X: smooth proj var. of dim D. $H = H^*(X,\Lambda)$. $\mathcal{H} = H((z^{-1})) = H[z] \oplus H[[z^{-1}]]$

$\Lambda = \mathbb{Q}, \mathbb{Q}[[Q]] \ (\otimes \mathbb{Q}(\lambda))$, $\quad H[z] = \mathcal{H}_+$, $\quad H[[z^{-1}]] = \mathcal{H}_-$

$$(q_\alpha q_\beta)^\wedge = \frac{q_\alpha q_\beta}{\hbar}, \quad (q_\alpha p_\beta)^\wedge = q_\alpha \frac{\partial}{\partial q_\beta}, \quad (p_\alpha p_\beta)^\wedge = \hbar \frac{\partial^2}{\partial q_\alpha \partial q_\beta}.$$

$$\Rightarrow [\hat{F}, \hat{G}] = \{F, G\}^\wedge + C(F,G), \qquad C(p_\alpha p_\beta, q_\alpha q_\beta) = \delta_{\alpha\beta} + (-1)^{\bar{q}_\alpha \bar{p}_\beta}.$$

$$\{F,G\} = \sum_\alpha \left((-1)^{\bar{q}_\alpha \bar{F}} \frac{\partial F}{\partial q_\alpha} \frac{\partial G}{\partial p_\alpha} - (-1)^{\bar{p}_\alpha \bar{F}} \frac{\partial F}{\partial p_\alpha} \frac{\partial G}{\partial q_\alpha} \right)$$

T infinitesimal symplectic trans $\rightsquigarrow h_T(f) = \frac{1}{2} \Omega(Tf, f)$

A, B self-adjoint op. on H $\Rightarrow$ A/z, Bz infinitesimal symplectic trans.

$$C(h_{A/z}, h_{Bz}) = C\Big(-\tfrac{1}{2} q_0^\mu q_0^\nu A_{\mu\nu} - \sum_{k\geq 0} q_{(k+1)}^\mu p_{k\nu} A_\mu^\nu, -\tfrac{1}{2} p_{0\mu} p_{0\nu} A^{\mu\nu} - \sum_{k\geq 0} q_k^\mu p_{(k+1)\nu} A_\mu^\nu\Big)$$
$$= \frac{1}{2} \operatorname{str}(AB).$$

Let $c(-) = \exp\left(\sum_{k=0}^\infty s_k ch_k(-)\right)$, E v.b. $/X$. $\rightsquigarrow E_{g,n,\beta} = R\pi_* e_{n+1}^* E / \overline{M}_{g,n}(X,\beta)$

Define
$$F^g_{c,E}(t_0, t_1, \dots) = \sum_{n,\beta} \frac{Q^\beta}{n!} \int_{c(E_{g,n,\beta}) \cap [\overline{M}_{g,n}(X,\beta)]} \bigwedge_{i=1}^n \left(\sum_{k=0}^\infty e_i^* t_k \cdot \psi_i^k \right)$$

$$D_{c,E}(t) = \exp\left(\sum_g \hbar^{g-1} F^g_{c,E}(t)\right), \qquad t = t_0 + t_1 z + t_2 z^2 + \cdots$$

twisted dilaton shift: $q(z) = \sqrt{c(E)} \cdot (t(z) - z) \in \mathcal{H}_+$

Define $\Delta : \mathcal{H} \to \mathcal{H}$ by an asymptotic expansion of $\sqrt{c(E)} \prod_{m=1}^\infty c(E \otimes L^{-m})$, where L is a line bundle with $c_1(L) = z$.

Thm (QRR) We have $\langle D_{c,E} \rangle = \hat{\Delta} \langle D_X \rangle$, where
$$\langle D_X \rangle = \exp\left(\sum_g \hbar^{g-1} F_X^g(t)\right)$$

$\rho_1 \dots \rho_r$: Chern roots of E. $\quad s(x) = \sum_{k\geq 0} s_k \frac{x^k}{k!}$ $\qquad e^{-mz\partial_x} s(x)\Big|_{x=\rho_i}$

$$\Rightarrow \log\Delta = \log\Big(\sqrt{c(E)} \prod_{m=1}^{\infty} c(E\otimes L^{-m})\Big) = \sum_i \Big(\frac{s(\rho_i)}{2} + \sum_{m=1}^{\infty} \underset{\|}{s(\rho_i - mz)}\Big)$$

$$\Big(\hat{\Delta} := e^{\widehat{\log\Delta}}\Big)$$

$$= \sum_i \Big(\frac{1}{1-e^{-z\partial_x}} - \frac{1}{2}\Big) s(x)\Big|_{x=\rho_i}$$

$$\sim \sum_i \Big(\sum_{m\geq 0} \frac{B_{2m}}{(2m)!} (z\partial_x)^{2m-1} s(x)\Big)_{x=\rho_i}$$

$$= \sum_{l,m\geq 0} \frac{B_{2m}}{(2m)!} s_{l+2m-1}\, ch_l(E)\, z^{2m-1}.$$

Thm (QRR')

$$\exp\Big(-\frac{1}{24}\sum_{l>0} s_{l-1}\int_X ch_l(E)\, c_{D-1}(X)\Big)\Big(\mathrm{sdet}\sqrt{c(E)}\Big)^{-\frac{1}{24}} \mathcal{D}_{C,E}$$

$$= \exp\Big(\sum_{m>0}\sum_{l\geq 0} \frac{B_{2m}}{(2m)!} s_{l+2m-1}\big(ch_l(E)\, z^{2m-1}\big)^{\wedge}\Big) \exp\Big(\sum_{l>0} s_{l-1}\Big(\frac{ch_l(E)}{z}\Big)^{\wedge}\Big) \mathcal{D}_X$$

pf. Consider

$$\begin{tikzcd}
D_i \arrow[r, phantom, "\subseteq"] & \overline{M}_{g,n+1}(X,\beta) \arrow[d, "\pi"] \arrow[r, phantom, "\supseteq"] & Z = \text{locus of nodes} \\
 & \overline{M}_{g,n}(X,\beta) \arrow[ul, "\sigma_i"] & \tilde{Z} \arrow[ul, "j"'] \arrow[u, "\text{double cover}"'] & L_\pm \arrow[l] \text{ cotangent bundles.}
\end{tikzcd}$$

$i = 1 \dots n$

$\psi_\pm := c_1(L_\pm)$

Claim. $ch_k(E_{g,n,\beta}) \cap [\overline{M}_{g,n}(X,\beta)] = \pi_*\Big(\sum_{r+l=k+1} \frac{B_r}{r!} ch_l(ev_{n+1}^* E)\cdot \Psi(r)\Big)$,

$$\Psi(r) = \psi_{n+1}^r \cap [\overline{M}_{g,n+1}(X,\beta)] - \sum_{i=1}^{n} (\sigma_i)_*\big(\psi_i^{r-1} \cap [\overline{M}_{g,n}(X,\beta)]\big)$$
$$+ \frac{1}{2} j_*\Big(\sum_{a+b=r-2} (-\psi_+)^a \psi_-^b \cap [\tilde{Z}]\Big).$$

pf of Claim. Assume that $\overline{M}_{g,n}(X,\beta)$, $\overline{M}_{g,n+1}(X,\beta)$, Z smooth, expected dim

$\pi(Z)$ normal crossing in $\overline{M}_{g,n}(X,\beta)$.

$$\mathrm{GRR} \Rightarrow \mathrm{ch}(E_{g,n,\beta}) = \pi_*\left(\mathrm{ch}(e_{n+1}^* E)\, \mathrm{Td}^\vee(\Omega_\pi)\right) \quad \left(\Omega_\pi = \Omega_{\overline{M}_{g,n+1}(X,\beta)/\overline{M}_{g,n}(X,\beta)}\right)$$

$$\begin{cases} 0 \to \Omega_\pi \to \omega_\pi \to \mathcal{O}_Z \to 0 \\ 0 \to \omega_\pi \to L_{n+1} \to \bigoplus D_i \to 0 \end{cases} \Rightarrow \Omega_\pi = L_{n+1} - \mathcal{O}_Z - \sum \mathcal{O}_{D_i}.$$

$$\Rightarrow \mathrm{Td}^\vee(\Omega_\pi) = \underbrace{\mathrm{Td}^\vee(L_{n+1})}_{=\,\psi_{n+1}/(e^{\psi_{n+1}}-1)} \mathrm{Td}^\vee(-\mathcal{O}_Z) \prod \mathrm{Td}^\vee(-\mathcal{O}_{D_i})$$

$\psi_{n+1}|_Z = \psi_{n+1}|_{D_i} = 0$. $Z, D_1 \cdots D_n$ disjoint.

$$\Rightarrow \mathrm{Td}^\vee(\Omega_\pi) - 1 = \left(\mathrm{Td}^\vee(L_{n+1}) - 1\right) + \left(\mathrm{Td}^\vee(-\mathcal{O}_Z) - 1\right) + \sum_i \left(\mathrm{Td}^\vee(\underbrace{-\mathcal{O}_{D_i}}_{=\,\mathcal{O}(-D_i)-\mathcal{O}}) - 1\right)$$

$$\mathrm{Td}^\vee(-\mathcal{O}_{D_i}) - 1 = \mathrm{Td}^\vee(\mathcal{O}(-D_i)) - 1 = \sum_{r\geq 1} \frac{B_r}{r!}(-D_i)^r$$

$$= -(\sigma_i)_* \sum_{r\geq 1} \frac{B_r}{r!} \psi_i^{r-1} \quad \left(\sigma_* \psi_i^{r-1} = \sigma_*\left(\sigma^*(-D_i)^{r-1} \cdot 1\right) = -(-D)^r\right)$$

$$N_{Z/\overline{M}_{g,n+1}(X,\beta)}\Big|_{\tilde{Z}} = L_+^\vee \oplus L_-^\vee \text{ and } \exists \text{ nbd } U \text{ of } \tilde{Z} \xrightarrow{2-1} \text{nbd of } Z$$

$$U \subset L_+^\vee \oplus L_-^\vee, \qquad Z \subset \overline{M}_{g,n+1}(X,\beta)$$

Koszul complex: $0 \to L_+ \otimes L_- \to L_+ \oplus L_- \to \mathcal{O}_U \to \mathcal{O}_{\tilde{Z}} \to 0$

$$\Rightarrow \mathrm{Td}^\vee(-\mathcal{O}_{\tilde{Z}}) - 1 = \frac{1 - e^{-(\psi_+ + \psi_-)}}{\psi_+ + \psi_-} \cdot \frac{\psi_+}{1 - e^{-\psi_+}} \cdot \frac{\psi_-}{1 - e^{-\psi_-}} - 1$$

$$= \frac{\psi_+ \psi_-}{\psi_+ + \psi_-}\left(\frac{1}{1-e^{-\psi_+}} - \frac{1}{\psi_+} - \frac{1}{2} + \frac{1}{1-e^{-\psi_-}} - \frac{1}{\psi_-} - \frac{1}{2}\right)$$

$$= \sum_{r\geq 2} \frac{B_r}{r!} \frac{\psi_+^{r-1} + \psi_-^{r-1}}{\psi_+ + \psi_-} \quad /\!/ \sum_{a+b=r-2} (-\psi_+)^a \psi_-^b.$$

$$\Rightarrow \mathrm{Td}^\vee(-\mathcal{O}_Z) - 1 = \frac{1}{2} j_* \left(\sum_{r\geq 2} \frac{B_r}{r!} \boxed{\frac{\psi_+^{r-1} + \psi_-^{r-1}}{\psi_+ + \psi_-}}\right)$$

