Mirror Principle I (LLY)

2022.4.14.

Torus action:

(1) $T=(S^1)^{n+1} \curvearrowright \mathbb{C}^n$ with weight $\lambda=(\lambda_0 \cdots \lambda_n)$.

$\rightsquigarrow \mathbb{P}^n \ni p_0, p_1, \dots, p_n$: fixed points. $[p]$: equivariant hyperplane class.

$\iota_{p_i}^* : H_T(\mathbb{P}^n) \longrightarrow H_T(p_i)$: restriction maps.

$\omega \longmapsto \omega(\lambda_i)$

(2) $G := S^1 \times T \curvearrowright \mathbb{C}^{(n+1)(d+1)}$ with weight $\underbrace{\lambda_i}_{T\text{-weight}} + r\underbrace{\alpha}_{S^1\text{-weight}}$ on ir-coordinate for $r=0,1,\dots,d$. $i=0,1,\dots,n$

$\rightsquigarrow \mathbb{P}^{nd+d+n} \ni p_{ir}$: fixed points. $[\kappa]$: equivariant hyperplane class.

$\iota_{p_{ir}}^* : H_G(\mathbb{P}^{nd+d+n}) \longrightarrow H_G(p_{ir}) = \mathbb{Q}[\alpha,\lambda]$: restriction maps.

$\omega \longmapsto \omega(\lambda_i + r\alpha)$

$N_d := \{ [\sum_{r=0}^{d} z_{0r} w_0^r w_1^{d-r}, \dots, \sum_{r=0}^{d} z_{nr} w_0^r w_1^{d-r}] \} \simeq \mathbb{P}^{nd+n+d}$ $\longleftarrow$ linear σ-model for $\mathbb{P}^n$.

(space of $(n+1)$-tuple of $\deg = d$ homogeneous polynomials in w_0, w_1 / scalar)

T-action on $(n+1)$-tuple & S^1-action on $\mathbb{P}^1 \ni [w_0, w_1]$ by weight $(\alpha, 0)$.

$\rightsquigarrow S^1 \times T$-action on N_d. $[f_0(w_0,w_1), \dots, f_n(w_0,w_1)] \longmapsto [e^{\lambda_0} f_0(e^{\alpha} w_0, w_1), \dots, e^{\lambda_n} f_n(e^{\alpha} w_0, w_1)]$.

Notations $R := \mathbb{Q}[\lambda](\alpha)$, $RH_G^*(N_d) := H_G^*(N_d) \otimes_{\mathbb{Q}[\lambda,\alpha]} R$

$R^{-1} := \mathbb{Q}(\lambda,\alpha)$, $R^{-1}H_G^*(N_d) := H_G^*(N_d) \otimes_{\mathbb{Q}[\lambda,\alpha]} R^{-1}$.

Fact All $\omega \in H_T^*(\mathbb{P}^n)$ with $\iota_{p_i}^*(\omega) \neq 0$ for all i are multiplicative closed.

$\rightsquigarrow$ localize get $H_T^*(\mathbb{P}^n)^{-1}$.

A class $\Omega \in H_T^*(\mathbb{P}^n)^{-1}$ with $\iota_{p_i}^*(\Omega) \neq 0$ for all i is called invertible.

Eulerity A sequence $Q = \{Q_d\}_{d=1}^{\infty}$, $Q_d \in RH_G^*(N_d)$ is an Ω-Euler data if $Q_0 = \Omega$ and $\forall\, d$, $r=0,\dots,d$, $i=0,\dots,n$, we have

$$\iota_{p_i}^*(\Omega)\, \iota_{p_{i,r}}^*(Q_d) = \overline{\iota_{p_{i,0}}^*(Q_r)} \cdot \iota_{p_{i,0}}^*(Q_{d-r}).$$

i.e. $\Omega(\lambda_i)\, Q_d(\lambda_i + r\alpha) = \overline{Q_r(\lambda_i)} \cdot Q_{d-r}(\lambda_i)$.

$\overline{}$: $N_d \longrightarrow N_d$ by $[f_0(w_0,w_1), \dots, f_n(w_0,w_1)] \longmapsto [f_0(w_1,w_0), \dots, f_n(w_1,w_0)]$.

Lemma (i) $Q_d(\lambda_i + d\alpha) = \overline{Q_d(\lambda_i)}$ $(r=d)$,

(ii) $Q_d(\lambda_j)\Big|_{\alpha = \frac{\lambda_j - \lambda_i}{d}} = Q_d(\lambda_i)\Big|_{\alpha = \frac{\lambda_i - \lambda_j}{d}}$

(iii) $\Omega(\lambda_i)\, Q_d(\lambda_j) = Q_r(\lambda_j)\, Q_{d-r}(\lambda_i)$ at $\alpha = \dfrac{\lambda_j - \lambda_i}{r}$ for $r \neq 0$.

Example (1) P: $P_d = \prod_{m=0}^{\ell d} (\ell\kappa - m\alpha) \in H_G^*(N_d)$. This is an $\ell \cdot p$-Euler data.

$\rightsquigarrow$ computing the Euler classes of obstruction bundle of $\begin{array}{c}\mathcal{O}(\ell)\\ \downarrow \\ \mathbb{P}^n\end{array}$.

(2) $\Omega = p^{-2}$, P: $P_d = \prod_{m=1}^{d-1} (\kappa - m\alpha)^2 \in H_G^*(N_d)$.

$\rightsquigarrow$ multiple cover formula, i.e. GW for $\mathbb{P}^1$.

(3) $M_d^\circ := M_{0,0}((1,d), \mathbb{P}^1 \times \mathbb{P}^n)$.

$\overset{\Psi}{f} : [w_0, w_1] \longmapsto [w_1, w_0] \times [f_0(w_0, w_1), \dots, f_n(w_0, w_1)]$, f_i: homogeneous degree d.

