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Week 1.

$$Z(\alpha,\varepsilon) = \int dX \sum_{n=0}^{\infty} e^{-\frac{\alpha}{2}X^2} \frac{(-i\varepsilon X^3)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-i\varepsilon)^n}{n!} \int X^{3n} \cdot e^{\frac{-\alpha}{2}X^2}\, dX$$

$$= \sum_{n=0}^{\infty} \frac{(-i\varepsilon)^{2n}}{(2n)!} \cdot \alpha^{-3n} \cdot \#\{\text{contrations}\}$$

$$= \sum_{n=0}^{\infty} \sum_{\substack{\Gamma:\,3\text{-regular graph}\\ \text{with } 2n \text{ vertices}}} \frac{(-i\varepsilon)^{2n}}{(2n)!} \cdot \alpha^{-3n} \cdot \underbrace{\#\{\text{ways}:\ \overbrace{Y \cdots Y}^{2n} \xrightarrow{\text{contract}} \Gamma\}}_{=:\, w(\Gamma)}$$

$$= \sum_{\Gamma:\,3\text{-reg. graph}} \frac{(-i\varepsilon)^{V(\Gamma)}}{V(\Gamma)!} \cdot \alpha^{-E(\Gamma)} \cdot \frac{V(\Gamma)!}{\prod_i (a_i V(\Gamma_i))!} \cdot \prod_i \left(\frac{(a_i V(\Gamma_i))!}{(V(\Gamma_i)!)^{a_i} a_i!} \cdot w(\Gamma_i)^{a_i} \right)$$

if $\Gamma = \bigsqcup (a_i \text{ copies of } \Gamma_i)$, Γ_i connected 3-reg.

$$= \sum_{\Gamma:\,3\text{-reg. graph}} (-i\varepsilon)^{V(\Gamma)} \cdot \alpha^{-E(\Gamma)} \cdot \prod_i \frac{w(\Gamma_i)^{a_i}}{(V(\Gamma_i)!)^{a_i} a_i!}$$

$$= \sum_{\Gamma:\,3\text{-reg. graph}} \prod_i \left(\frac{(-i\varepsilon)^{V(\Gamma_i)} w(\Gamma_i)}{\alpha^{E(\Gamma_i)} V(\Gamma_i)!} \right)^{a_i} \frac{1}{a_i!} \qquad \left(\leftarrow V(\Gamma_i) = \sum a_i V(\Gamma_i),\ E(\Gamma_i) = \sum_i a_i E(\Gamma_i)\right)$$

$$= \prod_{\Gamma:\,\text{connected 3-reg}} \exp\left(\frac{(-i\varepsilon)^{V(\Gamma)} w(\Gamma)}{\alpha^{E(\Gamma)} V(\Gamma)!} \right)$$

$$= \exp\left(\sum_{\Gamma:\,\text{connected 3-reg}} \frac{(-3!\,i\varepsilon)^{V}}{\alpha^{E} |Aut(\Gamma)|} \right) \qquad \left(\leftarrow w(\Gamma) = \frac{(3!)^V V!}{|Aut(\Gamma)|} \right)$$

Week 2.

· Change of Variable Formula :

Suppose $(x_i, \psi_j)_{\substack{i=1\sim n\\ j=1\sim m}}$ and $(y_i, \theta_j)_{\substack{i=1\sim n\\ j=1\sim m}}$ are two coordinate system in $\Lambda^{n|m}$

with the Jacobian $J = \frac{\partial(x,\psi)}{\partial(y,\theta)} = \left(\begin{array}{c|c} \frac{\partial x}{\partial y} & \frac{\partial x}{\partial \theta} \\ \hline \frac{\partial \psi}{\partial y} & \frac{\partial \psi}{\partial \theta}\end{array}\right)$ is invertible everywhere.

Then $\int_{\Lambda^{n|m}} f(x,\psi)\, dx\, d\psi = \int_{\Lambda^{n|m}} f(x(y,\theta), \psi(y,\theta)) \cdot \varepsilon \operatorname{Ber} J \, dy\, d\theta$

with $\varepsilon := \operatorname{sgn}\left(\det \frac{\partial x(y,0)}{\partial y}\right)$

pf: Note that J is inv. $\Leftrightarrow$ A and D are inv. $\quad \begin{pmatrix} A & B \\ C & D\end{pmatrix} := \left(\begin{array}{c|c} \frac{\partial x}{\partial y} & \frac{\partial x}{\partial \theta} \\ \hline \frac{\partial \psi}{\partial y} & \frac{\partial \psi}{\partial \theta}\end{array}\right)$

May assume $\frac{\partial \psi_m}{\partial \theta_m}$ is inv. near $p \in \mathbb{R}^n$.

Then $\psi_m(y,\theta) = a\,\theta_m + b$ with $a = \frac{\partial \psi_m}{\partial \theta_m}$ inv. ($\frac{\partial}{\partial \theta_m}$ is the "right" derivative)

$\xrightarrow{\text{IFT}}$ $(y,\theta) \longleftrightarrow (y, \theta^{\hat{m}}, \psi^m)$ bij. near p

Hence, $\int_{\Lambda^{n|m}} \underbrace{f(x(y,\theta), \psi(y,\theta)) \cdot \operatorname{Ber} J}_{:= F(y,\theta)} \, dy\, d\theta = \int F_1(a\theta^m + b) + F_0 \, dy\, d\theta$

$$= \int F_1 a \, dy\, d\theta^{\hat{m}} = \int (F_1 \psi^m + F_0) a \, dy\, d\theta^{\hat{m}} d\psi^m$$

$$= \int F(y, \theta^{\hat{m}}, \psi^m) \operatorname{Ber}\left(\frac{\partial(y,\theta)}{\partial(y,\theta^{\hat{m}},\psi^m)}\right) dy\, d\theta^{\hat{m}} d\psi^m$$

$$= \int f(y, \theta^{\hat{m}}, \psi^m) \operatorname{Ber} J \operatorname{Ber} J_m \, dy\, d\theta^{\hat{m}} d\psi^m$$

where we write $F(y, \theta^{\hat{m}}, \psi^m) = F_1(y, \theta^{\hat{m}})\psi^m + F_0(y, \theta^{\hat{m}})$

$$= F_1(a\theta^m + b) + F_0$$

$\left(J_k := \partial(y, \theta^{1\sim k}, \psi^{k+1\sim m}) \big/ \partial(y, \theta^{1\sim k-1}, \psi^{k\sim m}) \right)$

Inductively, we have $\int_{\Lambda^{n|m}} f(x(y,\theta), \psi(y,\theta)) \cdot \operatorname{Ber} J \, dy\, d\theta$

$$= \int f(y,\psi) \operatorname{Ber} J \operatorname{Ber} J_m \cdots J_1 \, dy\, d\psi.$$

$$= \int f(y,\psi) \operatorname{Ber}\left(\frac{\partial(x,\psi)}{\partial(y,\psi)}\right) dy\, d\psi.$$

Note $\int_{\Lambda^{n|m}} f(x,\psi)\,dx\,d\psi$

$$= \int f(x(y,0),\psi)\left|\frac{\partial x(y,0)}{\partial y}\right| dy\,d\psi \quad \text{by change of variables on } \mathbb{R}^n$$

$$= \int f(y,\psi)\cdot\left(\frac{\partial x(y,\psi)}{\partial y}\right)\varepsilon\, dy\,d\psi$$

$$= \int f(y,\psi)\,\varepsilon\,\mathrm{Ber}\left(\frac{\partial(x,\psi)}{\partial(y,\psi)}\right) dy\,d\psi$$

(The first equality is due to $\frac{\partial x(y,0)}{\partial y}$ inv.)

· $dX\,d\psi_1\,d\psi_2$ is invariant under
$$\begin{cases} \delta X = \varepsilon_1\psi_1 + \varepsilon_2\psi_2 \\ \delta\psi_1 = \varepsilon_2\partial h \\ \delta\psi_2 = -\varepsilon'\partial h \end{cases}$$

pf: $\delta\,\mathrm{Ber} = \begin{pmatrix} 0 & \varepsilon_1 & \varepsilon_2 \\ \varepsilon_2\ddot{h} & 0 & 0 \\ -\varepsilon_1\ddot{h} & 0 & 0 \end{pmatrix} = 0$

Week 3

$$L = \frac{1}{2} g_{ij}\, \dot{x}^i \dot{x}^j + \frac{i}{2} g_{ij} \left(\bar{\psi}^i \nabla_t \psi^j - \nabla_t \bar{\psi}^i \psi^j \right) - \frac{1}{2} R_{i\bar{j}k\bar{l}}\, \psi^{i\bar{j}k\bar{l}} \qquad \left(\nabla_t \psi^i = \partial_t \psi^i + \Gamma^i_{jk}\, \dot{x}^j \psi^k \right)$$

$$\text{"="}\ \frac{1}{2} g_{ij}\, \dot{x}^i \dot{x}^j + i g_{ij} \cdot \bar{\psi}^i \nabla_t \psi^j - \frac{1}{2} R_{i\bar{j}k\bar{l}}\, \psi^{i\bar{j}k\bar{l}}$$

$$\begin{cases} \delta x^i = \varepsilon \psi^{\bar{i}} - \bar{\varepsilon} \psi^i \\ \delta \psi^i = \varepsilon (i \dot{x}^i - \Gamma^i_{jk} \psi^{\bar{j}k}) \\ \delta \psi^{\bar{i}} = \bar{\varepsilon} (- i \dot{x}^i - \Gamma^i_{jk} \psi^{\bar{j}k}) \end{cases}$$