General case: consider
$$\begin{tikzcd} \overline{M}_{g,n+1}(X,\beta) \arrow[r, hook] \arrow[d, "\pi"'] & C \arrow[d, "\bar{\pi}"] \\ \overline{M}_{g,n}(X,\beta) \arrow[r, hook] & \mathcal{M} \end{tikzcd}$$

Define $\langle \underset{\sum a^i_j \psi^j}{\overset{}{a^1(\psi)}}, \dots, a^n(\psi) ; \gamma \rangle_{g,n,\beta} = \int_{[\overline{M}_{g,n}(X,\beta)]} \bigwedge_{i=1}^{n} \Big(\sum_{j\geq 0} e_i^*(a^i_j)\, \psi_i^j \Big) \wedge \gamma$

$$\Rightarrow D_{C,E} = \exp\Big(\sum_{g,n,\beta} \frac{\hbar^{g-1} Q^\beta}{n!} \langle t(\psi)^{\otimes n} ; c(E_{g,n,\beta}) \rangle \Big)$$

base cases:

- $\langle t(\psi)^{\otimes 2}, ch_{k+1}(E) ; c(E_{0,3,0}) \rangle_{0,3,0} = \int_X t_0^2 \cdot ch_{k+1}(E) \cdot c(E)$
- $\langle ch_k(E)\psi ; c(E_{1,1,0}) \rangle_{1,1,0} = \frac{1}{24} \int_X ch_k(E) \cdot e(X)$ $\quad \left(\begin{array}{l} [\overline{M}_{1,1}(X,0)] = e(T_X \otimes \mathcal{H}^\vee_{1,1}) \\ E_{1,1,0} = p_1^* E \otimes (1 - p_2^* \mathcal{H}^\vee_{1,1}) \end{array} \right)$
- $\langle ch_{k+1}(E) ; c(E_{1,1,0}) \rangle_{1,1,0} = \frac{1}{24} \int_X ch_{k+1}(E) \Big(e(X) \sum_{j\geq 1} s_j\, ch_{j-1}(E) - c_{D-1}(X) \Big)$

At $s = (0,0,\dots)$, QRR' is trivial: $D_{C,E} = D_X$.

So $\exp\Big(-\frac{1}{24} \sum_{l>0} s_{l-1} \int_X ch_l(E)\, c_{D-1}(X) \Big) \big(\mathrm{sdet} \sqrt{c(E)} \big)^{-\frac{1}{24}} D_{C,E}$

$$= \exp\Big(\sum_{m>0} \sum_{l\geq 0} \frac{B_{2m}}{(2m)!} s_{l+2m-1} \big(ch_l(E)\, z^{2m-1} \big)^\wedge \Big) \exp\Big(\sum_{l>0} s_{l-1} \Big(\frac{ch_l(E)}{z} \Big)^\wedge \Big) D_X$$

$\Leftrightarrow \forall k,\ -\frac{1}{24} \int_X ch_{k+1}(E)\, c_{D-1}(X) - \frac{1}{24} \partial_{s_k} \log \mathrm{sdet} \sqrt{c(E)} + D_{C,E}^{-1} \partial_{s_k} D_{C,E}$

$$= D_{C,E}^{-1} \sum_{l+2m=k+1} \frac{B_{2m}}{(2m)!} \big(ch_l(E)\, z^{2m-1} \big)^\wedge D_{C,E} + C\Big(\frac{B_2}{2!} \sum_{l\geq 0} s_{l-1} \big(ch_l(E)\, z \big)^\wedge, \Big(\frac{ch_{k+1}(E)}{z} \Big)^\wedge \Big)$$

$$\left(\begin{array}{l} \log \mathrm{sdet} \sqrt{c(E)} = \mathrm{str} \log \sqrt{c(E)} = \int_X e(X) \cdot \frac{1}{2} \sum_{l\geq 0} s_l\, ch_l(E) \\ C\Big(\frac{B_2}{2!} \sum_{l>0} s_{l-1} (ch_l(E)\, z)^\wedge, \Big(\frac{ch_{k+1}(E)}{z} \Big)^\wedge \Big) = \frac{1}{24} \int_X e(X) \Big(\sum_{l>0} s_{l-1}\, ch_l(E) \Big) ch_{k+1}(E) \end{array} \right)$$

$$\Leftrightarrow D_{C,E}^{-1} \partial_{s_k} D_{C,E} = D_{C,E}^{-1} \sum_{l+2m=k+1} \frac{B_{2m}}{(2m)!} \big(ch_l(E)\, z^{2m-1} \big)^\wedge D_{C,E} + \frac{1}{48} \int_X e(X)\, ch_k(E)$$

$$+ \frac{1}{24} \int_X ch_{k+1}(E) \Big(c_{D-1}(T_X) - e(X) \sum_{l\geq 0} s_{l+1}\, ch_l(E) \Big)$$

$$D_{C.E}^{-1}\partial_{s_k}D_{C.E} = \partial_{s_k}\sum_{g,n,\beta}\frac{\hbar^{g-1}Q^\beta}{n!}\langle t(\psi)^{\otimes n}; c(E_{g,n,\beta})\rangle$$

$$= \sum_{g,n,\beta}\frac{\hbar^{g-1}Q^\beta}{(n-1)!}\langle t(\psi)^{\otimes(n-1)}, \partial_{s_k}t(\psi); c(E_{g,n,\beta})\rangle$$

$$+ \frac{\hbar^{g-1}Q^\beta}{n!}\underbrace{\langle t(\psi)^{\otimes n}; ch_k(E_{g,n,\beta})\, c(E_{g,n,\beta})\rangle}_{\parallel}$$

$$\left(\begin{array}{l}\Psi(r) = \psi_{n+1}^r - \sum_i (\sigma_i)_* \psi_i^{r-1} \\ \quad + \frac{1}{2}j_*\left(\sum_{a+b=r-2}(-\psi_+)^a\psi_-^b\right)\end{array}\right) \qquad \int_{\pi_*\left(\sum_{r+l=k+1}\frac{B_r}{r!}ch_l(e_{n+1}^*E)\cdot\Psi(r)\right)}\left(\bigwedge_i e_i^* t_j\cdot\psi_i^j\right) c(E_{g,n,\beta})$$

$$= \sum\frac{B_r}{r!}\langle t(\psi)^{\otimes n}; \pi_*(e_{n+1}^* ch_l(E)\cdot\Psi(r))\, c(E_{g,n,\beta})\rangle.$$

$$r=0 \rightsquigarrow \sum_{g,n,\beta}\frac{\hbar^{g-1}Q^\beta}{n!}\underbrace{\langle t(\psi)^{\otimes n}; \pi_*(e_{n+1}^* ch_{k+1}(E))\, c(E_{g,n,\beta})}_{\parallel}\rangle \qquad \psi_i = \pi^*\psi_i + D_i$$

$$\langle t(\psi)^{\otimes n}, ch_{k+1}(E); c(E_{g,n+1,\beta})\rangle - \sum_{i=1}^n\langle t(\psi)^{\otimes(i-1)}, ch_{k+1}(E)\left(\frac{t(\psi)}{\psi}\right)_+, t(\psi)^{\otimes(n-i)}; c(E_{g,n,\beta})\rangle$$

$$= -\sum_{g,n,\beta}\frac{\hbar^{g-1}Q^\beta}{(n-1)!}\langle t(\psi)^{\otimes(n-1)}, ch_{k+1}(E)\left(\frac{t(\psi)-\psi}{\psi}\right)_+; c(E_{g,n,\beta})\rangle$$

$$-\frac{1}{2\hbar}\langle t(\psi)^{\otimes 2}, ch_{k+1}(E); c(E_{0,3,0})\rangle - \langle ch_{k+1}(E); c(E_{1,1,0})\rangle \qquad (*)$$

Calculate $D_{C.E}^{-1}\widehat{\left(\frac{ch_{k+1}(E)}{z}\right)}D_{C.E}$: $\qquad \sum q_i z^i = \sqrt{c(E)}\,(t(z)-z)$.

$h_{ch_{k+1}(E)/z}$ has pq, q^2 - terms : $\qquad \widehat{pq} \rightsquigarrow q(z) \mapsto \left(-\frac{ch_{k+1}(E)}{z}\cdot q(z)\right)_+$

$$\left(h_{A/z} = -\frac{1}{2}q_0^\mu q_0^\nu A_{\mu\nu} - \sum_{i=0}^\infty q_{i+1}^\mu p_{i\nu}A_\mu^\nu\right) \qquad \widehat{q^2} \rightsquigarrow -(ch_{k+1}(E)\,q_0\cdot q_0)/2$$

Get $-\sum_{g,n,\beta}\frac{\hbar^{g-1}Q^\beta}{(n-1)!}\langle t(\psi)^{\otimes(n-1)}, ch_{k+1}(E)\left(\frac{t(\psi)-\psi}{\psi}\right)_+; c(E_{g,n,\beta})\rangle$

$-\frac{1}{2\hbar}\int_X t_0^2\cdot ch_{k+1}(E)\,c(E) \rightsquigarrow$ 1st & 2nd terms of $(*)$.

3rd term of $(*)$ $= -\frac{1}{24}\int_X ch_{k+1}(E)\left(e(X)\sum_{l\geq 1}s_l\, ch_{l-1}(E) - c_{D-1}(X)\right)$

$$r=1 \ \rightsquigarrow \ B_1 \sum_{g,n,\beta} \frac{\hbar^{g-1}Q^{\beta}}{n!} \underbrace{\langle t(\psi)^{\otimes n} ; \pi_*\left(e_{n+1}^* ch_k(E)(\psi_{n+1} - \textstyle\sum D_i)\right) c(E_{g,n,\beta})\rangle}_{\shortparallel}$$

$$\shortparallel$$

$$-\frac{1}{2} \qquad \langle t(\psi)^{\otimes n}, ch_k(E)\psi ; c(E_{g,n+1,\beta})\rangle - n\langle t(\psi)^{\otimes(n-1)}, t(\psi) ch_k(E) ; c(E_{g,n,\beta})\rangle$$

$$= \frac{1}{2}\sum_{g,n,\beta} \frac{\hbar^{g-1}Q^{\beta}}{(n-1)!} \langle t(\psi)^{\otimes(n-1)}, ch_k(E)(t(\psi)-\psi) ; c(E_{g,n,\beta})\rangle$$

$$+ \frac{1}{4\hbar}\langle t(\psi)^{\otimes 2}, ch_k(E)\psi ; c(E_{0,3,0})\rangle + \frac{1}{2}\langle ch_k(E)\psi ; c(E_{1,1,0})\rangle$$

$$= -\sum_{g,n,\beta} \frac{\hbar^{g-1}Q^{\beta}}{(n-1)!} \langle t(\psi)^{\otimes(n-1)}, \partial_{s_k} t(\psi) ; c(E_{g,n,\beta})\rangle + \frac{1}{48}\int_X e(X)\, ch_k(E).$$

