Quantum Singularity Theory

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Reference: Fan, Jarvis, Ruan, The Witten Equation, MS, and Quantum Singularity Theory.

$W: \mathbb{C}^N \longrightarrow \mathbb{C}$ quasi-homo. poly. s.t. $W(\lambda^{n_1}x_1, \ldots, \lambda^{n_N}x_N) = \lambda^d W(x_1, \ldots, x_N)$.

weight (or charge) of $x_i = \frac{n_i}{d} =: q_i$

We always assume W is non-degenerate. i.e.

(i) W contains no monomial $x_i x_j$ $(i \neq j)$.

(ii) W has isolated singularity at 0.

$\Big\} \Rightarrow q_i \leq \frac{1}{2}$, unique!

e.g. ADE-examples:

A_n: $W = x^{n+1}$ $(n \geq 1)$ $\qquad E_6$: $W = x^3 + y^4$.

D_n: $W = x^{n-1} + xy^2$ $(n \geq 4)$ $\qquad E_7$: $W = x^3 + xy^3$.

$\qquad E_8$: $W = x^3 + y^5$.

Also, $J \in \mathrm{Aut}(W) = \{ \gamma \in \mathrm{Diag}_N(\mathbb{C}) \mid W(\gamma x) = W(x) \}$ (maximal diagonal symmetry group of W.)

$J = \mathrm{diag}(e^{2\pi i q_1} \ldots e^{2\pi i q_N})$

Choose $\langle J \rangle \leq G \leq \mathrm{Aut}(W)$.

For $\gamma \in G$, $\mathbb{C}^{N_\gamma} :=$ fixed point of γ, $W_\gamma := W|_{\mathbb{C}^{N_\gamma}}$.

$\rightsquigarrow$ G-inv. middle relative cohomology $\mathcal{H}_{\gamma, G} = H^{mid}(\mathbb{C}^{N_\gamma}, (\mathrm{Re}\, W)^{-1}(M, \infty), \mathbb{C})^G$.

$\rightsquigarrow$ State space $\mathcal{H}_{W,G} = \bigoplus_{\gamma \in G} \mathcal{H}_{\gamma, G}$.

$\rightsquigarrow$ For $\alpha_1, \ldots, \alpha_k \in$, $\ell_1, \ldots, \ell_k \in \mathbb{Z}_{\geq 0}$: genus g correlator $\langle \tau_{\ell_1}(\alpha_1) \cdots \tau_{\ell_k}(\alpha_k) \rangle_g^{G,W}$.

<u>Theorem</u> $\langle \tau_{\ell_1}(\alpha_1) \cdots \tau_{\ell_k}(\alpha_k) \rangle_g^{G,W}$ satisfy axioms in GW-theory but not divisor axiom.

Choose a basis $\{\alpha_i\}$ of $\mathcal{H}_{W,G}$, $\mathcal{F}_{g,W,G} := \sum_{k \geq 0} \langle \tau_{\ell_1}(\alpha_{i_1}) \cdots \tau_{\ell_k}(\alpha_{i_n}) \rangle_g^{W,G} \frac{t_{i_1}^{\ell_1} \cdots t_{i_n}^{\ell_n}}{n!}$

$\rightsquigarrow$ total potential function $\mathcal{D}_{W,G} = \exp\left(\sum_{g \geq 0} h^{2g-2} \mathcal{F}_{g,W,G} \right)$.

<u>Conjecture</u> (Witten) The total potential function of ADE with $G = \langle J \rangle$ are τ-function of ADE integrable hierarchies.

- A_n case: Generalized Witten conjecture, proved by Kontsevich.

<u>Main Theorem</u> (1) When $n \geq 6$, even or E_6, E_7, E_8, $G = \langle J \rangle$, $\mathcal{D}_{W,G}$ is τ-function of the Kac-Wakimoto / Drinfeld-Sokolov hierarchies

(2) For $n > 4$, $W = D_n$, $G = G_{D_n}$, $\mathcal{D}_{W,G}$ is τ-function of A_{2n-3}-hierarchies.

(3) For $n > 4$, $W = x^{n-1}y + y^2$ with $G = \mathrm{Aut}(W)$, $\mathcal{D}_{W,G}$ is τ-function of D_n-hierarchy.

Conjecture (ADE self-mirror conjecture)

W: simple singularity, $G = \langle J \rangle$. Then, the ring $\mathcal{H}_{W,\langle J\rangle} \simeq$ Milnor ring of W.

Main Theorem 2

(1) Except for D_n, n: odd, $\mathcal{H}_{W\langle J\rangle}$ of any ADE singularities is isomorphic to Milnor ring of the same singularity.

(2) $\mathcal{H}_{D_n, G_{D_n}} \simeq Q_{A_{2n-3}}$: Milnor ring of $W = x^{n-1}y + y^2$.

(3) $\mathcal{H}_{W, G_W}$ with $W = x^{n-1}y + y^2$ $(n \geq 4)$ is isomorphic to Q_{D_n} : Milnor ring of D_n.

- **Orbicurve and line bundle**

Orbicurve $\mathcal{C}$ is a nodal curve with mark points $p_1 \cdots p_k$ s.t.

(i) at each p_i, $\exists\, U \ni p_i$ s.t. U is uniformized by $z \longmapsto z^{m_i}$. (local group: $\mathbb{Z}/m_i$) G_{p_i}

(ii) at each node p, $\exists\, V \ni p$ s.t. V is uniformized by $(z,w) \longmapsto (z^{n_j}, w^{n_j})$.

$\{zw=0\} \subseteq \mathbb{C}^2$ (local group $G_p = \mathbb{Z}/n_j$ s.t. $e^{2\pi i/n_j}(z,w) = (e^{2\pi i/n_j}z, e^{-2\pi i/n_j}w)$)

- $p: \mathcal{C} \longrightarrow C$, log can. bundle $K_{C,\log} := K_C \otimes \mathcal{O}(p_1) \cdots \otimes \mathcal{O}(p_k)$

underlying Riemann surface. $K_{\mathcal{C},\log} := p^* K_{C,\log}$

- $\mathcal{L}$: orbifold line bundle on smooth $\mathcal{C}$ $\rightsquigarrow$ dual to a line bundle $|\mathcal{L}|$ on C.

$\mathcal{L} \supseteq \Delta \times \mathbb{C}$: local trivialization. (z,s) $\quad 1 \in G_p \simeq \mathbb{Z}/m$ acts on $\mathcal{L}$ by $(z,s) \mapsto (e^{2\pi i/m} z, e^{2\pi i v/m} s)$

$\downarrow$ $\quad \rightsquigarrow \Phi: (\Delta \setminus \{0\}) \times \mathbb{C} \longrightarrow \Delta \times \mathbb{C}$: $\mathbb{Z}/m$ - equivariant map

$\mathcal{C} \ni p$: orbifold point. $\quad (z,s) \longmapsto (z^m, z^{-v}s)$

$\overset{\text{use } \Phi}{\rightsquigarrow}$ $\mathbb{Z}/m$ acts on $\Delta \times \mathbb{C}$ trivially $\rightsquigarrow$ get line bundle $|\mathcal{L}|$.

If s: section of $|\mathcal{L}|$, locally $= g(u)$, then $(z, z^v g(z^m))$ is a local section of $\mathcal{L}$.

$\rightsquigarrow p^*: H^0(C, |\mathcal{L}|) \xrightarrow{\sim} H^0(\mathcal{C}, \mathcal{L})$, $\quad H^1(C, |\mathcal{L}|) \xrightarrow{\sim} H^1(\mathcal{C}, \mathcal{L})$.

$p^*: \Omega^{0,1}(|\mathcal{L}|) \longrightarrow \Omega^{0,1}(\mathcal{L})$ $\leftarrow$ orbifold $(0,1)$-form

(locally) $g(u)\,d\bar{u} \longmapsto z^v g(z^m)\, m \bar{z}^{m-1} d\bar{z}$.

- **Quasi-homogeneous W**

$W = \sum_{j=1}^{s} W_j$: non-degenerate, $W_j = c_j \prod_{\ell=1}^{N} x_\ell^{b_{j\ell}}$: monomials. $\rightsquigarrow B_W := (b_{j\ell})_{s\times N}$.

$\rightsquigarrow$ Smith normal form $B_W = V\underline{T}Q$.

diagonal $t_{ii} \mid t_{i+1\, i+1}$

Fact $G_W = \{ (\alpha_1 \ldots \alpha_N) \in (\mathbb{C}^*)^N \mid W(\alpha_1 x_1 \cdots \alpha_N x_N) = W(x_1, \ldots, x_N) \}$ is finite.

proof: $\alpha_j = \exp(u_j + i v_j)$. $\gamma \in G_W \Leftrightarrow \underline{B_W} \cdot (u + iv) = 0$. #

full rank !

For each $\gamma \in G_W$, write $\gamma = (\exp(2\pi i \theta_1^\gamma), \dots, \exp(2\pi i \theta_N^\gamma))$, $\theta_i^\gamma \in [0,1) \cap \mathbb{Q}$.

$J = (\exp(2\pi i q_1), \dots, \exp(2\pi i q_N)) \in G_W$.

<u>Lemma</u> W: non-degenerate, for any $\gamma \in G_W$, W_γ has no non-trivial critical points.

Thus, $\underline{W_\gamma}$ is a non-degenerate, quasi-homo. in variables x_i fixed by γ.

$W|_{\mathbb{C}^{N_\gamma}}$, $\mathbb{C}^{N_\gamma} = (\mathbb{C}^N)^\gamma$: fix points of γ in $\mathbb{C}^N$.

proof: $m \subseteq \mathbb{C}[x_1 \dots x_N]$: ideal generated by x_i not fixed by γ.

$\rightsquigarrow$ $W = W_\gamma + \underbrace{W_{moved}}_{\in\, m}$. $W_{moved} \in m^2$ (since γ fixed all monomials in W.)

Write $(\underbrace{x_1 \dots x_\ell}_{\text{fixed}}, \underbrace{x_{\ell+1} \dots x_N}_{\text{moved}})$. If $(\alpha_1 \dots \alpha_\ell)$: critical point of W_γ.

$\Rightarrow$ $(\alpha_1 \dots \alpha_\ell, 0 \dots 0)$: critical point of W.

$\Rightarrow$ $(\alpha_1 \dots \alpha_\ell) = 0$. #

- <u>W-structure on orbicurve</u>

$W \rightsquigarrow B_W = VTQ \rightsquigarrow A = (a_{j\ell}) := V^{-1}B = TQ$, $u_\ell :=$ sum of ℓ-th row of V^{-1}.