Note the change of variables formula is given by $x^i = \frac{\partial x^i}{\partial y^k} y^k$ and $\psi^i = \frac{\partial x^i}{\partial y^k} \theta^k$.
$(x, \psi) \longleftrightarrow (y, \theta)$

Therefore, L and δ is invariant under $(x, \psi) \longleftrightarrow (y, \theta)$.

$$\Big(\ g_{ij}(x) \psi^{\bar{i}} \nabla_t \psi^j = g_{ij}(x) \psi^{\bar{i}} \left(\dot{\psi}^j + \Gamma^j_{lk}(x)\, \dot{x}^l \psi^k \right)$$

$$= g_{st}(y) \frac{\partial y^s}{\partial x^i} \frac{\partial y^t}{\partial x^j} \cdot \frac{\partial x^i}{\partial y^p} \theta^{\bar{p}} \left(\left(\frac{\partial x^j}{\partial y^q} \theta^q \right)' + \left(\frac{\partial x^j}{\partial y^m} \frac{\partial y^n}{\partial x^k} \frac{\partial y^u}{\partial x^l} \Gamma^m_{nu}(y) + \frac{\partial^2 y^m}{\partial x^k \partial x^l} \cdot \frac{\partial x^j}{\partial y^m} \right) \frac{\partial x^l}{\partial y^p} \frac{\partial x^k}{\partial y^q} \dot{y}^p \theta^q \right)$$

$$= g_{st}(y) \frac{\partial y^t}{\partial x^j} \theta^{\bar{s}} \left(\left(\frac{\partial x^j}{\partial y^q} \theta^q \right)' + \frac{\partial x^j}{\partial y^m} \Gamma^m_{qp}(y)\, \dot{y}^p \theta^q - \frac{\partial x^j}{\partial y^m} \frac{\partial y^m}{\partial x^k} \frac{\partial}{\partial x^l} \left(\frac{\partial x^k}{\partial y^q} \right) \dot{x}^l \theta^q \right)$$

$$= g_{st}(y) \frac{\partial y^t}{\partial x^j} \theta^{\bar{s}} \left(\frac{\partial x^j}{\partial y^q} \dot{\theta}^q + \frac{\partial x^j}{\partial y^m} \Gamma^m_{qp}(y)\, \dot{y}^p \theta^q \right)$$

$$= g_{st}(y) \theta^{\bar{s}} \left(\dot{\theta}^t + \Gamma^t_{pq}(y)\, \dot{y}^p \theta^q \right) = g_{st} \theta^s \nabla_t \theta^t$$

The rest of the proofs are similar. $\Big)$

Now, take R.N.C.

$$\delta L = \delta\left(\frac{1}{2} g_{ij}\, \dot{x}^i \dot{x}^j \right) + \delta\left(i g_{ij} \cdot \bar{\psi}^i \nabla_t \psi^j \right) - \frac{1}{2} \delta\left(R_{i\bar{j}k\bar{l}}\, \psi^{i\bar{j}k\bar{l}} \right)$$

$$= g_{ij}\, \delta \dot{x}^i \dot{x}^j + i g_{ij} \left(\delta \psi^{\bar{i}} \dot{\psi}^j + \psi^{\bar{i}} \left(\delta \dot{\psi}^j + \delta \Gamma^j_{lk} \cdot \dot{x}^l \psi^k \right) \right) -$$
$$\frac{1}{2} \delta R_{i\bar{j}k\bar{l}}\, \psi^{i\bar{j}k\bar{l}} - \frac{1}{2} R_{i\bar{j}k\bar{l}} \left(\delta \psi^i \psi^{\bar{j}k\bar{l}} + \psi^i \delta \psi^{\bar{j}} \psi^{k\bar{l}} + \psi^{i\bar{j}} \delta \psi^k \psi^{\bar{l}} + \psi^{i\bar{j}k} \delta \psi^{\bar{l}} \right)$$

$$= g_{ij} \left(\varepsilon \dot{\psi}^{\bar{i}} - \cancel{\bar{\varepsilon} \dot{\psi}^i} \right) \dot{x}^j + i g_{ij} \left(\cancel{- i \bar{\varepsilon} \dot{x}^i \dot{\psi}^j} + \psi^{\bar{i}} \cdot \left(i \varepsilon \ddot{x}^i - \Gamma^j_{st,m}\, \varepsilon\, \dot{x}^m \psi^{\bar{s}t} \right) \right)$$

$-$ by parts

$$+ i g_{ij} \psi^{\bar{i}} \Gamma^j_{lk,m} \left(\varepsilon \psi^{\bar{m}} - \bar{\varepsilon} \psi^m \right) \dot{x}^l \psi^k - \underbrace{\frac{1}{2} R_{i\bar{j}k\bar{l};m} \left(\varepsilon \psi^{\bar{m}} - \bar{\varepsilon} \psi^m \right) \psi^{i\bar{j}k\bar{l}}}_{=\,0 \text{ by Bianchi.}}$$

$$- \frac{1}{2} R_{i\bar{j}k\bar{l}} \left(\varepsilon i \dot{x}^i \psi^{\bar{j}k\bar{l}} - \bar{\varepsilon} \psi^i i \dot{x}^j \psi^{k\bar{l}} + \varepsilon \psi^{i\bar{j}} i \dot{x}^k \psi^{\bar{l}} - \bar{\varepsilon} \psi^{i\bar{j}k} i \dot{x}^l \right) = 0$$

$$\left(i \varepsilon R_{i\bar{j}k\bar{l}} \left(\dot{x}^i \psi^{\bar{j}k\bar{l}} + \dot{x}^k \psi^{i\bar{j}\bar{l}} \right) \underset{\uparrow}{=} 2 i \varepsilon R_{i\bar{j}k\bar{l}}\, \dot{x}^i \psi^{\bar{j}k\bar{l}} = 2 i \varepsilon\, g_{\bar{l}m} \left(\Gamma^m_{jk,i} - \Gamma^m_{ik,j} \right) \dot{x}^i \psi^{\bar{j}k\bar{l}} \right.$$

$$R_{i\bar{j}k\bar{l}}\, \dot{x}^k \psi^{i\bar{j}\bar{l}} = R_{k\bar{l}i\bar{j}}\, \dot{x}^i \psi^{k\bar{l}\bar{j}} = R_{i\bar{j}k\bar{l}}\, \dot{x}^i \psi^{\bar{j}k\bar{l}} \qquad = 2 i \varepsilon \left(g_{ij} \Gamma^j_{st,m}\, \dot{x}^m \psi^{\bar{s}t\bar{i}} - g_{ij} \Gamma^j_{lk,m}\, \dot{x}^l \psi^{\bar{m}k\bar{i}} \right) \Big)$$

Now, let $\varepsilon = \varepsilon(t)$, $\bar{\varepsilon} = \bar{\varepsilon}(t)$. Then

$$\delta \int L = \int g_{ij}(\dot{\varepsilon}\psi^i - \dot{\bar{\varepsilon}}\psi^i)\dot{x}^j + \underbrace{i g_{ij}\psi^i \dot{\varepsilon}\, i\, \dot{x}^j - \dot{\varepsilon} g_{ij}\dot{x}^j\psi^i}_{=0}$$

$$= \int -i\dot{\varepsilon}\underbrace{(i g_{ij}\psi^i\dot{x}^j)}_{=Q} - i\dot{\bar{\varepsilon}}\underbrace{(-i g_{ij}\psi^i\dot{x}^j)}_{=\bar{Q}}$$

Week 4.

Prove $\langle e^{ik_1 x(t_1,s_1)} e^{ik_2 x(t_2,s_2)}\rangle = \langle 0| T[:e^{ik_1 x(t_1,s_1)}::e^{ik_2 x(t_2,s_2)}:]|0\rangle$

$$= 2\pi\delta(k_1+k_2)[(z_1-z_2)(\tilde{z}_1-\tilde{z}_2)]^{\frac{k_1k_2}{2}} \quad \text{where } z_j := e^{i(t_j-s_j)},\ \tilde{z}_j := e^{i(t_j+s_j)}.$$

pf: Note $:e^{ikx}:$ is $e^{ik\frac{i}{\sqrt{2}}\sum_{n\geq 1}\frac{-1}{n}(\alpha_n z^n + \tilde{\alpha}_n\tilde{z}^n)} e^{ikx_0} e^{iktp_0} e^{ik\frac{i}{\sqrt{2}}\sum_{n\geq 1}\frac{1}{n}(\alpha_n z^{-n}+\tilde{\alpha}_n\tilde{z}^{-n})}$

Assume $t_1 > t_2$, then $\langle 0|T[:e^{ik_1 x(t_1,s_1)}::e^{ik_2x(t_2,s_2)}:]|0\rangle$

$$= \langle 0| e^{ik_1\frac{i}{\sqrt{2}}\sum_{n\geq1}\frac{-1}{n}(\alpha_{-n}z_1^n+\tilde{\alpha}_{-n}\tilde{z}_1^n)} \underline{e^{ik_1x_0}}\,\underline{e^{ik_1tp_0}}\, e^{ik_1\frac{i}{\sqrt{2}}\sum_{n\geq1}\frac{1}{n}(\alpha_n z_1^{-n}+\tilde{\alpha}_n\tilde{z}_1^{-n})} e^{ik_2\frac{i}{\sqrt{2}}\sum_{n\geq1}\frac{-1}{n}(\alpha_{-n}z_2^n+\tilde{\alpha}_{-n}\tilde{z}_2^n)} \underline{e^{ik_2x_0}}\, \underbrace{|0\rangle}_{=|0\rangle_0\otimes\bigotimes_{n=1}^{\infty}|0\rangle_n}.$$

