{1}, \eta)$ is semisimple iff α is inverctible

$\Rightarrow$: Let e_μ be a semisimple basis, then

$\alpha = \sum_\mu \eta(e_\mu, \mathbb{1})^{-1} e_\mu$

(b.c. $\eta(e_\mu, e_\mu) = \eta(e_\mu \cdot e_\mu, \mathbb{1})$)

take $\alpha^{-1} = \sum_\mu \eta(e_\mu, \mathbb{1})^3 e_\mu$, then $\alpha \cdot \alpha^{-1} = \mathbb{1}$

$\Leftarrow$: omit

Lemma. Let $q: \overline{M}_{g-1, n+2} \to \overline{M}_{g,n}$ self-sewing, then

$$q^* \Omega_{1,1}(v) = \eta(v, \alpha)$$

Pf. $q^* \Omega_{1,1}(v) = \eta^{\mu\nu} \Omega_{0,3}(v \otimes e_\mu \otimes e_\nu) = \eta^{\mu\nu} \eta(v, e_\mu \cdot e_\nu) = \eta(v, \alpha)$

In particular, the degree 0 part of $\Omega_{1,1}(v)$ is $\eta(v, \alpha)$

Lemma. The degree 0 part of $\Omega_{1,2}(v \otimes w)$ is

$$\overline{\Omega}_{1,2}(v \otimes w)_0 = \eta(v \cdot w, \alpha)$$

Pf. Consider the sewing $s: \overline{M}_{0,3} \times \overline{M}_{1,1} \to \overline{M}_{1,2}$

As before, we have

$$s^* \overline{\Omega}_{1,2}(v \otimes w)_0 = \overline{\Omega}_{1,1}(v \cdot w)_0 = \eta(v \cdot w, \alpha)$$

The similar statement holds for any CohFT.

②

With euler class we can describe the effect of sewing a genus g'' curve to the curves in $\overline{\mathcal{M}}_{g',n}$.

Prop. Let $g = g' + g''$, $\Sigma \in \overline{\mathcal{M}}_{g'',2}$ fixed,

$S|_\Sigma : \overline{\mathcal{M}}_{g',n} \times \Sigma \to \overline{\mathcal{M}}_{g,n}$ sewing p_n on $\overline{\mathcal{M}}_{g',n}$ and p_2 on Σ

then $S|_\Sigma^* \bar{\Omega}_{g,n}(v_1 \otimes \cdots \otimes v_n) = \bar{\Omega}_{g',n}(v_1 \otimes \cdots \otimes v_{n-1} \otimes \alpha^{g''} \cdot v_n)$

Pf. For $g'' = 1$, by sewing axiom,

$$S^* \bar{\Omega}_{g,n}(v_1 \otimes \cdots \otimes v_n) = \eta^{\mu\nu} \bar{\Omega}_{g',n}(v_1 \otimes \cdots \otimes v_{n-1} \otimes e_\mu) \times \bar{\Omega}_{1,2}(v_n \otimes e_\nu)$$

Restricting this class (in $H^\cdot(\overline{\mathcal{M}}_{g',n} \times \overline{\mathcal{M}}_{1,1})$) to the subspace $\overline{\mathcal{M}}_{g',n} \times \Sigma$ is taking degree 0 part in the right and summing up, so

$$S|_\Sigma^* \bar{\Omega}_{g,n}(v_1 \otimes \cdots \otimes v_n) = \eta^{\mu\nu} \bar{\Omega}_{g',n}(v_1 \otimes \cdots \otimes v_{n-1} \otimes e_\mu) \cdot \underbrace{\bar{\Omega}_{1,2}(v_n \otimes e_\nu)_0}_{= \eta(v_n \cdot e_\nu, \alpha) \text{ by lemma}}$$

$$= \bar{\Omega}_{g',n}(v_1 \otimes \cdots \otimes v_{n-1} \otimes \eta^{\mu\nu} \eta(\alpha \cdot v_n, e_\nu) e_\mu)$$

$$= \bar{\Omega}_{g',n}(v_1 \otimes \cdots \otimes v_{n-1} \otimes (\alpha \cdot v_n))$$

For the case of general g'' we can sew on a genus g'' curve by sewing in a row g'' tori, hence by recursion. □

Similar statement holds for the other CohFTs

(Fixed bdy : $\sigma_\Sigma \tilde{\Omega}_{g,n}(\cdots \otimes v_n) = \tilde{\Omega}_{g',n}(\cdots \otimes (\alpha^{g''} \cdot v_n))$,

Free bdy : $\Omega_{g,n}(\cdots \otimes v_n)|_{\partial N_\Sigma} = v_\Sigma^* \Omega_{g',n}(\cdots \otimes (\alpha^{g''} \cdot v_n))$

$v_\Sigma : \partial N_\Sigma \to S_\Sigma = \operatorname{im} S|_\Sigma$)

As a conseq., if A is semisimple then

$$\bar{\Omega}_{g',n}(v_1 \otimes \cdots \otimes v_n) = S_\Sigma^* \bar{\Omega}_{g,n}(v_1 \otimes \cdots \otimes v_{n-1} \otimes (\alpha^{-g''} \cdot v_n))$$

Thus theory of $\mathcal{M}_{g,n}$ can be recovered from $\overline{\mathcal{M}}_{g',n} \times \overline{\mathcal{M}}_{g'',2}$

Cohomology of $H^\cdot(\mathcal{M}_{g,n})$

The part of $H^\cdot(\overline{\mathcal{M}}_{g,n})$ of degree $\leq g/3$ is called the stable range of $H^\cdot(\overline{\mathcal{M}}_{g,n})$

Thm (Looijenga) Let $g \geq 2$, $\mathcal{M}_g = \mathcal{M}_{g,0}$, then for d in stable range,

$$H^d(\mathcal{M}_{g,n}) \cong H^\cdot(\mathcal{M}_g)[\psi_1, \cdots, \psi_n])_d \leftarrow \text{deg } d \text{ part}$$

Consider the forgetful map $p: \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$

It induces proper pushforward

$$H^\cdot(\overline{\mathcal{M}}_{g,n+1}) \to H^{\cdot-2}(\overline{\mathcal{M}}_{g,n})$$

Define the m-th κ-class

$$\kappa_m = p_*(\psi_{n+1}^{m+1}) \ H^{2m}(\overline{\mathcal{M}}_{g,n})$$

Thm. (Madsen-Weiss) Let $g \geq 2$, $\mathcal{M}_g = \mathcal{M}_{g,0}$, then for d in the stable range, $H^d(\mathcal{M}_g) \cong (\mathbb{C}[\kappa_j]_{j\geq 1})_d$

Lemma. Let $M(z) = z(1-\exp(-a_1 z - a_2 z^2 - \cdots)) \in A[[z]]$.