For $\ell = 1 \sim N$, $A_\ell(\mathcal{L}_1, \dots, \mathcal{L}_N) := \mathcal{L}_1^{\otimes a_{\ell 1}} \otimes \dots \otimes \mathcal{L}_N^{\otimes a_{\ell N}}$.

A W-structure on $\mathcal{C}$ is an N-tuple $(\mathcal{L}_1, \dots, \mathcal{L}_N)$ of orbifold line bundles on $\mathcal{C}$ with iso. $\tilde{\varphi}_\ell : A_\ell(\mathcal{L}_1, \dots, \mathcal{L}_N) \xrightarrow{\sim} K_{\mathcal{C}.\log}^{u_\ell}$ for any $\ell = 1 \sim N$.

For each $j = 1 \sim s$, $\{\tilde{\varphi}_\ell\}$ gives $\varphi_j := \tilde{\varphi}_1^{v_{j1}} \otimes \dots \otimes \varphi_N^{v_{jN}} : \underbrace{W_j(\mathcal{L}_1, \dots, \mathcal{L}_N)}_{\mathcal{L}_1^{\otimes b_{j1}} \otimes \dots \otimes \mathcal{L}_N^{\otimes b_{jN}}} \xrightarrow{\sim} K_{\mathcal{C}.\log}$

$(V = (v_{j\ell}))$

Two W-structure $(\mathcal{L}_1 \dots \mathcal{L}_N, \tilde{\varphi}_1 \dots \tilde{\varphi}_N)$, $(\mathcal{L}_1' \dots \mathcal{L}_N', \tilde{\varphi}_1' \dots \tilde{\varphi}_N')$ are isomorphic if

$\exists\ \mathcal{L}_i \xrightarrow{\sim} \mathcal{L}_i'$ s.t. .... commutes!

<u>Proposition</u> Any two W-structure $(\mathcal{L}_1 \dots, \mathcal{L}_N, \tilde{\varphi}_1 \dots, \tilde{\varphi}_N)$ on $\mathcal{C}$ are isomorphic.

$(\mathcal{L}_1, \dots, \mathcal{L}_N, \tilde{\varphi}_1', \dots, \tilde{\varphi}_N')$

proof: $\tilde{\varphi}_j'^{-1} \circ \tilde{\varphi}_j$: iso of $K_{\log}^{u_j}$ for all j $\rightsquigarrow$ given by $\exp(\alpha_j) \in \mathbb{C}^\times$.

$$V^{-1}B_W = TQ = \left(\frac{C}{0} \right) \leftarrow \text{non-singular } N \times N. \quad \text{Let} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix} = C^{-1} \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix}$$

$\rightsquigarrow$ $\{ \exp(\beta_j) : \mathcal{L}_j \xrightarrow{\sim} \mathcal{L}_j' \}$ gives isomorphism.

#

<u>Remark</u> In $W = x^{r+1}$ (A_r case), W-structure is called an r-spin structure.

Note : for each $p \in \mathcal{C}$, we have $G_p \longrightarrow \mathrm{Aut}(\mathcal{L}) \underset{\text{W-structure}}{\rightsquigarrow} r_p : G_p \longrightarrow U(1)^N$

required : this is faithful.

Write $\gamma_i := \gamma_{p_i} := r_{p_i}(1) = (\exp(2\pi i \Theta_1^{\gamma}), \dots, \exp(2\pi i \Theta_N^{\gamma}))$.

<u>Lemma</u> $(\mathcal{L}_1, \dots, \mathcal{L}_N, \tilde{\varphi}_1, \dots, \tilde{\varphi}_N)$: W-structure of $\mathcal{C}$. Then, r_p factor through G_W.

Thus, $\gamma_i \in G_W$ for all i.

proof: Note $\mathcal{L}_1^{\otimes b_{1j}} \otimes \cdots \otimes \mathcal{L}_N^{\otimes b_{Nj}} \simeq K_{\log}$ for all j.

G_p acts trivially !

$\gamma_p \in G_p$ acts as $\exp(2\pi i \sum_i b_{ij} \Theta_i^{\gamma}) \Rightarrow \sum_i b_{ij} \Theta_i^{\gamma} \in \mathbb{Z}$ i.e. γ fixs W_j. #

• Moduli of stable W-orbicurves

stable W-orbicurve : $\underline{\mathcal{C}}$ with k mark points, W-structure, $r_p : G_p \longrightarrow G_W$ is faithful.

underlying curve is stable. (for all $p \in \mathcal{C}$).

Genus g, stable W-orbicurve over base T:

flat family: $\mathcal{C} \supseteq S_i$: marking $(\mathcal{L}_1, \dots, \mathcal{L}_N, \tilde{\varphi}_1, \dots, \tilde{\varphi}_N)$.

$\downarrow \;\; \uparrow \sigma_i$ mark section.

T

$\overline{W}_{g,k}(W)$ = stack of stable W-orbicurves. $st : \overline{W}_{g,k}(W) \longrightarrow \overline{M}_{g,k}$

forgetting W-structure.

<u>Theorem</u> $\overline{W}_{g,k}(W)$ is smooth, DM.

In particular, $st : \overline{W}_{g,k}(W) \longrightarrow \overline{M}_{g,k}$ is flat, proper, quasi-finite.

<u>Definition</u> For $\gamma = (\gamma_1, \dots, \gamma_k) \in G_W^k$, write $\overline{W}_{g,k}(\gamma) \subseteq \overline{W}_{g,k}$: open and closed substack.

at p_i, $r_{p_i}(1) = \gamma_i$.

$\rightsquigarrow \overline{W}_{g,k} = \sum_{\gamma} \overline{W}_{g,k}(\gamma)$.

<u>Proposition</u> $\overline{W}_{g,k}(\gamma) \neq \emptyset \iff q_j(2g-2+k) - \sum_{\ell=1}^{k} \Theta_j^{\gamma_\ell} \in \mathbb{Z}$.

proof: $\deg \mathcal{L}_j \in \mathbb{Q}$ but $\deg |\mathcal{L}_j| = \deg p_* \mathcal{L}_j \in \mathbb{Z}$.

$\sum_{j=1}^{N} b_{ij} \deg |\mathcal{L}_j| = 2g-2+k - \sum_{\ell=1}^{k} \sum_{j=1}^{N} b_{ij} \Theta_j^{\gamma_\ell}$. Also, $\sum_{j=1}^{N} b_{ij} q_j = 1$ for all $i = 1 \sim s$.

$\rightsquigarrow \deg |\mathcal{L}_j| = \left(q_j(2g-2+k) - \sum_{\ell=1}^{k} \Theta_j^{\gamma_\ell} \right) \in \mathbb{Z}$ for all $j = 1 \sim N$.

Conversely, for any smooth curve C, choose $E_1 \dots E_N$: line bundle on C s.t. $\deg(L_j) = q_j(2g-2+k) - \sum_{\ell=1}^{k} \theta_j^{\gamma_\ell}$ for each j. $A = (a_{ij}) = V^{-1}B$, $u = (u_i)$ as before.

$\rightsquigarrow X_i := E_1^{a_{i1}} \otimes \cdots \otimes E_N^{a_{iN}} \otimes K_{C.\log}^{-u_i} \otimes \mathcal{O}\left(\sum_{\ell=1}^{k}\sum_{j=1}^{N} a_{ij}\theta_j^{\gamma_\ell} p_\ell\right)$

$\rightsquigarrow \deg X_i = 0$ for all i.

A: full rank $\rightsquigarrow \exists\ (Y_1 \dots Y_N) \in \mathrm{Pic}^0(C)^N$ s.t.
$$Y_1^{a_{11}} \otimes \cdots \otimes Y_N^{a_{1N}} = X_1$$
$$\vdots$$
$$Y_1^{a_{N1}} \otimes \cdots \otimes Y_N^{a_{NN}} = X_n$$

$\rightsquigarrow L_j := Y_j^{-1} \otimes E_j$ satisfy $L_1^{a_{i1}} \otimes \cdots \otimes L_N^{a_{iN}} \simeq K_{C.\log}^{u_i} \otimes \mathcal{O}\left(-\sum_{\ell=1}^{k}\sum_{j=1}^{N} a_{ij}\theta_j^{\gamma_\ell} p_\ell\right)$.

$\rightsquigarrow$ Construct $\mathcal{C} \longrightarrow C$ s.t. $G_{p_\ell} = \langle \gamma_\ell \rangle$ and $\mathcal{L}_j := \rho^* L_j$. #

- Dual Graph

$(\mathcal{C}, p_1, \dots, p_k, \mathcal{L}_1 \dots \mathcal{L}_N, \tilde{\varphi}_1 \dots \tilde{\varphi}_N)$: W-stable curve $\rightsquigarrow$ graph Γ. $v \in V(\Gamma)$

↳ g_v : geometric genus, k_v : valance.

For each mark point (or tail τ on Γ), decorate $\gamma_\tau \in G_W$.

For each edge $e \in \Gamma$, decorate γ_+, γ_- on two flags s.t $\gamma_- = (\gamma_+)^{-1}$.

node! $\rightsquigarrow$ G_W-decrated graph.

$\rightsquigarrow \deg(|\mathcal{L}_j|_v) = q_j(2g_v - 2 + k_v) - \left(\sum_{\substack{\tau:\text{flag}\\ \text{on } v,}} \theta_j^{\gamma_\tau}\right) \in \mathbb{Z}$.

$\overline{W}(\Gamma)$ = closure in $\overline{W}_{g.k}$ of all stable W-curves with graph Γ.

$\overline{W}(\Gamma) \subseteq \overline{W}_{g.k}(\gamma) \subseteq \overline{W}_{g.k}$

$\gamma = (\gamma_1, \dots, \gamma_k)$ on tails of Γ.

- Morphisms

·· Forgetting tails : $\gamma = (\gamma_1, \dots, \underset{i\text{-th position}}{J}, \dots \gamma_k)$. Write $\gamma' = (\gamma_1, \dots, \hat{\gamma}_i, \dots, \gamma_k)$.

$\rightsquigarrow \vartheta : \overline{W}_{g.k}(\gamma) \longrightarrow \overline{W}_{g.k-1}(\gamma')$: forgetting mark point p_i and its orbifold structure.