$$\left(\partial_k t(z) = \partial_k\left(\frac{q(z)}{\sqrt{c(E)}} + z\right) = -\frac{1}{2} ch_k(E) \frac{q(z)}{\sqrt{c(E)}} = -\frac{1}{2} ch_k(E)(t(z)-z)\right)$$

$$r=2m \ \rightsquigarrow \ \frac{B_{2m}}{(2m)!}\sum_{g,n,\beta} \frac{\hbar^{g-1}Q^{\beta}}{n!} \langle t(\psi)^{\otimes n} ; \pi_*\left(e_{n+1}^* ch_{k+1-2m}(E)\, \Psi(2m)\right) c(E_{g,n,\beta})\rangle$$

$$= \frac{B_{2m}}{(2m)!}\sum_{g,n,\beta}\Bigg(-\frac{\hbar^{g-1}Q^{\beta}}{(n-1)!}\langle t(\psi)^{\otimes(n-1)}, ch_{k+1-2m}(E)\,\psi^{2m-1}(t(\psi)-\psi) ; c(E_{g,n,\beta})\rangle$$

$$+ \frac{1}{2}\underbrace{\frac{\hbar^{g-1}Q^{\beta}}{n!}\langle t(\psi)^{\otimes n} ; \pi_*\left(e_{n+1}^* ch_{k+1-2m}(E) j_* \textstyle\sum (-\psi_+)^a \psi_-^b\right) c(E_{g,n,\beta})\rangle}_{(*)}\Bigg)$$

$$\tilde{Z} = \bigsqcup_{g_i, I_i, \beta_i} \gamma_{red}\left(\overline{\mathcal{M}}_{g_1, I_1+\bullet}(X,\beta_1) \times_X \overline{\mathcal{M}}_{0,1+\bullet+\bullet}(X,0) \times_X \overline{\mathcal{M}}_{g_2, I_2+\bullet}(X,\beta_2)\right)$$

$$\sqcup\ \gamma_{irr}\left(\overline{\mathcal{M}}_{g-1, n+\bullet+\bullet}(X,\beta) \times_{X\times X} \overline{\mathcal{M}}_{0,1+\bullet+\bullet}(X,0)\right)$$

$$\begin{cases} \gamma_{red}^* j^* E_{g,n,\beta} = pr_1^* E_{g_1, I_1+\bullet, \beta_1} + pr_2^* E_{g_2, I_2+\bullet, \beta_2} - e_\Delta^* E \\ \gamma_{irr}^* j^* E_{g,n,\beta} = pr^* E_{g-1, n+\bullet+\bullet, \beta} - e_\Delta^* E \end{cases}$$

$$\Rightarrow (*) = \frac{1}{2}\sum_{a+b=2m-2}\sum_{g_i, n_i, \beta_i} \frac{\hbar^{g_1+g_2-1}Q^{\beta_1+\beta_2}}{n_1! n_2!}\langle t(\psi)^{\otimes n_1}, ch_{k+1-2m}(E) T_\mu (-\psi)^a ; \frac{c(E_{g_1,n_1+1,\beta_1})}{\sqrt{c(e_{n_1+1}^* E)}}\rangle$$

$$\langle t(\psi)^{\otimes n_2}, T^\mu \psi^b ; \frac{c(E_{g_2,n_2+1,\beta_2})}{\sqrt{c(e_{n_2+1}^* E)}}\rangle$$

$$+ \sum_{g,n,\beta}\frac{\hbar^{g-1}Q^{\beta}}{n!}\langle t(\psi)^{\otimes n}, ch_{k+1-2m}(E) T_\mu (-\psi)^a, T^\mu \psi^b ; \frac{c(E_{g-1,n+2,\beta})}{\sqrt{c(e_{n+1}^* E)}\sqrt{c(e_{n+2}^* E)}}\rangle$$

$$= \frac{\hbar}{2} \sum_{a+b=2m-2} \left(\sum_{g,n,\beta} \frac{\hbar^{g-1} Q^\beta}{(n-1)!} \left\langle t(\psi)^{\otimes(n-1)}, ch_{k+l-2m}(E)\, T_\mu (-\psi)^a ; \frac{c(E_{g,n,\beta})}{\sqrt{c(e_n^* E)}} \right\rangle \right)$$

$$\left(\sum_{g,n,\beta} \frac{\hbar^{g-1} Q^\beta}{(n-1)!} \left\langle t(\psi)^{\otimes(n-1)}, T^\mu \psi^b ; \frac{c(E_{g,n,\beta})}{\sqrt{c(e_n^* E)}} \right\rangle \right)$$

$$+ \sum_{g,n,\beta} \frac{\hbar^{g-1} Q^\beta}{(n-2)!} \left\langle t(\psi)^{\otimes(n-2)}, ch_{k+l-2m}(E)\, T_\mu (-\psi)^a, T^\mu \psi^b ; \frac{c(E_{g,n,\beta})}{\sqrt{c(e_n^* E)}\sqrt{c(e_{n-1}^* E)}} \right\rangle$$

Calculate $D_{c.E}^{-1} \left(ch_{k+l-2m}(E) z^{2m-1} \right)^\wedge D_{c.E}$:

$h_{ch_{k+l-2m} z^{2m-1}}$ has pq, p^2-terms. $\widehat{pq} \rightsquigarrow q(z) \mapsto - ch_{k+l-2m}(E)\, z^{2m-1} q(z)$.

$\hat{p}^2 \rightsquigarrow (*)$

$$\left(h_{B z^{2m-1}} = \frac{1}{2} \sum_{a+b=2m-2} (-1)^a p_{a\mu} p_{b\nu} B^{\mu\nu} - \sum_{i=0}^{\infty} q_i^\mu p_{(i+2m-1)\nu} B_\mu^\nu \right)$$

$$D_{c.E}^{-1} \frac{\partial^2}{\partial q_a \partial q_b} D_{c.E}$$

$$= \left(\partial_{q_a} \sum_{g,n,\beta} \frac{\hbar^{g-1} Q^\beta}{n!} \left\langle t(\psi)^{\otimes n} ; c(E_{g,n,\beta}) \right\rangle \right) \left(\partial_{q_b} \sum_{g,n,\beta} \frac{\hbar^{g-1} Q^\beta}{n!} \left\langle t(\psi)^{\otimes n} ; c(E_{g,n,\beta}) \right\rangle \right)$$

$$+ \partial_{q_a} \partial_{q_b} \sum_{g,n,\beta} \frac{\hbar^{g-1} Q^\beta}{n!} \left\langle t(\psi)^{\otimes n} ; c(E_{g,n,\beta}) \right\rangle$$

$$= \left(\sum_{g,n,\beta} \frac{\hbar^{g-1} Q^\beta}{(n-1)!} \left\langle t(\psi)^{\otimes(n-1)}, \psi^a ; \frac{c(E_{g,n,\beta})}{\sqrt{c(e_n^* E)}} \right\rangle \right) \left(\sum_{g,n,\beta} \frac{\hbar^{g-1} Q^\beta}{(n-1)!} \left\langle t(\psi)^{\otimes(n-1)}, \psi^b ; \frac{c(E_{g,n,\beta})}{\sqrt{c(e_n^* E)}} \right\rangle \right)$$

$$+ \sum_{g,n,\beta} \frac{\hbar^{g-1} Q^\beta}{(n-2)!} \left\langle t(\psi)^{\otimes(n-2)}, \psi^a, \psi^b ; \frac{c(E_{g,n,\beta})}{\sqrt{c(e_n^* E)}\sqrt{c(e_{n-1}^* E)}} \right\rangle \qquad \left(\partial_{q_c} t(z) = \frac{z^c}{\sqrt{c(E)}} \right)$$

$\square$

equivariant Euler class

$$e(E) = \prod_i \underbrace{(\lambda + p_i)}_{\parallel} = \exp\left(\log\lambda \cdot ch_0(E) + \sum_k \frac{(-1)^{k-1}(k-1)!}{\lambda^k} ch_k(E)\right)$$

$$\exp\left(\log\lambda - \sum_k \frac{(-p_i)^k}{k\lambda^k}\right)$$

Let $s_0 = \log\lambda$, $s_k = \dfrac{(-1)^{k-1}(k-1)!}{\lambda^k}$ $\rightsquigarrow$ $c(-) = \exp\left(\sum s_k ch_k(-)\right) = e(-)$.

$$D_{e,E} = \left(\operatorname{sdet}\sqrt{c(E)}\right)^{\frac{1}{24}} \exp\left(\frac{1}{24}\sum_{l>0} s_{l-1}\int_X ch_l(E)\, c_{D-1}(X)\right)$$

$$\exp\left(\sum_{m>0}\sum_{l\geq 0} \frac{B_{2m}}{(2m)!} s_{l+2m-1} \left(ch_l(E)\, z^{2m-1}\right)^\wedge\right) \exp\left(\sum_{l>0} s_{l-1}\left(\frac{ch_l(E)}{z}\right)^\wedge\right) D_X$$

$$= \prod_i \left(\operatorname{sdet}\sqrt{\lambda+p_i}\right)^{\frac{1}{24}} \exp\left(\frac{1}{24}\int_X (\lambda+p_i)(\log(\lambda+p_i)-1)\, c_{D-1}(X)\right)$$

$$\cdot \exp\left(\sum_{m>0} \frac{B_{2m}}{2m(2m-1)} \left(\left(\frac{z}{\lambda+p_i}\right)^{2m-1}\right)^\wedge\right) \exp\left(\left(\frac{(\lambda+p_i)(\log(\lambda+p_i)-1)}{z}\right)^\wedge\right) D_X$$

Let $c^\vee(-) = \exp\left(\sum (-1)^{k-1} s_k ch_k(-)\right)$ $\Rightarrow$ $c^\vee(E^\vee) = c(E)^{-1}$.