Fix g,n s.t. $2g-2+n>0$, $p_m: \mathcal{M}_{g,n+m} \to \mathcal{M}_{g,n}$ forgetful map, then in $H^\cdot(\mathcal{M}_{g,n})$ we have

$$\exp(a_1\kappa_1 + a_2\kappa_2 + \cdots) = \sum_{m=0}^{\infty} \frac{1}{m!}(p_m)_*(M(\psi_{n+1})\cdots M(\psi_{n+m}))$$

Pf. We first express $\kappa_{k_1,\cdots,k_m} = (p_m)_*(\psi_{n+1}^{k_1+1} \cdots \psi_{n+m}^{k_m+1})$ in terms of κ classes.

For each $\sigma \in S_m$ permutation, $\sigma = \rho_1 \cdots \rho_\ell$ write $k_{\rho_i} = \sum_{j \in \rho_i} k_j$, we show that

$$\kappa_{k_1,\cdots,k_m} = \sum_{\sigma \in S_m} \prod_{i=1}^{\ell} \kappa_{k_{\rho_i}} =: R_{k_1\cdots k_m}(\kappa)$$

(poly. of κ_j)

⑨

Suppose this is already proved for $m-1$ classes,

then for m, split p_m into $\overbrace{p_1 \circ \cdots \circ p_1}^{m}$

we have the following formulas for K_m:

1) $K_m = p_1^* K_m + \psi^m$ where ψ is the ψ class of the forgotten marked pt

2) $p_{1*}(\psi^{m+1} \cdot (\text{other } \psi \text{ classes})) = K_m \cdot (\text{other } \psi \text{ classes})$

Thus $p_{m*}(\psi_{n+1}^{k_1+1} \cdots \psi_{n+m}^{k_m+1})$

$= p_{1*}(\psi_{n+1}^{k_1+1} \cdots R_{k_2 \cdots k_m}(K))$ (Due to 2) ψ_{n+1} is not affected)

$= p_{1*}(\psi_{n+1}^{k_1+1} \cdot R_{k_2 \cdots k_m}(p_1^* K_j + \psi_{n+1}^j))$

$=$ (expand, and by proj. formula and some counting)

$= R_{k_1 \cdots k_m}(K)$

Now let $\phi : A[[z]] \to A[[z, K_i]]_{i \geq 1}$ be the A-linear map

sending z^i to $z^i K_i$

then the above computation shows that

$$\sum_{m=0}^{\infty} \frac{1}{m!} (p_m)_* (M(\psi_{n+1}) \cdots M(\psi_{n+m})) = F(1)$$

By exp we only count one cycle

$$= \exp\left(\phi\left(\sum_{m>0} \frac{(m-1)!}{m!} \left(\frac{M(z)}{z}\right)^m\right)\Big|_{z=1}\right)$$

there are $(m-1)!$ cycles in S_m

$$= \exp\left(\phi\left(-\log\left(1 - \frac{M(z)}{z}\right)\right)\Big|_{z=1}\right)$$

$$= \exp(a_1 K_1 z + a_2 K_2 z^2 + \cdots |_{z=1}) = \exp(a_1 K_1 + a_2 K_2 + \cdots)$$

(10)

Classification of s.s. CohFT

Fixed bdry theories

Let $\sigma_{g,T} : \widetilde{\mathcal{M}}_{g,1} \to \widetilde{\mathcal{M}}_{g+1,1}$ an embedding by

$$(C,[v]) \mapsto \sigma((C,[v]),(T,[v_1],[v_2]))$$

for a fixed $T \in \widetilde{\mathcal{M}}_{1,2}$

We have $\varprojlim_g H^\cdot(\widetilde{\mathcal{M}}_{g,1}) \cong \mathbb{C}[k_j]_{j\geq 1}$ by Madsen-Weiss

proj. system with maps $\sigma^*_{g,T}$

Given $\widetilde{\Omega}$ with s.s. base $(A,\mathbb{1},\eta)$ and s.s. basis $(e_\mu)_\mu$,

By Prop. in (9), we have $\quad$ ($e_\mu \cdot e_\mu = \theta_\mu^{-1} e_\mu$, where $\theta_\mu = \eta(e_\mu,\mathbb{1})$)

$$\sigma^*_{g,T}\widetilde{\Omega}_{g+1,1}(\alpha^{-g-1}\cdot v) = \widetilde{\Omega}_{g,1}(\alpha^{-g}\cdot v)$$

thus $\Omega^+ := \varprojlim \widetilde{\Omega}_{g,1}(\alpha^{-g}\cdot)$ defines a homom.

$\Omega^+ : A \to \varprojlim H^\cdot(\widetilde{\mathcal{M}}_{g,1})$ and can be view as

an element in $A^* \otimes \mathbb{C}[k_j]_{j\geq 1}$

Prop. Let $\widetilde{C} \in \widetilde{\mathcal{M}}_{0,3}$ and

$$m_{g,h} : \widetilde{\mathcal{M}}_{g,1}\times\widetilde{\mathcal{M}}_{h,1} \left(\to \widetilde{\mathcal{M}}_{g,1}\times\{\widetilde{C}\}\times\widetilde{\mathcal{M}}_{h,1} \xrightarrow{\sigma_1\times Id} \widetilde{\mathcal{M}}_{g,2}\times\widetilde{\mathcal{M}}_{h,1} \xrightarrow{\sigma_2}\right) \widetilde{\mathcal{M}}_{g+h,1}$$

then $m^*_{g,h}\widetilde{\Omega}_{g+h,1} = \widetilde{\Omega}_{g,1}\cdot\widetilde{\Omega}_{h,1} \in A^*\otimes H^\cdot(\widetilde{\mathcal{M}}_{g,1}\times\widetilde{\mathcal{M}}_{h,1})$

in terms of Frob. product and cross product of Coh.