Explictly : $\bar{\rho} \cdot \mathcal{C} \longrightarrow \overline{\mathcal{C}}$ forgetting p_i and its orbifold structure.

$$\bar{\rho}_* \mathcal{L}_1^{\otimes a_{j1}} \otimes \cdots \otimes \bar{\rho}_* \mathcal{L}_N^{\otimes a_{jN}} \xrightarrow[\tilde{\varphi}_j']{\sim} K_{\overline{\mathcal{C}}.\log}^{u_j} \otimes \mathcal{O}\Big(\big(-\sum_{\ell=1}^{N} a_{j\ell} \underbrace{\theta_\ell^{J}}_{= q_\ell}\big) p_i\Big) = K_{\overline{\mathcal{C}}.\log}^{u_j} \otimes \mathcal{O}(-u_j p_i)$$

$= K_{\overline{\mathcal{C}}.\log}^{u_j}$

Need $\sum_{\ell=1}^{N} a_{j\ell} q_\ell = u_j$

$\rightsquigarrow \vartheta$ send $(\mathcal{C}, p_1 \dots p_k, \mathcal{L}_1 \dots \mathcal{L}_N, \tilde{\varphi}_1 \dots \tilde{\varphi}_N)$ to $(\overline{\mathcal{C}}, p_1 .. \hat{p}_i .. p_k, \bar{\rho}_* \mathcal{L}_1 \dots \bar{\rho}_* \mathcal{L}_N, \tilde{\varphi}_1' \dots \tilde{\varphi}_N')$.

- ·· Gluing and Cutting.

$$\rho_{tree} : \overline{\mathcal{M}}_{g_1, k_1+1} \times \overline{\mathcal{M}}_{g_2, k_2+1} \longrightarrow \overline{\mathcal{M}}_{g_1+g_2, k_1+k_2}$$

$$\rho_{loop} : \overline{\mathcal{M}}_{g, k+2} \longrightarrow \overline{\mathcal{M}}_{g+1, k}.$$

To take care W-structure, we need

$$\tilde{\rho}_{tree,\gamma} : \overline{W}_{g_1, k_1+1}(\gamma) \times \overline{W}_{g_2, k_2+1}(\gamma^{-1}) \longrightarrow \overline{W}_{g_1+g_2, k_1+k_2}.$$

$$\tilde{\rho}_{loop,\gamma} : \overline{W}_{g, k+2}(\gamma, \gamma^{-1}) \longrightarrow \overline{W}_{g+1, k}.$$

- ·· Stabilization morphism : $st : \overline{W}_{g.k} \longrightarrow \overline{\mathcal{M}}_{g.k}$, $st_\gamma : \overline{W}_{g.k}(\gamma) \longrightarrow \overline{\mathcal{M}}_{g.k}$.

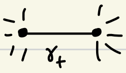
Automorphism of any W-structure over fixed $\mathscr{C}$ is G_W

$\rightsquigarrow \deg(st_\gamma) = |G_W|^{2g-1}$.

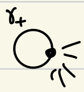
Also, there are $|G_W|^{k-1}$ choice of γ s.t. $\overline{W}_{g.k}(\gamma) \neq 0$

$\rightsquigarrow \deg(st) = |G_W|^{2g-2+k}$.

For decorated graph Γ, we have $st_\Gamma : \overline{W}(\Gamma) \longrightarrow \overline{\mathcal{M}}(\Gamma)$.

<u>Proposition</u> (1) If $\Gamma =$  , $\gamma = (\gamma_1 \dots \gamma_k) \in G_W^k$, then $\deg(st_\gamma) = |\langle \gamma_+ \rangle| \cdot \deg(st_\Gamma)$.

reason: generically, Aut of a point $\in \overline{W}(\Gamma) = G_W \underset{G_W/\langle \gamma_+ \rangle}{\times} G_W$.

(2) If $\Gamma =$  , $\gamma = (\gamma_1 \dots \gamma_k) \in G_W^k$, then $\deg(st_\Gamma) = \dfrac{|G_W|^{2g-2}}{|\langle \gamma_+ \rangle|}$.

- <u>ψ, κ, μ - classes</u>

universal orbit curve ↙ ← universal underlying stable curve

$$\begin{tikzcd} \mathscr{C}_{g.k} \arrow[r, "\rho"] \arrow[d, shift left, "\pi"] & C_{g.k} \arrow[dl, "\tilde{\pi}"] \\ \overline{W}_{g.k} \arrow[u, bend left, "\sigma_i"] & \end{tikzcd}$$

ψ classes : $\tilde{\psi}_i := c_1(\sigma_i^*(K_{\mathscr{C}}))$

$\psi_i := c_1(\bar{\sigma}_i^*(K_C)) = st^*(\psi_i)$.

<u>Proposition</u> If the orbifold structure along σ_i is γ_i, $|\langle \gamma_i \rangle| = m_i$, then $m_i \tilde{\psi}_i = st^* \psi_i$.

proof: D_i = image of σ_i. $\bar{\sigma}_i = \rho \circ \sigma_i$.

$x = z^m \Rightarrow dx = m z^{m-1} dz$

$$\bar{\sigma}_i^* K_{C_{g.k}} = \sigma_i^*(\rho^* K_{C_{g.k}}) = \sigma_i^*\left(K_{\mathscr{C}_{g.k}} \otimes \mathcal{O}(-(m_i-1)D_i)\right)$$

Residue map $\Rightarrow \sigma^* K_{log} = \mathcal{O} \Rightarrow \sigma_i^*(\mathcal{O}(-D_i)) = \sigma_i^*(K_{\mathscr{C}})$. #

κ-classes: In $\overline{M}_{g,k}$ case, $\kappa_a := \pi_*(c_1(K_{C,\log})^{a+1})$.

Now, define $\tilde{\kappa}_a := \pi_*(c_1(K_{\mathcal{C},\log})^{a+1})$.

Note that $K_{\mathcal{C},\log} = \rho^* K_{C,\log}$ and $\deg \rho = 1$ $\rightsquigarrow$ $\tilde{\kappa}_a = \kappa_a$.

μ-classes: In $\overline{M}_{g,k}$, Hodge class $\lambda_i = c_i(R\pi_* K_C)$.

ρ: finite $\Rightarrow$ $R\pi_* K_{\mathcal{C}} = R\pi_*(\rho_* K_{\mathcal{C}}) = R\pi_* K_C$.

$\pi : \mathcal{C}_{g,k} \longrightarrow \overline{M}_{g,k}$

In $\overline{W}_{g,k}$ case, $\mu_{ij} := \underline{Ch}_i(R\pi_* \mathcal{L}_j)$.

Chern character.

- <u>State space</u>

$W : \mathbb{C}^N \longrightarrow \mathbb{C}$ $\rightsquigarrow$ Milnor ring $Q_W := \mathbb{C}[x_1 \ldots x_N] / \langle \partial_1 W, \ldots, \partial_N W \rangle$.

$\dim Q_W = \prod \left(\frac{1}{q_i} - 1\right)$, central charge $\hat{c}_W := \sum_i (1 - 2q_i)$.

A-model: relative cohomology $H^N(\mathbb{C}^N, W^\infty, \mathbb{C})$, $W^\infty = (\mathrm{Re}\, W)^{-1}(M, \infty)$ for $M \gg 0$.

$\updownarrow$ dual $\qquad W^{-\infty} = (\mathrm{Re}\, W)^{-1}(-\infty, -M)$

$H_N(\mathbb{C}^N, W^\infty, \mathbb{Z})$

$\langle , \rangle : H_N(\mathbb{C}^N, W^\infty, \mathbb{Z}) \times H_N(\mathbb{C}^N, W^{-\infty}, \mathbb{Z}) \longrightarrow \mathbb{Z}$: topological intersection pairing, non-degenerate!

Write $q_i = \frac{n_i}{d}$, $\zeta^d = -1$, $I = (\zeta^{n_1}, \ldots, \zeta^{n_N}) : \mathbb{C}^N \xrightarrow{\sim} \mathbb{C}^N$ diagonal action.

$\rightsquigarrow I_* : H_N(\mathbb{C}^N, W^\infty, \mathbb{C}) \xrightarrow[W^\infty \longrightarrow W^{-\infty}]{\sim} H_N(\mathbb{C}^N, W^{-\infty}, \mathbb{C})$.

Define $\langle , \rangle$ on $H_N(\mathbb{C}^N, W^\infty, \mathbb{Z})$ by $\langle \Delta_i, \Delta_j \rangle := \langle \Delta_i, I_*(\Delta_j) \rangle$.

$\rightsquigarrow$ induce $\langle , \rangle$ on $H^N(\mathbb{C}^N, W^\infty, \mathbb{C})$.

For $\gamma \in G$, $\mathbb{C}^{N_\gamma}$:= fixed loci of $\mathbb{C}^N$ by γ

$W_\gamma := W|_{\mathbb{C}^{N_\gamma}}$.

γ-twisted sector : $\mathcal{H}_\gamma := H^{N_\gamma}(\mathbb{C}^{N_\gamma}, W_\gamma^\infty, \mathbb{C})^G$,

central charge of W_γ : $\hat{c}_\gamma := \sum_{i : \theta_i^\gamma = 0} (1 - 2q_i)$.

$\gamma = (\exp 2\pi i \theta_1^\gamma, \ldots, \exp 2\pi i \theta_N^\gamma)$ $\rightsquigarrow$ degree shifting number $\iota_\gamma := \sum_i (\theta_i^\gamma - q_i) = \frac{\hat{c}_\gamma - N_\gamma}{2} + \sum_{i : \theta_i^\gamma \neq 0} (\theta_i^\gamma - q_i)$.

$\rightsquigarrow$ for $\alpha \in \mathcal{H}_\gamma$, define $\deg_W(\alpha) = \deg(\alpha) + 2\iota_\gamma$.

Fact For $\gamma \in G_W$, $\iota_\gamma + \iota_{\gamma^{-1}} = \hat{c}_W - N_\gamma$.

For $\alpha \in \mathcal{H}_\gamma$, $\beta \in \mathcal{H}_{\gamma^{-1}}$, $\deg_W(\alpha) + \deg_W(\beta) = 2\hat{c}_W$.

State space : $\mathcal{H}_W := \bigoplus_{\gamma \in G} \mathcal{H}_\gamma \supseteq \underbrace{\mathcal{H}_J}_{J\text{-sector}} = \operatorname{span}\{\underbrace{e_1 := 1}_{\text{unit}}\}$: 1-dimension of $\deg = 0$.