$(p_0|k\rangle = k|k\rangle)$

$$= \langle 0| e^{ik_1\frac{i}{\sqrt{2}}\sum_{n\geq1}\frac{-1}{n}(\alpha_{-n}z_1^n+\tilde{\alpha}_{-n}\tilde{z}_1^n)} e^{ik_1\frac{i}{\sqrt{2}}\sum_{n\geq1}\frac{1}{n}(\alpha_n z_1^{-n}+\tilde{\alpha}_n\tilde{z}_1^{-n})} e^{ik_2\frac{i}{\sqrt{2}}\sum_{n\geq1}\frac{-1}{n}(\alpha_{-n}z_2^n+\tilde{\alpha}_{-n}\tilde{z}_2^n)} e^{ik_1k_2t}|k_1+k_2\rangle.$$

$$= \langle 0| e^{ik_1\frac{i}{\sqrt{2}}\sum_{n\geq1}\frac{-1}{n}(\alpha_{-n}z_1^n+\tilde{\alpha}_{-n}\tilde{z}_1^n)} e^{\sum_{n\geq1}\frac{-k_1k_2}{2n}\left(\left(\frac{z_2}{z_1}\right)^n+\left(\frac{\tilde{z}_2}{\tilde{z}_1}\right)^n\right)} e^{\sum_{n\geq1}\frac{k_2}{\sqrt{2}}(z_2^n\alpha_{-n}+\tilde{z}_2^n\tilde{\alpha}_{-n})} e^{ik_1k_2t}|k_1+k_2\rangle.$$

$$\Bigg(e^{ik_1\frac{i}{\sqrt{2}}\sum_{n\geq1}\frac{1}{n}\cdot\alpha_n z_1^{-n}} e^{ik_2\frac{i}{\sqrt{2}}\sum_{n\geq1}\frac{-1}{n}\cdot\alpha_{-n}z_2^n} = \left(\prod_{n\geq1}\sum_{m\geq0}\frac{1}{m!}\left(\frac{-k_1}{\sqrt{2}}\frac{\alpha_n z_1^{-n}}{n}\right)^m\right)\left(\prod_{n'\geq1}\sum_{\ell\geq0}\frac{1}{\ell!}\left(\frac{-k_2}{\sqrt{2}}\frac{-\alpha_{-n'}z_2^{n'}}{n'}\right)^\ell\right)$$

$$= \prod_{n\geq1}\sum_{m,\ell\geq0}\frac{1}{m!\,\ell!}\left(\frac{-k_1}{\sqrt{2}}\frac{z_1^{-n}}{n}\right)^m\left(\frac{k_2}{\sqrt{2}}\frac{z_2^n}{n}\right)^\ell\alpha_n^m\alpha_{-n}^\ell = \prod_{n\geq1}\sum_{m\leq\ell}\frac{1}{m!\,\ell!}\left(\frac{-k_1}{\sqrt{2}}\frac{z_1^{-n}}{n}\right)^m\left(\frac{k_2}{\sqrt{2}}\frac{z_2^n}{n}\right)^\ell\alpha_{-n}^{\ell-m}(\alpha_{-n}\alpha_n+\ell n)\cdots(\alpha_{-n}\alpha_n+(\ell-m+1)n)$$

$$\underset{\text{act on }|k_1+k_2\rangle}{``="}\prod_{n\geq1}\sum_{m\leq\ell}\frac{1}{m!\,\ell!}\left(\frac{-k_1}{\sqrt{2}}\frac{z_1^{-n}}{n}\right)^m\left(\frac{k_2}{\sqrt{2}}\frac{z_2^n}{n}\right)^\ell\alpha_{-n}^{\ell-m}\cdot\frac{\ell!}{(\ell-m)!}n^{\ell-m} = \prod_{n\geq1}\sum_{m,j\geq0}\frac{1}{m!\,j!}\left(\frac{-k_1k_2}{2n}\left(\frac{z_2}{z_1}\right)^n\right)^m\left(\frac{k_2}{\sqrt{2}}z_2^n\right)^j\alpha_{-n}^j.$$

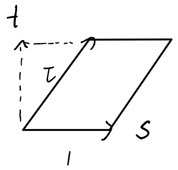
$$= \prod_{n\geq1}e^{\frac{-k_1k_2}{2n}\left(\frac{z_2}{z_1}\right)^n}e^{\frac{k_2}{\sqrt{2}}z_2^n\alpha_{-n}}\Bigg)$$

$$= 2\pi\, e^{k_1k_2\left[it-\sum_{n\geq1}\frac{1}{2n}\left(\left(\frac{z_2}{z_1}\right)^n+\left(\frac{\tilde{z}_2}{\tilde{z}_1}\right)^n\right)\right]}\delta(k_1+k_2) = 2\pi\, e^{\frac{1}{2}k_1k_2\left[\log\left(1-\frac{z_2}{z_1}\right)+\log\left(1-\frac{\tilde{z}_2}{\tilde{z}_1}\right)+\log z_1\tilde{z}_1\right]}\delta(k_1+k_2)$$

$$= 2\pi\left((z_1-z_2)(\tilde{z}_1-\tilde{z}_2)\right)^{\frac{k_1k_2}{2}}\delta(k_1+k_2)$$

$$\left(\sum_{n\geq1}\frac{-x^n}{n} = \log(1-x)\right)$$

Week 5.



$\tau = \tau_1 + i\tau_2$. $\quad (\xi, \bar\xi) = \left(\frac{s+it}{2\pi}, \frac{s-it}{2\pi}\right)$ with $\xi \equiv \xi + 1 \equiv \xi + \tau$.

Assume $\psi_-(t,s) = e^{-2\pi i a}\psi_-(t, s+2\pi) = e^{-2\pi i b}\psi_-(t+2\pi\tau_2, s+2\pi\tau_1)$

$\psi_+(t,s) = e^{2\pi i\tilde a}\psi_+(t, s+2\pi) = e^{2\pi i\tilde b}\psi_+(t+2\pi\tau_2, s+2\pi\tau_1)$.

· Consider $\psi_-'(t,s) := e^{ut+vs}\psi_-(t,s)$ where u, v will be determined.

Want: $\psi_-'(t,s) = \psi_-'(t, s+2\pi) = \psi_-'(t+2\pi\tau_2, s+2\pi\tau_1)$

i.e. $e^{ut+vs} = e^{ut+vs+2\pi v}\cdot e^{2\pi i a} = e^{ut+vs+2\pi\tau_2 u+2\pi\tau_1 v}\cdot e^{2\pi i b}$

$\longrightarrow v = -ia, \quad u = \frac{ia\tau_1 - ib}{\tau_2}$ $\qquad$ Therefore, $\psi_-'(t,s) = e^{\frac{ia\tau_1-ib}{\tau_2}t - ias}\psi_-(t,s)$.

Write $\psi_-'(t,s) = \sum_{r\in\mathbb{Z}} \psi_r(t)e^{irs}$ $\longrightarrow$ $\psi_-(t,s) = e^{\frac{ib-ia\tau_1}{\tau_2}t + ias}\psi_-'(t,s)$

$$= \sum_{r\in\mathbb{Z}+a}\psi_r(t)e^{\frac{ib-ia\tau_1}{\tau_2}t + irs}$$

Similarly, we have $\psi_+'(t,s) = e^{\frac{-i\tilde a\tau_1+i\tilde b}{\tau_2}t + i\tilde a s}\psi_+(t,s)$ and

$$\psi_+(t,s) = \sum_{\tilde r\in\mathbb{Z}+\tilde a}\tilde\psi_{\tilde r}(t)e^{\frac{i\tilde a\tau_1-i\tilde b}{\tau_2}t - i\tilde r s}$$

Also, $\bar\psi_-(t,s) = \sum_{r\in\mathbb{Z}-a}\bar\psi_r(t)e^{\frac{-ib+ia\tau_1}{\tau_2}t + irs}$; $\bar\psi_+(t,s) = \sum_{\tilde r\in\mathbb{Z}-\tilde a}\bar{\tilde\psi}_{\tilde r}(t)e^{\frac{-i\tilde a\tau_1+i\tilde b}{\tau_2}t - i\tilde r s}$

· Check $A^{0,1} = 2\pi i\frac{b-\tau a}{2\tau_2}d\bar\xi$, $\tilde A^{1,0} = 2\pi i\frac{\tilde b-\bar\tau\tilde a}{2\tau_2}d\xi$:

Note $\psi_-'(t,s) = e^{\frac{ia\tau_1-ib}{\tau_2}t - ias}\psi_-(t,s)$. $\qquad \left(s = \pi(\xi+\bar\xi),\ t = \frac{\pi}{i}(\xi-\bar\xi)\right)$

$$= e^{\frac{\pi}{\tau_2}(a\bar\tau-b)\xi - \frac{\pi}{\tau_2}(a\tau-b)\bar\xi}\psi_-(t,s)$$

$$\psi_+'(t,s) = e^{\frac{-i\tilde a\tau_1+i\tilde b}{\tau_2}t + i\tilde a s}\psi_+(t,s)$$

$$= e^{-\frac{\pi}{\tau_2}(\tilde a\bar\tau-\tilde b)\xi + \frac{\pi}{\tau_2}(\tilde a\tau-\tilde b)\bar\xi}\psi_+(t,s)$$