$\rightsquigarrow$ $\varphi : M_d^\circ \longrightarrow N_d$ which is $G = S^1 \times T$ equivariant.

$$f \longmapsto [f_0, \dots, f_n]$$

$\rightsquigarrow$ φ extend to $M_d := \overline{M_d^\circ} \longrightarrow N_d$: G equivariant map.

$$\begin{tikzcd}
\pi^* U_d = V_d \arrow[r] \arrow[d] & U_d \arrow[r] \arrow[d] & V = V^+ \oplus V^- \arrow[d] \\
N_d & M_d \arrow[l, "\varphi"'] \arrow[r, "\pi"] & \overline{M}_{0,0}(d, \mathbb{P}^n) \arrow[r] & \mathbb{P}^n
\end{tikzcd}$$

V^+: convex $H^1(C, f^* V^+) = 0$. V^-: concave $H^0(C, f^* V^-) = 0$.

(f, C)

$e_T(V_d)$ = equivariant Euler class. $\Omega^V = \dfrac{e_T(V^+)}{e_T(V^-)}$

Theorem $Q_d = \varphi_! \, e_T(V_d) \in H_G^*(N_d)$ is an Ω^V-Euler data.

proof: For $r > 0$, let $\{F_r\} \subseteq \overline{M}_{0,\underline{1}}(r, \mathbb{P}^n)$ be T-fixed components s.t. $x \mapsto p_i$.

x: marked point.

$N(F_r) := N_{F_r / \overline{M}_{0,1}(r, \mathbb{P}^n)}$: normal bundle.

$$\begin{array}{ccc} M_d & \xrightarrow{\varphi} & N_d \\ \cup\!| & & \Psi \\ & & p_{ir} \end{array}$$

$$\left\{ f : C \xrightarrow{(1,d)} \mathbb{P}^1 \times \mathbb{P}^n \;\middle|\; \begin{array}{l} C = C_1 \underset{x_1}{\cup} C_0 \underset{x_2}{\cup} C_2,\ C_0 \xrightarrow[\sim]{\pi_1 \circ f} \mathbb{P}^1 \\ \pi_2 \circ f(C_0) = p_i \in \mathbb{P}^n,\ \pi_1 \circ f(C_1) = 0,\ \pi_1 \circ f(C_2) = \infty \text{ in } \mathbb{P}^1. \end{array} \right\} : \begin{array}{l} G\text{-invariant} \\ \text{mapped to } p_{ir}. \end{array}$$

$$\pi_2 \circ f \big|_{C_1} \in \overline{M}_{0,1}(r, \mathbb{P}^n), \quad \pi_2 \circ f \big|_{C_2} \in \overline{M}_{0,1}(d-r, \mathbb{P}^n)$$

$$\rightsquigarrow \underset{G\text{-fixed}}{\{F_r \times F_{d-r}\}} \xrightarrow{\text{gluing}} M_d \xrightarrow{\varphi} N_d.$$

Consider a convex bundle V on $\mathbb{P}^n$ and $r \neq 0, d$.

$$Q_d(\lambda_i + r\alpha) = \int_{N_d} \varphi_{p_{ir}} Q_d = \int_{M_d} \varphi^*(\phi_{p_{ir}}) \, e_T(V_d)$$ ← localize to fixed components to compute!

$$N_{F_r \times F_{d-r} / M_d} = N(F_r) + N(F_{d-r}) + [H^0(C_0, (\pi_1 \circ f)^* T\mathbb{P}^1)]$$ ← deformation of $f|_{C_0}$

↑ in K-group

$$+ \underbrace{[L_r \otimes T_{x_1} C_0] + [L_{d-r} \otimes T_{x_2} C_0]}_{\text{deformation of nodes } x_1, x_2.} - \underbrace{[T_{p_i} \mathbb{P}^n]}_{\text{from gluing } x_1, x_2.} - \underbrace{[A_{C_0}]}_{\text{infinitesimal automorphism of } C_0 \text{ fixing } x_1, x_2.}$$

L_r : line bundle on $\overline{M}_{0,1}(r, \mathbb{P}^n)$ (at (f_1, C_1, x_1), fiber = tangent line at x_1).

- $e_T(T_{p_i}\mathbb{P}^n) = \prod_{j\neq i}(\lambda_i - \lambda_j)$
- $e_T(L_r \otimes \underbrace{T_{x_1}C_0}_{\text{weight} = \alpha}) = \alpha + c_1(L_r)$, $e_T(L_{d-r} \otimes \underbrace{T_{x_2}C_0}_{\text{weight} = -\alpha}) = -\alpha + c_1(L_{d-r})$.
- $e_T(H^0(C_0, (\pi_1\circ f)^* T\mathbb{P}^1)) \cdot e_T^{-1}(A_{C_0}) = -\alpha^2$

Euler sequence $\rightsquigarrow$ $0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)\otimes \mathbb{C}^2 \longrightarrow T\mathbb{P}^1 \longrightarrow 0$.

$\rightsquigarrow$ $0 \longrightarrow \underbrace{H^0(C_0, \mathcal{O})}_{wt = 0} \longrightarrow \underbrace{H^0(C_0, \mathcal{O}(1))}_{wt = \alpha, 0} \otimes \underbrace{\mathbb{C}^2}_{wt = -\alpha, 0} \longrightarrow \underbrace{H^0(C_0, (\pi_1\circ f)^* T\mathbb{P}^1)}_{\Rightarrow\ wt = \alpha, -\alpha, 0} \longrightarrow 0$

Also, from $0 \longrightarrow \underbrace{A_{C_0}}_{wt = 0} \longrightarrow \underbrace{H^0(C_0, (\pi_1\circ f)^* T\mathbb{P}^1)}_{wt = \alpha, -\alpha, 0} \longrightarrow \underbrace{T_{x_1}C_0}_{wt = \alpha} \oplus \underbrace{T_{x_2}C_0}_{wt = -\alpha} \longrightarrow 0$.