(of $(A^*,\eta^{-1},i(\mathbb{1}))$, by $e^\mu\cdot e^\nu = \delta_{\mu\nu}\theta_\mu^{-1}e^\mu$)

Pf. By sewing axiom,

$$\sigma_2^*\widetilde{\Omega}_{g+h,1}(v) = \sum_\mu \widetilde{\Omega}_{g,2}(v\otimes e_\mu)\times\widetilde{\Omega}_{h,1}(e_\mu)$$

$$\sigma_1^*\widetilde{\Omega}_{g,2}(v\otimes e_\mu) = \sum_\nu \eta(v\cdot e_\mu, e_\nu)\widetilde{\Omega}_{g,1}(e_\nu) = \eta(v, e_\mu\cdot e_\mu)\widetilde{\Omega}_{g,1}(e_\mu)$$

$v^\mu e_\mu \qquad$ deg 0 part of $H^\cdot(\widetilde{\mathcal{M}}_{0,3})$

On the other hand, $\widetilde{\Omega}_{g,1}\cdot\widetilde{\Omega}_{h,1}(v) = \theta_\mu^{-1}v^\mu\widetilde{\Omega}_{g,1}(e_\mu)\times\widetilde{\Omega}_{h,1}(e_\mu)$ $\diamond$

(11)

Write $i_{g,n} : \mathbb{C}[\kappa_j]_{j\geq 1} \to H^\cdot(\widetilde{\mathcal{M}}_{g,n})$

the $\mathbb{C}$-alg. homom. sending κ_j to j-th κ-class of $\widetilde{\mathcal{M}}_{g,n}$

then $m_{g,h}^* : H^\cdot(\widetilde{\mathcal{M}}_{g+h,1}) \to H^\cdot(\widetilde{\mathcal{M}}_{g,1}) \otimes H^\cdot(\widetilde{\mathcal{M}}_{h,1})$

pass to the limit $m^* : \varprojlim H^\cdot(\widetilde{\mathcal{M}}_{g,1}) \to \varprojlim H^\cdot(\widetilde{\mathcal{M}}_{g,1}) \otimes \varprojlim H^\cdot(\widetilde{\mathcal{M}}_{g,1})$

$$\mathbb{C}[\kappa_j]_{j\geq 1} \to \mathbb{C}[\kappa_j]_{j\geq 1} \otimes \mathbb{C}[\kappa_j]_{j\geq 1}$$

with $m_{g,h}^* \circ i_{g+h,1} = (i_{g,1} \otimes i_{h,1}) \circ m^*$

Define product on $A^* \otimes \mathbb{C}[\kappa_j]_{j\geq 1}$ by

$$X \cdot Y(v) = \theta_\mu^{-1} v^\mu X(e_\mu) Y(e_\mu)$$

Cor. $m^* \widetilde{\Omega}^+ = \widetilde{\Omega}^+ \cdot \widetilde{\Omega}^+$

Pf. $m^* \widetilde{\Omega}_{g+h,1}(\alpha^{-g-h} \cdot v) = \theta_\mu^{2(g+h)-1} v^\mu \widetilde{\Omega}_{g,1}(e_\mu) \times \widetilde{\Omega}_{h,1}(e_\mu)$

$= (\widetilde{\Omega}_{g,1}(\alpha^{-g}\cdot) \cdot \widetilde{\Omega}_{h,1}(\alpha^{-h}\cdot))v$

Prop. $m^* \kappa_j = \kappa_j \otimes 1 + 1 \otimes \kappa_j$

Pf. Consider the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_{g,2} \times \widetilde{\mathcal{M}}_{h,1} \cup \widetilde{\mathcal{M}}_{g,1} \times \widetilde{\mathcal{M}}_{h,2} & \xrightarrow{m'_{g,h}} & \widetilde{\mathcal{M}}_{g+h,2} \\ \downarrow p' & & \downarrow p \\ \widetilde{\mathcal{M}}_{g,1} \times \widetilde{\mathcal{M}}_{h,1} & \xrightarrow{m_{g,h}} & \widetilde{\mathcal{M}}_{g+h,1} \end{array}$$

p, p' forget the second marked pt

$m'_{g,h}$ sew $C \in \widetilde{\mathcal{M}}_{0,3}$ to the first marked pt at each curve

Then $m^* \kappa_j = m^* p_* \psi_2^{j+1} = p'_* m'^*(\psi_2^{j+1})$

$= p'_*((\psi_2 \otimes 1 + 1 \otimes \psi_2)^{j+1}) = p'_*(\psi_2^{j+1} \otimes 1 + 1 \otimes \psi_2^{j+1})$ ($\psi_2 \otimes \psi_2 = 0$ in $\widetilde{\mathcal{M}}_{g,2} \times \widetilde{\mathcal{M}}_{h,1} \cup \widetilde{\mathcal{M}}_{g,1} \times \widetilde{\mathcal{M}}_{h,2}$)

$= \kappa_j \otimes 1 + 1 \otimes \kappa_j$

(12)

A bialgebra $(A, \cdot, \mathbb{1}, \Delta, \theta)$ is a Hopf algebra

mult. unit comult counit

if there's $S: A \to A$ s.t.

$$\begin{tikzcd} & A\otimes A \arrow[r, "S\otimes id"] & A\otimes A \arrow[dr] & \\ A \arrow[ur, "\Delta"] \arrow[r, "\theta"] \arrow[dr, "\Delta"'] & \mathbb{C} \arrow[rr, "\mathbb{1}"] & & A \\ & A\otimes A \arrow[r, "id\otimes S"'] & A\otimes A \arrow[ur] & \end{tikzcd}$$

Given a Frob. alg. $(A, \cdot, \mathbb{1}, \eta)$, it is a bialgebra by equipping

$\Delta(v) = \eta(v, e_\mu \cdot e_\nu) e_\mu \otimes e_\nu$

$= \theta_\mu^{-1} v^\mu e_\mu \otimes e_\mu$ if semisimple

$\theta(v) = \eta(v, \mathbb{1})$ (Frob. trace)

$= \theta_\mu v^\mu$ if s.s.

In our case, $\mathbb{C}[k_j]_{j\geq 1}$ is a Hopf. alg.

with comult. $m^\vee$ and antipode $S(k_j) = -k_j$

Moreover, $A^* \otimes \mathbb{C}[k_j]$ is a bialg.

with comult. the $\otimes$ of comult. on both factor.

Thm: (Milnor - Moore) Any Hopf-algebra H gen. by primitive elements ($y \in H$ s.t. $\Delta(y) = y\otimes 1 + 1 \otimes y$) is the free alg. generated by primitive elements.

In our case, it shows primitive elements in $\mathbb{C}[k_j]_{j\geq 1}$ are linear combination of k_j.