$t^\vee(z) := c(E)\, t(z) + (1 - c(E))\, z$

$$\left(\begin{aligned} \Rightarrow\ q^\vee(z) &= \sqrt{c^\vee(E^\vee)}\,(t^\vee(z) - z) \\ &= \sqrt{c(E)}\,(t(z) - z) = q(z) \end{aligned}\right)$$

$$D_{c^\vee,E^\vee}(t^\vee) = \left(\operatorname{sdet}\sqrt{c^\vee(E^\vee)}\right)^{\frac{1}{24}} \exp\left(\frac{1}{24}\sum_{l>0}(-1)^l s_{l-1}\int_X (-1)^l ch_l(E)\, c_{D-1}(X)\right)$$

$$\exp\left(\sum_{\substack{m>0\\ l\geq 0}} \frac{B_{2m}}{(2m)!} (-1)^{l+2m} s_{l+2m-1}\left((-1)^l ch_l(E)\, z^{2m-1}\right)^\wedge\right)$$

$$\exp\left(\sum_{l>0} (-1)^l s_{l-1}\left(\frac{(-1)^l ch_l(E)}{z}\right)^\wedge\right) D_X$$

$$= \left(\operatorname{sdet} c(E)\right)^{-\frac{1}{24}} D_{c,E}(t). \qquad \text{(Quantum Serre)}.$$

Let $e^{-1}(E) = \prod(-\lambda+p_i)^{-1} = \exp\left(\underbrace{-\log(-\lambda)}_{-s_0-\pi i.} ch_0(E) + \sum (-1)^{k-1} s_k ch_k(E)\right)$

$t^\vee(z) = z + (-1)^{\dim E/2} e(E)\,(t(z)-z)$

$$D_{e^{-1},E^\vee}(t^\vee) = \left(\mathrm{sdet}\,(-1)^{\dim E/2} e(E)\right)^{-\frac{1}{24}}$$

$$\cdot\exp\left(\frac{1}{24}(-\pi i)\int_X ch_1(E)\,c_{D-1}(X)\right)\exp\left(-\pi i\left(ch_1(E)/2\right)^\wedge\right) D_{e.E}(t).$$

$\left(\gamma\in H^2(X) \Rightarrow \widehat{\left(\frac{\gamma}{z}\right)} D_X = \left(\delta_\gamma - \frac{1}{24}\int_X \gamma\, c_{D-1}(X)\right) D_X\right)$

$$= \left(\mathrm{sdet}\,(-1)^{\dim E/2} e(E)\right)^{-\frac{1}{24}} D_{e.E}(t,\pm Q) \quad \left(\pm Q^\beta = (-1)^{\beta\cdot ch_1(E)} Q^\beta\right)$$

Let $L_{c.E} = \{(p,q) \mid p = d_q F^0_{c.E}\}$ $\left(\Rightarrow p_{k\mu} = \partial_{q^\mu_k} F^0_{c.E}(q)\right)$

In general, if $\exp\left(\sum \hbar^{g-1}\tilde{f}^g(q)\right) = \hat{A}\exp\left(\sum \hbar^{g-1} f^g(q)\right)$,

then $L_{\tilde f} = A(L_f)$ $\quad \forall A: \mathcal{H}\to\mathcal{H}$ symplectic.

Suffices to check infinitesimal case and $A = p_a p_b,\ q_a p_b,\ q_a q_b$

- $A = p_a p_b$: $\widehat{p_a p_b}\exp\left(\sum \hbar^{g-1} f^g(q)\right) = \hbar\Big(\sum \hbar^{g-1} f^g_{ab}(q)$
$$+\left(\sum\hbar^{g-1} f^g_a(q)\right)\left(\sum \hbar^{g-1} f^g_b(q)\right)\Big)$$
$$\cdot\exp\left(\sum \hbar^{g-1} f^g(q)\right)$$
$\Rightarrow \tilde f^0 = f^0 + \varepsilon\,(f^0_a f^0_b)$,
$\tilde q_k = q_k + \varepsilon(\delta_{ka} p_b + \delta_{kb} p_a)$, $\tilde p_k = p_k$
$\Rightarrow \frac{\partial \tilde f^0}{\partial \tilde q_k} = \left(f^0_l + \varepsilon(f^0_{la} f^0_b + f^0_a f^0_{lb})\right)\left(\delta_{kl} - \varepsilon(\delta_{la} f^0_{kb} + \delta_{lb} f^0_{ka})\right) = f^0_k = \tilde p_k$
- $A = q_a p_b$: $\tilde f^0 = f^0 + \varepsilon(q_a + f^0_b)$, $\tilde q_k = q_k + \varepsilon\delta_{kb} q_a$, $\tilde p_k = p_k - \varepsilon \delta_{ka} p_b$
$\Rightarrow \frac{\partial\tilde f^0}{\partial \tilde q_k} = \left(f^0_l + \varepsilon(\delta_{la} f^0_b + q_a f^0_{lb})\right)\left(\delta_{kl} - \varepsilon\delta_{lb}\delta_{ka}\right) = f^0_k - \varepsilon\delta_{ka} f^0_b$
- $A = q_a q_b$: $\tilde f^0 = f^0 + \varepsilon(q_a q_b)$, $\tilde q_k = q_k$, $\tilde p_k = p_k + \varepsilon(\delta_{ka} q_b + \delta_{kb} q_a)$
$\Rightarrow \frac{\partial \tilde f^0}{\partial \tilde q_k} = f^0_k + \varepsilon(\delta_{ka} q_b + q_a\delta_{kb}) = \tilde p_k$

$$\exp\left(-\frac{1}{24}\sum_{\ell>0} s_{\ell-1}\int_X ch_\ell(E)\, c_{D-1}(X)\right)\left(\mathrm{sdet}\sqrt{c(E)}\right)^{-\frac{1}{24}} D_{c.E}$$

$$= \exp\left(\sum_{m>0}\sum_{\ell\geq 0}\frac{B_{2m}}{(2m)!}\, s_{\ell+2m-1}\left(ch_\ell(E)\, z^{2m-1}\right)^\wedge\right)\exp\left(\sum_{\ell>0} s_{\ell-1}\left(\frac{ch_\ell(E)}{z}\right)^\wedge\right) D_X$$

$$\Rightarrow \mathcal{L}_{c.E} = \exp\left(\sum_{\ell,m\geq 0}\frac{B_{2m}}{(2m)!}\, s_{\ell+2m-1}\left(ch_\ell(E)\, z^{2m-1}\right)\right)\mathcal{L}_X .$$

For $c=e$, $\mathcal{L}_{e.E} = \prod_i b_{\rho_i}(z)\,\mathcal{L}_X$, where

$$b_\rho(z) = \exp\left(\frac{1}{z}(\lambda+\rho)(\log(\lambda+\rho)-1) + \underbrace{\sum_{m>0}\frac{B_{2m}}{2m(2m-1)}\left(\frac{z}{\lambda+\rho}\right)^{2m-1}}_{=\, S\left(\frac{\lambda+\rho}{z}\right)}\right)$$

$$\Gamma(s) = \sqrt{\frac{2\pi}{s}}\exp\left(s\log s - s + S(s)\right)$$

$$\Rightarrow b_\rho(z) = \underbrace{\Gamma\left(\frac{\lambda+\rho}{z}\right)}_{=\,\frac{z}{\lambda+\rho}\int_0^\infty t^{(\lambda+\rho)/z}e^{-t}dt}\cdot\sqrt{\frac{\lambda+\rho}{2\pi z}}\;\underbrace{\exp\left(\frac{(\lambda+\rho)\log z}{z}\right)}_{=\, z^{(\lambda+\rho)/z}} \overset{x=tz}{=\!=} \frac{1}{\sqrt{2\pi z(\lambda+\rho)}}\int_0^\infty x^{\frac{\lambda+\rho}{z}}e^{-\frac{x}{z}}dx.$$

Define $J_X(t,z) = z + t + \sum_{n,\beta}\frac{Q^\beta}{n!}\left\langle t^{\otimes n}, \frac{T_\mu}{z-\psi_{n+1}}\right\rangle T^\mu = \sum_\beta J_\beta(t,z)\,Q^\beta$.

$$I_E(t,z) = \sum_\beta J_\beta(t,z)\,Q^\beta\prod_{i=1}^{r}\prod_{k=1}^{\rho_i\cdot\beta}(\lambda+\rho_i+kz) \quad (\text{assume } E=\oplus L_i)$$

Thm. For each t, $I_E(t,-z)\in\tilde{\mathcal{L}}_{e.E}\subset(\mathcal{H},\Omega_{e(E)})$.

$\tilde{\mathcal{L}}_{e.E} = \frac{1}{\sqrt{e(E)}}\mathcal{L}_{e.E}$ $\qquad (a.b)_{e(E)} = \int_X e(E)\,ab$ on $\mathcal{H}$

pf. $J_X\left(t+\sum(\lambda+\rho_i)\log x_i, z\right) \underset{SE+DIV}{=} \prod_i x_i^{(\lambda+\rho_i)/z}\sum_\beta J_\beta(t,z)\,Q^\beta\prod_i x_i^{\rho_i\cdot\beta}$

$$\Rightarrow \int_{(\mathbb{R}^+)^r} e^{-\sum x_i/z} J_X\left(t+\sum(\lambda+\rho_i)\log x_i, z\right)dx$$

$$= \sum_\beta J_\beta(t,z)\,Q^\beta\prod_i\underbrace{\int_0^\infty e^{-x_i/z}x_i^{\frac{\lambda+\rho_i}{z}+\rho_i\cdot\beta}dz}_{\overset{IBP}{=}\;\prod_{k=1}^{\rho_i\cdot\beta}(\lambda+\rho_i+kz)\int_0^\infty x_i^{\frac{\lambda+\rho_i}{z}}e^{-\frac{x_i}{z}}dz} = I_E(t,z)\underbrace{\prod_i\sqrt{2\pi z(\lambda+\rho)}}_{=\,(2\pi z)^{r/2}\sqrt{e(E)}}\;\underbrace{b_\rho(z)}_{=\,\prod_i b_{\rho_i}(z)}$$

$$\begin{array}{ccccc}
(\mathcal{H}, \Omega_{e(E)}) & \xrightarrow{\sqrt{e(E)}} & (\mathcal{H}, \Omega) & & \\
\cup\!\!\!\mid & & \cup\!\!\!\mid & & \\
\widetilde{\mathcal{L}}_{e,E} & \longrightarrow & \mathcal{L}_{e,E} & & \\
\ni & & \ni & & \\
I_E(t,-z) & \Longleftrightarrow & I_E(t,-z)\sqrt{e(E)} & \Leftrightarrow & I_E(t,-z)\sqrt{e(E)}\prod b_{\rho_i}(-z) \in \mathcal{L}_X
\end{array}$$

Suffices to show

$$(2\pi z)^{-\frac{r}{2}} \int_{(\mathbb{R}^+)^r} e^{-\Sigma x_i/z} J_X\left(t + \sum (\lambda+\rho_i)\log x_i, z\right) dx \in \mathcal{L}_X$$

$$= \prod_i \left(\frac{1}{\sqrt{2\pi z}} \int_0^\infty e^{-\frac{x_i}{z} + \underbrace{\log x_i(\lambda\partial^1 + \partial^{\rho_i})}_{= \frac{1}{z}\log x_i(\lambda + z\partial^{\rho_i})}} dx_i \right) J_X(t,z) \qquad (*)$$

$$z\partial_\alpha\partial_\beta J_X(t,z) = \sum_\gamma A^\gamma_{\alpha\beta}(t)\,\partial_\gamma J_X(t,z) \qquad \langle z\partial_\mu J_X(t,z)\rangle_{Z\Lambda[[z]]} \ni$$

$$\Rightarrow (z\partial^{T_{i_1}})\cdots(z\partial^{T_{i_n}}) J_X(t,z) = (z\partial^{T_{i_1}*\cdots*T_{i_n}}) J_X(t,z) + O(z)$$

$$\frac{1}{\sqrt{2\pi z}} \int_0^\infty e^{\frac{1}{z}(-x + \log x(\lambda + z\partial^\rho))} dx \cdot J_X(t,z)$$

$$= \sqrt{\lambda + z\partial^\rho} \cdot \exp\left(\frac{1}{z}(\lambda + z\partial^\rho)(\log(\lambda + z\partial^\rho) - 1) + \sum_{m>0} \frac{B_{2m}}{2m(2m-1)} \left(\frac{z}{\lambda + z\partial^\rho}\right)^{2m-1} \right) J_X(t,z)$$

$$= \underbrace{\sqrt{\lambda + z\partial^\rho}}_{= \sqrt{\lambda}\left(1 + \frac{z\partial^\rho}{2\lambda} + \cdots\right)} \cdot \exp\Big(\underbrace{\frac{1}{z}(\lambda + z\partial^\rho)(\log(\lambda + z\partial^\rho) - 1)}_{= \partial^{(\lambda+\rho *)(\log(\lambda+\rho *)-1)1}}\Big)\left(J_X(t,z) + O(z)\right)$$

$$\Rightarrow (*) = \lambda^{\dim E/2} J_X(t^*, z) + C^\mu(t^*, z)\, z\partial_\mu J_X(t^*, z),$$

where $t^* = t + \sum_i (\lambda + \rho_i *)(\log(\lambda + \rho_i *) - 1)1$.