V_d restrict to $F_r \times F_{d-r}$ $\rightsquigarrow$ $0 \longrightarrow V_d \longrightarrow V_r|_{F_r} \oplus V_{d-r}|_{F_{d-r}} \longrightarrow V|_{p_i} \longrightarrow 0$.

$\rightsquigarrow$ $\Omega^V(\lambda_i)\cdot e_T(V_d) = e_T(V_r)\cdot e_T(V_{d-r}) = \rho^* e_T(U_r)\cdot \rho^* e_T(U_{d-r})$

$\rho : \overline{M}_{0,1}(d, \mathbb{P}^n) \longrightarrow \overline{M}_{0,0}(d, \mathbb{P}^n)$: forgetful map.

$$\Rightarrow \Omega^V(\lambda_i)\cdot Q_d(\lambda_i + r\alpha) = \Omega^V(\lambda_i)\int_{M_d} \varphi^*(p_{ir})\, e_T(V_d)$$

$$= -\alpha^2 \prod_{j\neq i}(\lambda_i - \lambda_j)\, e_T(p_{ir}/N_d) \sum_{F_r}\int_{F_r} \frac{\rho^* e_T(U_r)}{e_T(N_{F_r})\cdot(\alpha + c_1(L_r))} \cdot \sum_{F_{d-r}}\int_{F_{d-r}} \frac{\rho^* e_T(U_{d-r})}{e_T(N_{F_{d-r}})\cdot(\alpha + c_1(L_{d-r}))}.$$

For $r = d$ or 0, we have $C = C_1 \cup_{x_1} C_0$ or $C_0 \cup_{x_2} C_2$.

$\rightsquigarrow$ $N_{F_d/M_d} = N(F_d) + [H^0(C_0, (\pi_1\circ f)^* T\mathbb{P}^1)] + [L_d \otimes T_{x_j}C_0] - [A_{C_0}]$

similarly $\Rightarrow$ $$Q_d(\lambda_i + d\alpha) = \alpha^{-1}\cdot e_T(p_{id}/N_d) \sum_{F_d}\int_{F_d} \frac{\rho^* e_T(U_d)}{e_T(N(F_d))\,(\alpha + c_1(L))}$$

$$Q_d(\lambda_i) = -\alpha^{-1}\cdot e_T(p_{i0}/N_d) \sum_{F_d}\int_{F_d} \frac{\rho^* e_T(U_d)}{e_T(N(F_d))(-\alpha + c_1(L))}$$

Finally, using $e_T(p_{ir}/N_d) = \prod_{\substack{j=0 \\ (j,m)\neq(i,r)}}^{n} \prod_{m=0}^{d} (\lambda_i - \lambda_j + (r-m)\alpha)$, $\bar{\alpha} = -\alpha$, $\bar{\lambda_i} = \lambda_i$, we get

$$\Omega^V(\lambda_i)\, Q_d(\lambda_i + r\alpha) = \overline{Q_r(\lambda_i)}\, Q_{d-r}(\lambda_i) \quad \text{for all } i = 0 \cdots n,\ r = 0 \cdots d.$$

For concave V, the only modification is gluing exact sequence

$$0 \longrightarrow V|_{p_i} \longrightarrow V_d \longrightarrow V_r|_{F_r} \oplus V_{d-r}|_{F_{d-r}} \longrightarrow 0.$$

$\rightsquigarrow$ $e_T(V_d) = e_T(V_r)\cdot e_T(V_{d-r})\cdot \iota_{p_i}^* e_T(V)$ $\rightsquigarrow$ OK!

For concavex V. OK! #

Definition Two sequence $P=\{P_d\}$, $Q=\{Q_d\}$ are linked if $\iota^*_{P_{i0}}(P_d-Q_d)\in R^{-1}$
$$\rotatebox{90}{$\subset$}\ R^{-1}H^*_G(N_d)$$
vanish at $\alpha=\dfrac{\lambda_i-\lambda_j}{d}$ for all $j\neq i$, $d>0$.

Theorem $Q_d := \varphi_! e_T(V_d)$ as before, $V|_{C\simeq\mathbb{P}^1}\simeq\bigoplus_a\mathcal{O}(\ell_a)\oplus\bigoplus_b\mathcal{O}(-k_b)$.
At $\alpha=\dfrac{\lambda_j-\lambda_i}{d}$, $(i\neq j)$, we have
$$\iota^*_{P_{j0}}(Q_d)=\prod_a\prod_{m=0}^{\ell_a\cdot d}\left(\ell_a\cdot\lambda_j-m\frac{\lambda_j-\lambda_i}{d}\right)\cdot\prod_b\prod_{m=1}^{k_b\cdot d-1}\left(-k_b\lambda_j+m\frac{\lambda_j-\lambda_i}{d}\right)$$
In particular, $Q\underset{\text{linked}}{\sim}P$: $P_d=\prod_a\prod_{m=0}^{\ell_a\cdot d}(\ell_a\cdot\kappa-m\alpha)\cdot\prod_b\prod_{m=1}^{k_b\cdot d-1}(-k_b\cdot\kappa+m\alpha)$.

proof: Just compute the weights of T-action on
$H^0(C,(\pi_2\circ f)^*\mathcal{O}(\ell))=H^0(C,\mathcal{O}(\ell d))$ (convex)
$H^1(C,(\pi_2\circ f)^*\mathcal{O}(-k))=H^1(C,\mathcal{O}(-kd))$ (concave) #

Theorem Suppose P, Q are any linked Ω-Euler data.
If $\deg_\alpha \iota^*_{P_{i0}}(P_d-Q_d)\le(n+1)d-2$ for all $i=0,\dots,n$ and $d\geq 1$, then $P=Q$.

proof: Prove $P_d=Q_d$ by induction.
The push-forward map $pf_d: RH^*_G(N_d)\longrightarrow RH^*_G(p_{ir})=\mathbb{Q}(\lambda)[\alpha]=R$
gives non-degenerate pairing $pf_d(u\cdot v)$. It suffices to show that
$L_s := pf_d(\kappa^s\cdot(P_d-Q_d))=0$ for all $s\geq 0$.