$\longrightarrow$ ψ_- corresponds to the line bundle $\mathcal{O}\otimes\mathcal{E}^{-\frac{\pi}{\tau_2}(a\bar\tau-b)\xi + \frac{\pi}{\tau_2}(a\tau-b)\bar\xi}$

with $\nabla = d + d\left(-\frac{\pi}{\tau_2}(a\bar\tau-b)\xi + \frac{\pi}{\tau_2}(a\tau-b)\bar\xi\right)$

$$= -\frac{\pi}{\tau_2}(a\bar\tau-b)d\xi + \frac{\pi}{\tau_2}(a\tau-b)d\bar\xi$$

ψ_+ corresponds to the line bundle $\mathcal{O}\otimes\mathcal{E}^{-\frac{\pi}{\tau_2}(\tilde a\bar\tau-\tilde b)\xi + \frac{\pi}{\tau_2}(\tilde a\tau-\tilde b)\bar\xi}$

with $\nabla = d + d\left(\frac{\pi}{\tau_2}(\tilde a\bar\tau-\tilde b)\xi - \frac{\pi}{\tau_2}(\tilde a\tau-\tilde b)\bar\xi\right)$

$$= \frac{\pi}{\tau_2}(\tilde a\bar\tau-\tilde b)d\xi - \frac{\pi}{\tau_2}(\tilde a\tau-\tilde b)d\bar\xi$$

Week 6

- $\bar{D}_\pm := -\partial_{\theta^\pm} + i\theta^\pm \partial_\pm \quad \left(\partial_\pm = \frac{1}{2}\partial_0 \pm \partial_1\right)$, $y^\pm = x^\pm - i\theta^\pm \bar\theta^\pm$

Solve $\bar{D}_\pm \Phi = 0$:

Let
$$\Phi = f + \theta^+ f_+ + \theta^- f_- + \theta^= f_= + \theta^\mp f_\mp$$
$$+ \theta^{+-} f_{+-} + \theta^{+=} f_{+=} + \theta^{+\mp} f_{+\mp} + \theta^{-=} f_{-=} + \theta^{-\mp} f_{-\mp} + \theta^{=\mp} f_{=\mp}$$
$$+ \theta^{+-=} f_{+-=} + \theta^{+-\mp} f_{+-\mp} + \theta^{+=\mp} f_{+=\mp} + \theta^{-=\mp} f_{-=\mp} + \theta^4 f_4$$

Abbrietion :
$$\theta^{=\mp} := \bar\theta^- \bar\theta^+$$
$$\theta^4 := \theta^{+-=\mp}$$
$${}_\pm f := \partial_\pm f$$

Then
$$\bar{D}_+ \Phi = -f_\mp + i\theta^+ {}_+f + \theta^+ f_{+\mp} + \theta^- f_{-\mp} + \theta^= f_{=\mp}$$
$$+ i\theta^+ (\theta^- {}_+f_- + \theta^\mp {}_+f_\mp + \theta^= {}_+f_=) - \theta^{+-} f_{+-\mp} - \theta^{+=} f_{+=\mp} - \theta^{-=} f_{-=\mp}$$
$$+ i\theta^+ (\theta^{-=} {}_+f_{-=} + \theta^{-\mp} {}_+f_{-\mp} + \theta^{=\mp} {}_+f_{=\mp}) + \theta^{+-=} f_4 + i\theta^4 {}_+f_{-=\mp}$$

$$\bar{D}_- \Phi = -f_= + i\theta^- {}_-f + \theta^+ f_{+=} + \theta^- f_{-=} - \theta^= f_{=\mp}$$
$$+ i\theta^- (\theta^- {}_-f_+ + \theta^\mp {}_-f_\mp + \theta^= {}_-f_=) - \theta^{+-} f_{+-=} + \theta^{+\mp} f_{+=\mp} + \theta^{-\mp} f_{-=\mp}$$
$$+ i\theta^+ (\theta^{-\mp} {}_-f_{-\mp} + \theta^{+\mp} {}_-f_{+\mp} + \theta^{=\mp} {}_-f_{=\mp}) - \theta^{+-\mp} f_4 + i\theta^4 {}_-f_{+=\mp}$$

$$\longrightarrow \begin{cases} 0 = f_\mp = f_= = f_{+=} = f_{-\mp} = f_{=\mp} = f_{+=\mp} = f_{-=\mp} \\ f_{+\mp} = -i\,{}_+f \ ; \ f_{-=} = -i\,{}_-f \ ; \ i\,{}_+f_- = f_{+-\mp} \ ; \ i\,{}_-f_+ = -f_{+-=} \ ; \ i\,{}_+f_{-=} = -f_4 = i\,{}_-f_{+\mp} \end{cases}$$

Therefore,
$$\Phi = f + \theta^+ f_+ + \theta^- f_- + \theta^{+-} f_{+-} + \theta^{+\mp} f_{+\mp} + \theta^{-=} f_{-=} + \theta^{+-=} f_{+-=} + \theta^{+-\mp} f_{+-\mp} + \theta^4 f_4$$

(Key formula)
$$= \left(f(y^\pm) + \cancel{i\,{}_+f\,\theta^{+\mp}} + \cancel{i\,{}_-f\,\theta^{-=}} + \cancel{{}_{-+}f\,\theta^4}\right) - \cancel{i\,{}_+f\,\theta^{+\mp}} - \cancel{i\,{}_-f\,\theta^{-=}} + \theta^{+-} f_{+-}(y^\pm) - \cancel{\theta^4\,{}_{-+}f}$$
$$\theta^+ f_+(y^\pm) + \theta^- f_-(y^\pm)$$

$\left(\theta^{+-} f_{+-}(x^\pm) = \theta^{+-} f_{+-}(y^\pm)\right)$

$\left(\theta^+ f_+(x^\pm) = \theta^+ f_+(y^\pm) + i\,{}_-f_+\theta^{+-=} \ , \ \theta^- f_-(x^\pm) = \theta^- f_-(y^\pm) + i\,{}_+f_-\theta^{-+\mp}\right)$

$$= f(y^\pm) + \theta^+ f_+(y^\pm) + \theta^- f_-(y^\pm) + \theta^{+-} f_{+-}(y^\pm)$$

- Key formula : $f(x^\pm) = f(y^\pm) + i\,{}_+f(x^\pm)\theta^{+\mp} + i\,{}_-f(x^\pm)\theta^{-=} + {}_{-+}f\,\theta^4$

pf : check for monomials !

· Compute the conserved currents and charges.

1. $G^0_\pm = 2\partial_\pm\bar\phi\,\psi_\pm \mp i\bar\psi_\mp \bar W'(\bar\phi)$, $\quad Q_\pm = \int dx^1 G^0_\pm$

$G^1_\pm = \mp 2\partial_\pm\bar\phi\,\psi_\pm - i\bar\psi_\mp \bar W'(\bar\phi)$ $\qquad \bar Q_\pm = \int dx^1 \bar G^0_\pm$

$\bar G^0_\pm = 2\bar\psi_\pm \partial_\pm\phi \pm i\psi_\mp W'(\phi)$

$\bar G^1_\pm = \mp 2\bar\psi_\pm \partial_\pm\phi \pm i\psi_\mp W'(\phi)$

2. Axial rotation : $\phi \longmapsto \phi$, $\psi_\pm \longmapsto e^{\mp i\alpha}\psi_\pm$

$\longrightarrow$ current : $J^0_A = \bar\psi_+\psi_+ - \bar\psi_-\psi_-$, $\quad J^1_A = -\bar\psi_+\psi_+ - \bar\psi_-\psi_-$

charge : $F_A = \int J^0_A\, dx^1$

3. Vector rotation : $(W(\Phi) = c\Phi^k)$ $\quad \phi \longmapsto e^{\frac{2}{k}i\alpha}\phi$, $\psi_\pm \longmapsto e^{(\frac{2}{k}-1)i\alpha}\psi_\pm$

$\longrightarrow$ current : $J^0_V = \frac{2i}{k}(\partial_0\bar\phi\phi - \bar\phi\partial_0\phi) - (\frac{2}{k}-1)(\bar\psi_+\psi_+ + \bar\psi_-\psi_-)$

$J^1_V = \frac{2i}{k}(-\partial_1\bar\phi\phi + \bar\phi\partial_1\phi) + (\frac{2}{k}-1)(\bar\psi_+\psi_+ - \bar\psi_-\psi_-)$

charge : $F_V = \int J^0_V\, dx^1$.

pf : Note $\delta = \varepsilon_+ Q_- - \varepsilon_- Q_+ - \bar\varepsilon_+\bar Q_- + \bar\varepsilon_-\bar Q_+$

and $Q_\pm := \frac{\partial}{\partial\theta^\pm} + i\bar\theta^\pm\partial_\pm$, $\quad \bar Q_\pm := -\frac{\partial}{\partial\bar\theta^\pm} - i\theta^\pm\partial_\pm$.