Note : $\mathbb{C}^{N_\gamma} = \mathbb{C}^{N_{\gamma^{-1}}} \Rightarrow \exists\ \varepsilon : \mathcal{H}_\gamma \longrightarrow \mathcal{H}_{\gamma^{-1}}$.

Define $\langle\,,\,\rangle_\gamma : \mathcal{H}_\gamma \otimes \mathcal{H}_{\gamma^{-1}} \longrightarrow \mathbb{C}$ by $\langle \alpha, \beta \rangle_\gamma = \langle \alpha, \varepsilon^* \beta \rangle$.

symmetric, non-degenerate.

$\rightsquigarrow$ Pairing $\langle\,,\,\rangle$ on $\mathcal{H}_W$ as direct sum of $\langle\,,\,\rangle_\gamma$.

Fact (1) $\langle\,,\,\rangle$ on $\mathcal{H}_W$ preserve $\deg_W$.

(2) $\mathcal{H}_\gamma^a := \{ x \in \mathcal{H}_\gamma \mid \deg_W(x) = a \}$. Then, $\langle\,,\,\rangle : \mathcal{H}_W^a \otimes \mathcal{H}_W^{2\hat{c}_W - a} \longrightarrow \mathbb{C}$.

- **CFT**

$st : \overline{W}_{g,k} \longrightarrow \overline{M}_{g,k}$ $\quad \gamma = (\gamma_1 \ldots \gamma_k) \in G^k$

$[\overline{W}_{g,k,G}(W,\gamma)]^{vir} \in H_*(\overline{W}_{g,k,G}(W,\gamma), \mathbb{Q}) \otimes \prod_i H_{N_{\gamma_i}}(\mathbb{C}^{N_{\gamma_i}}, W_{\gamma_i}^\infty, \mathbb{Q})^G$.

For W-graph Γ, write $D := -\sum_{i=1}^N \operatorname{index}(\mathcal{L}_i) = \hat{c}_W(g-1) + \sum_{j=1}^k \iota_{\gamma_j}$.

The virtual cycle $[\overline{W}(\Gamma)]^{vir}$ has dimension $6g-6+2k-2\#E(\Gamma)-2D$

$[\overline{W}(\Gamma)]^{vir} \in H_r(\overline{W}(\Gamma), \mathbb{Q}) \otimes \prod_{\tau \in \text{tails}} H_{N_{\gamma_\tau}}(\mathbb{C}^{N_{\gamma_\tau}}, W_{\gamma_\tau}^\infty, \mathbb{Q})$.

$= 2\left((\hat{c}-3)(1-g) + k - \#E(\Gamma) - \sum_{\tau \in \text{tails}} \iota_\tau\right)$

Define $\Lambda_{g,k}^W \in \operatorname{Hom}(\mathcal{H}_W^{\otimes k}, H^*(\overline{M}_{g,k}))$ by $\alpha = (\alpha_1 \ldots \alpha_k)$ with $\alpha_i \in H_{\gamma_i}$

$$\Lambda_{g,k}^W(\alpha) := \frac{|G|^g}{\deg(st)} \underbrace{PD}_{\text{Poincaré dual}}\, st_*\left([\overline{W}_{g,k}(W,\gamma)]^{vir} \cap \prod_{i=1}^k \alpha_i \right).$$

Theorem The collection $(\mathcal{H}_W, \underbrace{\langle\,,\,\rangle^W}_{\text{pairing on } \mathcal{H}_W}, \{\Lambda_{g,k}^W\}, \underbrace{e_1}_{\text{generator in } \mathcal{H}_J})$ is a CFT with flat identity.

sketch : Need to prove :

(C1) $\Lambda_{g,k}^W$ is invariant under S_k action.

(C2) $\rho_{tree} : \overline{M}_{g_1, k_1+1} \times \overline{M}_{g_2, k_2+1} \longrightarrow \overline{M}_{g_1+g_2, k_1+k_2}$. Then, $\Lambda_{g,k}^W$ satisfy :

$$\rho_{tree}^* \Lambda_{g_1+g_2, k_1+k_2}^W(\alpha_1 \ldots \alpha_{k_1+k_2}) = \sum_{\mu,\nu} \Lambda_{g_1,k_1+1}^W(\alpha_{i_1} \ldots \alpha_{i_{k_1}}, \mu)\, \eta^{\mu\nu} \otimes \Lambda_{g_2,k_2+1}^W(\nu, \alpha_{i_{k_1+1}} \cdots \alpha_{i_{k_1+k_2}})$$

where η, ν : run through basis of $\mathcal{H}_W$, $\eta^{\mu\nu} = \langle \eta, \nu \rangle^W$.

(C3) $\rho_{loop}: \overline{M}_{g-1,k+2} \longrightarrow \overline{M}_{g,k}$ Then, $\Lambda^{W}_{g,k}$ satisfy:

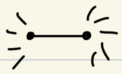
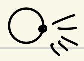
$$\rho_{loop}^{*} \Lambda^{W}_{g,k}(\alpha_1 \dots \alpha_k) = \sum_{\mu,\nu} \Lambda^{W}_{g-1,k+2}(\alpha_1 \dots \alpha_k, \mu, \nu)\, \eta^{\mu\nu}.$$

(C4a) $\Lambda^{W}_{g,k+1}(\alpha_1 \dots \alpha_k, e_1) = \vartheta^{*} \Lambda^{W}_{g,k}(\alpha_1 \dots \alpha_k)$, $\vartheta: \overline{M}_{g,k+1} \longrightarrow \overline{M}_{g,k}$ universal curve.

(C4b)
$$\int_{\overline{M}_{0,3}} \Lambda^{W}_{0,3}(\alpha_1, \alpha_2, e_1) = \langle \alpha_1, \alpha_2 \rangle^{W}.$$

(C1)(C4a)(C4b) follows from the axiom of $[\overline{W}_{g,k}(\Gamma)]^{vir}$.

For (C2)(C3), we need the lemma:

<u>Lemma</u> If $\Gamma =$  or , then for any $\alpha \in H_*(\overline{W}_{g,k}(\gamma))$, we have

$$\begin{tikzcd} \bigcup_{\varepsilon \in G} \overline{W}(\Gamma_\varepsilon) \arrow[r, "\tilde{i}"] \arrow[d, "\sum_\varepsilon st_{\Gamma_\varepsilon}"'] & \overline{W}_{g,k}(\gamma) \arrow[d, "st"] \\ \overline{M}(\Gamma) \arrow[r, "i"] & \overline{M}_{g,k} \end{tikzcd}$$

$$i^* st_* \alpha = \sum_{\varepsilon \in G} |\langle \varepsilon \rangle| \, (st_{\Gamma_\varepsilon})_* \tilde{i}^* \alpha.$$

Thus, $i^* st_* [\overline{W}_{g,k}(\gamma)]^{vir} = \sum_{\varepsilon \in G} |\langle \varepsilon \rangle| \, (st_{\Gamma_\varepsilon})_* \tilde{i}^* [\overline{W}_{g,k}(\gamma)]^{vir}$.

#

Define the correlators $\langle \tau_{\ell_1}(\alpha_1) \dots \tau_{\ell_k}(\alpha_k) \rangle^{W,G}_{g} = \int_{[\overline{M}_{g,k}]} \Lambda^{W,G}_{g,k}(\alpha_1, \dots, \alpha_k) \prod_{i=1}^{k} \psi_i^{\ell_i}$.

$\rightsquigarrow$ Potential function $\Phi^{W,G}(t) = \sum_{g \geq 0} \Phi^{W,G}_{g}(t) = \sum_{g \geq 0} \lambda^{2g-2} \sum_{k} \frac{1}{k!} \sum_{\ell_1 \dots \ell_k} \sum_{\alpha_1 \dots \alpha_k} \langle \tau_{\ell_1}(\alpha_1) \dots \tau_{\ell_k}(\alpha_k) \rangle^{W,G}_{g} \, t^{\alpha_1}_{\ell_1} \cdots t^{\alpha_k}_{\ell_k}$.

$\{\alpha_0 = 1, \alpha_1 \dots \alpha_s\}$: basis of $\mathcal{H}_W$, $t = (t_0, t_1, \dots)$, $t_\ell = (t^{\alpha_0}_{\ell} \dots t^{\alpha_s}_{\ell})$ formal variables.

<u>Theorem</u> (Manin) The genus $= 0$ theory defines a formal Frobenius manifold structure on $\mathbb{Q}[[\mathcal{H}_W^*]]$ with pairing $\langle \, , \rangle^{W}$ and potential $\Phi^{W}_{0}(t)$.

<u>Theorem</u> The potential $\Phi^{W,G}(t)$ satisfies the analogues of string and dilton equation and TRR.

proof: Just use $\vartheta^* \psi_i = \psi_i + D_{i|k+1}$.

Also, $\psi_i = \sum_{A \cup B = \{1..k\}} D_{i|k-1,k}$ on $\overline{M}_{0,k}$.

#

Quantum Singularity Theory 2022. 6. 27.

Main Theorem !

(1) Except for D_n, n: odd, $\mathcal{H}_{W\langle J\rangle}$ of any ADE singularities is isomorphic to Milnor ring of the same singularity.

(2) $\mathcal{H}_{D_n, G_{D_n}} \simeq Q_{A_{2n-3}}$: Milnor ring of $W = x^{n-1}y + y^2$.

(3) $\mathcal{H}_{W, G_W}$ with $W = x^{n-1}y + y^2$ $(n \geq 4)$ is isomorphic to Q_{D_n} : Milnor ring of D_n.

B-model : $Q_W = \mathbb{C}[x_1 \cdots x_N] / \langle \partial_1 W \ldots \partial_N W \rangle$, $\omega = dx_1 \wedge \cdots \wedge dx_N$.

$$\langle f, g \rangle = \mathrm{Res}_{x=0} \frac{fg\, dx_1 \wedge \cdots \wedge dx_N}{\partial_1 W \cdots \partial_N W}.$$

A-model : $\mathcal{H}_{W,G}$, $\langle\, ,\, \rangle^W$

Wall's isomorphism : $(H^N(\mathbb{C}^N, W^\infty, \mathbb{C})^{\langle J\rangle}, \langle\, ,\, \rangle) \simeq ((Q_W \omega)^{\langle J\rangle}, \mathrm{Res})$

$$\rightsquigarrow \mathcal{H}_{W,G} = \bigoplus_{\gamma \in G} (H^{mid}(\mathbb{C}^{N_\gamma}, W_\gamma^\infty, \mathbb{Q}))^G \simeq \bigoplus_{\gamma \in G} (Q_{W_\gamma} \omega_\gamma)^G$$

volume form to $\mathbb{C}^{N_\gamma}$.