By localization, we have
$$L_s=\sum_{i=0}^n\sum_{r=0}^d(\lambda_i+r\alpha)^s\frac{\iota^*_{P_{ir}}(P_d-Q_d)}{\prod\limits_{\substack{k=0\\(k,m)\neq(i,r)}}^n\prod\limits_{m=0}^d(\lambda_i-\lambda_k+(r-m)\alpha)}\quad\leftarrow e_T(P_{ir}/N_d)$$

P, Q : Euler data, for each $r=1,2\dots d-1$, $\iota^*_{P_{ir}}(P_d)$ and $\iota^*_{P_{ir}}(Q_d)$ are expressible in $P_1,\dots,P_{d-1}$.
By induction, we have $L_s=\displaystyle\sum_{i=0}^n\frac{\lambda_i^s A_i(\alpha)}{\alpha^d}+\frac{(\lambda_i+d\alpha)^s A_i(-\alpha)}{(-\alpha)^d}$ (only $r=0,d$)

where
$$A_i(\alpha)=\frac{(-1)^d\,\iota^*_{P_{i0}}(P_d-Q_d)}{d!\prod_{k\neq i}(\lambda_i-\lambda_k)\prod_{k\neq i}\prod_{m=1}^d(\lambda_i-\lambda_k-m\alpha)}.$$

Since $P\underset{\text{linked}}{\sim}Q$, $\iota^*_{P_{i0}}(P_d-Q_d)=0$ at $\alpha=\dfrac{\lambda_i-\lambda_k}{d}$, $(k\neq i)$.
By induction and $\Omega(\lambda_i)Q_d(\lambda_j)=Q_r(\lambda_j)Q_{d-r}(\lambda_i)$ at $\alpha=\dfrac{\lambda_j-\lambda_i}{r}$ for $r\neq 0$,

we get $\iota^*_{p_{i,0}}(P_d - Q_d) = 0$ at $\alpha = \frac{\lambda_i - \lambda_k}{m}$ for all $k \neq i$, $m = 1, 2, \dots, d$.

$\Rightarrow A_i(\alpha) \in \mathbb{Q}(\lambda)[\alpha]$ for all i.

Assumption $\Rightarrow \deg_\alpha A_i \leq (n+1)d - 2 - nd = d-1$.

But $L_s \in \underbrace{\mathbb{Q}(\lambda)[\alpha]}_{\text{polynomial in } \alpha}$ for all s, we have $A_i \equiv 0$.

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<u>Lemma</u> $Q_d := \varphi_! e_T(V_d)$ as before, $V|_{C \simeq \mathbb{P}^1} \simeq \bigoplus_a \mathcal{O}(l_a) \oplus \bigoplus_b \mathcal{O}(-k_b)$ concavex bundle.

$$I : N_{d-1} \longrightarrow N_d \qquad \Rightarrow \quad I^d : N_0 = \mathbb{P}^n \longrightarrow N_d .$$
$$[f_0, \dots, f_n] \longmapsto [w_1 f_0, \dots, w_1 f_n] \qquad\qquad p_i \longmapsto p_{i,0}$$

Then, the restriction $I_d^*(Q_d) \in H_G^*(\mathbb{P}^n)$ has $\deg_\alpha I_d^*(Q_d) \leq (n+1)d - 2$.

proof: As in proof of Eulerity of $\varphi_! e_T(V_d)$, we have

$$Q_d(\lambda_i) = \underbrace{\prod_{j \neq i}(p - \lambda_j) \cdot \prod_{j=0}^{n} \prod_{m=1}^{d} (p - \lambda_j - m\alpha)}_{e_T(p_{i,0}/N_d)} \sum_{F_d} \int_{F_d} \frac{\rho^* e_T(U_d)}{e_T(N(F_d))[\alpha(\alpha - c_1(L))]}$$

localization

$$= \prod_{j=0}^{n} \prod_{m=1}^{d} (p - \lambda_j - m\alpha) \cdot \underbrace{\int_{\overline{M}_{0,1}(d, \mathbb{P}^n)} \frac{\rho^* e_T(U_d)}{\alpha(\alpha - c_1(L))} ev^*\Big(\prod_{j \neq i}(p - \lambda_j)\Big)}$$

$$= \int_{\mathbb{P}^n} ev_!\left(\frac{\rho^* e_T(U_d)}{\alpha(\alpha - c_1(L))}\right) \cdot \prod_{j \neq i}(p - \lambda_j) = \iota^*_{p_i} ev_!\left(\frac{\rho^* e_T(U_d)}{\alpha(\alpha - c_1(L))}\right).$$

$$\Rightarrow I_d^*(Q_d) = \prod_{j=0}^{n} \prod_{m=1}^{d} (p - \lambda_j - m\alpha) \cdot ev_!\left(\frac{\rho^* e_T(U_d)}{\alpha(\alpha - c_1(L))}\right).$$

$$\Rightarrow \deg_\alpha I_d^*(Q_d) \leq (n+1) \cdot d - 2.$$

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Mirror Principle I (LLY) 2022. 4. 18.