Hence, $\delta(\Phi = \phi + \theta^\pm\psi_\pm + \theta^2 F)$

$$= \varepsilon_+\Big(\cancel{\partial_{y^-}\phi(-i\bar\theta^-)} + \psi_- \underline{-\theta^-\partial_{y^-}\psi_-(-i\bar\theta^-) - \theta^+\partial_{y^-}\psi_+(-i\bar\theta^-)} - \theta^+F + \cancel{\theta^2\partial_{y^-}F(-i\bar\theta^-)} + i\bar\theta^-\big(\cancel{\partial_{y^-}\phi} + \underline{\theta^\pm\partial_{y^-}\psi_\pm} + \cancel{\theta^2\partial_{y^-}F}\big)\Big)$$

$$-\varepsilon_-\Big(\cancel{\partial_{y^+}\phi(-i\bar\theta^\mp)} + \psi_+ \underline{-\theta^+\partial_{y^+}\psi_+(-i\bar\theta^\mp) - \theta^-\partial_{y^+}\psi_-(-i\bar\theta^\mp)} + \theta^-F + \cancel{\theta^2\partial_{y^+}F(-i\bar\theta^\mp)} + i\bar\theta^\mp\big(\cancel{\partial_{y^+}\phi} + \underline{\theta^\pm\partial_{y^+}\psi_\pm} + \cancel{\theta^2\partial_{y^+}F}\big)\Big)$$

$$+\bar\varepsilon_+\Big(\partial_{y^-}\phi(+i\theta^-) \cancel{-\theta^-\partial_{y^-}\psi_-(+i\theta^-)} - \theta^+\partial_{y^-}\psi_+(+i\theta^-) + \cancel{\theta^2\partial_{y^-}F(-i\theta^-)} + i\theta^-\big(\partial_{y^-}\phi + \theta^+\partial_{y^-}\psi_+ + \cancel{\theta^-\partial_{y^-}\psi_-} + \cancel{\theta^2\partial_{y^-}F}\big)\Big)$$

$$-\bar\varepsilon_-\Big(\partial_{y^+}\phi(+i\theta^+) + \cancel{\theta^+\partial_{y^+}\psi_+(+i\theta^+)} + \theta^-\partial_{y^+}\psi_-(+i\theta^+) + \cancel{\theta^2\partial_{y^+}F(-i\theta^+)} + i\theta^+\big(\partial_{y^+}\phi + \cancel{\theta^+\partial_{y^+}\psi_+} + \theta^-\partial_{y^+}\psi_- + \cancel{\theta^2\partial_{y^+}F}\big)\Big)$$

$$= \underbrace{\varepsilon_+\psi_- - \varepsilon_-\psi_+}_{=\,\delta\phi} + \theta^\pm\underbrace{(\pm 2i\bar\varepsilon_\mp\partial_\pm\phi \pm \varepsilon_\pm F)}_{=\,\delta\psi_\pm} + \theta^2\underbrace{(-2i\bar\varepsilon_+\partial_-\psi_+ - 2i\bar\varepsilon_-\partial_+\psi_-)}_{=\,\delta F}$$

Now, consider $\delta S = \delta\int d^2x\big(|\partial_0\phi|^2 - |\partial_1\phi|^2 - |W'(\phi)|^2 + 2i\bar\psi_-\partial_+\psi_- + 2i\bar\psi_+\partial_-\psi_+ - W''(\phi)\psi_{+-} - \bar W''(\bar\phi)\bar\psi_{-+}\big)$

$\longrightarrow$ terms involving ε_- $= \int d^2x\big(\partial_0(-\varepsilon_-\psi_+)\partial_0\bar\phi - \partial_1(-\varepsilon_-\psi_+)\partial_1\bar\phi \cancel{- W''(\phi)(-\varepsilon_-\psi_+)\bar W'(\bar\phi)}$

$+2i\bar\psi_-\partial_+(\varepsilon_-\underline{F}) + 2i(-2i\bar\varepsilon_-\partial_+\bar\phi)\partial_-\psi_+ \cancel{- W'''(\phi)(-\varepsilon_-\psi_+)\psi_{+-}}$

$\cancel{- W''(\phi)\psi_+(\varepsilon_-F)} \underline{- \bar W''(\bar\phi)\bar\psi_-(-2i\bar\varepsilon_-\partial_+\bar\phi)}\big)$

$$
= \int d^2x \Big(-(\partial_0\varepsilon_-)\psi_+\partial_0\bar\phi - \varepsilon_-\partial_0\psi_+\partial_0\bar\phi + (\partial_1\varepsilon_-)\psi_+\partial_1\bar\phi + \varepsilon_-(\partial_1\psi_+)\partial_1\bar\phi
$$

$$
+ 2i\bar\psi_-(\partial_+\varepsilon_-)F + \underline{4\varepsilon_-\partial_+\bar\phi\,\partial_-\psi_+} \Big) \rightarrow \varepsilon_-(\partial_0+\partial_1)\bar\phi(\partial_0-\partial_1)\psi_+
$$

$$
= \int d^2x \Big[\partial_0\varepsilon_-\big(\cancel{-\psi_+\partial_0\bar\phi + \psi_+\partial_0\bar\phi} - i\bar\psi_-\bar W'(\bar\phi) - \partial_0\bar\phi\psi_+ - \partial_1\bar\phi\psi_+
$$

$$
\partial_1\varepsilon_-\big(\cancel{\psi_+\partial_1\bar\phi - \psi_+\partial_1\bar\phi} - i\bar\psi_-\bar W'(\bar\phi) + \partial_0\bar\phi\psi_+ + \partial_1\bar\phi\psi_+
$$

$$
+ \cancel{\varepsilon_-\psi_+\partial_0\partial_0\bar\phi} - \underline{\varepsilon_-\psi_+\partial_1\partial_1\bar\phi} - \cancel{\varepsilon_-(\partial_0\partial_0\bar\phi)\psi_+} + \varepsilon_-\partial_1\partial_1\bar\phi\psi_+
$$

$$
- \varepsilon_-\partial_0\partial_1\bar\phi\psi_+ + \underline{\varepsilon_-(\partial_0\partial_1\bar\phi)\psi_+} \Big]
$$

$$
= \int d^2x \Big[\partial_0\varepsilon_-\big(\underbrace{-2(\partial_+\bar\phi)\psi_+ - i\bar\psi_-\bar W'(\bar\phi)}_{G_+^0}\big) + \partial_1\varepsilon_-\big(\underbrace{2(\partial_+\bar\phi)\psi_+ - i\bar\psi_-\bar W'(\bar\phi)}_{G_+^1}\big) \Big]
$$

The computations of other G's are similar.

2. $\delta_A^c\phi = 0$, $\quad \delta_A^c\psi_\pm = \frac{\partial}{\partial\sigma}\left(e^{\mp ic\sigma}\psi_\pm\right)\Big|_{\sigma=0} = \mp ic\psi_\pm$, $\quad \delta_A^c\bar\psi_\pm = \pm ic\bar\psi_\pm$

$$
\rightarrow \delta_A^c S = \int d^2x \Big(2i(-ic\bar\psi_-)\partial_+\psi_- + 2i\bar\psi_-\partial_+(ic\psi_-) + 2i(ic\bar\psi_+)\partial_-\psi_+ + 2i\bar\psi_+\partial_-(-ic\psi_+)
$$

$$
- W''(\phi)(-ic\psi_{+-} + ic\psi_{+-}) - \bar W''(\bar\phi)(-ic\bar\psi_{-+} + ic\bar\psi_{-+}) \Big)
$$

$$
= \int d^2x \left(-2(\partial_+c)\bar\psi_-\psi_- + 2(\partial_-c)\bar\psi_+\psi_+\right)
$$

$$
= \int d^2x \Big(\partial_0 c\,\underbrace{(\bar\psi_+\psi_+ - \bar\psi_-\psi_-)}_{J_A^0} + \partial_1 c\,\underbrace{(-\bar\psi_+\psi_+ - \bar\psi_-\psi_-)}_{J_A^1} \Big)
$$

3. $\delta_V^c\phi = \frac{\partial}{\partial\sigma} e^{\frac{2}{k}ic\sigma}\phi\Big|_{\sigma=0} = \frac{2ic}{k}\phi$.

$\delta_V^c\phi = \frac{\partial}{\partial\sigma} e^{(\frac{2}{k}-1)ic\sigma}\phi\Big|_{\sigma=0} = (\frac{2}{k}-1)ic\psi_\pm$

$$
\rightarrow \delta S = \int d^2x \Big(\partial_0(\tfrac{2ic}{k}\phi)\partial_0\bar\phi + \partial_0\phi\,\partial_0(\tfrac{-2ic}{k}\bar\phi) - \partial_1(\tfrac{2ic}{k}\phi)\partial_1\bar\phi - \partial_1\phi\,\partial_1(\tfrac{-2ic}{k}\bar\phi)
$$

$$
\cancel{- W''(\phi)(\tfrac{2ic}{k}\phi)\bar W'(\bar\phi) - W'(\phi)\bar W''(\bar\phi)(\tfrac{-2ic}{k}\bar\phi)}
$$

$$
+ 2i\left(-(\tfrac{2}{k}-1)ic\bar\psi_-\right)\partial_+\psi_- + 2i\bar\psi_-\partial_+\left((\tfrac{2}{k}-1)ic\psi_-\right)
$$

$$
+ 2i\left(-(\tfrac{2}{k}-1)ic\bar\psi_+\right)\partial_-\psi_+ + 2i\bar\psi_+\partial_-\left((\tfrac{2}{k}-1)ic\psi_+\right)
$$

$$
\cancel{- W'''(\phi)(\tfrac{2ic}{k})\phi\psi_+\psi_- - W''(\phi)(\tfrac{2}{k}-1)ic\psi_+\psi_- - W''(\phi)\psi_+(\tfrac{2}{k}-1)ic\psi_-}
$$

$$
\cancel{- \bar W'''(\bar\phi)(\tfrac{-2ic}{k})\bar\phi\bar\psi_-\bar\psi_+ + \bar W''(\bar\phi)(\tfrac{2}{k}-1)ic\bar\psi_-\bar\psi_+ + \bar W''(\bar\phi)\bar\psi_-(\tfrac{2}{k}-1)ic\bar\psi_+} \Big)
$$

$$
= \int d^2x\, \partial_0 c \left(\tfrac{2i}{k}\phi(\partial_0\bar\phi) - \tfrac{2i}{k}(\partial_0\phi)\bar\phi - (\tfrac{2}{k}-1)\bar\psi_-\psi_- - (\tfrac{2}{k}-1)\bar\psi_+\psi_+ \right) = J_V^0
$$

$$
\partial_1 c \left(\tfrac{-2i}{k}\phi(\partial_1\bar\phi) + \tfrac{2i}{k}(\partial_1\phi)\bar\phi - (\tfrac{2}{k}-1)\bar\psi_-\psi_- - (\tfrac{2}{k}-1)\bar\psi_+\psi_+ \right) = J_V^1
$$