In the following computation, we will use this identification.

Self-mirror case

(1) A_n cases : $W = x^{n+1}$, $G_{A_n} = \langle J \rangle$, $H^{mid}(\mathbb{C}^{N_{J^k}}, W_{J^k}^\infty, \mathbb{Q})^G = \mathrm{span}\{e_k\}$.

$$Q_{A_n} = \mathbb{C}[X] / X^n \xrightarrow{\sim} \mathcal{H}_{A_n, G_{A_n}} \qquad \alpha \in \mathbb{C} \text{ s.t. } \alpha^{n-1} = \frac{1}{n+1}.$$

$$X^i \longmapsto \alpha^i e_{i+1}$$

(2) E_7 cases : $W = x^3 + xy^3 \rightsquigarrow q_x = 1/3$, $q_y = 2/9$, $\hat{c}_{E_7} = 8/9$.

$\zeta = \exp(\frac{2\pi i}{9})$, J acts as (ζ^3, ζ^2), i.e. $\theta_x^J = 1/3$, $\theta_y^J = 2/9$, $G_{E_7} = \langle J \rangle \simeq \mathbb{Z}/9\mathbb{Z}$.

$$\begin{cases} e_0 := dx \wedge dy \in H^{mid}(\mathbb{C}^{N_{J^0}}, W_{J^k}^\infty, \mathbb{Q}) \\ e_k := dx \quad \in H^{mid}(\mathbb{C}^{N_{J^k}}, W_{J^k}^\infty, \mathbb{Q}) \text{ for } k = 3, 6. \\ e_k := 1 \quad \in H^{mid}(\mathbb{C}^{N_{J^k}}, W_{J^k}^\infty, \mathbb{Q}) \text{ for } 3 \nmid k. \end{cases}$$

$1 := e_1$.

$$\rightsquigarrow H^{mid}(\mathbb{C}^{N_{J^k}}, W_{J^k}^\infty, \mathbb{Q}) = \begin{cases} \mathrm{span}\{e_0, xe_0, x^2e_0, ye_0, y^2e_0, xye_0, x^2ye_0\} & \text{if } k = 0 \\ \mathrm{span}\{e_k, xe_k\} & \text{if } k = 3, 6, \\ \mathrm{span}\{e_k\} & \text{if } 3 \nmid k. \end{cases}$$

$\rightsquigarrow G_{E_7}$-invariant space : $\mathcal{H}_{E_7, G_{E_7}} = \mathrm{span}\{y^2e_0, 1, e_2, e_4, e_5, e_7, e_8\}$.

We compute $g=0$, 3-point correlators :

degree shift : $\iota_{J^k} = \sum_{i=1}^{N} (\theta_i^{J^k} - q_i)$

$$\deg_W(x^i y^j e_k) = \deg(x^i y^j e_k) + 2\iota_{J^k} = N_{J^k} + 2\iota_{J^k}.$$

$\rightsquigarrow$

k	0	1	2	3	4	5	6	7	8
ι_{J^k}	$-5/9$	0	$5/9$	$1/9$	$6/9$	$2/9$	$-2/9$	$3/9$	$8/9$
$\deg_W(x^i y^j e_k)$	$8/9$	0	$10/9$	$11/9$	$12/9$	$4/9$	$5/9$	$6/9$	$16/9$

$\langle a e_{k_1}, b e_{k_2}, c e_{k_3} \rangle_0^{E_7}$. $\quad d^{\mathrm{vir}} = -2\hat{c}_{E_7} + \sum_{j=1}^{3} \deg_W(e_{k_j}) = 0 = \dim \overline{M}_{0,3}$.

$\rightsquigarrow$ The only non-trivial $g=0$, 3-point correlators :

$\langle y^2 e_0, y^2 e_0, 1 \rangle$, $\langle y^2 e_0, e_5, e_5 \rangle$,

$\langle 1, 1, e_8 \rangle$, $\langle 1, e_2, e_7 \rangle$, $\langle 1, e_4, e_5 \rangle$, $\langle e_5, e_7, e_7 \rangle$.

- $\deg(|\mathcal{L}_x|) = q_x(2g-2+k) - \sum_{\ell=1}^{k} \theta_x^{\gamma_\ell} = 1/3 - \sum_{\ell=1}^{3} \theta_x^{J^{k_\ell}}$

  $\deg(|\mathcal{L}_y|) = q_y(2g-2+k) - \sum_{\ell=1}^{k} \theta_y^{\gamma_\ell} = 2/9 - \sum_{\ell=1}^{3} \theta_y^{J^{k_\ell}}$

  Since $g=0$, $\deg(|\mathcal{L}_x|), \deg(|\mathcal{L}_y|) < 0$: concave $\Rightarrow$ virtual cycle

  $\updownarrow$ Poincaré

  top Chern class of $R^1\pi_*(\mathcal{L}_x \oplus \mathcal{L}_y) = 0$.

  $\rightsquigarrow$ $\langle 1, 1, e_8 \rangle, \langle 1, e_2, e_7 \rangle, \langle 1, e_4, e_5 \rangle, \langle e_5, e_7, e_7 \rangle = 1$.

- $\langle y^2 e_0, y^2 e_0, 1 \rangle_0$: residue pairing of y^2 and y^2 in $H^{\mathrm{mid}}(\mathbb{C}^{N_{J^0}}, W_{J^0}^{\infty}, \mathbb{Q}) = \mathcal{Q}_{E_7}$.

  $$\rightsquigarrow \langle y^2 e_0, y^2 e_0, 1 \rangle_0 = \langle y^2, y^2 \rangle_{\mathcal{Q}_{E_7}} = \operatorname{Res}_{x,y=0} \frac{y^4\, dx \wedge dy}{(3x^2+y^3)\cdot 3xy^2} = -\frac{1}{3}.$$

- Compute $\langle y^2 e_0, e_5, e_5 \rangle$ using composition :

  $[\overline{W}_{0,4}(E_7, J^5, J^5, J^5, J^5)]^{\mathrm{vir}} = \underline{-3} \cdot (\text{fundamental cycle})$

  degree of $H^0(|\mathcal{L}_x| \oplus |\mathcal{L}_y|) \longrightarrow H^1(|\mathcal{L}_x| \oplus |\mathcal{L}_y|)$

  $\qquad\qquad\quad \overset{\shortparallel}{0 \oplus \mathbb{C}} \ni (x,y) \longmapsto (3x^2+y^3, 2xy) \in \overset{\shortparallel}{\mathbb{C} \oplus 0}$

  $\rightsquigarrow \Lambda_{0,4}^{E_7}(e_5, e_5, e_5, e_5) = -3$

  $$\overset{\shortparallel}{\sum_{i,j} \Lambda_{0,3}^{E_7}(e_5, e_5, \alpha_i)\, \eta^{ij}\, \Lambda_{0,3}^{E_7}(\alpha_j, e_5, e_5)} = -3 \left(\Lambda_{0,3}^{E_7}(y^2 e_0, e_5, e_5) \right)^2$$

  $$\Rightarrow \langle e_5, e_5, y^2 e_0 \rangle_0 = \int_{\overline{M}_{0,3}} \Lambda_{0,3}^{E_7}(y^2 e_0, e_5, e_5) = \pm 1.$$

- Using $\langle e_k, e_{9-k} \rangle_{\mathcal{H}_{E_7}} = 1$, define $\mathbb{C}[X,Y] \xrightarrow{\phi_\alpha} \mathcal{H}_{E_7}$:

$X \longmapsto \alpha^3 e_7$, $1 \longmapsto 1 = e_1$, $XY \longmapsto \alpha^5 e_2$, $Y^2 \longmapsto \mp 3\alpha^4 y^2 e_0$.

$Y \longmapsto \alpha^2 e_5$, $X^2 \longmapsto \alpha^6 e_4$, $X^2Y \longmapsto \alpha^8 e_8$.

Also, $\phi(X) * \phi(Y)^2 = 0$.

$$\phi(Y)^3 = \phi(Y) * (\mp 3\alpha^4 y^2 e_0) = \mp 3\alpha^6 \sum_{i,j} \langle e_5, y^2 e_0, \alpha_i \rangle \eta^{ij} \alpha_j = -3\alpha^6 e_4 = -3\phi(X)^2.$$

$\rightsquigarrow \phi_\alpha : Q_{E_7} = \mathbb{C}[X,Y] / \langle XY^2, Y^3 + 3X^2 \rangle \xrightarrow{\sim} (\mathcal{H}_{E_7}, *)$ as graded algebra.

Preserving pairing: $\langle 1, X^2Y \rangle_{Q_{E_7}} = \frac{1}{9}$, $\langle Y^2, Y^2 \rangle_{Q_{E_7}} = -\frac{1}{3}$

$\langle 1, e_8 \rangle_{\mathcal{H}_{E_7}} = 1$, $\langle \mp y^2 e_0, \mp y^2 e_0 \rangle_{\mathcal{H}_{E_7}} = -3$.

$\rightsquigarrow$ Choose α s.t. $\alpha^8 = 1/9$.

$\rightarrow \phi_\alpha$: isomorphism as Frobenius algebra $Q_{E_7} \simeq (\mathcal{H}_{E_7}, *)$.

(3) E_6, E_8 case:

$$\mathcal{H}_{H_6, G_{E_6}} \simeq \mathcal{H}_{A_2, G_{A_2}} \otimes \mathcal{H}_{A_3, G_{A_3}} \simeq Q_{A_2} \otimes Q_{A_3} \simeq Q_{E_6}$$
$$\mathcal{H}_{H_8, G_{E_8}} \simeq \mathcal{H}_{A_2, G_{A_2}} \otimes \mathcal{H}_{A_4, G_{A_4}} \simeq Q_{A_2} \otimes Q_{A_4} \simeq Q_{E_8}.$$

<u>Explicitly</u>: ① E_6 : $x^3 + y^4$.

$\mathcal{H} = \mathrm{span}\{e_1, e_2, e_5, e_7, e_{10}, e_{11}\}$, $e_i = 1 \in H^{mid}(\mathbb{C}^{N_{J^i}}, W^\infty_{J^i}, \mathbb{Q})$.