Lagrange Map and Mirror Transform

Fix an invertible class $\Omega \in H_T^*(\mathbb{P}^n)^{-1}$. $\mathcal{A} :=$ the set of all Ω-Euler data.

<u>Definition</u> An invertible map $\mu : \mathcal{A} \longrightarrow \mathcal{A}$ is called a mirror transform if for any $P \in \mathcal{A}$, $\mu(P) \underset{\text{linked}}{\sim} P$.

<u>Notation</u> $I : N_{d-1} \longrightarrow N_d$ gives $N_0 = \mathbb{P}^n \xrightarrow{I_d} N_d$.

$[f_0, \dots, f_n] \mapsto [w_1 f_0, \dots, w_1 f_n]$ compose I d times (S, S_0 : all such sequences.)

For a sequence $\underset{S}{\underset{\cap}{P}} = \{ \underset{R^{-1}H_G^*(N_d)}{\underset{\cap}{P_d}} \}$, define $\underset{S_0}{\underset{\cap}{I(P)}} = \{ \underset{R^{-1}H_G^*(N_0)}{\underset{\cap}{B_d}} \}$ where $B_d := I_d^*(P_d)$.

Recall : $w \in R^{-1}H_G^*(N_d)$ is 1-1 correspondence to $\{ \iota_{p_{ir}}^*(w) \}_{\substack{i=0\dots n \\ r=0\dots d}} \subset R^{-1}$.

$\rightsquigarrow$ Given any $B = \{B_d\} \subset R^{-1}H_G^*(N_0)$, $\exists !$ $P = \{ \underset{R^{-1}H_G^*(N_d)}{\underset{\cap}{P_d}} \}$ s.t.

$$\iota_{p_{ir}}^*(P_d) = \underset{B_0}{\underbrace{\iota_{p_i}^*(\Omega)}}^{-1} \, \overline{\iota_{p_i}^*(B_r)} \, \iota_{p_i}^*(B_{d-r}) \quad \text{for } i = 0, \dots, n, \ r = 0, \dots, d.$$

This gives $\mathcal{L}_\Omega : S_0 \longrightarrow S$. Lagrange map.

$B \longmapsto \mathcal{L}_\Omega(B) = P$

<u>Observation</u> • $r=0$ case $\Rightarrow$ $\iota_{p_i}^*(B_d) = \iota_{p_{i0}}^*(P_d) = \iota_{p_i}^* I_d^*(P_d)$ for all d

$\Rightarrow$ $I : S \longrightarrow S_0$

$\mathcal{L}_\Omega$ is a section.

• We have $\iota_{p_i}^*(\Omega)\, \iota_{p_{ir}}^*(P_d) = \overline{\iota_{p_{i0}}^*(P_r)}\, \iota_{p_{i0}}^*(P_{d-r})$ for all i, r.

(A) If all $P_d \in RH_G^*(N_d)$, then $P = \mathcal{L}(B)$ is an Ω-Euler data.

• Any map $\mu_0 : S_0 \longrightarrow S_0$ can be lifted to $\underline{\mu = \mathcal{L} \circ \mu_0 \circ I} : S \longrightarrow S$.

If $Q \in \mathcal{A}$ is an Euler data, then $Q = \mathcal{L} \circ I(Q)$. Lagrange lift

(B) If $\mu_0 : S_0 \longrightarrow S_0$ is invertible with inverse ν_0. Then, the Lagrange lifts μ, ν are inverse to each other when restricted on Euler data.

<u>Definition</u> Given $B = \{B_d\} \in S_0$, define

$$\underline{HG[B](t)} := e^{-pt/\alpha} \left(\Omega + \sum_{d>0} \frac{B_d \cdot e^{dt}}{\prod_{k=0}^{n} \prod_{m=1}^{d} (p - \lambda_k - m\alpha)} \right)$$

cohomology valued power series

p = equivariant hyperplane class of $N_0 = \mathbb{P}^n$. $\in H_G^*(N_0)$

Proposition Let $B \in S_0$, $\Omega := B_0$.

(1) Given any $g \in e^t \cdot R[[e^t]]$, $\exists!\ \widetilde{B} \in S_0$ such that $HG[B](t+g) = HG[B](t)$.

(2) Given any $f \in e^t \cdot R[[e^t]]$, $\exists!\ \widetilde{B} \in S_0$ such that $e^{f/\alpha} HG[B](t) = HG[\widetilde{B}](t)$.

proof: (1) Expand $HG[B](t+g) = e^{-pt/\alpha} \cdot e^{-pg/\alpha} \sum_{d \geq 0} \dfrac{B_d \cdot e^{dt} \cdot e^{dg}}{\prod_{k=0}^{n} \prod_{m=1}^{d} (p - \lambda_k - m\alpha)}$

Write $e^{dg} = \sum_{s \geq 0} g_{d,s} e^{st}$, $g_{d,s} \in R$ and $e^{-pg/\alpha} = \sum_{s \geq 0} g_s' e^{st}$, $g_s' \in R[p/\alpha]$.

Compare coefficients $\Rightarrow$ $\widetilde{B}_d = B_d' + \sum_{r=0}^{d-1} g_{d-r}' B_r' \prod_{j=0}^{n} \prod_{m=r+1}^{d} (p - \lambda_j - m\alpha)$

$$B_d' = B_d + \sum_{r=0}^{d-1} g_{r,d-r} B_r \prod_{j=0}^{n} \prod_{m=r+1}^{d} (p - \lambda_j - m\alpha)$$

(2) Similarly, expand $e^{f/\alpha} HG[B](t)$ by $e^{f/\alpha} = \sum_{s \geq 0} f_s e^{st}$, $f_s \in R[\alpha^{-1}]$.