Let $X: A \to \mathbb{C}[k_j]_{j\geq 1}$, X is called group-like if

$X_0 = \theta$ and $m^\vee X = X \otimes X$

X is called primitive if $m^\vee X = X\otimes 1 + 1 \otimes X$

($\Leftrightarrow$ $X = \sum_j \phi_j k_j$ for some $\phi_j \in A^*$)

PAGE
DATE

(13)

Lemma. $X : A \to \mathbb{C}[[k_j]]_{j\geq 1}$

X is group-like iff $\exists x : A \to \mathbb{C}[[k_j]]_{j\geq 1}$ prim.

s.t. $X = \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Pf. We first show that $m^*(\exp(x)) = \exp(m^*(x))$ $\forall x \in A^* \otimes \mathbb{C}[[k_j]]_{j\geq 1}$

From definition,

$x^{\cdot n}(v) = \underbrace{x \cdot x \cdots x}_{n}(v) = \theta_\mu^{-n+1} v^\mu (x(e_\mu))^n$

thus $m^*(x \cdots x)(v) = \theta_\mu^{-n+1} v^\mu (m^*(x(e_\mu)))^n$

$= (m^*(x)^{\cdot n})(v)$

hence $m^*(\exp(x)) = \exp(m^*(x))$

$= \exp(x \otimes 1 + 1 \otimes x)$ if x primitive

$= \exp(x \otimes 1) \cdot \exp(1 \otimes x)$ $\exp(x)$ gp-like

$= (\exp(x) \otimes 1) \cdot (1 \otimes \exp(x)) = \exp(x) \otimes \exp(x)$

conversely, if X is group-like

then let $x = \log(X) := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(X-\theta)^n}{n}$

similarly $m^* x = \log(m^* X) = \log(X \otimes X)$

$= \log(X \otimes 1) + \log(1 \otimes X)$

so x is primitive and $\exp(x) = X$

Cor. $\exists \phi_j \in A^*$ s.t. $\tilde{\Omega}^+ = \exp(\phi_j k_j)$

since $\tilde{\Omega}^+$ is gp-like, by cor. in (11)

(14)

Thm. Let $\tilde{\Omega}_{g,n} : A^{\otimes n} \to H^{\cdot}(\tilde{\mathcal{M}}_{g,n})$ fixed bdry CohFT

then $\exists$ some $\phi_j \in A^*$, $j \geq 1$ s.t. $\left(\tilde{\Omega}^{+} = \exp(\sum_{j\geq 1} \phi_j \kappa_j)\right)$

$\tilde{\Omega}_{g,n}(v_1 \otimes \cdots \otimes v_n) = i_{g,n} \tilde{\Omega}^{+}(\alpha^g . v_1 \cdots v_n) \quad \forall g \geq 1$

Conversely any Frob. alg. $(A, \cdot, \mathbb{1}, \eta)$ with some $\phi_j \in A^*$, $j>1$

determine a fixed bdry CohFT by above and

$\tilde{\Omega}_{0,n}(v_1 \otimes \cdots \otimes v_n) = s^*(\tilde{\Omega}_{1,n}(v_1 \otimes \cdots \otimes (\alpha^{-1}. v_n))$

$s : \tilde{\mathcal{M}}_{0,n} \times \Sigma \to \tilde{\mathcal{M}}_{1,n}$ for a fixed Σ

Pf. From definition $\tilde{\Omega}^{+} = \varprojlim \tilde{\Omega}_{g,1}(\alpha^{-g}.)$

$\Rightarrow \tilde{\Omega}_{g,1}(v) = i_{g,1} \tilde{\Omega}^{+}(\alpha^g, v)$

For $\tilde{\Omega}_{g,n}$, let $\varphi_{g,n} : \tilde{\mathcal{M}}_{g,1} \to \tilde{\mathcal{M}}_{g,n}$ sewing-smoothing a fixed element in $\tilde{\mathcal{M}}_{0,n+1}$,

$s_{g,n} : \tilde{\mathcal{M}}_{g,n} \to \tilde{\mathcal{M}}_{g+g',n}$ sewing-smoothing a fixed element in $\tilde{\mathcal{M}}_{g',2}$

Then by Prop in (9),

$\tilde{\Omega}_{g,n}(v_1 \otimes \cdots \otimes v_n) = s^*_{g,n} \tilde{\Omega}_{g+g',n}(v_1 \otimes \cdots \otimes (\alpha^{-g'}. v_n))$

$= s^*_{g,n}(\varphi^*_{g+g',n})^{-1} i_{g+g',1} \tilde{\Omega}^{+}(\alpha^g . v_1 \cdots v_n)$, see (5)

for $(g+g')/3 > 3g-3+n$ by Harer's stability (Choose such g')

Since the inverse of $\varphi^*_{g+g',n}$ is p^*

by $p \circ s_{g,n} = s_{g,1} \circ p$ and $s^*_{g,1} i_{g+g',1} = i_{g,1}$

$$\begin{tikzcd} \tilde{\mathcal{M}}_{g,n} \arrow[r, "s_{g,n}"] \arrow[d, "p"] & \tilde{\mathcal{M}}_{g+g',n} \arrow[d, "p"] \\ \tilde{\mathcal{M}}_{g,1} \arrow[r, "s_{g,1}"] & \tilde{\mathcal{M}}_{g+g',1} \end{tikzcd}$$

We get $\tilde{\Omega}_{g,n}(v_1 \otimes \cdots v_n)$

$= p^* i_{g,1} \tilde{\Omega}^{+}(\alpha^g . v_1 \cdots v_n)$

$= i_{g,n} \tilde{\Omega}^{+}(\alpha^g . v_1 \cdots v_n)$

15

For the converse we must check the axioms:

2. $\tilde{\Omega}_{0,3}(v_1 \otimes v_2 \otimes \mathbb{1}) = s^* \Omega_{1,3}(v_1 \otimes v_2 \otimes (\alpha^{-1} \cdot \mathbb{1}))$

$= s^* p^* i_{1,1} \tilde{\Omega}^+(v_1, v_2)$

$= s^* p^* (\theta(v_1 \cdot v_2) + \phi_1(v_1 \cdot v_2) K_1)$

Note the $p^* K_1 = K_1 - \psi_2 - \psi_3$ $\quad$ $K_j = 0$ for $j \geq 2$ in $\tilde{M}_{1,1}$