Fact. $\mathcal{L}_X$ is a cone ruled by $zT_f\mathcal{L}_X$. ($\Leftrightarrow$ $F^0_{e,E}$ satisfies TRR, SE and DE)

By def. $J_X(t,-z) \in \mathcal{L}_X$ $\left(P_{k\mu} = \sum \frac{Q^\beta}{n!}\langle t^{\otimes n}, T_\mu\psi^k\rangle = \partial_{q^\mu_k} F^0_X\right)$

$$\Rightarrow \lambda^{\dim E/2} J_X(t^*, -z) - C^\mu(t^*, -z)\, z\partial_\mu J_X(t^*, -z) \in \mathcal{L}_X \qquad \square$$

Thm $\mathcal{L} = \{(p,q) \mid p = d_q F\}$ is ruled by $zT_f\mathcal{L}$ iff

F satisfies TRR : $\partial_\alpha^{k+1}\partial_\beta^l\partial_\gamma^m F = (\partial_\alpha^k\partial^{0\mu}F)(\partial_\gamma^0\partial_\beta^l\partial_\gamma^m F)$,

SE : $\partial^{01}F = \frac{1}{2}(t_0,t_0) + \sum_{k\geq 0} t_{k+1}^\mu \partial_\mu^k F$,

DE : $\partial^{11}F = \sum_{k\geq 0} t_k^\mu \partial_\mu^k F - 2F$. $\quad (\partial_\mu^n := \partial_{t_n^\mu})$

$$\langle T_\alpha\psi^{k+1}, T_\beta\psi^l, T_\gamma\psi^m, t^{\otimes n}\rangle_\beta = \sum_{n_i,\beta_i}\frac{n!}{n_1!n_2!}\langle T_\alpha\psi^k, T^\mu, t^{\otimes n_1}\rangle_{\beta_1}\langle T_\mu, T_\beta\psi^l, T_\gamma\psi^m, t^{\otimes n_2}\rangle_{\beta_2}$$

$$\langle 1, t^{\otimes n}\rangle_\beta = \langle \left(\tfrac{t(\psi)}{\psi}\right)_+, t^{\otimes n}\rangle_\beta + \delta_{(n,\beta)(2,0)}\cdot\tfrac{1}{2}(t_0,t_0)$$

$$\langle \psi, t^{\otimes n}\rangle_\beta = (n-2)\langle t^{\otimes n}\rangle_\beta$$

pf. $(\Rightarrow)$ $F(q) = \underset{\substack{\shortparallel \\ 0}}{F(0)} + \int_0^1 \underset{\substack{\shortparallel \\ q_k^\mu \partial_{q_k^\mu} F(sq)}}{\underline{\partial_s F(sq)}}\, ds = \frac{1}{2}p\cdot q$. homog of deg 2

$\Rightarrow 2F = \sum q_k^\mu \partial_{q_k^\mu} F = \sum t_k^\mu \partial_\mu^k F - \partial^{11}F$ (DE).

$f \in zT_f\mathcal{L} \Rightarrow z^{-1}f \in T_f\mathcal{L} \Rightarrow \mathcal{L} \subseteq \{f \mid h_{z^{-1}}(f) = 0\}$.

$\Rightarrow 0 = -\frac{1}{2}q_0^\mu q_0^\nu g_{\mu\nu} - \sum_{k\geq 0} q_{k+1}^\mu p_{k\mu} = -\frac{1}{2}(t_0,t_0) - \sum_{k\geq 0} t_{k+1}^\mu \partial_\mu^k F + \partial^{01}F$ (SE)

Let $\tau^\delta = \partial^{0\delta}\partial^{01}F \overset{(SE)}{=} \partial^{0\delta}\left(\frac{1}{2}(t_0,t_0) + \sum_{k\geq 0} t_{k+1}^\mu\partial_\mu^k F\right) = t_0^\delta + \sum_{k\geq 0} t_{k+1}^\mu \partial_\mu^k\partial^{0\delta}F$.

$\mathcal{L}$ has constant tangent space L along zL and $\operatorname{codim}_L zL = \dim H$.

$L \pitchfork \mathcal{H}_- \Rightarrow zL \pitchfork z\mathcal{H}_- \Rightarrow (zL)_+ \pitchfork (q_1 = -1, q_{\geq 2} = 0)$, $(q_0 = \tau)$.

$\Rightarrow \{\tau^\delta\}$ gives a coor on $\mathcal{H}_+/(zL)_+ \Rightarrow \partial_\beta^l\partial_\gamma^m F(t) = \partial_\beta^l\partial_\gamma^m F\big|_{(\tau(t),0)}$

Get $\partial_\alpha^{k+1}\partial_\beta^l\partial_\gamma^m F(t) = \partial_\alpha^{k+1}\left(\partial_\beta^l\partial_\gamma^m F\big|_{(\tau,0)}\right) = \partial_\alpha^{k+1}\tau^\delta\cdot\partial_\delta^0\partial_\beta^l\partial_\gamma^m F\big|_{(\tau,0)}$

$= \partial_\alpha^k\partial^{0\delta}F(t)\cdot\partial_\delta^0\partial_\beta^l\partial_\gamma^m F(t)$. (TRR)

($\Leftarrow$) Define $\tau(t)$ by $G^\delta(\tau,t) = \tau^\delta - t_0^\delta - \sum_{k\geq 0} t_{k+1}^\mu \partial_\mu^k \partial^{0\delta} F|_{(\tau,0)} = 0$

$\left(\frac{\partial G^\delta}{\partial \tau^\lambda} = \delta_\lambda^\delta - \sum t_{k+1}^\mu \partial_\lambda^0 \partial_\mu^k \partial^{0\delta} F|_{(\tau,0)} \right.$ is inv near $\left. 0 \right)$

$\partial_\alpha^{k+1} (\partial_\beta^l \partial_\gamma^m F|_{(\tau,0)}) \overset{TRR}{=} \partial_\alpha^{k+1} \partial_\beta^l \partial_\gamma^m F(t) \Rightarrow \partial_\beta^l \partial_\gamma^m F|_{(\tau,0)} = \partial_\beta^l \partial_\gamma^m F(t)$

$\Rightarrow$ Along the fibers of $[t \mapsto \tau(t)]$, the tangent spaces of $\mathcal{L}$ are const.

DE $\Rightarrow$ F is homog of deg 2 $\Rightarrow$ $\mathcal{L}$ is homog of deg 1

So $\mathcal{L}$ is a cone and $f \in T_f\mathcal{L} =: L$

SE $\Rightarrow$ $z^{-1} f \in L$ $\Rightarrow$ $f \in L \cap zL$

$L \cap zL \supseteq \{ f \in \mathcal{L} \mid T_f \mathcal{L} = L \}$ $\Rightarrow$ $\underbrace{(zL)_+}_{} \supseteq \underbrace{\text{some fiber of } [t \mapsto \tau(t)]}_{}$

both codim = dim$H$ in $\mathcal{H}_+$

So $zL = \{ f \in \mathcal{L} \mid T_f \mathcal{L} = L \}$. $\square$

$$T_{J_X(t,-z)} \mathcal{L}_X = \langle v^k_\mu \rangle$$

$$\parallel$$

$$T_\mu z^k + \sum \partial_{q^\mu_k} \partial_{q^\nu_\ell} F^0_X \Big|_{q=t-z} \cdot \frac{T^\nu}{(-z)^{\ell+1}}$$

$\Rightarrow$ ruling is spanned by $z v^k_\mu = T_\mu z^{k+1} + O(1)$

$\partial_{t^\mu} J_X(t,-z) = T_\mu + O(z^{-1}) \Rightarrow t \mapsto J_X(t,-z)$ is transverse to the ruling

Cor. Let $J_{e.E}(t,z) = z + t + \sum_{n,\beta} \frac{Q^\beta}{n!} \langle t^{\otimes n}, \frac{T_\mu}{z-\psi_{n+1}} ; e(E_{0,n,\beta}) \rangle g^{\mu\nu}_{e(E)} T_\nu$.

Then $J_{e.E}(t,-z) \in \widetilde{\mathcal{L}}_{e.E}$ and

$J_{e.E}(\tau,z) = I_E(t,z) + c^\mu(t,z)\, z\partial_\mu I_E(t,z)$ for some $c^\mu(t,-) \in \Lambda\{z\}$.

pf. $I_E(t,z) \equiv J_0(t,z) \equiv J_X(t,z) \pmod{Q^\beta}$

$\Rightarrow t \mapsto I_E(t,-z)$ is transverse to the ruling $z\, \boxed{\underset{\substack{\parallel \\ L_t}}{T_{I_E(t,-z)} \widetilde{\mathcal{L}}_{e,E}}} \subseteq \widetilde{\mathcal{L}}_{e.E}$

$\Rightarrow zL_t \cap (-z + z\mathcal{H}_-)$ at 1 point $(p_t, q_t) \in \widetilde{\mathcal{L}}_{e.E} \subseteq (\mathcal{H}, \Omega_{e(E)})$.

Let $\tau = q_t + z \Rightarrow J_{e.E}(\tau,-z) = (p_t, q_t)$ $\left(J_{e.E}(\tau,-z) \in \widetilde{\mathcal{L}}_{e.E} \cap (-z+\tau+\mathcal{H}_-)\right)$

So $J_{e.E}(\tau,-z) = I_E(t,z) + c^\mu(t,z)\, z\partial_\mu I_E(t,z)$. $\square$

As $\lambda \to 0$, $J_{e.E} \to z + t + \sum_{n,\beta} \frac{Q^\beta}{n!} (e_{n+1})_* \left(\bigwedge_{i=1}^n e_i^* t \wedge \frac{e(E'_{0,n+1,\beta})}{z-\psi_{n+1}} \right)$,

where $E'_{0,n+1,\beta} = \ker(E_{0,n+1,\beta} \to e^*_{n+1} E)$.

$$\left(\int_{\overline{M}_{0,n+1}(X,\beta)} (*) \wedge \frac{e^*_{n+1} T_\mu}{z-\psi_{n+1}} \wedge e(E_{0,n+1,\beta}) = \int_X (e_{n+1})_* \left((*) \wedge \frac{e(E'_{0,n+1,\beta})}{z-\psi_{n+1}} \right) T_\mu \cdot e(E) \right)$$

Assume E is convex $\Rightarrow I_E \to \sum_\beta J_\beta(t,z)\, Q^\beta \prod_i \prod_{k=1}^{\rho_i\cdot\beta} (\rho_i + kz)$.