Compare coefficients $\Rightarrow$ $\widetilde{B}_d = B_d + \sum_{r=0}^{d-1} f_{d-r} B_r \prod_{j=0}^{n} \prod_{m=r+1}^{d} (p - \lambda_j - m\alpha)$.

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Note that $\prod_{j=0}^{n} (p - \lambda_j - m\alpha)$ vanishes when restricted on p_i at $\alpha = \dfrac{\lambda_i - \lambda_j}{d}$.

$\Rightarrow$ $\iota_{p_i}^*(B_d) = \iota_{p_i}^*(\widetilde{B}_d)$ at $\alpha = \dfrac{\lambda_i - \lambda_j}{d}$.

(c) Given $f, g \in e^t R[[e^t]]$, $\mu_0 : S_0 \longrightarrow S_0$, $B \longmapsto \widetilde{B}$ invertible such that

$e^{f/\alpha} HG[B](t+g) = HG(\widetilde{B})[t]$. If $\iota_{p_i}^*(B_d)$ is <u>well-defined</u> at $\alpha = \dfrac{\lambda_j - \lambda_i}{d}$ $(j \neq i)$,

then $\iota_{p_i}^*(B_d) = \iota_{p_i}^*(\widetilde{B}_d)$ for all d.

e.g. $B = I(P)$ for some Euler data P.

$\rightsquigarrow$ $\iota_{p_i}^*(B_d) \in R$: polynomial in α.

Lemma Let $\mu = \mathcal{L} \circ \mu_0 \circ I$ be the Lagrange lift of above μ_0 in (c). Then, μ is a mirror transform. In particular, if P is an Euler data, then $\widetilde{P} = \mu(P)$ is also an Euler data with $e^{f/\alpha} HG[I(P)](t+g) = HG[\widetilde{P}](t)$.

proof: Only need to consider two cases : $f=0$, $g=0$ separately.

Let P be an Euler data. $\widetilde{P} := \mu(P)$, $B := I(P)$, $\widetilde{B} := \mu_0(B)$.

It suffices to prove $\widetilde{P}$ is also an Euler data.

$$\left(\begin{array}{l} (B) \Rightarrow \mu \text{ is invertible on } \mathcal{A} : \text{the set of Euler data.} \\ (c) \Rightarrow \iota_{p_i}^*(B_d) = \iota_{p_{i,0}}^*(P_d) \text{ and } \iota_{p_i}^*(\widetilde{B}_d) = \iota_{p_{i0}}^*(\widetilde{P}_d) \text{ at } \alpha = \frac{\lambda_i - \lambda_j}{d}\ (j \neq i). \end{array} \right) \Rightarrow \widetilde{P} \underset{\text{linked}}{\sim} P.$$

P : Euler data $\Rightarrow$ $\iota_{p_{ir}}^*(P_d) = \iota_{p_i}^*(\Omega)^{-1} \overline{\iota_{p_i}^*(B_r)\ \iota_{p_i}^*(B_{d-r})}$ for $i = 0 \ldots n$, $r = 0 \ldots d$.

Multiply the identity:

$$e^{d\tau}\frac{e^{(\lambda_i+r\alpha)(t-\tau)/\alpha}}{\prod\limits_{j=0}^{n}\prod\limits_{\substack{m=0\\(j,m)\neq(i,r)}}^{d}(\lambda_i+r\alpha-\lambda_j-m\alpha)} = \frac{1}{\prod\limits_{j\neq i}(\lambda_i-\lambda_j)}\cdot e^{\lambda_i t/\alpha}\cdot\frac{e^{rt}}{\prod\limits_{j=0}^{n}\prod\limits_{m=1}^{r}(\lambda_i-\lambda_j+m\alpha)}\cdot e^{-\lambda_i\tau/\alpha}\cdot\frac{e^{(d-r)\tau}}{\prod\limits_{j=0}^{n}\prod\limits_{m=1}^{d-r}(\lambda_i-\lambda_j-m\alpha)}$$

$\uparrow$ $e_T(p_{ir}/N_d)$ $\qquad\uparrow$ $e_T(p_i/\mathbb{P}^n)$

sum over $i=0\dots n$, $r=0,\dots,d$, get

$$e^{d\tau}pf_d\left(P_d\cdot e^{\kappa(t-\tau)/\alpha}\right)$$

$$=\sum_{r=0}^{d}pf\left(\Omega^{-1}\,\overline{e^{-pt/\alpha}\cdot\frac{B_r\cdot e^{rt}}{\prod\limits_{j=0}^{n}\prod\limits_{m=1}^{r}(p-\lambda_j-m\alpha)}}\cdot e^{-p\tau/\alpha}\frac{B_{d-r}\,e^{(d-r)\tau}}{\prod\limits_{j=0}^{n}\prod\limits_{m=1}^{d-r}(p-\lambda_j-m\alpha)}\right)$$

$\overline{}: N_d\longrightarrow N_d$

$\rightsquigarrow R^{-1}H_G^*(N_d)\longrightarrow R^{-1}H_G^*(N_d)$

$\alpha\longmapsto-\alpha$, $\lambda_i\longmapsto\lambda_i$, $\kappa\longmapsto\kappa-d\alpha$.

sum over $d=0,1,2\dots$, get

$$(*)\quad \sum_{d\geq0}e^{d\tau}pf_d\left(P_d\cdot e^{\kappa(t-\tau)/\alpha}\right)=pf\left(\Omega^{-1}\,\overline{HG[B](t)}\;HG[B](\tau)\right).$$

$pf: H_T(\mathbb{P}^n)\longrightarrow H_T(p_i)$

$pf_d: H_G(N_d)\longrightarrow H_G(p_{ir})$

push-forward map.

Similar for $\tilde{P}$, $\tilde{B}$.