$\Rightarrow s^* p^* K_1 = 0$

and $s^* p^* \theta(v_1 \cdot v_2) = \theta(v_1 \cdot v_2) = \eta(v_1, v_2)$

3. Let $\tau : \tilde{M}_{g-1,n+2} \to \tilde{M}_{g,n}$

Want to check $\quad$ for s.s basis e_μ

$\tau^* \Omega_{\tilde{g},n}(v_1 \otimes \cdots v_n) = \sum_\mu \Omega_{\tilde{g-1},n+2}(v_1 \otimes \cdots v_n \otimes e_\mu \otimes e_\mu)$

$\parallel$

$\tau^* p^* i_{g,1} \tilde{\Omega}^+(\alpha^g \cdot v_1 \cdots v_n)$

$\parallel$

$p^*_{1,n+1,n+2} \tau^* i_{g,1} \tilde{\Omega}^+(\alpha^g \cdot v_1 \cdots v_n)$ $\quad$ ($p_{1,n+1,n+2}$ forget pts other than p_1, p_{n+1}, p_{n+2})

$\parallel$

$p^*_{1,n+1,n+2} p^*_{2,3} i_{g-1,1} \tilde{\Omega}^+(\alpha^g \cdot v_1 \cdots v_n)$

since $\tau^* \psi_i = \psi_i \ \forall i$
$\Rightarrow \tau^* K_i = K_i$

$\parallel$

$p^* i_{g-1}^* \sum_\mu \tilde{\Omega}^+(\alpha^g \cdot v_1 \cdots v_n e_\mu e_\mu) = RHS.$

4. Similar to 3.

5. obvious.

(Thm. (Harer's Stability) Let $s_0 : \tilde{M}_{g,n} \cong \tilde{M}_{g,n} \times \underset{\tilde{M}_{0,3}}{\Sigma} \to \tilde{M}_{g,n+1}$

then $s_0^* : H^k(\tilde{M}_{g,n+1}) \to H^k(\tilde{M}_{g,n})$

is an isom. for $k \leq g/3$ (stable range)

with inverse p^*, where $p : \tilde{M}_{g,n+1} \to \tilde{M}_{g,n}$ forgetful map)

This is the origin of the term of "stable range"

PAGE
DATE

Ⓐ

Verifying $R\overline{\Omega}_{g,n}$ is a CohFT:

(1) Let $T\overline{\Omega}_{g,n}(v_1\otimes\cdots\otimes v_n)$ $\quad (T = z(\mathbb{1}-R(z)\mathbb{1}) \in z^2 \mathrm{Aut}(A)[[z]])$

$$= \sum_{m\geq 0} \frac{1}{m!} (p_m)_* \overline{\Omega}_{g,n+m}(v_1\otimes\cdots\otimes v_n \otimes T(\psi_{n+1}) \otimes \cdots \otimes T(\psi_{n+m}))$$

$p_m : \mathcal{M}_{g,n+m} \to \mathcal{M}_{g,n}$ $\quad$ distribution of m_1+m_2 legs

$$s^* T\overline{\Omega}_{g,n}(\cdots) = \sum_{m\geq 0} \frac{1}{m!} s^* p_{m*}(\cdots) = \sum_{m\geq 0} \frac{1}{m!}\binom{m}{m_1,m_2}(p_{m_1}\times p_{m_2})_* s'^*(\cdots)$$

$$\begin{tikzcd}
\coprod_{m_1+m_2=m} \overline{\mathcal{M}}_{g_1,n_1+m_1+1}\times\overline{\mathcal{M}}_{g_2,n_2+m_2+1} \arrow[r, "s'"] \arrow[d, "p_{m_1}\times p_{m_2}"'] & \overline{\mathcal{M}}_{g,n+m} \arrow[d, "p_m"] \\
\overline{\mathcal{M}}_{g_1,n_1+1}\times\overline{\mathcal{M}}_{g_2,n_2+1} \arrow[r, "s"] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

$$= \sum_{m_1,m_2} \eta_{\mu\nu} \frac{1}{m_1!}\frac{1}{m_2!} p_{m_1*}(v_1\otimes\cdots v_{n_1}\otimes T\psi_{n_1+1}\otimes\cdots T\psi_{n_1+m_1}\otimes e_\mu)$$
$$\times\, p_{m_2*}(v_{n_1+1}\otimes\cdots v_n\otimes T\psi_{n+m_1+1}\otimes\cdots T\psi_{n+m}\otimes e_\nu)$$
$$= T\overline{\Omega}_{g,n_1}(\cdots)\times T\overline{\Omega}_{g,n_2}(\cdots)$$

similar holds for q^*, thus $T\overline{\Omega}$ is a CohFT

(2) Let $\underline{R}\Omega_{g,n} = \sum_\Gamma \frac{1}{|\mathrm{Aut}\,\Gamma|}\underline{\mathrm{Cont}}_\Gamma$, $\quad R^*(-z)R(z) = \mathrm{id}$ (symplectic) on $H^*(\overline{\mathcal{M}}_{g,n})$

where $\underline{\mathrm{Cont}}_\Gamma \in H^*(\overline{\mathcal{M}}_{g,n},\mathbb{Q})\otimes(A^*)^n$

by 1) place $\Omega_{g(v),n(v)}$ at $v\in V$ $\quad$ (ψ act by product)

2) place $R^{-1}(\psi_\ell)$ at $\ell\in L$ $\quad$ well-def since R symplectic

3) place $(\eta^{-1} - R^{-1}(\psi_e')\eta^{-1}R^{-1}(\psi_e'')^t)/(\psi_e'+\psi_e'')$ at $e\in E$

$s^*\underline{R}\Omega_{g,n}$ consists of two parts

1) If $\Gamma_s = \Gamma_s' \xrightarrow{e_0} \Gamma_s''$ contains $e_0\in E$ corresponding to s

then it contributes $\frac{-(\psi_1+\psi_2)}{|\mathrm{Aut}\,\Gamma_s|} \underline{\mathrm{Cont}}_{\Gamma_s}(\cdots)$ $\quad$ ← excess bundle

i.e. placing $-\eta^{-1}+R^{-1}(\psi_e')\eta^{-1}R^{-1}(\psi_e'')^t$ at e_0

2) If Γ not containing e_0, then the fiber consists of $\mathcal{M}_{\Gamma'}$

with Γ obtain by Γ' contracting e_0, so the contribution

is equal to placing η^{-1} at e_0, by sewing axiom of $\Omega_{g,n}$

thus the total contribution of e_0 is $R^{-1}(\psi_e')\eta^{-1}R^{-1}(\psi_e'')^t$,

coincide the sewing axiom of $(R\Omega)_{g,n}$

$$\begin{tikzcd}
(\text{some strata}) \arrow[r] \arrow[d] & \coprod_\Gamma \overline{\mathcal{M}}_\Gamma \arrow[d, "p_\Gamma"] \\
\overline{\mathcal{M}}_{g_1,n_1+1}\times\overline{\mathcal{M}}_{g_2,n_2+1} \arrow[r, "s"] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