$Q_{E_6} \longrightarrow \mathcal{H}_{E_6, G_{E_6}}$: $Y \longmapsto \alpha^3 e_5$, $X \longmapsto \alpha^4 e_{10}$ with $\alpha^{10} = 1/12$.

② E_8 : $x^3 + y^5$.

$\mathcal{H} = \mathrm{span}\{e_1, e_2, e_4, e_7, e_8, e_{11}, e_{13}, e_{14}\}$, $e_i = 1 \in H^{mid}(\mathbb{C}^{N_{J^i}}, W^\infty_{J^i}, \mathbb{Q})$.

$Q_{E_8} \longrightarrow \mathcal{H}_{E_8, G_{E_8}}$: $Y \longmapsto \alpha^3 e_7$, $X \longmapsto \alpha^5 e_{11}$ with $\alpha^{14} = 1/15$.

(4) D_{n+1} case with n : odd and $G = \langle J \rangle$.

$$\mathcal{H}_{D_{n+1}, \langle J \rangle} = \mathrm{span}\{x^{(n-1)/2} e_n, y e_n, e_1, \dots, e_{n-1}\}.$$

$\rightsquigarrow \mathbb{C}[X,Y] \longrightarrow \mathcal{H}_{D_{n+1}, \langle J \rangle}$ $(n \geq 5)$

$X \longmapsto e_3$

$Y \longmapsto \alpha(x^{\frac{n-1}{2}} e_n) + \beta(y e_n)$, $\alpha = \mp 2n \cdot \langle e_3, e_{n-2}, y e_n \rangle_0$

$\beta = \pm 2n \cdot \langle e_3, e_{n-2}, x^{(n-1)/2} e_n \rangle_0$.

$\boxed{n=3}$: $X \longmapsto x e_3$, $Y \longmapsto y e_3$.

Non-self mirror case

(1) D_{n+1} with $G = G_{D_{n+1}}$. (When n is odd, $[G_{D_{n+1}} : \langle J \rangle] = 2$)

$$\mathcal{H}_{D_{n+1}, G} = \mathrm{span}\{ye_0, e_1, \dots, e_{n-1}, e_{n+1}, \dots, e_{2n-1}\}$$

$$\rightsquigarrow \phi : Q_W = \mathbb{C}[X,Y] \big/ \langle X^{n-1}Y, X^n + 2Y \rangle \longrightarrow (\mathcal{H}_{D_{n+1}, G}, *), \quad W = x^n y + y^2 \overset{\text{''}}{\cdot\cdot} D_{n+1}^T.$$

$$X^i \longmapsto \alpha^i \cdot \begin{cases} e_{n+1+i}, & 0 \le i < n-1 \\ \mp 2ye_0, & i = n-1 \\ e_{i-n+1}, & n \le i < 2n-1 \end{cases} \qquad Y \longmapsto -\frac{\alpha^n}{2} e_1$$

$\alpha \in \mathbb{C}$ s.t. $\alpha^{2n-2} = -\frac{1}{n}$.

(2) $W = x^n y + y^2$. $\mathcal{H}_{W, G_W} \simeq Q_{D_{n+1}}$.

• Results on Integrable Hierachies

W : non-deg. quasi-homo. $\{\phi_1 = 1, \phi_2 \dots \phi_\mu\}$: monomial basis of Q_W.

Miniversal deformation $\mathbb{C}^\mu \ni t = (t_1 \dots t_\mu) \leftrightarrow W + \underline{t_1\phi_1} + \cdots + t_\mu \phi_\mu$.

$\deg t_i = 1 - \deg \phi_i$.

Tagent space $T_t \mathbb{C}^\mu$ have $\underset{\text{associative multiplication}}{\circ}$, $\underset{\text{Euler vector field}}{E = \sum_i \deg t_i \cdot \partial_{t_i}}$, $\underset{\text{residue pairing}}{\langle\ ,\ \rangle}$, $\mathcal{F}$ $\rightsquigarrow$ Frobenius manifold structure near $0 \in \mathbb{C}^\mu$.

Givental constructe "formal GW potential function" on any semi-simple Frobenius manifold

$$\mathcal{D}_{W, \text{formal}} := \exp\Big(\sum_{g \ge 0} h^{2g-2} \mathcal{F}^g_{\text{formal}}\Big).$$

It satisfies: (1) $\mathcal{F}^0_{\text{formal}} = \mathcal{F}$

(2) $\mathcal{D}_{W, \text{formal}}$ satisfies all formal axioms in GW.

Main Theorem 2 (1) Except for D_n, n:odd or $n=4$, all ADE-singularities have $\Phi^{W, \langle J \rangle} = \mathcal{D}_{W, \text{formal}}$ up to a linear change of variables.

(2) $\Phi^{D_n, G_{D_n}} = \mathcal{D}_{A_{2n-3}, \text{formal}}$ up to a linear change of variables.

(3) For $D_n^T = x^{n-1}y + y^2$ $(n > 4)$, $\Phi^{D_n^T, G_{D_n^T}} = \mathcal{D}_{D_n, \text{formal}}$.

Define the ancestor correlator $\langle \tau_{\ell_1}(\alpha_1) \dots \tau_{\ell_n}(\alpha_n) \rangle_g^{W,G}(t) = \sum_k \frac{1}{k!} \langle \tau_{\ell_1}(\alpha_1) \dots \tau_{\ell_n}(\alpha_n), \overbrace{t \dots t}^{k} \rangle_g^{W,G}$

Lemma $\{T^i\}$: basis of $\mathcal{H}_{W,G}$. $t = \sum t_i T^i$. For ADE-singularities, $\langle \tau_{\ell_1}(\alpha_1) \dots \tau_{\ell_n}(\alpha_n) \rangle_g^{W,G}(t)$ is just a polynomial in t_i. Moreover, if $\ell_1 = \dots = \ell_n = 0$, it is just a polynomial in α_i.

(Just dimension constraint!)

Lemma $\Rightarrow$ We only need to consider $\mathcal{F}_t^{W.G}$, $\mathcal{A}_t^{W.G}$ for semi-simple point $t \neq 0$.

Quantum singularity theory (A) Saito singularity theory (B)

- Reconstruction Theorem

Theorem If $\hat{c} < 1$, the ancestor potential function is uniquely determined by genus 0 primary potential.

If $\hat{c} = 1$, the ancestor potential function is uniquely determined by genus 0 and 1 primary potential.

Lemma $\alpha_i \in \mathcal{H}_{\gamma_i, G}$, $\beta = \prod \psi_i$. Assume $\deg \beta < g$ for $g \geq 1$ $(\hat{c} < 1)$ or $\deg \beta < g$ for $g \geq 2$ $(\hat{c} = 1)$.

Then, $\displaystyle\int_{\overline{M}_{g.n+k}} \beta \cdot \Lambda^W_{g.n+k}(\alpha_1 \ldots \alpha_n, T_{i_1} \ldots T_{i_k}) = 0$.

proof: It $\neq 0$ only if $\deg \beta = 3g - 3 + n + k - \hat{c}(g-1) - \sum_{\tau} \underbrace{\iota_{\gamma_\tau}}_{\sum_{i=1}^{N}(\theta_i^\gamma - q_i)} - \sum_{\tau=1}^{n+k} N_{\gamma_\tau}/2$.

Note: $\iota_{\gamma_\tau} + \frac{N_{\gamma_\tau}}{2} \leq \hat{c}$ for ADE-singularity.

$\rightsquigarrow \deg \beta \geq (3 - \hat{c})(g-1) + (n+k)(1-\hat{c})$.

Hence, if $g \geq 2$, we have $\deg \beta \geq g$.

If $g = 1$, then $\deg \beta > 0$ for $\hat{c} < 1$

$\deg \beta \geq 0$ for $\hat{c} = 1$. #

Lemma (Faber - Shadrin - Zvonkine) (g-reduction)

P = monomial in ψ, κ-classes in $\overline{M}_{g.k}$ of $\deg \geq g$ $(g \geq 1)$.

≥ 1 $(g = 0)$

Then, P = linear combination of dual graphs.

with at least one edge.

proof of thm: $\langle \tau_{d_1}(\alpha_1) \cdots \tau_{d_n}(\alpha_n), T_{i_1} \cdots T_{i_k} \rangle_{g.n+k} = \displaystyle\int_{\overline{M}_{g.n+k}} \psi_1^{d_1} \cdots \psi_n^{d_n} \Lambda^W_{g\,n+k}(\alpha_1 \ldots \alpha_n, T_{i_1} \ldots T_{i_k})$.

$\deg \psi_1^{d_1} \cdots \psi_n^{d_n}$ small $\Rightarrow$ vanishes

large $\Rightarrow$ change integral over boundary class,

then apply splitting formula! #

Theorem (Reconstruction lemma)

$$\langle \gamma_1, \dots, \gamma_{k-3}, \alpha, \beta, \varepsilon * \phi \rangle_0 = S + \langle \gamma_1 \dots \gamma_{k-3}, \alpha, \varepsilon, \beta * \phi \rangle_0$$
$$- \langle \gamma_1 \dots \gamma_{k-3}, \alpha * \varepsilon, \beta, \phi \rangle_0$$
$$+ \langle \gamma_1 \dots \gamma_{k-3}, \alpha * \beta, \varepsilon, \phi \rangle_0$$

where S = combination of genus 0 correlators with fewer than k insertions.

Moreover, all k-point, genus 0 correlators are determined by pairing, 3-point correlators, and $\langle \underline{\alpha_1 \dots \alpha_{k'-2}}, \alpha_{k'-1}, \alpha_{k'} \rangle_0$ for $k' \le k$.

all primitive (means $\alpha_i \neq \alpha * \beta$).

proof: Choose basis $\delta_0 = \varepsilon * \phi$, $\delta_1 \dots$ and δ_i' : dual basis.