Y = zero locus of some global section of E

$\Rightarrow [\overline{\mathcal{M}}_{0,n+1}(Y,\beta)] = e(E_{0,n+1,\beta}) \cap [\overline{\mathcal{M}}_{0,n+1}(X,\beta)]$.

So $\underset{\underset{\lim_{\lambda\to 0} J_{e,E}}{\shortparallel}}{e(E)\, J_{X,Y}(t,z)} = j_* J_Y(j^* t, E)$ $\quad \left(I_{X,Y} := \lim_{\lambda\to 0} I_{e,E} \right)$

Cor. $I_{X,Y}(t,-z)$, $J_{X,Y}(\tau,-z)$ determine the same cone.

Restricting $I_{X,Y}$, $J_{X,Y}$ to $H^{\le 2}(X;\Lambda)$

Prop. If $c_1(E) \le c_1(X)$, then for $t \in H^{\le 2}(X,\Lambda)$,

$$I_{X,Y}(t,z) = z \underbrace{F(t)}_{\text{invertible}} + \sum_i G^i(t)\, T_i + O(z^{-1}).$$

pf. $I_{X,Y}(t,z) \equiv J_X(t,z) \pmod{Q^\beta}$

$$\Rightarrow I_{X,Y}(t,z) = z + t + \sum_{\beta\ne 0} \underbrace{J_\beta(t,z)}_{\sum_{n,k} \frac{1}{n!} \langle t^{\otimes n}, \frac{T_\mu \psi^k}{z^{k+1}} \rangle T^\mu} Q^\beta \prod_i \prod_{k=1}^{\rho_i\cdot\beta} (\rho_i + kz) + O(z^{-1}).$$

highest power of z in $J_\beta(t,z)$ is $-(k+1) = 1 - c_1(X)\cdot\beta$

$\left(n + D + k = \dim [\overline{\mathcal{M}}_{0,n+1}(X,\beta)] = c_1(X)\cdot\beta + D + (n+1) - 3 \right)$

$\Rightarrow$ highest power of z in $J_\beta(t,z) \prod_i \prod_{k=1}^{\rho_i\cdot\beta} (\rho_i + kz)$ is $1 + \underbrace{(c_1(E) - c_1(X))\cdot\beta}_{\le 0} \le 1$

"$=1$" $\Rightarrow \deg T_\mu = 2D$, $\deg T^\mu = 0$

"$=0$" $\Rightarrow \deg T_\mu \ge 2D - 2$, $\deg T^\mu \le 2$.

So $I_{X,Y}(t,z) = z \underset{\underset{1}{\equiv}}{F(t)} + \sum_i \underset{\underset{t^i}{\equiv}}{G^i(t)}\, T_i + O(z^{-1})$. $\pmod{Q^\beta}$ $\quad \square$

Cor. If $c_1(E) \le c_1(X)$, then $\forall t \in H^{\le 2}(X,\Lambda)$,

$$J_{X.Y}(\tau,z) = \frac{I_{X.Y}(t,z)}{F(t)}, \quad \tau = \sum_i \frac{G^i(t)}{F(t)} T_i .$$

pf. $$\frac{I_{X.Y}(t,-z)}{F(t)} = -z + \underbrace{\sum_i \frac{G^i(t)}{F(t)} T_i}_{=\tau} + O(z^{-1}) \in zL_t \cap (-z + z\mathcal{H}_-)$$

$$zL_t \supseteq \Lambda \cdot I_{X.Y}(t,-z)$$

$\Rightarrow$ " $= J_{X.Y}(\tau, z)$" $\square$

Example: $Y = (\ell) \subset X = \mathbb{P}^{n-1}$

$$\begin{cases} J_X(t_0+th, z) = z e^{(t_0+th)/z} \sum_{d\ge 0} Q^d e^{dt} \Big/ \prod_{k=1}^{d} (h+kz)^n \\ I_{X.Y}(t_0+th, z) = z e^{(t_0+th)/z} \sum_{d\ge 0} Q^d e^{dt} \prod_{k=1}^{\ell d} (\ell h + kz) \Big/ \prod_{k=1}^{d} (h+kz)^n \end{cases}$$

$\Rightarrow$ $\ell < n-1$: $J_{X,Y}(t_0+th, z) = I_{X.Y}(t_0+th, z)$

$\ell = n-1$: $J_{X.Y}(\tau_0+th, z) = I_{X.Y}(t_0+th, z)$, $\tau_0 = t_0 + \ell! Q e^t$

$\ell = n$: $J_{X.Y}(t_0+\tau h, z) = I_{X.Y}(t_0+th, z)/F(t)$, $\tau = G(t)/F(t)$

where $I_{X.Y} = e^{t_0/z}\left(z F(t) + G(t) h + O(z^{-1})\right)$.

Serre duality in genus 0.

$$\underbrace{D_{c^\vee.E^\vee}(t^\vee)}_{c(E)(t(z)-z)+z} = (\mathrm{sdet}\, c(E))^{-\frac{1}{24}} D_{c.E}(t) \Rightarrow (\mathcal{H}, \Omega_{c^\vee(E^\vee)}) \xrightarrow{c^\vee(E^\vee)} (\mathcal{H}, \Omega_{c(E)})$$

$$\begin{array}{ccc} (\mathcal{H}, \Omega_{c^\vee(E^\vee)}) & \xrightarrow{c^\vee(E^\vee)} & (\mathcal{H}, \Omega_{c(E)}) \\ \cup\!\shortmid & & \cup\!\shortmid \\ L_{c^\vee.E^\vee} & \longrightarrow & L_{c.E} \end{array}$$

So $\tau \mapsto c^\vee(E^\vee) J_{c^\vee.E^\vee}(\tau, -z)$ generates $L_{c.E}$. ($\tau \in H$)

Cor. $$J_{c.E}(\tau,z) = c^\vee(E^\vee)\, z\, \partial^{c(E)} J_{c^\vee.E^\vee}(\tau^\vee, z)$$

$$\parallel$$

$$\partial_\mu \partial^{c(E)} F^0_{c^\vee,E^\vee}(\tau^\vee, 0, 0, \dots) T^\mu$$

pf. Let $J = J_{c,E}$, $J^\vee = J_{c^\vee, E^\vee}$, $c = c(E)$, $c^\vee = c^\vee(E^\vee) = c^{-1}$.

$\exists\, C^\mu$ s.t. $c^\vee\left(J^\vee(\tau^\vee, z) + C^\mu z \partial_\mu J^\vee(\tau^\vee, z)\right) = J(\tau, z)$ for some τ.

$$\left(J^\vee(\tau^\vee, -z) \in \mathcal{L}_{c^\vee, E^\vee} \cap (-z + \tau^\vee + \mathcal{H}_-) \Rightarrow \begin{cases} c^\vee J^\vee = \mathcal{L}_{c,E} \cap (-c^\vee z + \tau^\vee + \mathcal{H}_-) \\ c^\vee z \partial_\mu J^\vee \in \mathcal{L}_{c,E} \cap (z T^\mu + z\mathcal{H}_-) \end{cases} \right)$$

Comparing the z-term $\Rightarrow c^\vee(1 + C^\mu T_\mu) = 1 \Rightarrow C^\mu T_\mu = c - 1$

So $J(\tau, z) = c^\vee\left(z \partial^1 J^\vee(\tau^\vee, z) + C^\mu z \partial_\mu J^\vee(\tau^\vee, z)\right) = c^\vee z \partial^c J^\vee(\tau^\vee, z)$.

Comparing the z^0-term $\Rightarrow \tau = \Omega_c(J(\tau, -z)/(-z), T_\mu)\, g_c^{\mu\nu} T_\nu$ $\square$

$$\Omega_c(J(\tau,-z)/(-z), T_\mu) = \Omega_{c^\vee}(\partial^c J^\vee(\tau^\vee, -z), T_\mu) = \partial^c \partial_\mu F^0_{c^\vee, E^\vee}(\tau^\vee, 0, 0, \dots)$$

Similarly, we have

$$\begin{array}{ccc} (\mathcal{H}, \Omega_{e^{-1}(E^\vee)}) & \xrightarrow[Q \mapsto \pm Q]{(-1)^{\dim E}\, e^{-1}(E)} & (\mathcal{H}, \Omega_{e(E)}) \\ \cup\!\mid & & \cup\!\mid \\ \mathcal{L}_{e^{-1}, E^\vee} & \longrightarrow & \mathcal{L}_{e, E} \end{array}$$

$$e(E)\, J_{e,E}(\tau, z; Q) = (-1)^{\dim E} z\, \partial^{e(E)} J_{e^{-1}, E^\vee}(\tau^\vee, z; \pm Q).$$

$$e(E)\, J_{e,E}(\tau, z; Q) \text{ term } = \partial_\mu \partial^{e(E)} F^0_{e^{-1}, E^\vee}(\tau^\vee, 0, 0, \dots)\, g^{\mu\nu}_{e(E)} T_\nu.$$

Toric fibrations

w : regular value

$$T^K \hookrightarrow T^N \curvearrowright \mathbb{C}^N \text{ diagonally} \rightsquigarrow \text{moment map}$$

$$\begin{tikzcd} \mathbb{C}^N \arrow[r, "\mu"] \arrow[dr] & \mathrm{Lie}(T^N)^\vee \arrow[r, two heads, "m"] \arrow[d, phantom, "\cong"] & \mathrm{Lie}(T^K)^\vee \ni w \arrow[d, phantom, "\cong"] \\ & \mathbb{R}^N \arrow[r, "(m^i_j)"] & \mathbb{R}^K \end{tikzcd}$$

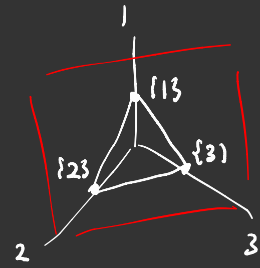
$$(z^j) \mapsto (|z^j|^2)$$

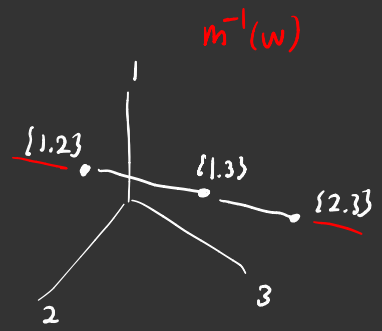
Get $T^N \curvearrowright X = \mathbb{C}^N /\!\!/_w T^K := (m\circ\mu)^{-1}(w)/T^K$ smooth

$$\left(T^K = T^K_w = \{ g \mid Ad^*_g w = w \} \right)$$

$$\begin{array}{c} (m\circ\mu)^{-1}(w) \\ \downarrow \\ X \end{array} \quad T^K\text{-fibration} \rightsquigarrow -P_1 \cdots -P_K \text{ generates } H^2(X,\mathbb{R})$$

fixed pts of X : $\alpha = \{ j_1 < \cdots < j_K \}$.





B Kähler mfd, $L_1, \ldots, L_N$ l.b. $/B$. $\ell_j = -c_1(L_j)$.

$$T^K \hookrightarrow T^N \curvearrowright \bigoplus L_j \Rightarrow E = \bigoplus L_j /\!\!/_w T^K \qquad H^*(E) = H^*(B)[P_1 \ldots P_K]$$

$$\phi \downarrow \; \uparrow \alpha : \text{fixed section}$$

$$B$$

$\mathcal{B} \in H_2(E) \rightsquigarrow \beta = \phi_* \mathcal{B} \in H_2(B)$, $d_i = P_i \cdot \mathcal{B} \in \mathbb{Z} \rightsquigarrow Q^\beta q^d$.