⑯

Free bdry CohFT

Let $\Omega_{g,n}$ be a free bdry CohFT, then

$\tilde{\Omega} = \pi^* \Omega$ is a fixed bdry CohFT, $\pi : \tilde{M}_{g,n} \to M_{g,n}$

thus $\pi^* \Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) = i_{g,n} \tilde{\Omega}^+(\alpha^g . v_1 \cdots v_n)$

for some $\tilde{\Omega}^+ : A \to \mathbb{C}[k_j]_{j\geq 1}$

Let $\pi_{I=\{2\}} : \tilde{M}^{I}_{g,2} \to M_{g,2}$ and $s|^{\{2\}}_{\Sigma} : \tilde{M}^{\{2\}}_{g',2} \to \tilde{M}_{g'',2}$

by sewing a genus g'' curve Σ, then as in ②,

$$s|^{\{2\}}_{\Sigma}{}^* \Omega^{\{2\}}_{g,2}(v_1 \otimes v_2) = \Omega^{\{2\}}_{g',2}(v_1 \otimes (\alpha^{g'} . v_2))$$

where $\Omega^{\{2\}}_{g,n} = \pi^{\{2\}*} \Omega_{g,n}$

Denote $\Omega^+ = \varinjlim \Omega^{\{2\}}_{g,n}(\cdot \otimes (\alpha^{-g} .)) : A \otimes A \to \mathbb{C}[\psi, k_j]_{j\geq 1}$ ψ-class of free pt on $\tilde{M}^{\{2\}}_{g,2}$

and let $Z(k_j, \psi)_{j\geq 1}$

$:= (id \times i^{-1})\, \Omega^+ \in (A^* \otimes A)[[k_j, \psi]]_{j\geq 1}$

(adjoint wrt η)   as an element in $A^* \otimes A^*[[k_j, \psi]]_{j\geq 1}$

$R(\psi) = Z(0, -\psi)^* \in End(A)[[\psi]]$

From definition we have ψ-classes are taken out

$(**)$ $\eta(Z(k,\psi)v, w) = \Omega^+(v \otimes w)(k, \psi) \in \mathbb{C}[\psi, k_j]_{j\geq 1}$

$= \Omega^+(v_1 \otimes R(-\psi)^{-1} v_2)$

Lemma (1) $\tilde{\Omega}^+(v_1 \cdot v_2)(k) = \Omega^+(R(\psi) v_1, v_2)$

(2) $R(\psi)^* = R(-\psi)^{-1}$ (symplectic) wrt. η

Pf. By sewing axiom,

$$\Omega_{g,2}(\alpha^{-g_1} . v_1 \otimes \alpha^{-g_2} . v_2)\Big|_{\partial N_s} = v^*_{g_1, g_2} \sum_{\mu} \Omega_{g_1,2}(e_\mu \otimes \alpha^{-g_1} . v_1) \times \Omega_{g_2,2}(e_\mu \otimes \alpha^{-g_2} . v_2)$$

where $v_{g_1,g_2} : \partial N_s \cong \tilde{M}^{\{2\}}_{g_1,2} \times \tilde{M}^{\{2\}}_{g_2,2} \to M_{g_1,2} \times M_{g_2,2}$

sewing p_2 on each curves

(17)

Pulling back by the bundle map $\pi : \widetilde{\mathcal{M}}_{g,n} \to \widetilde{\mathcal{M}}_{g,n}$

we have $\widetilde{\Omega}_{g,2}(\alpha^{-g_1} \cdot v_1 \otimes \alpha^{-g_2} \cdot v_2)|_{\pi^{-1}(\partial N_c)}$

$= \nu^*_{g_1,g_2} \sum_\mu \Omega^{\{2\}}_{g_1,2}(e_\mu \otimes \alpha^{-g_1} \cdot v_1)\, \Omega^{\{2\}}_{g_2,2}(e_\mu \otimes \alpha^{-g_2} \cdot v_2)$

$\overset{\lim}{\Leftarrow\Rightarrow} \widetilde{\Omega}^+(v_1 \cdot v_2)(k) = \nu^* \sum_\mu \underbrace{\Omega^+(e_\mu \otimes v_1)(k', \psi')}_{\in H^\cdot(\widetilde{\mathcal{M}}^{\{2\}}_{g_1,2})} \underbrace{\Omega^+(e_\mu \otimes v_2)(k'', \psi'')}_{\in H^\cdot(\widetilde{\mathcal{M}}^{\{2\}}_{g_2,2})}$, $k = k_1 + k_2$

$(\psi = \nu^*_{g_1 g_2}\psi' = \nu^*_{g_1 g_2}\psi''\ \forall g_1, g_2) = \sum_\mu \Omega^+(e_\mu \otimes v_1)(k', -\psi)\, \Omega^+(e_\mu \otimes v_2)(k'', \psi)$

(set $k' = 0$, $k'' = k$) $= \sum_\mu \Omega^+(e_\mu \otimes v_1)(0, -\psi)\, \Omega^+(e_\mu \otimes v_2)(k, \psi)$

$\overset{(*)}{=} \sum_\mu \eta(R(\psi)v_1, e_\mu)\, \eta(v_2, Z(k,\psi)e_\mu)$

$= \eta(R(\psi)v_1, Z(k,\psi)^* v_2)$

Set $k = 0$, then

$\eta(v_1, v_2) = \widetilde{\Omega}^+(v_1 \cdot v_2)(0) = \eta(R(\psi)v_1, R(\psi)v_2)$

$\Rightarrow R(\psi)^* = R(-\psi)^{-1}$ which proves (2)

On the other hand since $Z(k,\psi)$ and $R(\psi)$ commute,

$\eta(R(\psi)v_1, Z(k,\psi)^* v_2) = \eta(Z(k,\psi)v_1, R(-\psi)^{-1} v_2)$

$\overset{(*)}{=} \Omega^+(v_1 \otimes R(-\psi)^{-1} v_2)$

Similarly, $\widetilde{\Omega}^+(v_1 \cdot v_2) = \eta(\underbrace{R(\psi) Z(k,\psi)}_{\text{commute}} v_1, v_2) \overset{(*)}{=} \Omega^+(R(\psi)v_1 \otimes v_2)$

which proves (1)

PAGE

DATE

(18)

Thm. Let $i_{g,n} : \mathbb{C}[k_j]_{j\geq 1} \to H^\cdot(\mathcal{M}_{g,n})$ as before

then $\Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) = i_{g,n}\, \tilde{\Omega}^+(\alpha^g . R(\psi_1)^{-1} v_1 \cdots R(\psi_n)^{-1} v_n)(\kappa)$

Pf Consider the contiguous sewing $(G = \sum g_i)$

$S : \mathcal{M}_{g,n} \times \bigtimes_{i=1}^{n} \tilde{\mathcal{M}}_{g_i,2}^{\{2\}} \to \overline{\mathcal{M}_{g+G,n}}$ curve.