<u>Case 1</u> $f=0$. $HG[\tilde{B}](t)=HG[B](t+g)$.

$$(*)\Rightarrow pf\left(\Omega^{-1}\,\overline{HG[B](t+g(e^t))}\;HG[B](\tau+g(e^\tau))\right)=\sum_{d\geq0}e^{d(\tau+g(e^\tau))}pf_d\left(P_d\cdot e^{\kappa(t+g(e^t)-\tau-g(e^\tau))/\alpha}\right).$$

$$\xrightarrow[\ ]{\substack{q=e^\tau\\ \zeta=(t-\tau)\alpha}}\sum_{d\geq0}q^d pf_d(\tilde{P}_d e^{\zeta\kappa})=\sum_{d\geq0}q^d e^{dg(q)}pf_d\left(P_d\,e^{\kappa\zeta}\,e^{\kappa(g_+(qe^{\zeta\alpha})-g_+(q))/\alpha}\;e^{-\kappa(g_-(qe^{\zeta\alpha})+g_-(q))/\alpha}\right).$$

$g=g_++g_-$

$\overline{g_\pm}=\pm g_\pm$

since $\overline{}: N_d\to N_d$ is involution.

For $g(q)\in R[[q]]$, $g_+(qe^{\zeta\alpha})-g_+(q)\in\alpha\cdot R[[q,\zeta]]$

Also, $\omega\longmapsto\bar{\omega}$ only change α into $-\alpha$, g_- : odd $\Rightarrow g_-(q),\ g_-(qe^{\zeta\alpha})\in\alpha\cdot R[[q]]$

$P_d\in RH_G^*(N_d)$

$\Rightarrow$ RHS $\in R[[q,\zeta]]$ $\Rightarrow$ $pf_d(\tilde{P}_d\cdot\kappa^s)\in R$ in LHS for all s.

Also, $\tilde{P}_d\in R^{-1}H_G^*(N_d)$ is of the form $\tilde{P}_d=a_N\cdot\kappa^N+\dots+a_0$, $a_i\in R^{-1}$, $N=(n+1)d+n$.

Since $pf_d(\kappa^N)=1$, we get $a_N,\dots,a_0\in R$. (By $pf_d(\tilde{P}_d\cdot\kappa^s)\in R$)

$\Rightarrow\tilde{P}_d\in RH_G^*(N_d)$ which is an Euler data.

<u>Case 2</u> $g=0$. $HG[\tilde{B}](t)=e^{f/\alpha}HG[B](t)$. Similar as above write $f=f_++f_-$, $\overline{f_\pm}=f_\pm$.

$$\sum_{d\geq0}q^d pf_d(\tilde{P}_d e^{\zeta\kappa})=e^{-(f_+(qe^{\zeta\kappa})-f_+(q))/\alpha}\cdot e^{(f_-(qe^{\zeta\kappa})+f_-(q))/\alpha}\cdot\sum_{d\geq0}q^d pf_d(P_d\cdot e^{\zeta\kappa})$$

in $R[[q,\zeta]]$. Similar get $\tilde{P}_d\in RH_G^*(N_d)\Rightarrow$ Euler data.

#

Example Consider the convex bundle $V = \mathcal{O}(\ell) = \mathcal{O}(n+1)$. $\Omega^\vee = \ell \cdot p$.

$$\begin{array}{c} V \\ \downarrow \\ \mathbb{P}^n \end{array}$$

Two Euler data $P: P_d = \prod_{m=0}^{\ell d}(\ell \kappa - m\alpha)$, $Q: Q_d = \varphi_! e_T(V_d)$ are linked.

$$\overset{\lambda\to 0}{\rightsquigarrow} HG[I(P)](t) = e^{-Ht/\alpha} \sum_{d\geq 0} \frac{\prod_{m=0}^{\ell d}(\ell H - m\alpha)}{\prod_{m=1}^{d}(H-m\alpha)^{n+1}} e^{dt} = \ell H \left(f_0 - f_1 \frac{H}{\alpha} + f_2 \frac{H}{\alpha} - \cdots \right)$$

$f_0, f_1, \dots, f_{n-1}$: basis of solutions of hypergeometric differential equation.

Lemma In the non-equivariant limit $\lambda \to 0$, the mirror map $T(t) := \frac{f_1}{f_0} = t + \frac{g_1}{f_0}$ makes $HG[I(Q)](T(t)) = \frac{1}{f_0} HG[I(P)](t)$,

where $f_0 = \sum_{d\geq 0} \frac{(\ell d)!}{(d!)^{n+1}} e^{dt}$, $g_1 = \sum_{d\geq 1} \frac{(\ell d)!}{(d!)^{\ell}} \sum_{m=d+1}^{\ell d} \frac{\ell}{m} e^{dt}$.

proof: Expand $HG[I(P)](t) = e^{-pt/\alpha} \sum_{d\geq 0} \frac{\prod_{m=0}^{\ell d}(\ell p - m\alpha)}{\prod_{k=0}^{n}\prod_{m=1}^{d}(p-\lambda_k - m\alpha)} e^{dt}$

$$= \ell p \cdot \left(f_0 + \alpha^{-1}\left(p f_1 + g_2 \sum_{k=0}^{n} \lambda_k\right) + \cdots \right).$$

Let $f = (\log f_0)\alpha + \frac{g_2}{f_0} \sum_{k=0}^{n} \lambda_k \in e^t R[[e^t]]$.

$\rightsquigarrow \exists$ mirror transform μ s.t. $HG[I(\tilde{P})](t) = e^{f/\alpha} HG[I(P)](t)$, where $\tilde{P} = \mu(P)$.

$\Rightarrow HG[I(\tilde{P})](t) = \ell p - \alpha^{-1} \ell p^2 \frac{f_1}{f_0} + \cdots$ $\quad (*)$

Again, let $g = \frac{g_1}{f_0} \in e^t R[[e^t]]$.