Let $U_j = \sum P_i m^i_j - \ell_j$. $P^\alpha = (m_\alpha^{-1})^j_i \ell_j$ s.t. $U_j^\alpha := U_j(P^\alpha) = 0 \quad \forall j \in \alpha$

$$\Rightarrow e(N^\alpha) = \prod_{j \notin \alpha} U_j^\alpha$$

$$N^\alpha = N_{\alpha(B)/E}$$

Bott residue formula : $$\phi_* f = \sum_\alpha \frac{c_\alpha^* f}{e(N^\alpha)} = \sum_\alpha \mathrm{Res}_\alpha f(P) \cdot \frac{dP_1 \wedge \cdots \wedge dP_K}{U_1(P) \cdots U_N(P)}$$

$$c_\alpha^* f = f(P^\alpha)$$

Let $I_E(z,t,\tau) = e^{Pt/z} \sum_{\beta, d} \dfrac{J_\beta(z,\tau)\, Q^\beta q^d e^{dt}}{\prod_{j=1}^{N} \prod_{m=1}^{U_j \cdot \beta} (U_j + mz)}$.

Thm (Brown). For each (t,τ), $I_E(-z) \in \mathcal{L}_E$.

Let $F(t,-z) = -z + t(z) + \sum_{n,\beta,d} \dfrac{Q^\beta q^d}{n!} \left\langle t(\psi)^{\otimes n}, \dfrac{T_\mu}{-z-\psi} \right\rangle T^\mu$

$F^\alpha := \alpha^* F \in \mathcal{H}(B) \Rightarrow F = \sum_\alpha \alpha_* \left(\dfrac{F^\alpha}{e^\alpha} \right)$ (Atiyah - Bott).

$F^\circ_{N^\alpha} = \sum_{n,\beta} \dfrac{Q^\beta q^{p^\alpha \cdot \beta}}{n!} \langle t(\psi)^{\otimes n} ; e_T^{-1}(N^\alpha_{0,n,\beta}) \rangle$

For $\alpha, \alpha' \in m^{-1}(w)$, α, α' connected by a line $\Leftrightarrow \#(\alpha \cup \alpha') = K+1$.

$$\begin{cases} j_+(\alpha,\alpha') \in \alpha' \setminus \alpha \\ j_-(\alpha,\alpha') \in \alpha \setminus \alpha' \end{cases} \Rightarrow \begin{aligned} &\chi_{\alpha,\alpha'} := c_1(\alpha^* T_{\overline{\alpha\alpha'}}) = U^\alpha_{j_+(\alpha,\alpha')} = -U^{\alpha'}_{j_-(\alpha,\alpha')}, \\ &d_{\alpha,\alpha'} := [\overline{\alpha\alpha'}] \in H_2(X, \mathbb{R}), \\ &d_{\alpha,\alpha'} \cdot \chi_{\alpha,\alpha'} = p^\alpha - p^{\alpha'}, \\ &U_j(d_{\alpha,\alpha'}) = \frac{U_j^\alpha - U_j^{\alpha'}}{\chi_{\alpha,\alpha'}} = \begin{cases} 1, & \text{if } j = j_\pm(\alpha,\alpha') \\ 0, & \text{if } j \in \alpha \cap \alpha'. \end{cases} \end{aligned}$$

$F^\alpha(t,-z) = -z + \alpha^* t(z) + \alpha^* \sum_{n,\beta,d} \dfrac{Q^\beta q^d}{n!} (e_{n+1})_* \left(\bigwedge_i (e_i^* t)(\psi_i) \wedge \dfrac{1}{-z-\psi_{n+1}} \right)$

$(\Sigma, x_1 \cdots x_{n+1}) \xrightarrow{f} E$ T-fixed stable $\Rightarrow f(x_{n+1}) \in \alpha(B) \Rightarrow$ (1) $x_{n+1} \in$ a leg, or

(2) $x_{n+1} \in C \longrightarrow \alpha(B)$

(1) $\Sigma = \Sigma' \cup \mathbb{P}' \ni x_{n+1}$

$$\begin{array}{ccc} x_0 & & \\ \big\downarrow{\scriptstyle \deg k} & & \big] \\ X \supset \overline{\alpha\alpha'} & & \alpha \end{array}$$

$\text{Cont} = \sum_{\alpha', k} \dfrac{q^{k d_{\alpha,\alpha'}} e_T^{-1}(N_{\alpha,\alpha'}(k))}{k\left(-z + \frac{\chi_{\alpha,\alpha'}}{k}\right)} F^{\alpha'}\left(-\dfrac{\chi_{\alpha,\alpha'}}{k}\right)$

$\gamma \cap [\overline{M}_{gm}(E,\beta)]$
$= \sum_\Gamma \dfrac{\gamma \cap [\overline{M}_\Gamma]}{|\mathrm{Aut}\,\Gamma|\, e(N_\Gamma^{vir})}$

$\dfrac{Q^\beta q^d}{n!} (e_{n+1})_* \left(\bigwedge_i (e_i^* t)(\psi) \wedge \dfrac{1}{-z-\psi} \right)$

$\psi = -e_{n+1}^*(\chi_{\alpha,\alpha'}/k)$

$|\mathrm{Aut}\,\Gamma| = k \cdot |\mathrm{Aut}\,\Gamma'|$

$q^d = q^{k d_{\alpha,\alpha'}} \cdot q^{d'}$

$e^{-1}(N_\Gamma) = e^{-1}(N_{\alpha\alpha'}(k))\, e^{-1}(N_{\Gamma'})$

$$e_T(N_{\alpha,\alpha'}(k)) = \prod_{m=1}^{k-1}\left(U^\alpha_{j+(\alpha,\alpha')} - \frac{m}{k}\chi_{\alpha,\alpha'}\right) \prod_{j\notin\alpha'} \prod_{m=1}^{k\cdot U_j(d_{\alpha,\alpha'})} \left(U_j^\alpha - \frac{m}{k}\chi_{\alpha,\alpha'}\right).$$

$$\left(R\Gamma(\mathbb{P}^1, f^*T_X) = R\Gamma\Big(\mathbb{P}^1, \bigoplus_j \mathcal{O}\big(\underset{\underset{f^*\mathcal{O}_X(D_{\rho_j})|_{\mathbb{P}^1}}{\parallel}}{\underline{U_j\cdot kd_{\alpha\alpha'}}}\big)\Big) - \mathrm{Pic}(X)^\vee_{\mathbb{C}} \right)$$

(2) $\Sigma = C \cup \Sigma_1 \cup \cdots \cup \Sigma_b$

$$\overline{M}_\rho \subseteq \overline{M}_{0,n_0+b}(\alpha(B),\beta_0) \times_{E^b} \prod_{i=1}^{b} \overline{M}_{0,n_i+1}(E,\beta_i)$$

$$\mathrm{Cont} = \sum_{n,\beta} \frac{Q^\beta q^{p^\alpha\cdot\beta}}{n!} (e_{n+1})_*\left(\bigwedge_i (e_i^* t^\alpha)(\psi_i) \wedge \frac{e_T^{-1}(N^\alpha_{0,n+1,\beta})}{-z-\psi_{n+1}} \right), \text{ where}$$

$$t^\alpha(z) = \alpha^* t(z) + \sum_{\alpha',k} \frac{q^{kd_{\alpha\alpha'}}\, e_T^{-1}(N_{\alpha,\alpha'}(k))}{k(-z+\chi_{\alpha,\alpha'}/k)} F^{\alpha'}\left(-\frac{\chi_{\alpha,\alpha'}}{k}\right).$$

So $$F^\alpha(t,-z) = -z + t^\alpha(z) + \sum_{n,\beta}\frac{Q^\beta q^{p^\alpha\cdot\beta}}{n!}\left\langle t^\alpha(\psi)^{\otimes n}, \frac{T_\mu}{-z-\psi}; e_T^{-1}(N^\alpha_{0,n+1,\beta})\right\rangle T^\mu \in \mathcal{L}^\alpha.$$

$\Rightarrow$ $F^\alpha(z)$ power series in Q, q with coeff ess. sing at $z=0$

pole at $z=\infty$

simple pole at $z = -\frac{\chi_{\alpha\alpha'}}{k}$

$$\mathrm{Res}_{-\frac{\chi_{\alpha\alpha'}}{k}} F^\alpha(z)\, dkz = \frac{q^{kd_{\alpha\alpha'}}}{e_T(N_{\alpha,\alpha'}(k))} F^{\alpha'}\left(-\frac{\chi_{\alpha\alpha'}}{k}\right)$$

(*)

Thm. $\{F^\alpha(-z)\}$ are characterized by (i) $F^\alpha(-z) \in \mathcal{L}^\alpha$, and (ii) (*)

pf. $F^\alpha(-z) \in \mathcal{L}^\alpha \subset (\mathcal{H}^\alpha, \Omega^\alpha) \Rightarrow \exists!\, u^\alpha \in H^\alpha$ s.t. $-z+u^\alpha+O(z^{-1}) \in zL \underset{\parallel}{} T_{F^\alpha(-z)}\mathcal{L}^\alpha$.

$$\begin{tikzcd} L \arrow[r, "(\ )_+", "\sim"'] & \mathcal{H}^\alpha_+ \\ & H^\alpha \arrow[u, phantom, "\cup" rotate=90] \arrow[ul, "1+O(z^{-1})"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathcal{H}^\alpha_+ \arrow[r, "1+O(z^{-1})", "\sim"'] & L \arrow[l, bend left, "S_{u^\alpha}(-z)"] \end{tikzcd}$$

Let $G^\alpha(-z) = S_{u^\alpha}(-z) F^\alpha(-z) \in z\mathcal{H}^\alpha_+$.

$$G^\alpha(z) = \sum_{\alpha', k} S_{u^\alpha}\left(\frac{\chi_{\alpha\alpha'}}{k}\right) \frac{q^{kd_{\alpha\alpha'}} e_T^{-1}(N_{\alpha\alpha'}(k))}{kz + \chi_{\alpha\alpha'}} S_{u^{\alpha'}}\left(\frac{\chi_{\alpha\alpha'}}{k}\right)^{-1} G^{\alpha'}\left(-\frac{\chi_{\alpha\alpha'}}{k}\right)$$

$$+ q^\alpha(z)$$ ← (Q.q)-series with coeff. $H[z]$

$$\left(\frac{z^{-m} - (-\chi_{\alpha\alpha'}/k)^{-m}}{kz + \chi_{\alpha\alpha'}} \in O(z^{-1}) \Rightarrow \text{diff. of both sides} \in O(z^{-1}) \right)$$

Given $\{q^\alpha\}$ and $\{u^\alpha\}$, $\{G^\alpha\}$ can be computed (uniquely) $\rightsquigarrow F^\alpha = S_{u^\alpha}^{-1} G^\alpha$.