WDVV $\Rightarrow \langle \gamma_1, \dots, \gamma_{k-3}, \alpha, \beta, \varepsilon * \phi \rangle_0 = \langle \gamma_1 \dots \gamma_{k-3}, \alpha, \beta, \varepsilon * \phi \rangle_0 \langle \delta_0', \varepsilon, \phi \rangle_0$

$$= \sum_{I \cup J = k-3} \sum_{\ell} \langle \gamma_I, \alpha, \varepsilon, \delta_\ell \rangle_0 \langle \delta_\ell', \phi, \beta, \gamma_J \rangle_0 - \sum_{\substack{I \cup J = k-3 \\ J \neq \phi}} \sum_{\ell} \langle \gamma_I, \alpha, \beta, \delta_\ell \rangle_0 \langle \delta_\ell', \phi, \varepsilon, \gamma_J \rangle_0$$

k-point terms: $\sum_\ell \langle \gamma_{i \le k-3}, \alpha, \varepsilon, \delta_\ell \rangle \langle \delta_\ell', \phi, \beta \rangle + \sum_\ell \langle \alpha, \varepsilon, \delta_\ell \rangle \langle \delta_\ell', \phi, \beta, \gamma_{j \le k-3} \rangle$

$$- \sum_\ell \langle \alpha, \beta, \delta_\ell \rangle \langle \delta_\ell', \phi, \varepsilon, \gamma_{j \le k-3} \rangle.$$

#

Lemma If $\deg(a) \le \hat{c}$ for all class a, $P = \max\deg\{\text{primitive class}\}$, then all genus 0 correctors are determined by pairing and k-point correlators with $k \le 2 + \frac{1+\hat{c}}{1-P}$

proof: $\langle \underline{a_1 \dots a_{k-2}}, a_{k-1}, a_k \rangle$. dimension count: $\deg a_i \le P \quad i = 1 \sim k-2$
primitive
$\le \hat{c} \quad i = k-1, k$

$$\rightsquigarrow \hat{c} + k - 3 \le (k-2)P + 2\hat{c}.$$

#

Lemma All genus 0 correlators for A_n, D_{n+1}, E_6, E_7, E_8, D_{n+1}^T are uniquely determined by pairing, 3-point, 4-point correlators.

proof: In each case

	A_n	E_6	E_7	E_8	D_{n+1} (n: even)	D_{n+1} (n: odd)	D_{n+1}^T
P	$\frac{1}{n+1}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{n}$	$\frac{n-1}{2n}$	$\frac{n-1}{2n}$
$\hat{c}$	$\frac{n-1}{n+1}$	$\frac{5}{6}$	$\frac{8}{9}$	$\frac{14}{15}$	$\frac{n-1}{n}$	$\frac{n-1}{n}$	$\frac{n-1}{n}$
lemma $\Rightarrow$	$k \le 4$					$k \le 5$	

For D_{n+1} (n: odd), D_{n+1}^T, use $(\mathcal{H}_{D_{n+1}, \langle J \rangle}, *) \simeq Q_{D_{n+1}}$ to compute non-trivial
$(\mathcal{H}_{D_{n+1}^T, G}, *) \simeq Q_{D_{n+1}}$

5-point correlators. Then use reconstruction formula. #

Theorem (6.2.10) only state E_7 case :

In quantum singularity theory (A) with maximal symmetric group G_{E_7} and in Saito singularity theory (B), all genus 0 correlators are determined by the pairing, 3-point correlators, and $\langle X, X, X^2, XY\rangle_0$, $\langle X, Y, X^2, X^2\rangle_0$, $\langle Y, Y, XY, X^2Y\rangle_0$.

proof: $\deg X = \frac{1}{3}$, $\deg Y = \frac{2}{9}$, $\hat{c} = \frac{8}{9}$.

Dimension constraint $\Rightarrow$ $\langle X, X, X^2, XY\rangle_0$, $\underline{\langle X, X, X, X^2Y\rangle_0}$, $\langle X, Y, X^2, X^2\rangle_0$, $\underline{\langle X, Y, Y^2, X^2Y\rangle_0}$, $\langle Y, Y, XY, X^2Y\rangle_0$ are non-trivial.

$$\langle X, \underset{\alpha}{X}, \underset{\beta}{X}, \underset{\varepsilon*\phi}{X^2Y}\rangle_0 = S + \langle X, \underset{\alpha}{X}, \underset{\varepsilon}{X^2}, \underset{\beta*\phi}{XY}\rangle + \langle X, \underset{\substack{\alpha*\varepsilon\\ \shortparallel \\ 0}}{X^3}, \underset{\beta}{X}, \underset{\phi}{Y}\rangle - \langle X, \underset{\alpha*\beta}{X^2}, \underset{\varepsilon}{X^2}, \underset{\phi}{Y}\rangle$$

$$\langle X, Y, Y^2, X^2Y\rangle_0 = \langle Y, X, Y^2, \underbrace{X^2Y}_{=X^2*Y}\rangle$$

$$= S + \langle Y, X, X^2, \underset{\substack{\shortparallel\\ -3X^2}}{Y^3}\rangle + \langle Y, \underset{\substack{\shortparallel\\ 0}}{X^3}, Y^2, Y\rangle - \langle Y, \underset{\substack{\shortparallel\\ 0}}{XY^2}, X^2, Y\rangle$$

#

To prove Main Theorem 2, we already have Main Theorem 1, i.e. have matched unit, pairing, multiplication, 3-point functions. The remaining is the 4-point functions.

- 4-point correlators in A-model

$\Gamma_{g.k.W} = \{$ connected, one-edge W-graph with genus g, k tails $\}$

$\Gamma_{g.k.W.cut} = \{$ W-graph without edge but one pair of tails decorated by $\gamma_+, \gamma_- = \gamma_+^{-1}$ $\}$

$\Gamma_{g.k.W}(\gamma_1 \dots \gamma_k) \subseteq \Gamma_{g.k.W}$, for any $\Gamma_{cut} \in \Gamma_{g.k.W.cut}$, $\exists!$ $\underline{\Gamma} \in \Gamma_{g.k.W}$

$\Gamma_{g.k.W.cut}(\gamma_1 \dots \gamma_k) \subseteq \Gamma_{g.k.W.cut}$ gluing tails γ_+, γ_-.

Theorem Assume the W-structure is concave, i.e. $\pi_*(\oplus \mathcal{L}_i) = 0$ with all marking narrow, i.e. $\gamma \in G_W$ at each marking has trivial fixed locus.

If $D = \hat{c}_W(g-1) + \sum_{j=1}^{k} \iota_{\gamma_j} = 1$, then

$$\Lambda^W_{g.k}(e_{\gamma_1} \dots e_{\gamma_k}) = \sum_{\ell=1}^{N} \Bigg[\left(\frac{q_\ell^2}{2} - \frac{q_\ell}{2} + \frac{1}{12}\right)\kappa_1 - \sum_{i=1}^{k}\left(\frac{1}{12} - \frac{1}{2}\theta_\ell^{\gamma_i}(1-\theta_\ell^{\gamma_i})\right)\psi_i$$
$$+ \frac{1}{2} \sum_{\Gamma_{cut} \in \Gamma_{g.W.k.cut}(\gamma_1 \dots \gamma_k)} \left(\frac{1}{12} - \frac{1}{2}\theta_\ell^{\gamma_+}(1-\theta_\ell^{\gamma_+})\right)[\overline{M}(\Gamma)] \Bigg].$$

The proof using orbifold-GRR. (omit!)

$\underline{E_7 \text{ case}}$: $\langle X, X, X^2, XY\rangle_0$, $\langle X, Y, X^2, X^2\rangle_0$, $\langle Y, Y, XY, X^2Y\rangle_0$.

$\rightsquigarrow$ $\underline{\langle e_7, e_7, e_4, e_2\rangle}$, $\langle e_7, e_5, e_4, e_4\rangle$, $\langle e_5, e_5, e_2, e_8\rangle$.

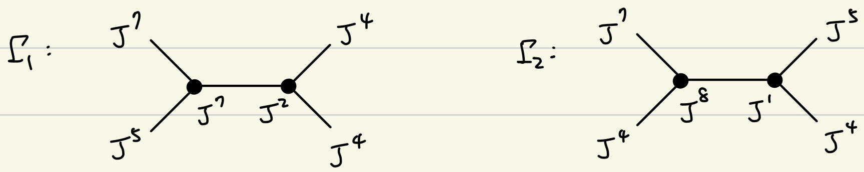
in reference !

Use above theorem to compute them !

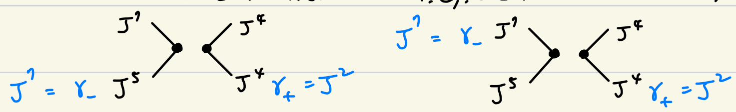
Note: $\displaystyle\int_{\overline{M}_{0,4}} K_1 = \int_{\overline{M}_{0,4}} \psi_1 = \int_{\overline{M}_{0,4}} [\overline{M}(\Gamma)] = 1$.

e.g. we compute $\langle e_7, e_5, e_4, e_4\rangle_0^{E_7}$:

There are two graphs in $\Gamma_{0,4,E_7}(J^7, J^5, J^4, J^4)$:



$\rightsquigarrow \Gamma_{1,cut}$, $\Gamma_{1,cut}' \in \Gamma_{0,4,E_7,cut}(J^7, J^5, J^4, J^4)$ $\qquad \searrow \Gamma_{2,cut}$ $\boxed{\times 4}$ in $\Gamma_{0,4,E_7,cut}(J^7, J^5, J^4, J^4)$



$$\Rightarrow \langle e_7, e_5, e_4, e_4\rangle_0^{E_7} = \int_{\overline{M}_{0,4}} \Lambda_{0,4}^{E_7}(e_7, e_5, e_4, e_4)$$

$$= \left(\frac{g_x^2}{2} - \frac{g_x}{2} + \frac{1}{12}\right) - \left(\frac{1}{12} - \frac{1}{2}\cdot\overset{\theta_x^{\gamma_1}}{\overset{\|}{\frac{1}{3}}}\cdot\frac{2}{3}\right) - \left(\frac{1}{12} - \frac{1}{2}\cdot\overset{\theta_x^{\gamma_2}}{\overset{\|}{\frac{1}{3}}}\cdot\frac{2}{3}\right)$$

$$- \left(\frac{1}{12} - \frac{1}{2}\cdot\overset{\theta_x^{\gamma_3}}{\overset{\|}{\frac{1}{3}}}\cdot\frac{2}{3}\right) - \left(\frac{1}{12} - \frac{1}{2}\cdot\overset{\theta_x^{\gamma_4}}{\overset{\|}{\frac{1}{3}}}\cdot\frac{2}{3}\right)$$

$$+ \left(\frac{1}{12} - \frac{1}{2}\cdot\overset{\theta_x^{\gamma_+}(\Gamma_1)}{\overset{\|}{\frac{2}{3}}}\cdot\frac{1}{3}\right) + 2\left(\frac{1}{12} - \frac{1}{2}\cdot\overset{\theta_x^{\gamma_+}(\Gamma_2)}{\overset{\|}{\frac{1}{3}}}\cdot\frac{2}{3}\right)$$

$$+ \left(\frac{g_y^2}{2} - \frac{g_y}{2} + \frac{1}{12}\right) - \left(\frac{1}{12} - \frac{1}{2}\cdot\overset{\theta_x^{\gamma_1}}{\overset{\|}{\frac{5}{9}}}\cdot\frac{4}{9}\right) - \left(\frac{1}{12} - \frac{1}{2}\cdot\overset{\theta_x^{\gamma_2}}{\overset{\|}{\frac{1}{9}}}\cdot\frac{8}{9}\right)$$

$$- \left(\frac{1}{12} - \frac{1}{2}\cdot\overset{\theta_x^{\gamma_3}}{\overset{\|}{\frac{8}{9}}}\cdot\frac{1}{9}\right) - \left(\frac{1}{12} - \frac{1}{2}\cdot\overset{\theta_x^{\gamma_4}}{\overset{\|}{\frac{8}{9}}}\cdot\frac{1}{9}\right)$$

$$+ \left(\frac{1}{12} - \frac{1}{2}\cdot\overset{\theta_x^{\gamma_+}(\Gamma_1)}{\overset{\|}{\frac{5}{9}}}\cdot\frac{4}{9}\right) + 2\left(\frac{1}{12} - \frac{1}{2}\cdot\overset{\theta_x^{\gamma_+}(\Gamma_2)}{\overset{\|}{\frac{7}{9}}}\cdot\frac{2}{9}\right) = -\frac{1}{9} .$$