$S = S_n \circ \cdots \circ S_1$, S_i : sews the ith marked pt to a genus g_i

By sewing axiom $\times n$,

$\Omega_{g+G,n}((\alpha^{-g_1}.v_1)\otimes \cdots \otimes(\alpha^{-g_n}.v_n))|_{\partial N_S}$

$= \nu^n \sum_{\mu_1,\ldots,\mu_n} \Omega_{g,n}(e_{\mu_1}\otimes \cdots \otimes e_{\mu_n}) \times \bigtimes_{i=1}^{n} \Omega_{g_i,2}^{\{2\}}(e_{\mu_i} \otimes \alpha^{-g_i}.v_i)$

$\overset{\text{thm}}{\Leftrightarrow} \tilde{\Omega}^+(\alpha^g . v_1 \cdots v_n) = \sum_{\mu_1,\ldots,\mu_n} \Omega_{g,n}(e_{\mu_1}\otimes \cdots \otimes e_{\mu_n}) \prod_{i=1}^{n} \Omega^+(e_{\mu_i}, v_i)$

$= \sum_{\mu_1\cdots\mu_n} \Omega_{g,n}(e_{\mu_1}\otimes\cdots\otimes e_{\mu_n}) \prod_{i=1}^{n} \eta(\mathcal{Z}(\kappa_1^{(i)} - \underset{\shortparallel}{\psi_i})e_{\mu_i}, v_i)$

$\nu^n \psi_i'$ if ψ_i' is the ψ-class of free pt of $\tilde{\mathcal{M}}_{g_i,2}^{\{2\}}$

The kappa class on the left

is $\kappa^{(g,n)} + \kappa^{(1)} + \cdots + \kappa^{(i)} + \cdots + \kappa^{(n)}$

$\underset{H^\cdot(\mathcal{M}_{g,n})}{\Uparrow} \qquad \underset{H^\cdot(\mathcal{M}_{g_i,2})}{\Uparrow}$

Setting $\kappa^{(i)} = 0$ and we have

(RHS =) $\tilde{\Omega}^+(\alpha^g \cdot R(\psi_1)^{-1} v_1 \cdots \cdot R(\psi_n)^{-1} v_n)(\kappa^{(g,n)})$

$= \Omega_{g,n}(v_1 \otimes \cdots \otimes v_n)$

PAGE
DATE

(19)

Prop. Let $\widetilde{\Omega}^{+} : A \to \mathbb{C}[k_j]_{j\geq 1}$ group-like

$R(\psi) \in \mathrm{End}(A)[[\psi]]$ symplectic $(R(0) = id,\ R(\psi)^{*} = R(-\psi)^{-1})$

then Thm in (18) determines a free bdry CohFT

iff $\log \widetilde{\Omega}^{+}(v) = -\eta(\beta(\log R(\psi)^{-1}\mathbb{1}), v) \quad \forall v \in A$

where $\beta : A[[\psi]] \xrightarrow{A\text{-linear}} A((k_j))_{j\geq 1}$ by $\psi^j \mapsto k_j$

Pf. $\Rightarrow$: By forgetful map axiom,

$p^{*} i_{g,1} \widetilde{\Omega}^{+}(v) = i_{g,2} \widetilde{\Omega}^{+}(v \cdot R(\psi)^{-1}\mathbb{1})$ by the last thm

Since $p^{*}k_j = k_j - \psi_2^j$, we have

$p^{*} \exp(\sum_j \phi_j k_j) = \exp(\sum_j \phi_j k_j) \cdot \exp(-\sum_j \phi_j \psi_2^j)$

Thus $(\widetilde{\Omega}^{+} \cdot \exp(-\sum_j \phi_j \psi_2^j))(v) = \widetilde{\Omega}^{+}(v \cdot R(\psi_2)^{-1}\mathbb{1})$

$\Rightarrow R(\psi_2)^{-1}\mathbb{1} = \exp(-\sum_j \phi_j \psi_2^j)^{*}$

$\Rightarrow \eta(\beta \log R(\psi_2)^{-1}\mathbb{1}, v) = -\sum_j \phi_j k_j(v) = -\log \widetilde{\Omega}^{+}(v)$

$\Leftarrow$: Define $w_{g,n}(v_1 \otimes \cdots \otimes v_n) = \theta(\alpha^g . v_1 \cdots v_n)$

then w is a nodal CohFT (or any type).

If $\widetilde{\Omega}^{+} = \exp(\sum \phi_i k_i)$, then

$R(\psi)^{-1}\mathbb{1} = \exp(-\sum a_i \psi^i)$ with $a_i \in A$

s.t. $a_i = i^{-1}\phi_i \quad (\eta(a_i, v) = \theta(a_i \cdot v) = \phi_i(v))$

Givental's R action

Now by the lemma of κ class in (8)

$Rw|_{M_{g,n}} = \theta(\alpha^g \cdot R(\psi_1)^{-1}v_1 \cdots R(\psi_n)^{-1}v_n \cdot \sum_{m\geq 0} \frac{1}{m!}(p_m)_*(T(\psi_{n+1}) \cdots T(\psi_{n+m})))$

$= i_{g,n}\widetilde{\Omega}^{+}(\alpha^g \cdot R(\psi_1)^{-1}v_1 \cdots R(\psi_n)^{-1}v_n)$

thus this is a free bdry CohFT. $\quad T(z) = z(\mathbb{1} - R(z)^{-1}\mathbb{1})$

PAGE
DATE

(20) Nodal CohFT

Stratification of $\overline{\mathcal{M}}_{g,n}$

$\forall\, C \in \overline{\mathcal{M}}_{g,n}$, we call the comp. containing p_n special