Week 7

· Compute $L_{Kin} := \int d\theta^4 K(\Phi,\bar{\Phi}) = -g_{i\bar{j}}\partial^\mu\phi^i\partial_\mu\phi^{\bar{j}} + i g_{i\bar{j}}\bar{\psi}^{\bar{j}}_{\mp} D_{\pm}\psi^i_{\mp} + R_{i\bar{j}k\bar{l}}\psi^i_+\psi^k_-\bar{\psi}^{\bar{j}}_-\bar{\psi}^{\bar{l}}_+$

$$+ g_{i\bar{j}}(F^i - \Gamma^i_{lk}\psi^l_+\psi^k_-)(\bar{F}^{\bar{j}} - \Gamma^{\bar{j}}_{\bar{l}\bar{k}}\bar{\psi}^{\bar{l}}_-\bar{\psi}^{\bar{k}}_+)$$

sol : Note

$$\Phi = \phi(x^\pm) - i\partial_+\phi(x^\pm)\theta^{+\bar{+}} - i\partial_-\phi(x^\pm)\theta^{-\bar{-}} - \partial_{-+}\phi(x^\pm)\theta^4$$

$$+ \theta^+\psi_+(x^\pm) - i\theta^{+-\bar{-}}\partial_-\psi_+(x^\pm) + \theta^-\psi_-(x^\pm) - i\theta^{-+\bar{+}}\partial_+\psi_-(x^\pm) + \theta^2 F(x^\pm)$$

$$\bar{\Phi} = \bar{\phi}(x^\pm) + i\partial_+\bar{\phi}(x^\pm)\theta^{+\bar{+}} + i\partial_-\bar{\phi}(x^\pm)\theta^{-\bar{-}} - \partial_{-+}\bar{\phi}(x^\pm)\theta^4$$

$$- \theta^{\bar{+}}\bar{\psi}_+(x^\pm) - i\theta^{-\bar{-}\bar{+}}\partial_-\bar{\psi}_+(x^\pm) - \theta^{\bar{-}}\bar{\psi}_-(x^\pm) - i\theta^{+\bar{+}\bar{-}}\partial_+\bar{\psi}_-(x^\pm) + \bar{\theta}^2\bar{F}(x^\pm)$$

$\longrightarrow$ Taylor expansion of K at ϕ :

$$K(\Phi,\bar{\Phi}) = K(\phi,\bar{\phi}) + (\partial_i K)(\Phi^i - \phi^i) + (\partial_{\bar{j}}K)(\bar{\Phi}^{\bar{j}} - \phi^{\bar{j}})$$

$$+ (\partial_{i\bar{j}}K)(\Phi^i - \phi^i)(\bar{\Phi}^{\bar{j}} - \phi^{\bar{j}}) + \frac{1}{2}\left[(\partial_{ij}K)(\Phi^i - \phi^i)^2 + (\partial_{\bar{i}\bar{j}}K)(\bar{\Phi}^{\bar{j}} - \phi^{\bar{j}})^2\right] \quad A$$

B

$$+ \frac{1}{3!}\Big[3\partial_k g_{i\bar{j}}(\Phi^i - \phi^i)(\bar{\Phi}^{\bar{j}} - \phi^{\bar{j}})(\Phi^k - \phi^k) + 3\partial_{\bar{k}} g_{i\bar{j}}(\Phi^i - \phi^i)(\bar{\Phi}^{\bar{j}} - \phi^{\bar{j}})(\bar{\Phi}^{\bar{k}} - \phi^{\bar{k}})$$

$$+ (\partial_{ijk}K)(\Phi^i - \phi^i)(\bar{\Phi}^j - \phi^j)(\Phi^k - \phi^k) + (\partial_{\bar{i}\bar{j}\bar{k}}K)(\Phi^{\bar{i}} - \phi^{\bar{i}})(\bar{\Phi}^{\bar{j}} - \phi^{\bar{j}})(\Phi^{\bar{k}} - \phi^{\bar{k}})\Big]$$

C

$$+ \frac{1}{4!}\Big[6\partial_{\bar{l}}\partial_{\bar{k}} g_{i\bar{j}}(\Phi^i - \phi^i)(\bar{\Phi}^{\bar{j}} - \phi^{\bar{j}})(\Phi^{\bar{k}} - \phi^{\bar{k}})(\Phi^l - \phi^l)$$

$$+ 4\partial_l\partial_k g_{i\bar{j}}(\Phi^i - \phi^i)(\bar{\Phi}^{\bar{j}} - \phi^{\bar{j}})(\Phi^k - \phi^k)(\Phi^l - \phi^l)$$

$$+ 4\partial_{\bar{l}}\partial_{\bar{k}} g_{i\bar{j}}(\Phi^i - \phi^i)(\bar{\Phi}^{\bar{j}} - \phi^{\bar{j}})(\Phi^{\bar{k}} - \phi^{\bar{k}})(\Phi^{\bar{l}} - \phi^{\bar{l}})$$

$$+ \partial_l\partial_k g_{ij}(\Phi^i - \phi^i)(\bar{\Phi}^j - \phi^j)(\Phi^k - \phi^k)(\Phi^l - \phi^l)$$

$$+ \partial_{\bar{l}}\partial_{\bar{k}} g_{\bar{i}\bar{j}}(\Phi^{\bar{i}} - \phi^{\bar{i}})(\bar{\Phi}^{\bar{j}} - \phi^{\bar{j}})(\Phi^{\bar{k}} - \phi^{\bar{k}})(\Phi^{\bar{l}} - \phi^{\bar{l}})\Big]$$

$$\cdot\int d\theta^4 A = -\partial_i K\,\partial_{-+}\phi^i - \partial_{\bar{j}}K\,\partial_{-+}\phi^{\bar{j}} + \frac{1}{2}\left[-(\partial_{ij}K)\cdot 2\partial_-\phi^i\partial_+\phi^j - (\partial_{\bar{i}\bar{j}}K)\cdot 2\partial_-\phi^{\bar{i}}\partial_+\phi^{\bar{j}}\right]$$

$$+ \partial_{i\bar{j}}K\cdot(\partial_+\phi^i\partial_-\bar{\phi}^{\bar{j}} + \partial_-\phi^i\partial_+\bar{\phi}^{\bar{j}} + \underline{i\psi^i_+\partial_-\bar{\psi}^{\bar{j}}_+ - i\partial_-\psi^i_+\bar{\psi}^{\bar{j}}_+} + \underline{i\psi^i_-\partial_+\bar{\psi}^{\bar{j}}_- - i\partial_+\psi^i_-\bar{\psi}^{\bar{j}}_-} + F^i\bar{F}^{\bar{j}})$$

$$\overset{\text{(by parts)}}{=} g_{i\bar{j}}(2\partial_-\phi^{\bar{j}}\partial_+\phi^i + 2\partial_-\phi^i\partial_+\phi^{\bar{j}} + 2i\bar{\psi}^{\bar{j}}_+\partial_-\psi^i_+ + 2i\bar{\psi}^{\bar{j}}_-\partial_+\psi^i_- + F^i\bar{F}^{\bar{j}})$$

$$-i\partial_k g_{i\bar{j}}(\partial_-\phi^k\psi^i_+\bar{\psi}^{\bar{j}}_+ + \partial_+\phi^k\psi^i_-\bar{\psi}^{\bar{j}}_-) - i\partial_{\bar{k}} g_{i\bar{j}}(\partial_-\phi^{\bar{k}}\psi^i_+\bar{\psi}^{\bar{j}}_+ + \partial_+\phi^{\bar{k}}\psi^i_-\bar{\psi}^{\bar{j}}_-)$$

$$= -g_{i\bar{j}}\partial^\mu\phi^i\partial_\mu\phi^{\bar{j}} + g_{i\bar{j}}F^i\bar{F}^{\bar{j}} + i g_{i\bar{j}}\bar{\psi}^{\bar{j}}_\pm(D_\mp\psi^i_\pm - 2\partial_\mp\phi^j\Gamma^i_{jk}\psi^k_\pm)$$

$$-i\partial_k g_{i\bar{j}}(\partial_-\phi^k\psi^i_+\bar{\psi}^{\bar{j}}_+ + \partial_+\phi^k\psi^i_-\bar{\psi}^{\bar{j}}_-) - i\partial_{\bar{k}} g_{i\bar{j}}(\partial_-\phi^{\bar{k}}\psi^i_+\bar{\psi}^{\bar{j}}_+ + \partial_+\phi^{\bar{k}}\psi^i_-\bar{\psi}^{\bar{j}}_-)$$

$$\cdot \int d\theta^4 B = \frac{1}{2}\partial_k g_{i\bar j}\left(i\partial_+\phi^i\bar\psi_-^{\bar j}\psi_-^k + i\partial_-\phi^i\bar\psi_+^{\bar j}\psi_+^k - \psi_+^i\bar F^{\bar j}\psi_-^k - i\psi_+^i\bar\psi_+^{\bar j}\partial_-\phi^k + \psi_-^i\bar F^{\bar j}\psi_+^k - i\psi_-^i\bar\psi_-^{\bar j}\partial_+\phi^k\right)$$