Similarly, $\langle e_7, e_7, e_4, e_2\rangle_0^{E_7} = \frac{1}{9}$, $\langle e_5, e_5, e_2, e_8\rangle_0^{E_7} = \frac{1}{3}$.

Note: Since the correspondence of $\mathbb{C}[X,Y]/I \xrightarrow{\sim} \mathcal{H}_{E_7}^{G}$, $\rightsquigarrow \langle X, Y, X^2, X^2\rangle_0^{E_7} = -\frac{1}{9}\cdot\alpha^{17}$

$X \longmapsto \alpha^3 e_7$ $\qquad$ etc.

#

• 3. 4-point correlators in B-model

Noumi - Yamada formula for flat coordinates on Saito's Frobenius manifold of ADE.

Theorem (N-Y) (only state E_7-case!)

$\mathcal{N} = \{(v_1, v_2) \in \mathbb{N}^2 \mid 0 \le v_1 \le 2,\ 0 \le v_2 \le 1\} \cup \{(0,2)\}$. $W_\lambda = W + \sum_{v \in \mathcal{N}} t_v \cdot x^v$.

The flat coordinate for E_7 with primitive form $dx_1 \wedge dx_2$ is

$$s_v = t_0 \cdot \delta_{0,v} + \sum_{\substack{\sigma \in \mathbb{N}^{\mathcal{N}} \\ \sum_v \alpha_v \sigma_v = 1 - v_1 g_1 - v_2 g_2}} C_v(\ell(\alpha)) \frac{t^\alpha}{\alpha!}$$

where $C_v : \mathbb{N}^2 \longrightarrow \mathbb{C}$ is given by:

For any $v \in \mathcal{N}$, $L(v) := \mathbb{N}^2 \cap (\mathrm{span}\{(3,0),(1,3)\} + v) = \{(v_1 + 3k_1 + k_2,\ v_2 + 3k_2) \mid \substack{k_2 \ge 0 \\ k_1 \ge -(v_2 + k_1)/3}\}$.

$$C_v(\alpha) := \begin{cases} (-1)^{k_1+k_2} \left(\frac{v_1+1}{3} - \frac{v_2+1}{9} ; k_1 \right) \left(\frac{v_2+1}{3} ; k_2 \right) & \text{for } \alpha \in L(v). \\ 0 & \text{otherwise}. \end{cases}$$

where $(z;k) = \Gamma(z+k)/\Gamma(z)$: shifted factorial function.

E_7 case: $W = x_1^3 + x_1 x_2^3 + t_1 x_1^2 x_2 + t_3 x_1^2 + t_4 x_1 x_2 + t_5 x_2^2 + t_6 x_1 + t_7 x_2 + t_9$.

coordinate (up to 2^{nd} order):

$t_1 = s_1$ $\qquad$ $t_3 = s_3$

$t_4 = s_4 + \frac{4}{9} s_1 s_3$ $\qquad$ $t_5 = s_5$

$t_6 = s_6 + \frac{1}{3} s_1 s_5 + \frac{5}{18} s_3^2$ $\qquad$ $t_7 = s_7 + \frac{1}{9} s_1 s_6 + \frac{1}{9} s_3 s_4$.

$t_9 = s_9 + \frac{2}{9} s_3 s_6 + \frac{1}{3} s_4 s_5$

$\rightsquigarrow \partial_{x_1} W = 3x_1^2 + x_2^3 + 2 s_1 x_1 x_2 + 2 s_3 x_1 + (s_4 + \frac{4}{9} s_1 s_3) x_2 + s_6 + \frac{1}{3} s_5 s_1 + \frac{5}{18} s_3^2$

$\partial_{x_2} W = 3 x_1 x_2^2 + s_1 x_1^2 + (s_4 + \frac{4}{9} s_1 s_3) x_1 + 2 s_5 x_2 + s_7 + \frac{1}{9} s_1 s_6 + \frac{1}{9} s_3 s_4$.

$$C_{ijk}(s) := \mathrm{Res}_{x=0} \left. \frac{\partial_{s_i} W \cdot \partial_{s_j} W \cdot \partial_{s_k} W \cdot dx_1 \wedge dx_2}{\partial_{x_1} W \cdot \partial_{x_2} W} \right|_{s=0}$$

$\rightsquigarrow C_{991}(0) = 1/9$, $C_{946}(0) = 1/9$, $C_{577}(0) = -1/3$ (all non-trivial 3-point function)

$C_{559}(0) = -1/3$, $C_{667}(0) = 1/9$, $C_{937}(0) = 1/9$

Scaling the primitive form to $9\, dx_1 \wedge dx_2$

$\rightsquigarrow F_3^{prim} = \frac{1}{2} s_1 s_9^2 + s_4 s_6 s_9 - \frac{3}{2} s_5 s_7^2 - \frac{3}{2} s_5^2 s_9 + \frac{1}{2} s_6^2 s_7 + s_3 s_7 s_9$.

Had proved A-model $\longleftrightarrow$ B-model in 3-point correlators.

Expect: $\langle X, X, X^2, XY \rangle_0 \longleftrightarrow s_6^2 s_3 s_4$

$\langle X, Y, X^2, X^2 \rangle_0 \longleftrightarrow s_6 s_7 s_3^2$

$\langle Y, Y, XY, X^2 Y \rangle_0 \longleftrightarrow s_7^2 s_4 s_1$

4-point correlators: $C_{ijk\ell} = \frac{\partial}{\partial s_\ell} C_{ijk}\Big|_{s=0} = \frac{\partial}{\partial s_\ell} \mathrm{Res}_{x=0} \frac{\partial_{s_i} W \cdot \partial_{s_j} W \cdot \partial_{s_k} W}{\partial_{x_1} W \cdot \partial_{x_2} W} \cdot 9\, dx_1 \wedge dx_2 \Big|_{s=0}$

$\rightsquigarrow C_{6634} = -\frac{1}{9}, \quad C_{6733} = \frac{1}{9}, \quad C_{7741} = -\frac{1}{3}.$

$\rightsquigarrow F_4^{prim} = -\frac{1}{18} s_3 s_4 s_6^2 + \frac{1}{18} s_3^2 s_6 s_7 - \frac{1}{6} s_1 s_4 s_7^2.$

#

proof of Main Theorem 2: (E_7 case)

$F_3^A,\ F_4^A,\ F_3^B,\ F_4^B.$

Key tool:

Suppose $F_3^A = F_3^B$.

Rescaling the primitive form by c and change of variables $s_i \longmapsto \lambda^{1-\deg(s_i)} s_i$.

preserve unit e and Frobenius algebra.

This gives $F_3^B \longmapsto c \cdot \lambda^{\hat{c}_W} F_3^B$. Choose $c = \lambda^{-\hat{c}_W}$, then $F_3^B \longmapsto F_3^B$

$F_4^B \longmapsto c \cdot \lambda^{\hat{c}_W+1} F_4^B$ $F_4^B \longmapsto \lambda F_4^B$.

(E_7 case) Choose primitive form $9\, dx_1 \wedge dx_2$ on B-model.

A-model: Take flat coordinate

$T_1 \longleftrightarrow e_8 = X^2Y,\quad T_3 \longleftrightarrow e_4 = X^2,\quad T_4 \longleftrightarrow e_2 = XY,\quad T_5 \longleftrightarrow \pm y^2 e_0 = Y^2,$

$T_6 \longleftrightarrow e_7 = X,\quad T_7 \longleftrightarrow e_5 = Y,\quad T_9 \longleftrightarrow e_1 = 1.$ differ a constant 3.

$\rightsquigarrow F_3^A = \frac{1}{2} T_9^2 T_1 + T_9 T_7 T_3 + T_9 T_6 T_4 - \frac{3}{2} T_9 T_5^2 - \frac{3}{2} T_7^2 T_5 + \frac{1}{2} T_7 T_6^2 = F_3^B\ !$

Recall: $\mathbb{C}[X,Y] \xrightarrow{\phi_\alpha} \mathcal{H}_{E_7}$:

$X \longmapsto \alpha^3 e_7,\quad 1 \longmapsto 1 = e_1,\quad XY \longmapsto \alpha^5 e_2,\quad Y^2 \longmapsto \mp 3\alpha^4 y^2 e_0.$

$Y \longmapsto \alpha^2 e_5,\quad X^2 \longmapsto \alpha^6 e_4,\quad X^2Y \longmapsto \alpha^8 e_8,$

$F_4^{A.prim} = \frac{1}{18} T_6^2 T_3 T_4 + \frac{1}{6} T_7^2 T_4 T_1 - \frac{1}{18} T_6 T_7 T_3^2 = -F_4^B$!!!

Now, choose $\lambda = -1$, $c = (-1)^{-\frac{8}{9}}$, i.e. change the primitive form to $(-1)^{-\frac{8}{9}} \cdot 9\, dx_1 \wedge dx_2$.

Thus, $T_i \longmapsto (-1)^{1-\deg(s_i)} s_i$ gives the isomorphism between state spaces.

#