The datum of (1) topological type, (2) # of marked pts and (3) nodes linking to other comp., of the special comp. is called the special type of C

Then $\overline{\mathcal{M}}_{g,n}$ is stratified by special types τ:

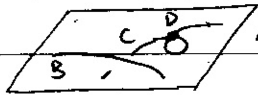
$$\overline{\mathcal{M}}_{g,n} = \coprod \mathcal{M}^{g,n}_{\tau}, \quad {}^{*}\mathcal{M}^{g,n}_{\tau} = \coprod_{i\in I} (\overline{\mathcal{M}}_i \times \mathcal{M}_{\gamma, v+k})/F_i$$

autom. — some spaces — genus — marked pts — outer nodes — if the special comp is smooth

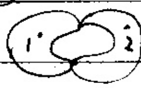
Define $\tau' > \tau$ if $\overline{\mathcal{M}}_{\tau'} \cap \mathcal{M}_{\tau} \neq \phi$

$\mathcal{U}_\tau = \bigcup_{\tau' \geq \tau} \mathcal{M}_{\tau'}$ open in $\overline{\mathcal{M}}_{g,n}$

Example. $\overline{\mathcal{M}}_{1,2}$:



A =    C =  $B \cap \overline{C}$ =  $\in B$

B =    D =  $\begin{matrix} B \leq \\ D \leq \end{matrix} C \leq A$

Lemma. Below degree $r/3$, the chern class of the normal bundle V_τ of $\mathcal{M}_\tau \subset \mathcal{U}_\tau$ is not a zero-divisor (not ann. by deg $\leq r/3$ element) in stable range

Pf. By Looijenga's thm, $H^{\cdot}(\mathcal{M}_{\gamma, v+k}) \cong H^{\cdot}(\mathcal{M}_\gamma)[\psi_1, \cdots, \psi_{v+k}]$ free

thus by bdry lemma $c(v_\tau) = -\sum_{i\in I} - \sum_{j=1}^{k} (\psi_i' \otimes 1 + 1 \otimes \psi_{v+i})$

by the above decomposition*, Thus $c(v_\tau)$ is non-div. in stable range

Lemma. If $j: S \to M$ closed smooth cpx sbmfld, $v_S: N \to S$ normal bundle

then if $[\alpha_1][\alpha_2] \in H^k(M)$ s.t. $[\alpha_1]|_S = [\alpha_2]|_S$, $[\alpha_1]|_{S^c} = [\alpha_2]|_{S^c}$,

then $[\alpha_1] - [\alpha_2] \in j_* \operatorname{Ann}_{H^{k-2d}(S)}(c(v_S))$

(omit) d codim of j

PAGE
DATE

21

Thm. A nodal CohFT $\bar{\Omega}$ is uniquely determined by its restriction Ω to $M_{g,n}$.

Pf. We do induction on $d = 3g-3+n$

$d=0$: $\bar{\Omega}_{0,3}(v_1 \otimes v_2 \otimes v_3) = \eta(v_1 \cdot v_2, v_3) = \Omega_{0,3}(v_1 \otimes v_2 \otimes v_3)$

$d \geq 1$: For g,n with $3g-3+n=d$, let $G > 9g-9+n$

Consider $s: \bar{M}_{g,n} \times M_{G,2} \to \bar{M}_{g+G,n}$

sewing p_n on the first and p_2 on the second

$N \to S = \operatorname{im} s$ the tubular nbhd of S

I be the set of strata τ s.t. $M_\tau^{g+G,n} \cap \partial N \neq \phi$

$\mathcal{U} = \bigcup_{\tau \in I} M_\tau^{g+G,n} \subset \bar{M}_{g+G,n}$

then $\mathcal{U}$ has a stratification with every stratum has special comp. (comp with p_n) of genus $\geq g$

and $\mathcal{U} \subset \mathcal{U}_{\tau_0}$ where τ_0 is the stratum corr. to $(r, v, k) = (G, 1, 1)$.

Lemma. If τ_0 is any stratum on $M_{g,n}$, and res. of $\bar{\Omega}_{g,n}$ on $M_\tau \subseteq \bar{M}_{g,n}$ with $\tau \geq \tau_0$ is known

then " on $U_\tau \subseteq \bar{M}_{g,n}$ " is known in deg $\leq r/3$

Pf. Induction on order of τ

If τ is biggest, then $M_\tau = \mathcal{U}_\tau$ done

If proved for $\tau \in I$ any union of intervals containing the biggest type, then for $\tau' \notin I$ maximal,

by hypothesis, res. of $\bar{\Omega}$ on $\mathcal{U}_{\tau'} \setminus M_{\tau'}$ is determined up to deg $r/3$

we then patch the theory on $M_{\tau'}$ and $\mathcal{U}_{\tau'} \setminus M_{\tau'}$ by the previous two lemmas in 20 *

PAGE
DATE

(22)

Now by the lemma, since $\mathcal{M}_{\tau}^{g+G,n}$ is determined for any $\tau \geq \tau_0$ (Since they are product of a smooth moduli some $\overline{\mathcal{M}}_{g',n'}$ with $3g'-3+n'<d$ (the decomp. ✿), by sewing axiom),

$\Omega_{\mathcal{U}}$ (restriction to $\mathcal{U} \subset \mathcal{H}_{\tau_0}$) is also determined in $\deg < G/3$

$\Rightarrow$ Restriction of $\overline{\Omega}_{g+G,n}$ to $\partial N \subset U$ is also uniquely determined

Moreover $H^{\bullet}(\partial N) \overset{\iota^*}{\underset{\leftarrow}{\cong}} H^{\bullet}(\overline{\mathcal{M}}_{g,n}) \otimes H^{\bullet}(\mathcal{M}_G)[\psi_1']$ in $\deg < G/3$ by Gysin sequence

Thus by sewing axiom apply to $\overline{\Omega}_{g+G,n}(v_1 \otimes \cdots \otimes v_n)|_{\partial N}$

$\Rightarrow \eta^{\mu\nu} \overline{\Omega}_{g,n}(v_1 \otimes \cdots \otimes e_\mu) \times \Omega_{G,2}(v_n, e_\nu)$ is determined in $\deg < G/3$

Restricting to any $\Sigma \in \mathcal{M}_{G,2}$, we have $\overline{\Omega}_{g,n}(v_1 \otimes \cdots \otimes \alpha^G \cdot v_n)$ is determined in $\deg < G/3$ as before, Since α is invertible and $G/3 > 3g-3+n$, we're done. ∎