$$+\frac{1}{2}\partial_{\bar k} g_{i\bar j}\left(i\psi_+^i\partial_-\bar\phi^{\bar j}\bar\psi_+^{\bar k} + i\psi_+^i\bar\psi_+^{\bar j}\partial_-\bar\phi^{\bar k} + i\psi_-^i\partial_+\bar\phi^{\bar j}\bar\psi_-^{\bar k} + i\psi_-^i\bar\psi_-^{\bar j}\partial_+\bar\phi^{\bar k} + F^i\bar\psi_+^{\bar j}\bar\psi_-^{\bar k} - F^i\bar\psi_-^{\bar j}\bar\psi_+^{\bar k}\right)$$

$$= \partial_k g_{i\bar j}\left(i\partial_+\phi^i\bar\psi_-^{\bar j}\psi_-^k + i\partial_-\phi^i\bar\psi_+^{\bar j}\psi_+^k + \psi_-^i\bar F^{\bar j}\psi_+^k\right) + \partial_{\bar k} g_{i\bar j}\left(i\psi_+^i\bar\psi_+^{\bar j}\partial_-\bar\phi^{\bar k} + i\psi_-^i\bar\psi_-^{\bar j}\partial_+\bar\phi^{\bar k} + F^i\bar\psi_+^{\bar j}\bar\psi_-^{\bar k}\right)$$

$$\cdot \int d\theta^4 C = \partial_l\partial_{\bar k} g_{i\bar j}\left(\psi_+^i\bar\psi_+^{\bar j}\psi_-^l\bar\psi_-^{\bar k}\right)$$

$$= \left(g_{s\bar p}\Gamma^{\bar p}_{\bar k\bar j}\Gamma^s_{il} + R_{i\bar j l\bar k}\right)\cdot\psi_+^i\bar\psi_+^{\bar j}\psi_-^l\bar\psi_-^{\bar k}$$

Therefore,

$$L_{Kin} = -g_{i\bar j}\partial^\mu\phi^i\partial_\mu\bar\phi^{\bar j} + g_{i\bar j}F^i\bar F^{\bar j} + ig_{i\bar j}\bar\psi_\pm^{\bar j}\left(D_\mp\psi_\pm^i - 2\partial_\mp\phi^j\Gamma^i_{jk}\psi_\pm^k\right)$$

$$-i\partial_k g_{i\bar j}\left(\partial_-\phi^k\psi_+^i\bar\psi_+^{\bar j} + \partial_+\phi^k\psi_-^i\bar\psi_-^{\bar j}\right) - i\partial_{\bar k} g_{i\bar j}\left(\cancel{\partial_-\bar\phi^{\bar k}\psi_+^i\bar\psi_+^{\bar j} + \partial_+\bar\phi^{\bar k}\psi_-^i\bar\psi_-^{\bar j}}\right)$$

$$+\partial_k g_{i\bar j}\left(i\partial_+\phi^i\bar\psi_-^{\bar j}\psi_-^k + i\partial_-\phi^i\bar\psi_+^{\bar j}\psi_+^k + \psi_-^i\bar F^{\bar j}\psi_+^k\right)$$

$$+\partial_{\bar k} g_{i\bar j}\left(\cancel{i\psi_+^i\bar\psi_+^{\bar j}\partial_-\bar\phi^{\bar k}} + i\psi_-^i\bar\psi_-^{\bar j}\partial_+\bar\phi^{\bar k} + F^i\bar\psi_+^{\bar j}\bar\psi_-^{\bar k}\right)$$

$$+\left(g_{s\bar p}\Gamma^{\bar p}_{\bar k\bar j}\Gamma^s_{il} + R_{i\bar j l\bar k}\right)\cdot\psi_+^i\bar\psi_+^{\bar j}\psi_-^l\bar\psi_-^{\bar k}$$

$$= -g_{i\bar j}\partial^\mu\phi^i\partial_\mu\bar\phi^{\bar j} + ig_{i\bar j}\bar\psi_\pm^{\bar j}D_\mp\psi_\pm^i + R_{i\bar j k\bar l}\psi_+^i\psi_-^k\bar\psi_-^{\bar j}\bar\psi_+^{\bar l}$$

$$+g_{i\bar j}\left(F^i - \Gamma^i_{lk}\psi_+^l\psi_-^k\right)\left(\bar F^{\bar j} - \Gamma^{\bar j}_{\bar l\bar k}\bar\psi_-^{\bar l}\bar\psi_+^{\bar k}\right)$$

$$\partial_k g_{i\bar j} = g_{l\bar j}\Gamma^l_{ki}$$

$$\partial_{\bar k} g_{i\bar j} = g_{i\bar l}\Gamma^{\bar l}_{\bar k\bar j}$$

Week 8

$$Z(M,C) := \int d^nX \cdot \exp\left(\tfrac{-1}{2} X^i M_{ij} X^j + C_{ijkl} X^{ijkl}\right)$$
$$X^{ijkl} = X^iX^jX^kX^l$$

Compute $\langle X^iX^j\rangle_{(1)}$, $\langle X^{ijkl}\rangle_{(1)}$:

· Consider $\int d^nX \cdot \exp\left(\tfrac{-1}{2} X^i M_{ij} X^j + J_k X^k\right) = \int d^nX \cdot \exp\left(\tfrac{-1}{2} X^T \underbrace{P^T P}_{=M} X + J\cdot X\right)$

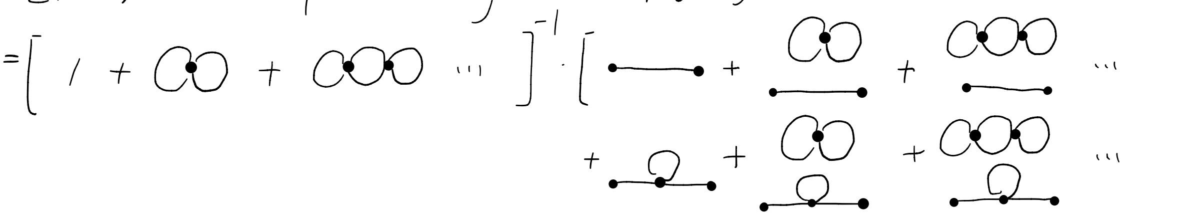
$$= \int d^nY \cdot \exp\left(\tfrac{-1}{2} Y^T Y + JP^{-1}Y\right) \det P^{-1} \quad , \quad Y := PX.$$
$$= \sqrt{2\pi}^{\,n} \cdot \exp\left(\tfrac{1}{2} JP^{-1}(JP^{-1})^T\right)(\det M)^{-1/2}$$
$$= (2\pi)^{n/2}(\det M)^{-1/2} \exp\left(\tfrac{1}{2} J M^{-1} J\right).$$

Then $\int d^nX \cdot \exp\left(\tfrac{-1}{2} X^i M_{ij} X^j\right) X^{\alpha\beta\cdots} = (2\pi)^{n/2}(\det M)^{-1/2}\,\#\{\text{contraction } \alpha\beta\cdots\}.$

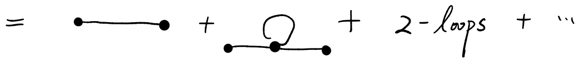
(a propagator connecting X^α, X^β carries a factor of $(M^{-1})_{\alpha\beta}$)

· $\langle X^\alpha X^\beta\rangle = \frac{1}{Z(M,C)} \int d^nX \cdot \exp\left(\tfrac{-1}{2} X^i M_{ij} X^j + C_{ijkl} X^{ijkl}\right) X^\alpha X^\beta$

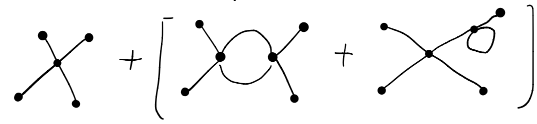
$$= \frac{1}{Z(M,C)} \int d^nX \cdot \exp\left(\tfrac{-1}{2} X^i M_{ij} X^j\right) X^{\alpha\beta} \sum_{r=0}^{\infty} (C_{ijkl}X^{ijkl})^r \cdot \frac{1}{r!}$$



$\langle X^\alpha X^\beta\rangle_{(1)}$ 2-loops + ⋯]



Similarly, $\langle X^{\alpha\beta\gamma\delta}\rangle =$



$= \langle X^{\alpha\beta\gamma\delta}\rangle_{(1)}$ + ⋯

$+\frac{1}{2}\binom{4}{2}\langle X^\alpha X^\beta\rangle^2$ (← disconnected diagrams)

· $\langle X^\alpha X^\beta\rangle_{(1)} = \sum_{P_2^4} M^{\alpha i} M^{\beta j} M^{kl} C_{ijkl}$

· $\langle X^{\alpha\beta\gamma\delta}\rangle_{(1)} = \sum_{P_2^4 \times P_2^4 \times 2} M^{\alpha i} M^{\beta j} M^{\gamma p} M^{\delta q} M^{kr} M^{ls} C_{ijkl} C_{pqrs}$

$$+ \sum_{P_3^4 \times P_2^4} M^{\alpha i} M^{\beta j} M^{\gamma k} M^{lp} M^{qr} M^{\delta s} C_{ijkl} C_{pqrs}$$

Week 9

· Wess-Zumino gauge :

$$V = \theta^{-\bar{-}}(v_0 - v_1) + \theta^{+\bar{+}}(v_0 + v_1) - \theta^{-\bar{+}}\sigma - \theta^{+\bar{-}}\bar{\sigma}$$
$$+ i\theta^{-+}(\theta^{\bar{-}}\bar{\lambda}_- + \theta^{\bar{+}}\bar{\lambda}_+) + i\theta^{\bar{+}\bar{-}}(\theta^{-}\lambda_- + \theta^{+}\lambda_+) + \theta^4 D.$$

pf : Choose suitable A s.t. $\sum \theta^\alpha f_\alpha + i(\bar{A} - A) =$ ______ .