It remains to show $\{q^\alpha\}$ and $\{u^\alpha\}$ are uniquely determined by $\{\alpha^* t\}$

Under (mod Q.q). $G^\alpha \equiv q^\alpha$. $S_{u^\alpha}(-z) \equiv e^{u^\alpha/z}$

$$\left(\begin{aligned} & -z + u^\alpha + \sum_{n\ge 2} \frac{1}{n!} \left\langle (u^\alpha)^{\otimes n}, \frac{T_\mu}{-z-\psi} ; e_T^{-1}(N^\alpha_{0,n+1,0}) \right\rangle g_\alpha^{\mu\nu} T_\nu \\ & = -z + u^\alpha + \sum_{n\ge 2} \frac{1}{n!} (u^\alpha)^n (-z^{-1})^{n-1} = -z e^{-u^\alpha/z} \in zL \end{aligned} \right)$$

$$\Rightarrow q^\alpha(-z) = e^{u^\alpha/z} F^\alpha(-z) = \left[e^{u^\alpha/z}(-z + \alpha^* t(z)) \right]_+$$

$$G^\alpha(-z) \in z\mathcal{H}_+ \Rightarrow q^\alpha(0) = 0 \Rightarrow \sum \frac{(u^\alpha)^m}{m!}(t_m - \delta_{m1}) = 0 \quad \left(\alpha^* t(z) = \sum t_m z^m\right)$$

$$\Rightarrow u^\alpha = \sum t_m \frac{(u^\alpha)^m}{m!} \rightsquigarrow \exists! \text{ formal solution } u^\alpha.$$

$$\rightsquigarrow q^\alpha(-z)$$

$\rightarrow \exists!$ formal solution u^α without (mod Q, q) $\square$

To prove Brown's Theorem, it suffices to show $\{\alpha^* I_E\}$ satisfies (i) and (ii).

(ii) $$I_E^\alpha(z) := \alpha^* I_E(z) = e^{p^\alpha t/z} \sum_{\beta, d} \frac{J_\beta(z,\tau) Q^\beta q^d e^{dt}}{\prod_{j=1}^N \prod_{m=1}^{U_j d - l_j\cdot\beta} (U_j^\alpha + mz)}$$

$d_i \mapsto d_i + p_i^\alpha \cdot \beta$

$(m_j p^\alpha \cdot \beta - l_j \cdot \beta = \alpha^* U_j \cdot \beta)$

$$= e^{p^\alpha t/z} \sum_{\beta, d} \frac{J_\beta(z,\tau) (Q^\beta q^{p^\alpha\cdot\beta}) q^d e^{dt} \cdot e^{(p^\alpha\cdot\beta)t}}{\prod_{j\in\alpha} \prod_{m=1}^{U_j d} (mz) \prod_{j\notin\alpha} \prod_{m=1}^{U_j d + U_j^\alpha \cdot\beta} (U_j^\alpha + mz)}$$

For $j \notin \alpha$, $\exists!\ \alpha'$ s.t. $j = j_+(\alpha, \alpha')$

$$\Rightarrow \left(\underbrace{U^\alpha_{j_+(\alpha,\alpha')}}_{\substack{\| \\ \chi_{\alpha\alpha'}}} + kz\right) \cdot \left. \frac{e^{p^\alpha t/z} \cdot (Q^\beta q^{p^\alpha \cdot \beta})\, q^d e^{dt} \cdot e^{(p^\alpha \cdot \beta)t}}{\prod_{j \in \alpha} \prod_{m=1}^{U_j d} (mz) \prod_{j \notin \alpha} \prod_{m=1}^{U_j d + U^\alpha_j \beta} (U^\alpha_j + mz)} \right|_{z = -\frac{\chi_{\alpha\alpha'}}{k}}$$

$$= q^{k d_{\alpha\alpha'}} \Big/ \underbrace{\prod_{m=1}^{k-1} \left(U^\alpha_{j_+(\alpha\alpha')} - \tfrac{m}{k}\chi_{\alpha\alpha'}\right) \prod_{j \notin \alpha'} \prod_{m=1}^{kU_j \cdot d_{\alpha\alpha'}} \left(U^\alpha_j - \tfrac{m}{k}\chi_{\alpha\alpha'}\right)}_{e_T(N_{\alpha\alpha'}(k))}$$

$$\left((p^\alpha - p^{\alpha'})/\chi_{\alpha\alpha'} = d_{\alpha\alpha'}\right)$$

$$\cdot \frac{e^{-p^{\alpha'} tk/\chi_{\alpha\alpha'}} \left(Q^\beta q^{p^{\alpha'} \cdot \beta} q^{\overset{d'}{\overset{\|}{\boxed{d - k d_{\alpha\alpha'} + p^\alpha \cdot \beta - p^{\alpha'} \cdot \beta}}}}\right) e^{d't}\ e^{(p^{\alpha'} \cdot \beta)t}}{\prod_{j \in \alpha'} \prod_{m=1}^{\substack{U_j \cdot (d - k d_{\alpha\alpha'}) \\ + U^\alpha_j \cdot \beta - U^{\alpha'}_j \cdot \beta}} \left(-\tfrac{m}{k}\chi_{\alpha\alpha'}\right) \prod_{j \notin \alpha'} \prod_{m=1}^{\substack{U_j \cdot (d - k d_{\alpha\alpha'}) \\ + U^\alpha_j \cdot \beta}} \left(U^{\alpha'}_j - \tfrac{m}{k}\chi_{\alpha\alpha'}\right)}$$

$$\left(U_j \cdot d_{\alpha\alpha'} = \frac{U^\alpha_j - U^{\alpha'}_j}{\chi_{\alpha\alpha'}} = \begin{cases} 1 & j = j_\pm \\ 0 & j \in \alpha \cap \alpha' \end{cases}\right)$$

$$= \frac{q^{k d_{\alpha\alpha'}}}{e_T(N_{\alpha\alpha'}(k))} I^{\alpha'}_{E,(\beta,d')}\left(-\frac{\chi_{\alpha\alpha'}}{k}\right) (Q^\beta q^{p^{\alpha'} \cdot \beta})\, q^{d'},$$ as desired.

For (i), we need : $\prod x_j^{m^i_j} = q_i e^{t^i} \quad \forall i$

$$\downarrow$$

Thm. Put $\mathcal{J}_\alpha(z, t, \lambda) = \int_{\mathbb{R}_+^{N-k}} e^{\sum (x_j + \lambda_j \log x_j)/z} \bigwedge_{j \notin \alpha} d\log x_j$.

Then $q^{-\frac{p^\alpha}{z}} \mathcal{J}_\alpha(z, t, z\partial^d) J(z, \tau) = I^\alpha_E(z, t, \tau) \prod_{j \notin \alpha} \int_0^\infty e^{(x - U^\alpha_j \log x)/z}\, d\log x$.

$$\left(\begin{array}{l} \Rightarrow q^{-\frac{p^\alpha}{z}} \hat{\mathcal{J}}_\alpha J = I^\alpha_E \cdot \prod_{j \notin \alpha} \hat{\Gamma}(-z, U^\alpha_j), \\ \text{where } \hat{\Gamma}(z, v) = \sqrt{\frac{2\pi z}{v}} \exp\left(\frac{v(\log v - 1)}{z} + \sum_{m \geq 1} \frac{B_{2m}}{2m(2m-1)} \left(\frac{z}{v}\right)^{2m-1}\right) \end{array} \right)$$

pf. $\mathcal{J}_\alpha(z,t,z\partial^l)\, J(z,\tau) = \int_{\mathbb{R}_+^{N-k}} \underbrace{e^{\sum(x_j + z\partial^{l_j}\log x_j)/z}\, J(z,\tau)}_{\displaystyle \| \atop \displaystyle e^{\sum x_j/z}\, J(z,\tau + \sum l_j \log x_j)} \bigwedge_{j\notin\alpha} d\log x_j$

$$= \sum_\beta J_\beta(z,\tau)\, Q^\beta \int_{\mathbb{R}_+^{N-k}} e^{\sum x_j/z}\, e^{\sum l_j \log x_j/z} \prod_j x_j^{l_j\cdot\beta} \bigwedge_{j\notin\alpha} d\log x_j \qquad (*)$$

$$(qe^t)^d = \prod_j x_j^{\overset{U_j d}{\overset{\|}{\boxed{m_j^i d_i}}}} \Rightarrow e^{\sum_{j\in\alpha} x_j/z} = \prod_{j\in\alpha} e^{x_j/z} \underset{\substack{\uparrow \\ \mathbb{Z}^k \xrightarrow{\sim} \mathbb{Z}^k \\ d \mapsto (U_j d)_{j\in\alpha}}}{=} \sum_d \frac{(qe^t)^d}{\prod_{j\in\alpha} z^{U_j d}(U_j\cdot d)! \prod_{j\notin\alpha} x_j^{U_j d}}$$

$$l_j = \sum p_i^\alpha m_j^i - U_j^\alpha$$

$$\Rightarrow \begin{cases} e^{\sum l_j \log x_j/z} = \prod x_j^{-U_j^\alpha/z} \prod (x_j^{m_j^i})^{p_i^\alpha/z} = \prod_{j\notin\alpha} x_j^{-U_j^\alpha/z} \prod_i (q_i e^{t^i})^{p_i^\alpha/z} \\ \prod_j x_j^{l_j\cdot\beta} = \prod x_j^{-U_j^\alpha\cdot\beta} \prod (x_j^{m_j^i})^{p_i^\alpha\cdot\beta} = \prod_{j\notin\alpha} x_j^{-U_j^\alpha\cdot\beta} \prod_i (q_i e^{t^i})^{p_i^\alpha\cdot\beta} \end{cases}$$

So $q^{-\frac{p^\alpha}{z}}\,(*)$

$$\int_0^\infty e^{(x - U_j^\alpha \log x)/z}\, d\log x \Big/ \prod_{m=1}^{U_j d + U_j^\alpha\cdot\beta} (U_j^\alpha + mz)$$
$$\|$$

$$= e^{p^\alpha t/z} \sum_\beta J_\beta(z,\tau)\, Q^\beta (qe^t)^{p^\alpha\cdot\beta} \sum_d (qe^t)^d \frac{\prod_{j\notin\alpha}\int_0^\infty e^{x/z}\, x^{-U_j d - U_j^\alpha/z - U_j^\alpha\cdot\beta - 1}\, dx}{\prod_{j\in\alpha}\prod_{m=1}^{U_j d}(mz)}$$

$= \text{RHS}$

$\square$

(i): $I_E^\alpha(-z) \in \mathcal{L}^\alpha \overset{QRR}{\iff} \prod \dfrac{\hat\Gamma(z,U_j^\alpha)}{\sqrt{2\pi z}}\, I_E^\alpha(-z) \in \alpha^* \mathcal{L}_{\alpha(B)}$

$$\|$$

$$(2\pi z)^{-\frac{\dim N^\alpha}{2}}\, \underline{q^{\frac{p^\alpha}{z}}}\, \hat{\mathcal{J}}_\alpha(-z,-z\partial^l)\, \underline{\sum Q^\beta J_\beta(-z,\tau)}$$

$$Q^\beta \mapsto Q^\beta q^{-p^\alpha\cdot\beta}$$

$\iff (2\pi z)^{-\frac{\dim N^\alpha}{2}}\, \hat{\mathcal{J}}_\alpha(-z,-z\partial^l)\, J(-z,\tau - p^\alpha \log q) \in \mathcal{L}_B$, which is true.