$\rightsquigarrow \exists$ mirror transform ν s.t. $HG[I(\tilde{Q})](t) = HG[I(Q)](t + \frac{g_1}{f_0})$, where $\tilde{Q} = \nu(Q)$.

$\Rightarrow HG[I(\tilde{Q})](t) = e^{-p(t+\frac{g_1}{f_0})/\alpha} (\ell p + \cdots) = \ell p - \alpha^{-1} \ell p^2 (t + \frac{g_1}{f_0}) + \cdots$ $\quad (*)$

Now, from $(*)$, we have $\frac{I_d^*(\tilde{P}_d - \tilde{Q}_d)}{\prod_{k=0}^{n}\prod_{m=1}^{d}(p-\lambda_k - m\alpha)} \equiv 0 \pmod{\alpha^{-2}}$

$\Rightarrow \deg_\alpha \iota_{p_i,0}^* (\tilde{P}_d - \tilde{Q}_d) \leq (n+1)d - 2$. Also, $\tilde{P} \underset{\text{linked}}{\sim} P \sim Q \sim \tilde{Q} \overset{\text{uniqueness}}{\Longrightarrow} \tilde{P} = \tilde{Q}$.

$\Rightarrow HG[I(Q)](T(t)) = e^{f/\alpha} HG[I(P)](t)$.

Take non-equivariant limit $\lambda \to 0 \Rightarrow HG[I(Q)](T(t)) = \frac{1}{f_0} HG[I(P)](t)$.

#

Theorem $V = \bigoplus_{a=1}^{N^+} \mathcal{O}(\ell_a) \oplus \bigoplus_{b=1}^{N^-} \mathcal{O}(-k_b)$
$$\downarrow$$
$$\mathbb{P}^n$$

(1) If $d(\sum \ell_a + \sum k_b) - N^- \le d(n+1) - 2$ for all $d > 0$, then $Q = P$.

(2) If $d(\sum \ell_a + \sum k_b) - N^- \le d(n+1)$ for all $d > 0$, then there exists a mirror transform μ (depend on ℓ_a, k_b) such that $Q = \mu(P)$.

proof: (1) By definition, $\deg_\alpha I_d^*(P_d) = d(\sum \ell_a + \sum k_b) - N^-$.

Also, $\deg_\alpha I_d^*(Q_d) \le d(n+1) - 2$ by localization computation.

Then, $\deg_\alpha (I_d^*(P_d - Q_d)) \le d(n+1) - 2 \Rightarrow P = Q$.

(2) The only remaining case are (i) $N^- = 0$, $\sum \ell_a = n+1$

(ii) $N^- = 1$, $\sum \ell_a + k_1 = n+1$

(iii) $N^- = 0$, $\sum \ell_a = n$.

Then, immitate $\mathcal{O}(n+1) \longrightarrow \mathbb{P}^n$ construction.

#

Toric Case

$X = X_\Sigma$: toric variety. $X = (\mathbb{C}^N - Z)/(\mathbb{C}^\times)^m$, $m = \operatorname{rk} H^2(X, \mathbb{Z})$.

$\mathbb{C}^N$ coordinate $z_i \rightsquigarrow$ a section $z_i \curvearrowright \begin{matrix} L_i \\ \downarrow \\ X \end{matrix}$, $D_a = c_1(L_a)$: T-invariant divisor.

$a = 1, \dots, N$.

For $d \in H^2(X, \mathbb{Z})$, define $\Gamma_d = \bigoplus_a H^0(\mathbb{P}^1, \mathcal{O}(\langle D_a, d\rangle))$ with $\underline{(\mathbb{C}^\times)^m \text{ action}}$.

same as action on $\mathbb{C}^N$.

$Z_d = \{\phi \in \Gamma_d \mid \phi(z, w) \in Z \text{ for all } (z, w) \in \mathbb{C}^2\}$

$\rightsquigarrow W_d := (\Gamma_d - Z_d)/(\mathbb{C}^\times)^m$: linear σ-module of X

S^1-action on $\mathbb{P}^1$ of weight α gives S^1-action on W_d.

S^1-fixed components of W_d is

$z_i = 0$ if $\langle D_i, r\rangle < 0$ or $\langle D_i, d-r\rangle < 0$.

$$X_r = \left\{ \phi = [z_1 w_0^{\langle D_1, r\rangle} w_1^{\langle D_1, d-r\rangle}, \dots, z_N w_0^{\langle D_N, r\rangle} w_1^{\langle D_N, d-r\rangle}] \,\middle|\, (z_1 \dots z_N) \in \mathbb{C}^N \right\} /\!/ (\mathbb{C}^\times)^m.$$

$\rightsquigarrow$ equivariant Euler class $e_G(X_r/W_d) = \prod_{a=1}^{N} \prod_{\substack{k=0 \\ k \ne \langle D_a, r\rangle}}^{\langle D_a, d\rangle \ge 0} (D_a + \langle D_a, r\rangle\alpha - k\alpha)$.

$$O(t) = e^{-Ht/\alpha} \sum_d \underbrace{\frac{\prod_{\langle D_a, d\rangle < 0} \prod_{k=0}^{-\langle D_a, d\rangle - 1} (D_a + k\alpha)}{\prod_{\langle D_a, d\rangle \ge 0} \prod_{k=1}^{\langle D_a, d\rangle} (D_a - k\alpha)}}_{B_d} e^{dt}$$ is an Euler series.

means $\sum_{0 \le r \le d} \int_X \Omega^{-1} \cap \overline{B}_r \cdot B_{d-r} e^{(H + r\alpha)\cdot\xi} \in R[[\xi]]$ for all d.