Write $A = \phi - i\theta^{+\bar{+}}\partial_+\phi - i\theta^{-\bar{-}}\partial_-\phi - \theta^4\partial_{+-}\phi$

$$+ \theta^{\pm}\psi_{\pm} - i\theta^{+-\bar{-}}\partial_-\psi_+ - i\theta^{-+\bar{+}}\partial_+\psi_- + \theta^2 F.$$

$$\longrightarrow i(\bar{A} - A) = 2\,\mathrm{Im}A = 2\,\mathrm{Im}\phi - 2\theta^{+\bar{+}}\mathrm{Re}(\partial_+\phi) - 2\theta^{-\bar{-}}\mathrm{Re}(\partial_-\phi) - 2\theta^4\,\mathrm{Im}(\partial_{+-}\phi)$$
$$- i\theta^\beta\psi_\beta - \theta^{+-\bar{-}}\partial_-\psi_+ - \theta^{-+\bar{+}}\partial_+\psi_- + \theta^{\bar{+}-\bar{-}}\partial_-\bar{\psi}_+ + \theta^{\bar{-}+\bar{+}}\partial_+\bar{\psi}_-$$
$$- i\theta^{+-}F + i\theta^{\bar{-}\bar{+}}\bar{F}.$$

Take $\mathrm{Im}\phi = \frac{-1}{2}f_0$, $\psi_\pm = -if_\pm$ $\left(\bar{\psi}_\pm = i\bar{f}_\pm = -if_{(\bar{\pm})}\right)$

V real $\Rightarrow -\theta^{\bar{+}}\bar{f}_+ = \theta^{\bar{+}}f_{\bar{+}}$, ...

$$\longrightarrow \deg_\theta\left(\theta^\alpha f_\alpha + i(\bar{A} - A)\right) \geq 2.$$

$$\| $$

$$\sum_{|\alpha|\geq 2}\theta^\alpha f_\alpha - 2\theta^{+\bar{+}}\mathrm{Re}(\partial_+\phi) - 2\theta^{-\bar{-}}\mathrm{Re}(\partial_-\phi) - 2\theta^4\,\mathrm{Im}(\partial_{+-}\phi)$$
$$- \theta^{+-\bar{-}}\partial_-\psi_+ - \theta^{-+\bar{+}}\partial_+\psi_- + \theta^{\bar{+}-\bar{-}}\partial_-\bar{\psi}_+ + \theta^{\bar{-}+\bar{+}}\partial_+\bar{\psi}_- - i\theta^{+-}F + i\theta^{\bar{-}\bar{+}}\bar{F}$$

$$= \theta^{-\bar{-}}(v_0 - v_1) + \theta^{+\bar{+}}(v_0 + v_1) - \theta^{-\bar{+}}\sigma - \theta^{+\bar{-}}\bar{\sigma}$$
$$+ i\theta^{-+}(\theta^{\bar{-}}\bar{\lambda}_- + \theta^{\bar{+}}\bar{\lambda}_+) + i\theta^{\bar{+}\bar{-}}(\theta^{-}\lambda_- + \theta^{+}\lambda_+) + \theta^4 D.$$

(

$$v_0 = \frac{1}{2}\left(f_{+\bar{+}} + f_{-\bar{-}} - 2\mathrm{Re}(\partial_+\phi) - 2\mathrm{Re}(\partial_-\phi)\right)$$

$$v_1 = \frac{1}{2}\left(f_{+\bar{+}} - f_{-\bar{-}} - 2\mathrm{Re}(\partial_+\phi) + 2\mathrm{Re}(\partial_-\phi)\right)$$

$$\sigma = -f_{-\bar{+}} , \quad \bar{\sigma} = -\bar{f}_{-\bar{+}} = -f_{+\bar{-}}$$

$$\lambda_- = i(\partial_-\bar{\psi}_+ + f_{\bar{+}-\bar{-}}) , \quad \lambda_+ = -i(\partial_+\bar{\psi}_- + f_{\bar{-}+\bar{+}})$$

$$D = f_4 - 2\,\mathrm{Im}(\partial_{+-}\phi)$$

)

· $L = -\sum_{i=1}^{N} |D_\mu \phi_i|^2 - U(\phi)$, $U(\phi) = \frac{e^2}{2}\left(\sum_{i=1}^{N} |\phi_i|^2 - v\right)^2$

$(D_\mu \phi = (\partial_\mu + i v_\mu)\phi$, $D_\mu \bar\phi = (\partial_\mu - i v_\mu)\bar\phi$

Find v_μ so that L is inv. under $\phi \to e^{i\gamma(x)}\phi$.

sol: Note U is inv. under $\phi \to e^{i\gamma(x)}\phi$.

Therefore L is inv. under $\phi \to e^{i\gamma(x)}\phi$ only if $\delta \sum |D_\mu \phi_i|^2 = 0$

Since $\begin{cases} \delta\phi = i\gamma\phi \\ \delta\bar\phi = -i\gamma\bar\phi \end{cases}$, $0 = \delta \sum |D_\mu\phi_i|^2 = \sum (D_\mu\phi_i)[\partial_\mu \delta\bar\phi_i - i\delta v_\mu \bar\phi_i - i v_\mu \delta\bar\phi_i]$

$+ [\partial_\mu \delta\phi_i + i\delta v_\mu \phi_i + i v_\mu \delta\phi_i](D_\mu \bar\phi_i)$

implies
$$0 = \sum_i (D_\mu\phi_i)\left[-i(\partial_\mu\gamma + \delta v_\mu)\bar\phi_i - \cancel{i\gamma(\partial_\mu - i v_\mu)\bar\phi_i}\right]$$
$$+ \left[i(\partial_\mu\gamma + \delta v_\mu)\phi_i + \cancel{i\gamma(\partial_\mu + i v_\mu)\phi_i}\right](D_\mu\bar\phi_i)$$
$$= \sum\left[(D_\mu\bar\phi_i)\phi_i - \bar\phi_i(D_\mu\phi_i)\right]\cdot i(\partial_\mu\gamma + \delta v_\mu)$$

equation of motion

Solve $\sum (D_\mu\bar\phi_i)\phi_i - \bar\phi_i(D_\mu\phi_i) = 0$, we get $v_\mu = \frac{i}{2}\frac{\sum(\bar\phi_i\partial_\mu\phi_i - \partial_\mu\bar\phi_i\phi_i)}{\sum|\phi_i|^2}$.

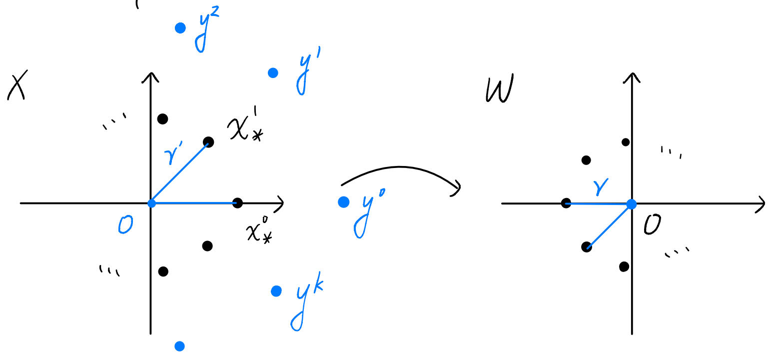
From $\phi \to e^{i\gamma(x)}\phi$, we have

$$v_\mu \longrightarrow \frac{i}{2\sum|\phi_i|^2}\left(\sum \bar\phi_i\partial_\mu\phi_i + i(\partial_\mu\gamma)|\phi_i|^2 - \partial_\mu\bar\phi_i\phi_i + i(\partial_\mu\gamma)|\phi_i|^2\right) = v_\mu - \partial_\mu\gamma$$

Week 10

· Consider $W(X) = \frac{1}{k+2} X^{k+2} - X$. Show that there is exactly one soliton connecting each pair of critical points.

pf: Critical pts : $x_*^n = e^{\frac{2\pi i n}{k+1}}$ for $n = 0, \dots, k$.



$W(x_*^n) = -\frac{k+1}{k+2} x_*^n$

Note $W^{-1}(0) = \{0, (k+2)^{\frac{1}{k+1}} x_*^n\}$

$\longrightarrow$ The pre-image of γ that connecting x_*^0 and x_*^1 is $\overset{\angle}{\gamma'}$

$\longrightarrow$ exactly 1 soliton.

(Suppose a path from x_*^i to y^j maps to $\overline{W(x_*^i)O}$ with $i \neq j$, then the path must contain a pt $= r e^{\frac{2\pi\sqrt{-1}m}{2(k+1)}}$ for some $m = 2i \pm 1$, $r \in \mathbb{R}$

However, $W(r e^{\frac{2\pi\sqrt{-1}m}{2(k+1)}}) = \left(\frac{r^{k+1}}{k+2} \underbrace{e^{\frac{2\pi\sqrt{-1}m}{2}}}_{= \pm 1} - 1\right) r e^{\frac{2\pi\sqrt{-1}m}{2(k+1)}} \notin \overline{W(x_*^i)O}$ $\rightarrow\!\!\!\times$)